Summing Up All Genus Free Energy of ABJM Matrix Model

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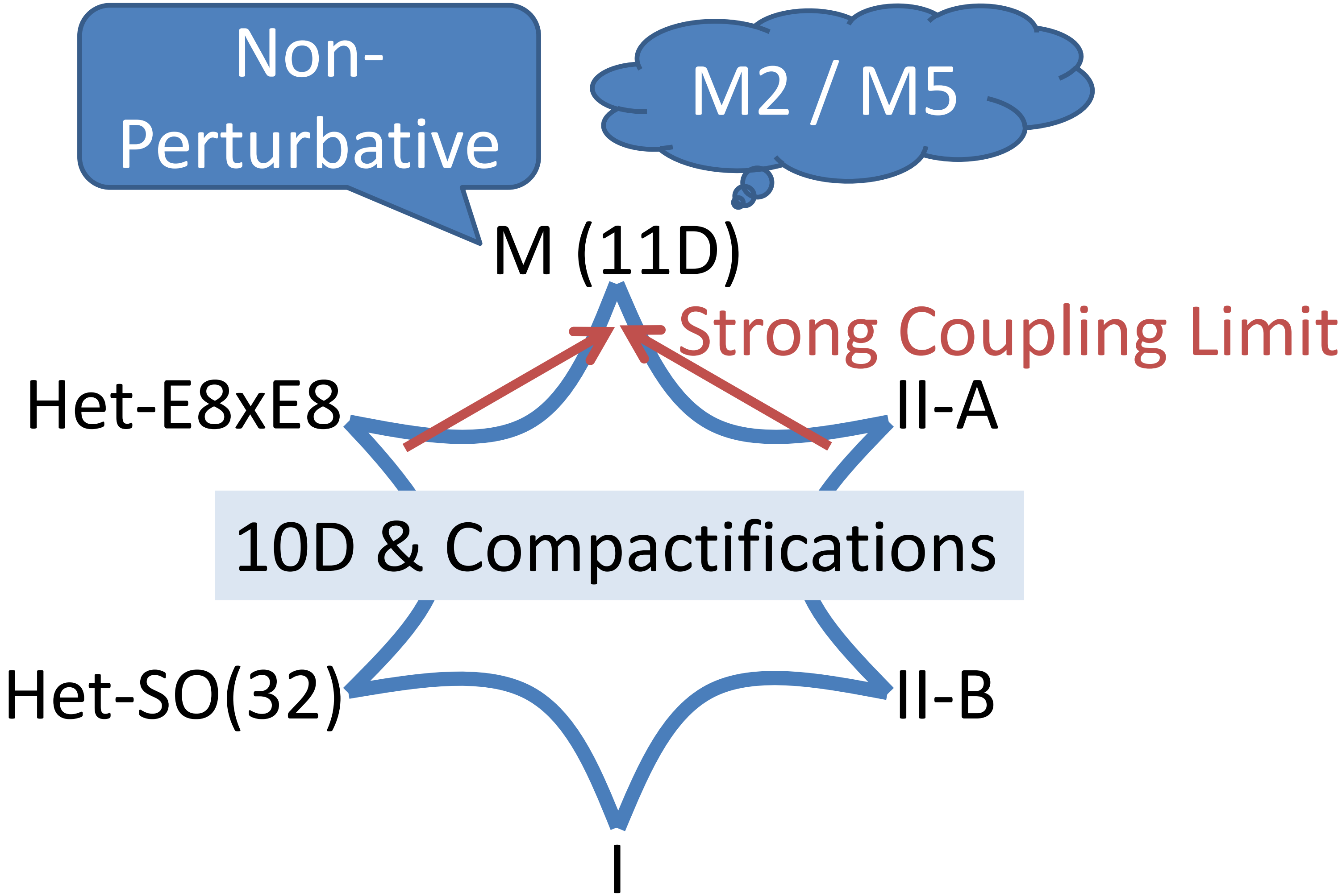
with H.Fuji and S.Hirano

M is for Mother



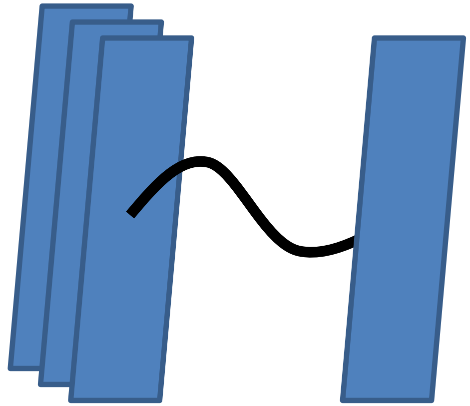
M is for Mystery

Little was known before ABJM

- 11D Supergrav as LowEnergyTheory
- Supergrav Sol for M2- & M5-branes
- Near Horizon Geometry
- Superconformal Symmetry osp(8|4)
- Action for Single M2-/M5-brane
- DOF $N^{3/2}$/N^3 for N M2-/M5-branes

cf. D-branes

DOF N^2 for D-branes



Described by $\begin{pmatrix} \text{Matrix} \end{pmatrix}$

M is for Matrix

A breakthrough by ABJM

- Non-Abelian M2-brane theory by ABJM
  [Aharony-Bergman-Jafferis-Maldacena]
- Partition Function Localized to CS Matrix
  [Kapustin-Willett-Yaakov, Hama-Hosomichi-Lee]
- Planar $N^{3/2}$ Behavior Reproduced
  [Drukker-Marino-Putrov]
- Today: All Genus Sum

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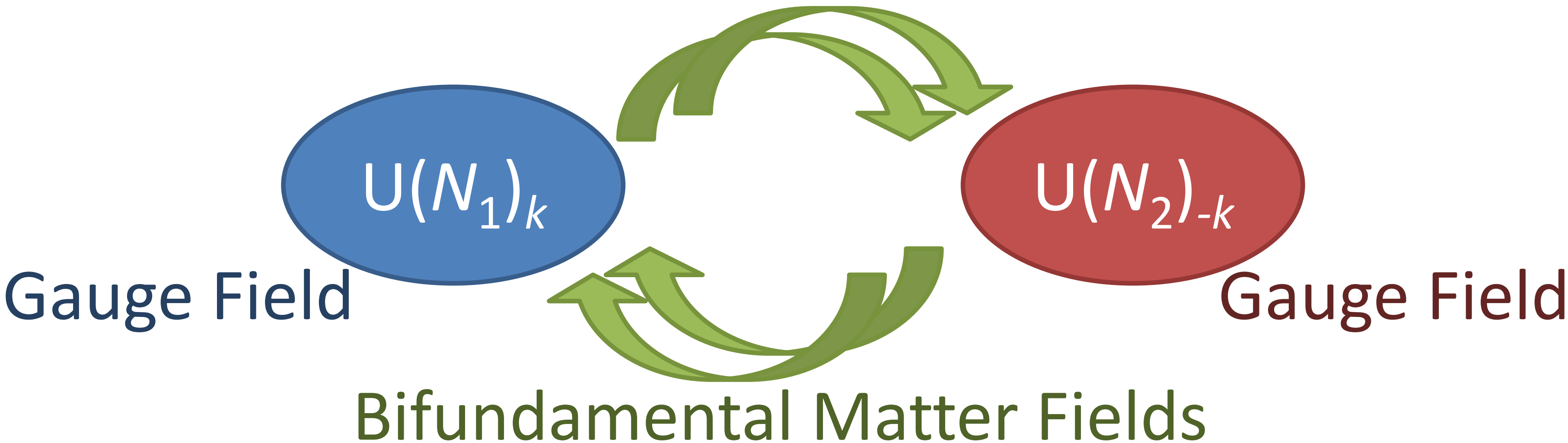
ABJ(M) theory



$N_1 = N_2$: No Fractional M2-branes (ABJM Slice)

Coincident M2-branes on S^7 / Z_k

Computation of Partition Function?

Localization

$$Z = \int D\Phi e^{-S[\Phi]}$$

- Grassmann-odd Symmetry $\delta : \ \delta S = 0$
- Grassmann-odd Quantity $V \ : \ \delta^2 V = 0 \quad \delta V \geq 0$

$$Z(t) = \int D\Phi e^{-S[\Phi]-t\delta V}$$

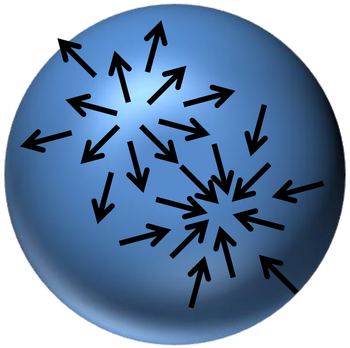
$$\frac{d}{dt} Z(t) = -\int D\Phi \delta\left(V e^{-S[\Phi]-t\delta V}\right) = 0$$

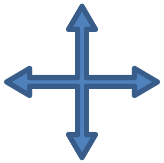
$$Z(0) = Z(\infty)$$

- Integration Localized to $\delta V = 0$

$$Z(\infty) = \int d\sigma \frac{\det \Delta_F(\sigma)}{\det \Delta_B(\sigma)} e^{-S_{\text{cl}}(\sigma)}$$

Example



$$2 - 2g = \int_{\mathrm{Mfd}} e^{-t\delta V} = \#$$ 

(Poincare-Hopf Theorem)

Application to ABJM

- Gauge Field

$$\left.\frac{\det \Delta_F(\sigma)}{\det \Delta_B(\sigma)}\right|_{\text{gauge}} = \frac{\det(-\Delta + \alpha(\sigma)^2)}{\text{Pf}\ (i\gamma^\mu \nabla_\mu + i\alpha(\sigma) - 1/2)}$$

$$= \cdots \sim \prod_{\alpha>0} \prod_{n=1}^{\infty} \left(1 + \frac{\alpha(\sigma)^2}{n^2}\right)^2 = \prod_{\alpha>0} \left(\frac{\sinh \pi\alpha(\sigma)}{\pi\alpha(\sigma)}\right)^2$$

- Matter Field

$$\ldots = \cosh^{-2}$$

Finally, Localized to Matrix Model

$$Z = \frac{1}{N_1!N_2!}\int \prod_{i=1}^{N_1}\frac{d\mu_i}{2\pi}\prod_{k=1}^{N_2}\frac{d\nu_k}{2\pi}\, e^{-(\sum \mu_i^2 - \sum \nu_k^2)/2g_s}$$

$$\times \frac{\prod_{i<j}\left(2\sinh\frac{\mu_i-\mu_j}{2}\right)^2 \prod_{k<l}\left(2\sinh\frac{\nu_k-\nu_l}{2}\right)^2}{\prod_{i,k}\left(2\cosh\frac{\mu_i-\nu_k}{2}\right)^2}$$

$$g_s = 2\pi i/k$$

If, instead

$$Z = \frac{1}{N_1!N_2!}\int \prod_{i=1}^{N_1}\frac{d\mu_i}{2\pi}\prod_{k=1}^{N_2}\frac{d\nu_k}{2\pi}\, e^{-(\sum \mu_i^2+\sum \nu_k^2)/2g_s}$$

$$\times \prod_{i<j}\left(2\sinh\frac{\mu_i-\mu_j}{2}\right)^2 \prod_{k<l}\left(2\sinh\frac{\nu_k-\nu_l}{2}\right)^2$$

$$\times \prod_{i,k}\left(2\cosh\frac{\mu_i-\nu_k}{2}\right)^2$$

Lens Space Matrix Model

If, further simplified

$$Z = \frac{1}{N!} \int \prod_{i=1}^{N} \frac{d\mu_i}{2\pi} \, e^{-\sum \mu_i^2 / 2g_s}$$

$$\times \prod_{i<j} \left(2 \sinh \frac{\mu_i - \mu_j}{2} \right)^2$$

Chern-Simons Matrix Model

The simplest one

$$Z = \frac{1}{N!} \int \prod_{i=1}^{N} \frac{d\mu_i}{2\pi} \, e^{-\sum \mu_i^2 / 2g_s} \times \prod_{i<j} (\mu_i - \mu_j)^2$$

Gaussian Matrix Model

Matrix Model

- Eigenvalue Density Wanted

Eigenvalues

$$\rho(z) = \frac{1}{N}\sum_{i=1}^{N}\left\langle \delta(z - z_i) \right\rangle$$

- Definition - Resolvent -

$$\omega(z) = \frac{1}{N}\sum_{i=1}^{N}\left\langle \frac{1}{z - z_i} \right\rangle$$

- Planar Limit $\omega(z) \to \omega_0(z) \quad F \to F_0$

Properties of Resolvent

1. $z \to \infty$ Behavior $\qquad \omega(z) \sim \frac{1}{z}$

2. Dispersion Relation $\qquad \frac{1}{z+i\epsilon} = \frac{1}{z} - i\pi\delta(z)$

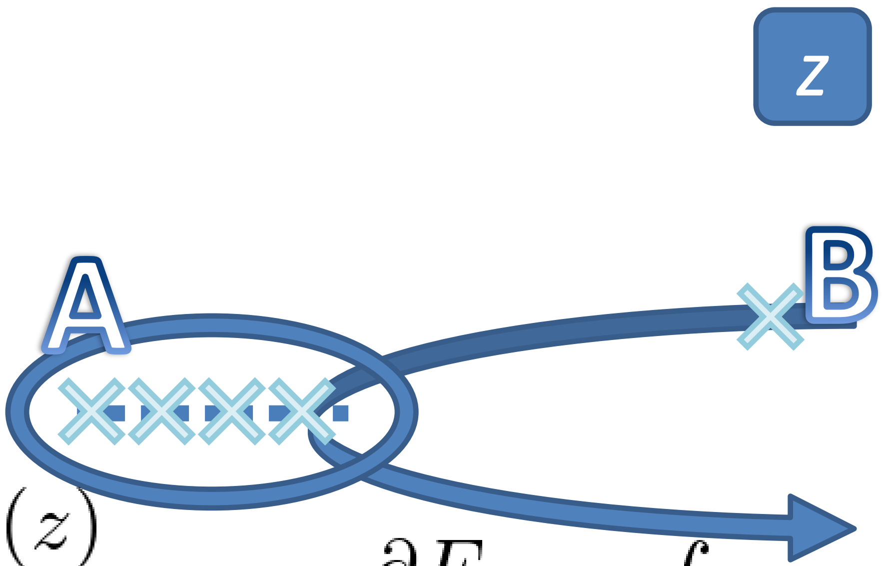
$$\rho(z) = -\frac{1}{2\pi i}\left(\omega_0(z+i\epsilon) - \omega_0(z-i\epsilon)\right)$$

3. EOM ・・・ Discontinuity Eq

$$\text{Force} = \frac{1}{2}\left(\omega_0(z+i\epsilon) + \omega_0(z-i\epsilon)\right)$$

Integration

Integration Contour



$$\lambda \sim \oint_A dz\,\omega_0(z)$$

$$\frac{\partial F_0}{\partial \lambda} \sim \oint_B dz\,\omega_0(z)$$

Resolvent $\omega_0(z)$ ⇨ Partition Func $F_0(\lambda)$

Chern-Simons Matrix Model

- Resolvent

$$\omega(z) = g_s \left\langle \sum_{i=1}^{N} \coth \frac{z - z_i}{2} \right\rangle$$

- Asymptotic Behavior

$$\omega_0(z) \to \pm 2\pi i \lambda \qquad \text{as} \qquad z \to \pm\infty$$

- Discontinuity Eq from EOM

$$z = \frac{1}{2}\big(\omega_0(z + i\epsilon) + \omega_0(z - i\epsilon)\big)$$

Chern-Simons Matrix Model

- A Regular Function [Halmagyi-Yasnov]

$$g(Z) = e^{\omega_0/2} + Ze^{-\omega_0/2} \qquad Z = e^z$$

- Asymptotic Behavior

$$\lim_{Z\to\infty} g(Z) = Ze^{-\pi i\lambda} \qquad \lim_{Z\to 0} g(Z) = e^{-\pi i\lambda}$$

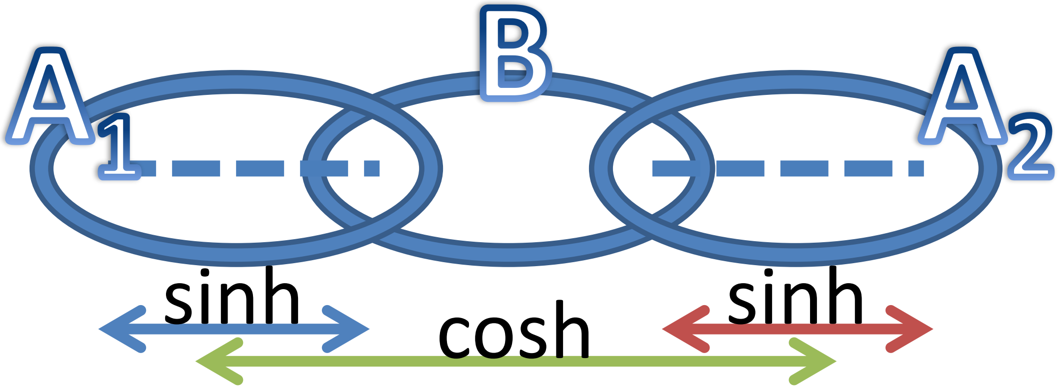
- Determined!

$$g(Z) = (Z+1)e^{-\pi i\lambda}$$

Lens Space Matrix Model

- Same Tech as Chern-Simons Matrix Model
- Two Cuts Instead

$$g_{\mathrm{LS}}(Z) = Z^2 - \zeta Z + 1$$



ABJM Matrix Model

- Analytic Cont from Lens Space Matrix Model

$$\lambda_1 = N_1/k \quad \lambda_2 = -N_2/k$$

- Result (for ABJM Slice)

$$\lambda = \frac{\kappa}{8\pi} {}_3F_2\Big(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; -\frac{\kappa^2}{16}\Big)$$

$$= \frac{\log^2 \kappa}{2\pi^2} + \frac{1}{24} + \mathcal{O}(1/\kappa^2)$$

$$\partial_\lambda F_0 = 2\pi^2 \log \kappa + \frac{4\pi^2}{\kappa^2} {}_4F_3\Big(1, 1, \frac{3}{2}, \frac{3}{2}; 2, 2; -\frac{16}{\kappa^2}\Big)$$

$$= 2\pi^2 \log \kappa + \mathcal{O}(1/\kappa^2)$$

Result

- Result

$$Z_0 = g_s^{-2} F_0 = \frac{\sqrt{2}\pi}{3} k^2 \left(\lambda - \frac{1}{24}\right)^{3/2} + \mathcal{O}(e^{-\sqrt{\lambda - 1/24}})$$

- Neglecting Worldsheet Instanton

$$Z_0 = \frac{\sqrt{2}\pi}{3} k^{1/2} \left(N - \frac{1}{24} k\right)^{3/2}$$

- $N^{3/2}$ Reproduced and More

Interpretation

- Charge Shift [Aharony-Hashimoto-Hirano-Ouyang]

$$N \to N - \frac{1}{24}\left(k - \frac{1}{k}\right)$$

from Euler Coupling

$$\frac{1}{24}\int C_3 \wedge I_8 = \frac{1}{24}\chi(S^7/Z_k) = \frac{1}{24}\left(k - \frac{1}{k}\right)$$

- Match in Planar Case

Furthermore, Non-Planar Prediction

- Renormalization of 't Hooft coupling

$$\lambda_{\rm ren}^{-1} = \frac{(\lambda - 1/24)^{-1}}{1 + (1/24)k^{-2}(\lambda - 1/24)^{-1}}$$

- Or in terms of $x = 1/\log \kappa$

$$g_s y = \frac{g_s x}{\sqrt{1 + g_s^2 x^2 / 48}}$$

Contents

Higher Genus Behavior from Duality

- Chern-Simons Theory on Lens Space S^3/Z_2
  (String Completion)
- Open Top A-model on $T^*(S^3/Z_2)$
  (Large N Duality)
- Closed Top A on Hirzebruch Surface $F_0 = P^1 \times P^1$
  (Mirror Symmetry)
- Closed Top B on Spectral Curve $u\,v = H(x,y)$

Holomorphic Anomaly Eq

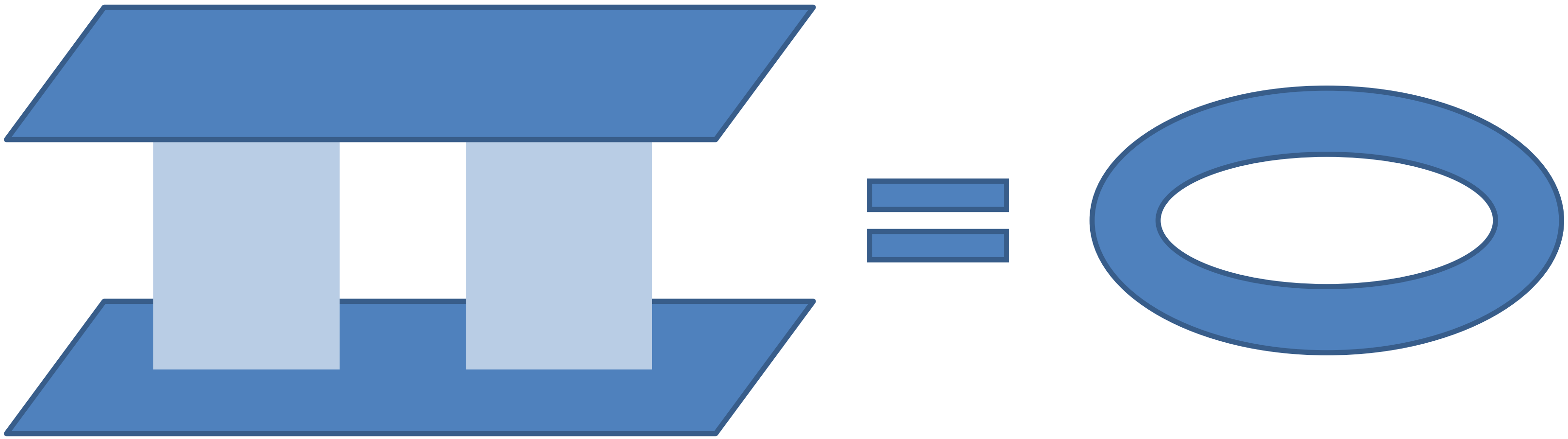
- General Case [Bershadsky-Cecotti-Ooguri-Vafa]

Determined by F_0

$$\partial_{\bar{I}} F_g = \frac{1}{2} C_{\bar{I}\bar{J}\bar{K}} G^{J\bar{J}} G^{K\bar{K}} \times \left[D_J D_K F_{g-1} + \sum_{r=1}^{g-1} D_J F_r D_K F_{g-r} \right]$$

Application to ABJM

- One Modulus λ = Ordinary Differential Eq
- 2 Cuts = Torus



Derivative

- 2nd Derivative

Elliptic F

Torus Modulus

$$\frac{i}{4\pi^3}\partial_\lambda^2 F_0(\lambda) + 1 = i\frac{K'(i\kappa/4)}{K(i\kappa/4)} = \tau$$

- 3rd Derivative (Yukawa Coupling)

$$\frac{1}{4}C_{\lambda\lambda\lambda} = \partial_\lambda^3 F_0(\lambda) = (2\pi i)^3 \frac{2}{\vartheta_2(\tau)^2\vartheta_4(\tau)^4} = (2\pi i)^3 \xi$$

Ansatz & Reduction

- Ansatz for Partition Function

$$F_g(\tau) = \xi^{2g-2} f_g(\tau) \qquad f_g(\tau) = f_g[E_2, E_4, E_6]$$

- Eisenstein Series

$$E_2(\tau,\bar\tau) = E_2(\tau) - \frac{3}{\mathrm{Im}\,\tau} \quad E_4 = E_4(\tau) \quad E_6 = E_6(\tau)$$

- Reduced HAE [Drukker-Marino-Putrov]

$$\frac{df_g}{dE_2} = -\frac{1}{3}\left[d_\xi^2 f_{g-1} + \frac{1}{3}\frac{\partial_\tau \xi}{\xi} d_\xi f_{g-1} + \sum_{r=1}^{g-1} d_\xi f_r d_\xi f_{g-r} \right]$$

- Covariant Derivative $\quad d_\xi = \partial_\tau + (k/3)\partial_\tau \xi/\xi$

Higher Genus Behavior

$$F_0 = \frac{2}{3x^3}$$

$$F_1 = \frac{\log x'}{2} + \frac{1}{6x} \qquad x' = \frac{\pi}{2}x$$

$$F_2 = \frac{5x^3}{48} - \frac{x^2}{24} + \frac{x}{144}$$

$$F_3 = \frac{5x^6}{64} - \frac{5x^5}{192} + \frac{x^4}{288} - \frac{x^3}{5184}$$

$$F_4 = \frac{1105x^9}{9216} - \frac{5x^8}{128} + \frac{25x^7}{4608} - \frac{x^6}{2592} + \frac{x^5}{82944}$$

$$\cdots$$

Resum 1

$$F'_0 = \frac{2}{3y^3} = \frac{2}{3x^3} + g_s^2\frac{1}{6x} + g_s^4\frac{x}{144} - g_s^6\frac{x^3}{5184} + \cdots$$

$$F'_1 = \frac{\log y'}{2} = \frac{\log x'}{2} - g_s^2\frac{x^2}{24} + g_s^4\frac{x^4}{288} - g_s^6\frac{x^6}{2592} + \cdots$$

$$F'_2 = \frac{5y^3}{48} = \frac{5x^3}{48} - g_s^2\frac{5x^5}{192} + g_s^4\frac{25x^7}{4608} + \cdots$$

$$F'_3 = \frac{5y^6}{64} = \frac{5x^6}{64} - g_s^2\frac{5x^8}{128} + \cdots$$

$$F'_4 = \frac{1105y^9}{9216} = \frac{1105x^9}{9216} + \cdots$$

$\cdots$

$$g_s y = \frac{g_s x}{\sqrt{1+(g_s x)^2/6}}$$

Discrepancy Discussed Later

Resum 2

- The Top Term

$$F_g \sim \frac{A_g}{3g-3} x^{3g-3} + \cdots$$

- Recursion Relation (cf. SW theory [Huang-Klemm])

$$A_g \sim (3g-3)A_{g-1} + \sum_{r=2}^{g-2} A_r A_{g-r}$$

- Translated into Differential Eq
- Solved by Modified Bessel Functions

Airy Function Result

- Result [Fuji-Hirano-M]

$$Z = g_s^{-2} F_0' + F_1' + g_s^2 F_2' + \cdots$$
$$= \mathrm{Ai}\left(\left[\frac{\pi}{\sqrt{2}} \frac{1}{k^2} \right]^{2/3} \lambda_{\mathrm{ren}} \right)$$

- Airy Function

$$\mathrm{Ai}\left(z\right) = \frac{1}{2\pi i} \int_C \exp\left(-zt + \frac{1}{3} t^3 \right)$$

Contents

Discrepancy?

- Lots of Conventional Factors?

  ex: Yukawa Coupling / 4
- SU(N) x SU(N) instead of U(N) x U(N)?

  Another Superconformal Theory by ABJM

  SU(N) instead of U(N) for D3-branes

Airy Function

- Modified Bessel Functions Everywhere as for Loop Operator VEV [Drukker-Gross]
- Cubic Integral Reminiscent of Chern-Simons
- Airy Function in Other Setups of M theory "Wave Function of the Universe" [Ooguri-Vafa-Verlinde]
- Corrections to Quantum Gravity? Max SUSY ・・・ No Corrections?

Future Directions [work in progress]

- Relation to Superalgebra U($N|N$)
- Airy Function Beyond ABJM Slice
- Other Analysis (Surgery?) for WS Instanton