Affine SU(N) algebra from wall-crossings

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arXiv: 1107.4762 : T. N. and Satoshi Yamaguchi

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Counting BPS D-branes on CY3

- Microstates of black holes in R^4
- Instanton counting on D-branes
- Topological string
- Quiver gauge theory, Brane tilings
- Wall-crossing phenomena

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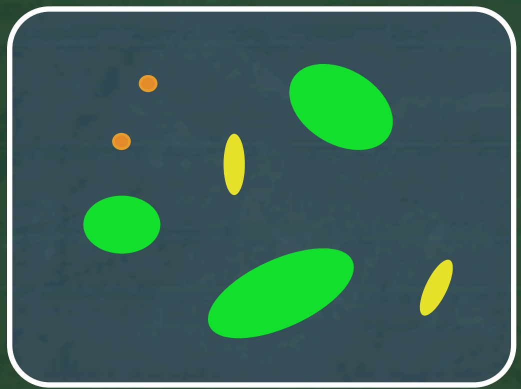
D4-D2-D0 on CY3

D4-D2-D0 bound states on CY3

6-dim

(Calabi-Yau)



D2/D0 branes on a single D4

4-dim

(R^4)

• BPS particle

$\mathcal{N} = 2$ supersymmetry

D-brane bound states?

The degeneracy of BPS D-branes depends on the moduli parameters

||

sizes and B-fields of compact cycles

Wall-crossing phenomena

D4-D2-D0 on CY3

Large radii limit

D0-brane ⟶ instanton on D4

D2-brane ⟶ magnetic flux on D4

Chern-Simons interaction on D4

D0-charge = # instantons

$$\int_{\mathrm{D4}} C^{(1)} \wedge F \wedge F \longrightarrow Q_0 = \frac{1}{8\pi^2} \int F \wedge F,$$

D2-charge = # magnetic flux

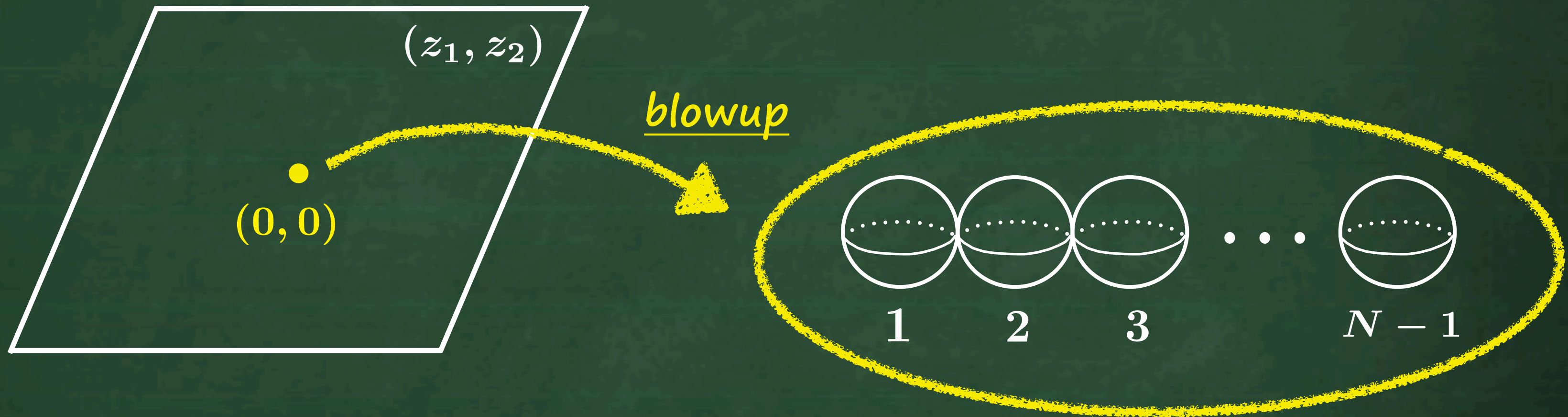
$$\int_{\mathrm{D4}} C^{(3)} \wedge F \longrightarrow Q_2 = \frac{1}{2\pi} \int F \wedge k,$$

k : unit volume form on 2-cycle

ALE instantons

If a D4-brane is wrapped on ALE space...

A_{N-1} -ALE space: minimal resolution of $\mathbb{C}^2/\mathbb{Z}_N$



$$(z_1, z_2) \longrightarrow (\omega z_1, \overline{\omega} z_2)$$

where $\omega = e^{2\pi i/N}$

The instanton partition function on a D4-brane wrapping A_{N-1} ALE space

=

character of affine SU(N) algebra

[Nakajima]
[Vafa-Witten]

ALE instantons

q : Boltzmann weight of D0

Q : Boltzmann weight of D2

Affine SU(N) character

$$\chi_r^{\widehat{su}(N)_1} = \frac{1}{\eta(q)^{N-1}} \sum_{n_1,\cdots,n_{N-1}\in\mathbb{Z}} q^{\sum_{i=1}^{N-1} n_i^2 - \sum_{i=1}^{N-2} n_i n_{i+1} + n_1 r + \frac{r^2}{2}\frac{N-1}{N}} \prod_{j=1}^{N-1} {Q_j}^{n_j + \frac{N-i}{N} r}$$

ex.) affine SU(2)

$$\chi_r^{\widehat{su}(2)_1} = \frac{1}{\eta(q)} \sum_{n=-\infty}^{\infty} q^{n^2+nr+\frac{r^2}{4}} Q^{n+\frac{r}{2}}$$

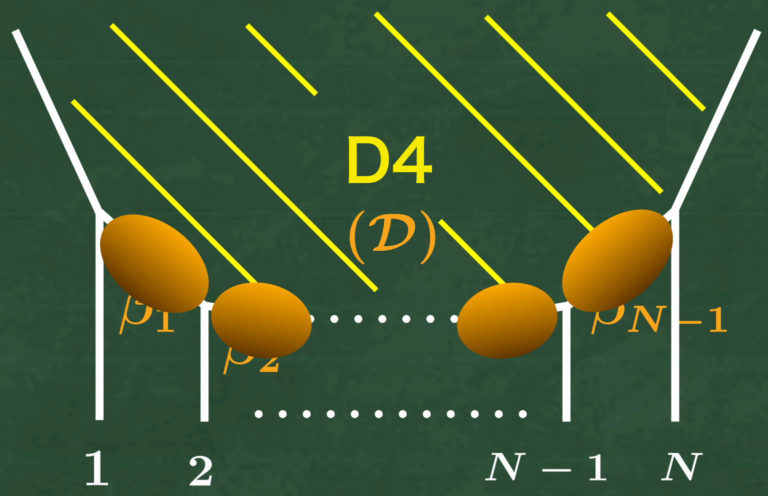
What in this talk?

Relation between ALE instantons
and D4-D2-D0 wall-crossings

ALE $\times \mathbb{C}$

A_{N-1} -ALE $\times \mathbb{C}$

$\mathcal{D} \simeq A_{N-1}$-ALE space



A single D4-brane wrapped on $\mathcal{D}$

D2-branes wrapped on
$\beta_1 \sim \beta_{N-1}$: $N-1$ blowup two-cycles

D0-branes localized in CY3

moduli

II

B-fields and radii of
$\beta_1 \sim \beta_{N-1}$

Outline of the rest of this talk

1. Wall-crossings in d=4, N=2 SUSY theory

2. Apply it to our case.

3. Summary

Wall-crossings in d=4, N=2 SUSY theory

Wall-crossing of BPS states

BPS index

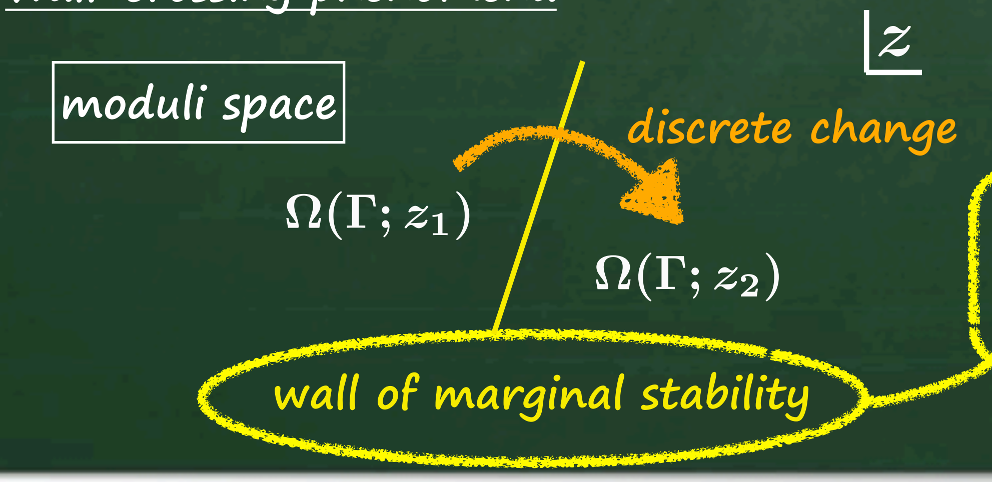
The trace is taken over all one-particle states with charge Γ .

$$\Omega(\Gamma; z) := -\frac{1}{2}\mathrm{Tr}_{\Gamma}[(-1)^{2J}(2J)^2] \quad (z : \text{vacuum moduli parameters})$$

$$= \#\ \text{“bosonic” BPS multiplets} \;-\; \#\ \text{“fermionic” BPS multiplets}$$

This index is piecewise constant in the moduli space of vacua, but not globally constant.

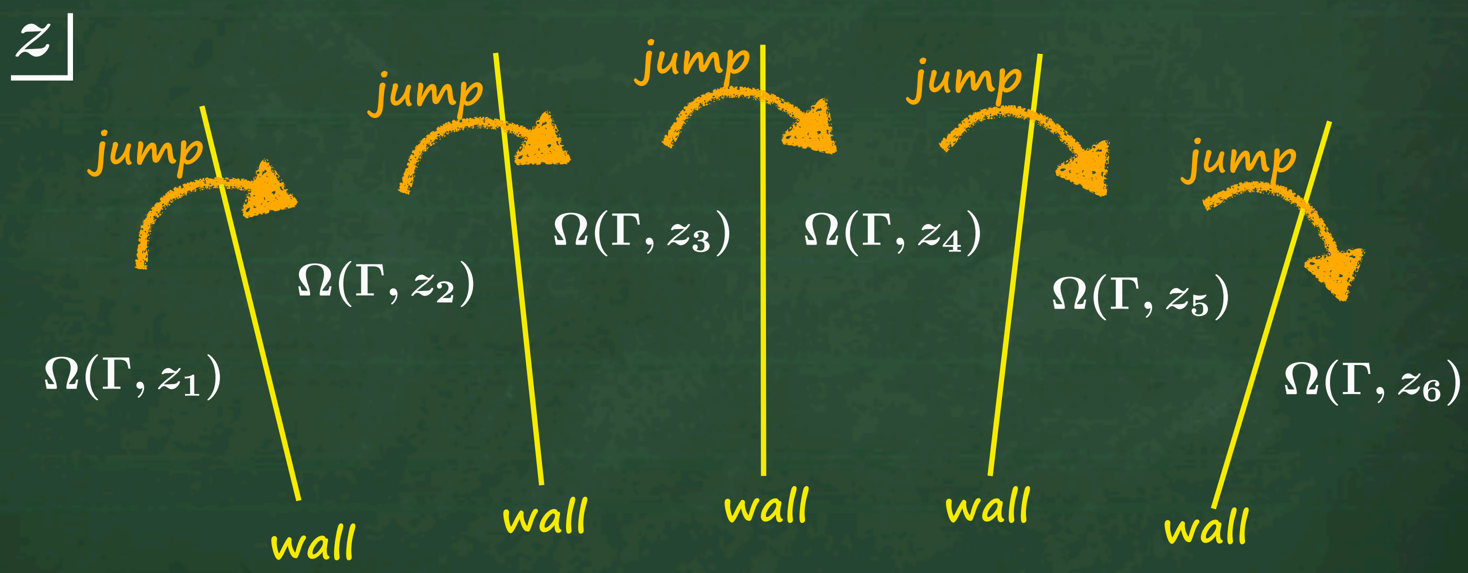
Wall-crossing phenomena



Some decay $\overset{\text{BPS}}{\Gamma} \longrightarrow \overset{\text{BPS}}{\Gamma_1} + \overset{\text{BPS}}{\Gamma_2}$ occurs.

Wall-crossing of BPS states

So the moduli space is divided into chambers by the walls of marginal stability.



The BPS index is exactly constant in each chamber.

Wall-crossing of BPS states

Γ : primitive

There is no positive integer that can divide out Γ.
(except for one)

Semi-primitive wall-crossing formula

Suppose the wall-crossing is associated to a BPS decay channel

$$\Gamma \to \Gamma_1 + n\Gamma_2, \qquad n \in \mathbb{N}$$

where Γ_1, Γ_2 are primitive. Then the BPS partition function behaves as

$$\mathcal{Z} \to \mathcal{Z} \times \prod_{j=1}^{\infty} \left(1 + (-1)^{j\langle\Gamma_2,\Gamma\rangle} e^{j\Gamma_2}\right)^{\pm j\langle\Gamma_2,\Gamma\rangle\Omega(j\Gamma_2)}$$

[Denef-Moore]
[Kontsevich-Soibelman]

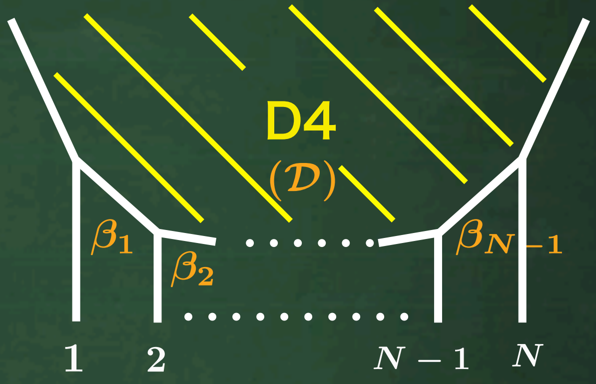
through the wall-crossing. Here we defined

$$\mathcal{Z} = \sum_{\Gamma} \Omega(\Gamma; z) e^{\Gamma}, \qquad e^{\Gamma_1} e^{\Gamma_2} = e^{\Gamma_1 + \Gamma_2}$$

partition function Boltzmann weight

Apply it to our case!

ALE × $\mathbb{C}$



Wall-crossing?

For $\Gamma = \mathrm{D4} + k^i \mathrm{D2}_{(i)} + l \mathrm{D0}$, possible decays are

$$\Gamma \to \Gamma_1 + \Gamma_2$$

$$\Gamma_2 = m^i \mathrm{D2}_{(i)} + n\mathrm{D0} \quad (m^i = \pm 1)$$

$$(\Gamma_1 = \Gamma - \Gamma_2)$$

⟶ $\langle \Gamma_2, \Gamma \rangle = 0$

There is no wall-crossing!!

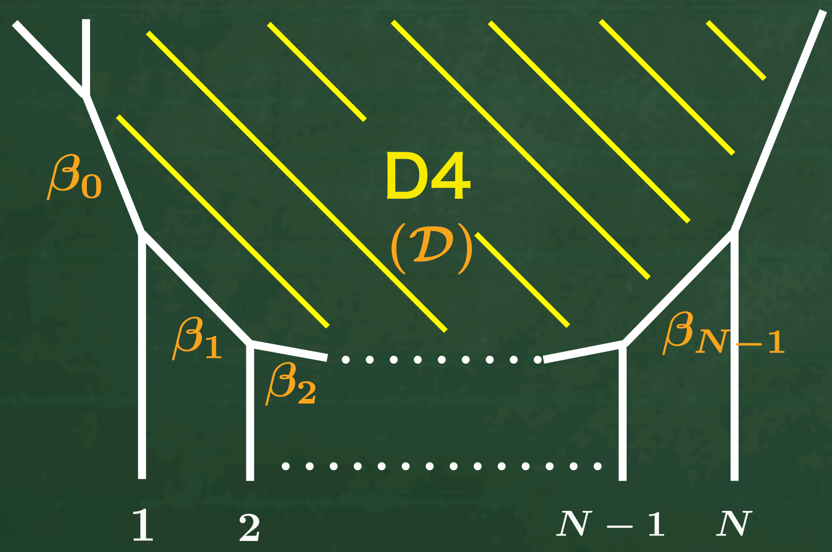
cf) wall-crossing formula

$$\mathcal{Z} \to \mathcal{Z} \times \prod_{j=1}^{\infty} \left(1 + (-1)^{j\langle \Gamma_2, \Gamma\rangle} e^{j\Gamma_2}\right)^{\pm j \langle \Gamma_2, \Gamma \rangle \Omega(j\Gamma_2)}$$

"Add" dummy cycle

Dummy cycle and flops

We "add" a dummy two-cycle.



In the large radius limit of β_0, we should recover the original CY3.

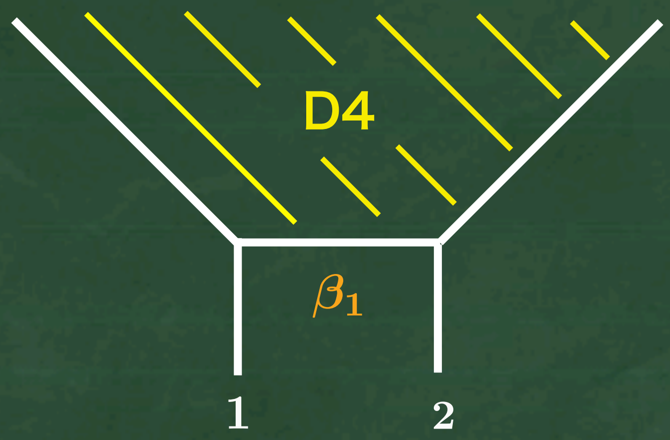
Now, we have a non-vanishing intersection product:

$$\langle \beta_0, \mathcal{D} \rangle = 1$$

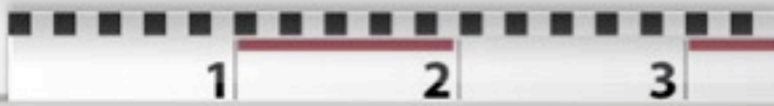
→ Non-trivial wall-crossings!

“Add” dummy cycle

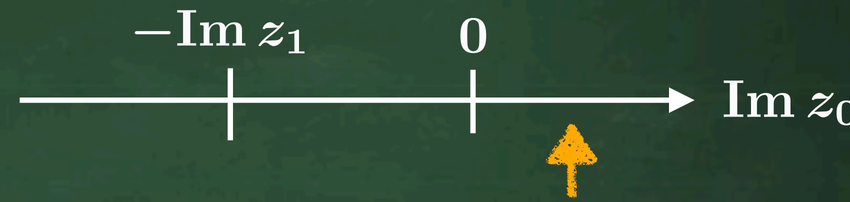
ex.) A_1-ALE $\times \mathbb{C}$



$D4$ is wrapped on A_1-ALE space

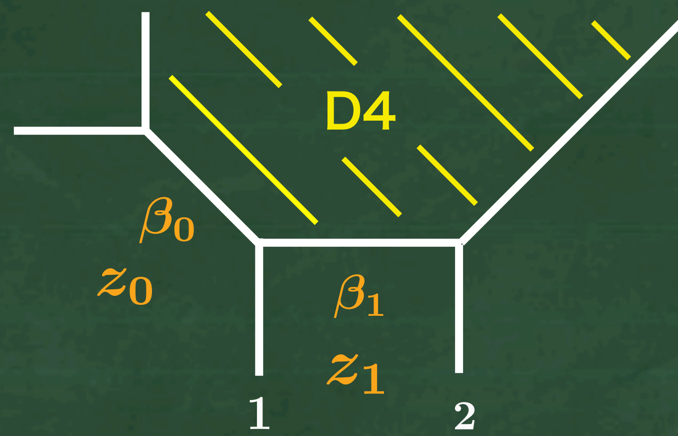


"Add" dummy cycle



$(0 < \mathrm{Im}\, z_0)$

ex.) A_1-ALE $\times\ \mathbb{C}$

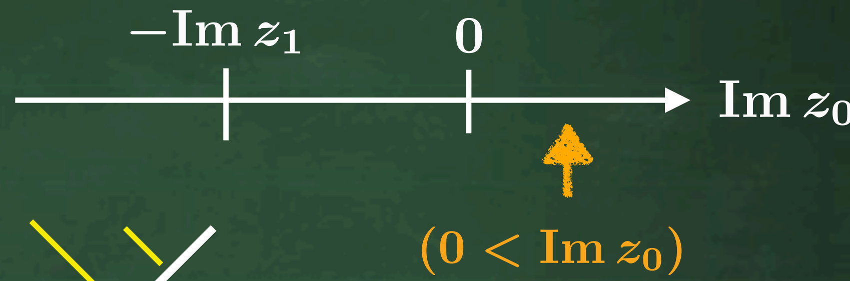


moduli parameters

$z_0,\ z_1 \in \ \mathbb{C}$

$\mathrm{Im}\, z_i \sim$ *area of i-th cycle*

"Add" dummy cycle



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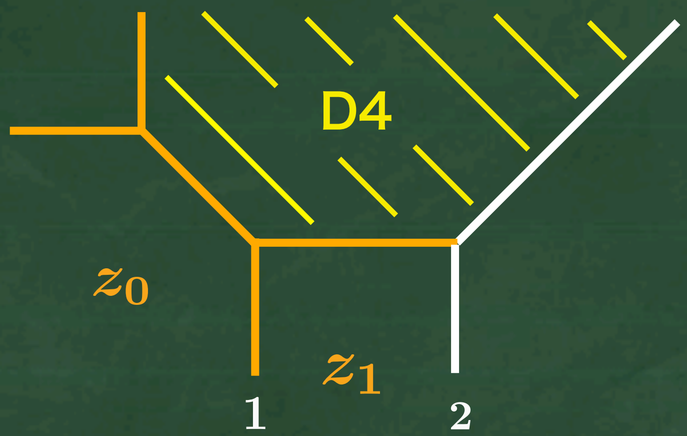

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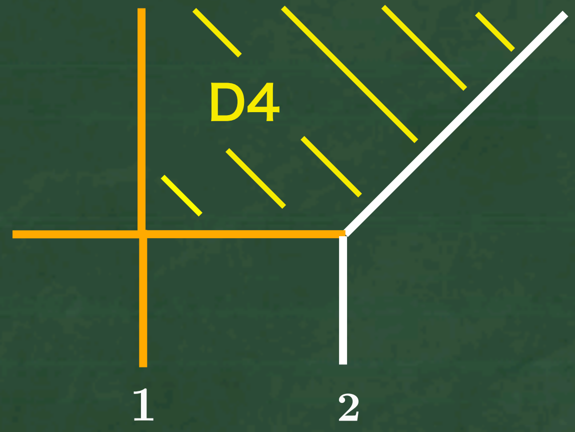

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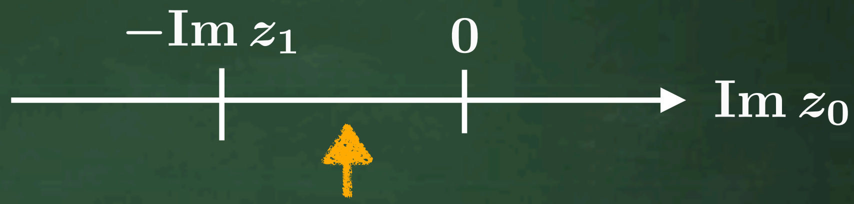
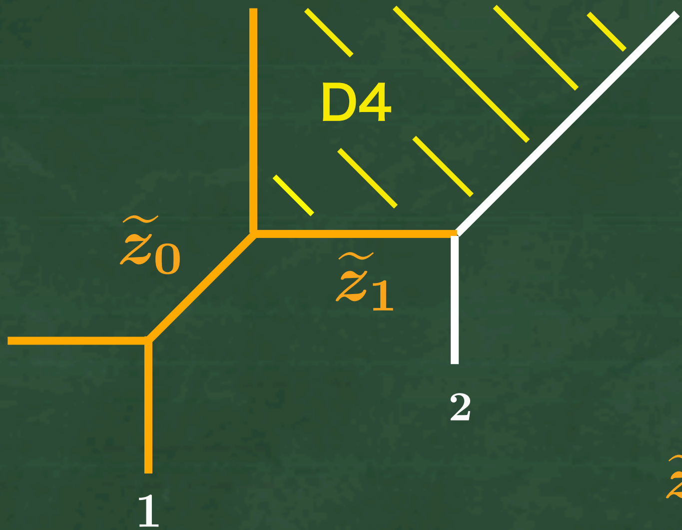


ex.) A_1-ALE $\times \mathbb{C}$

$D4$ is wrapped

on $\mathcal{O}(-1) \to \mathbb{P}^1$



moduli parameters

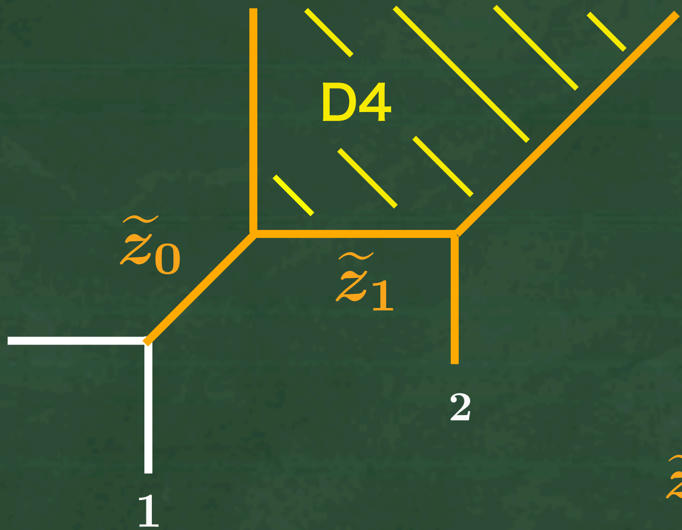
$z_0, z_1 \in \mathbb{C}$

$\mathrm{Im}\, \widetilde{z}_i \sim$ area of i-th cycle

$\widetilde{z}_0 = -z_0, \; \widetilde{z}_1 = z_0 + z_1$

"Add" dummy cycle

ex.) A_1-ALE $\times \mathbb{C}$



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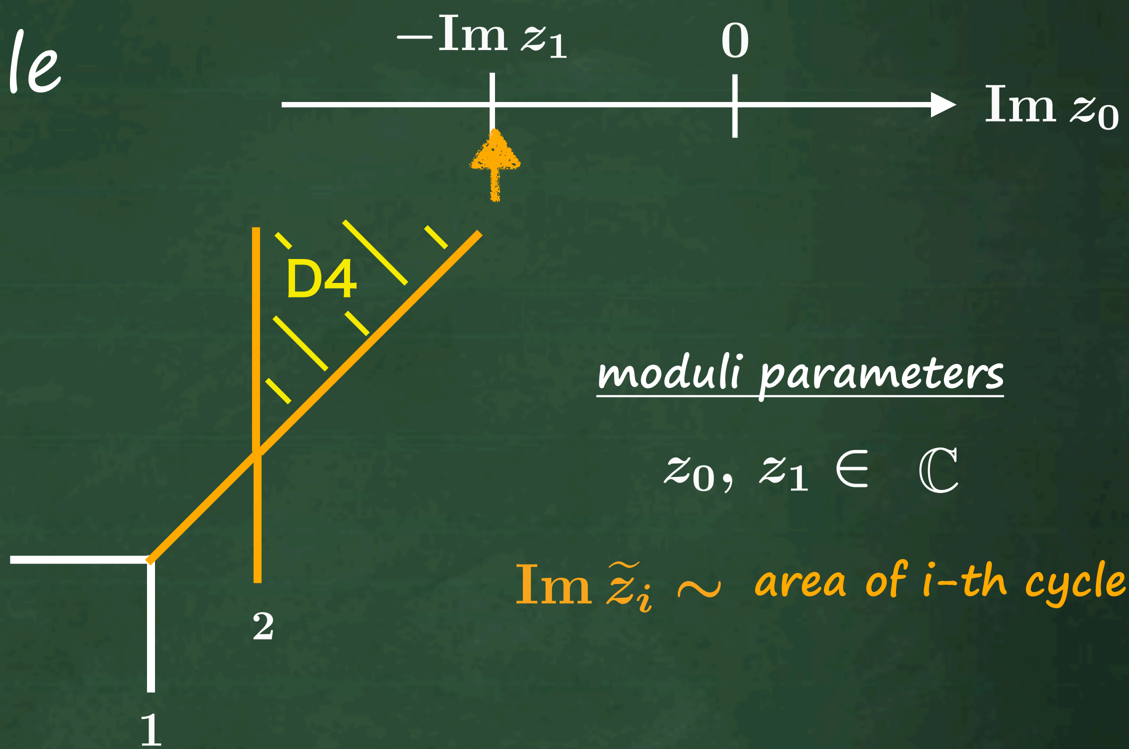


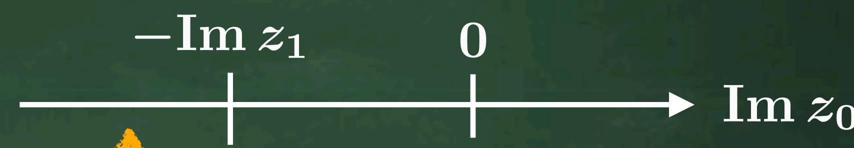
moduli parameters

$z_0,\ z_1 \in \mathbb{C}$

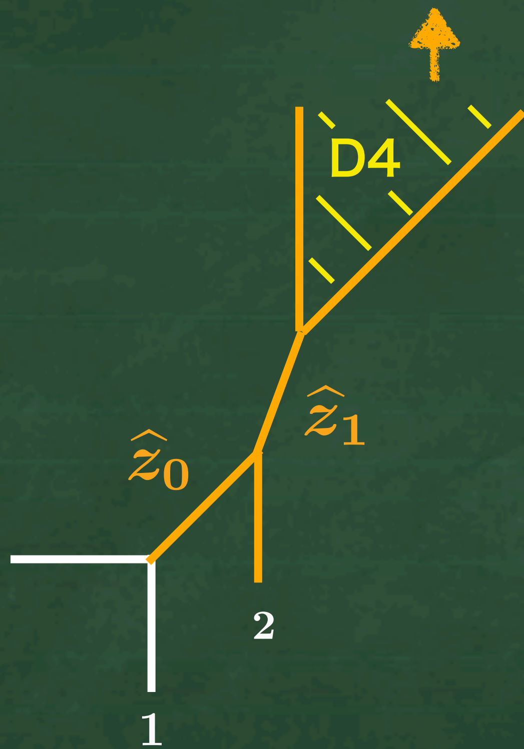
$\mathrm{Im}\, \tilde{z}_i \sim$ area of i-th cycle

"Add" dummy cycle

ex.) A_1-ALE $\times \mathbb{C}$



$D4$ is wrapped on $\mathbb{C}^2$

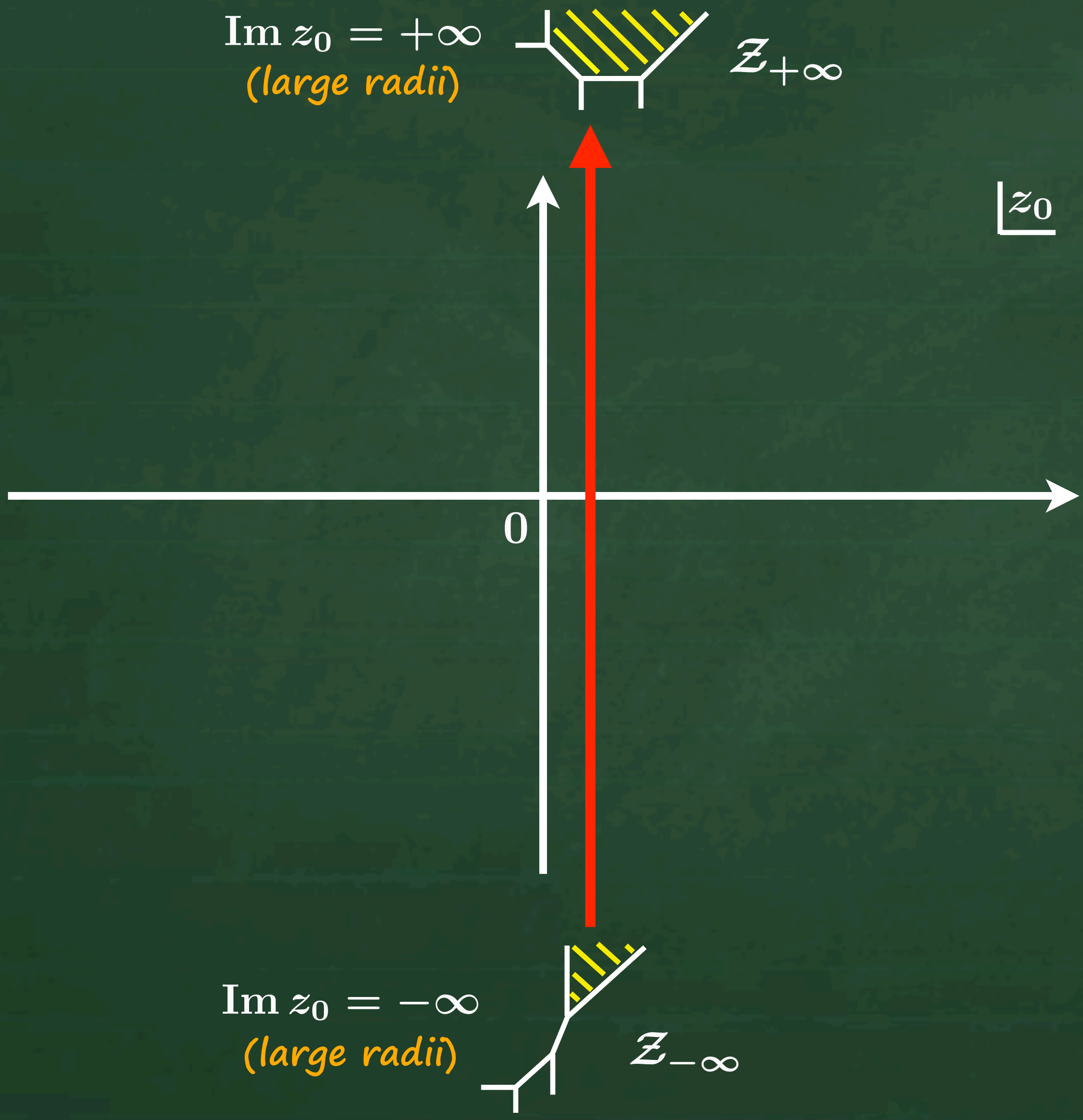


moduli parameters

$z_0,\ z_1 \in \ \mathbb{C}$

$\mathrm{Im}\, \widehat{z}_i \sim$ area of i-th cycle

$\widehat{z}_0 = z_1, \ \widehat{z}_1 = -z_0 - z_1$



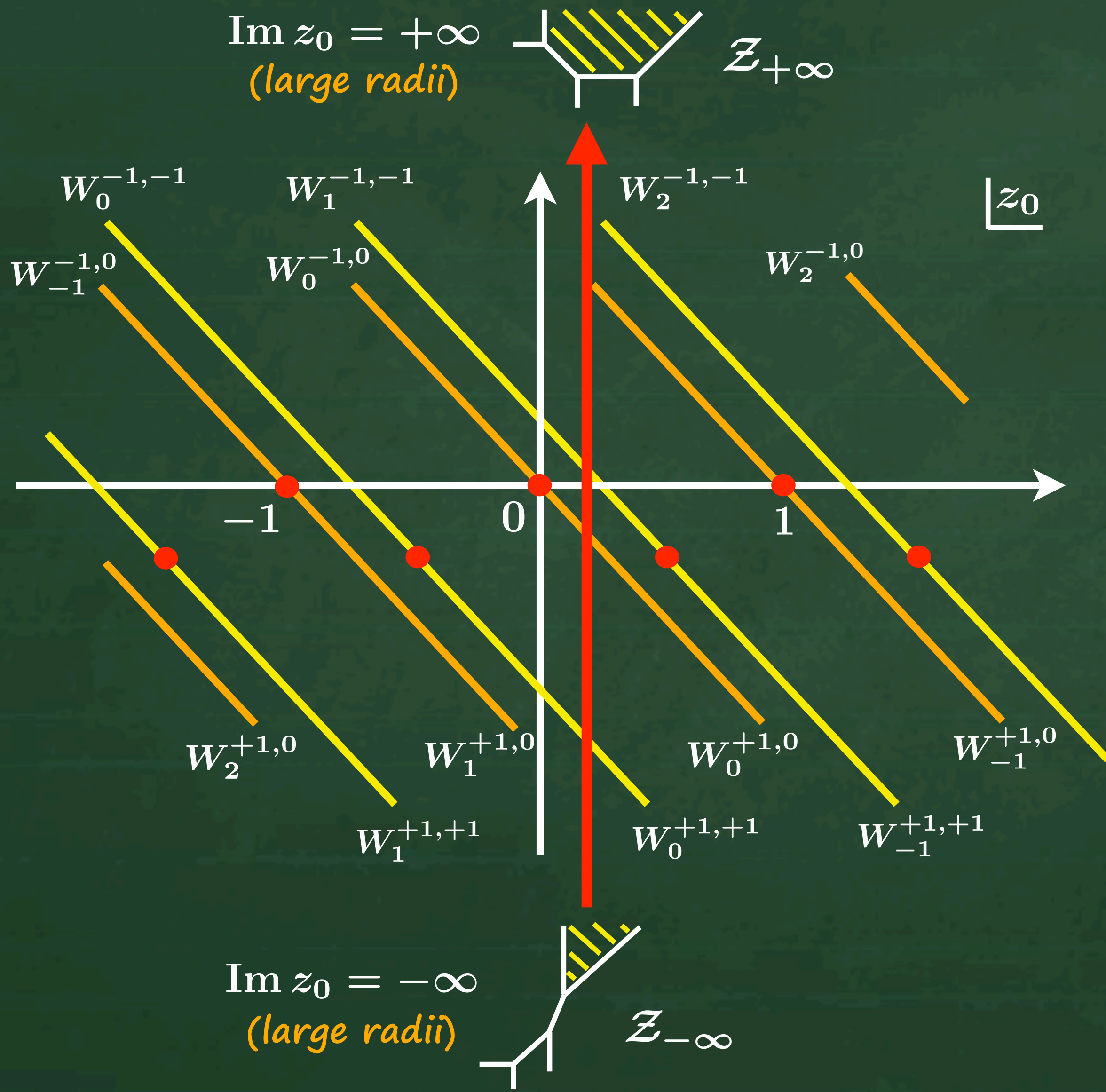
Walls of MS

$$\Gamma \to \Gamma_1 + \Gamma_2$$

$$\Gamma = \mathrm{D4} + l^i \mathrm{D2}_{(i)} + k\mathrm{D0}$$

$W_n^{m^0, m^1}$:

$$\Gamma_2 = \sum_{i=0,1} m^i \mathrm{D2}_{(i)} + n\mathrm{D0}$$



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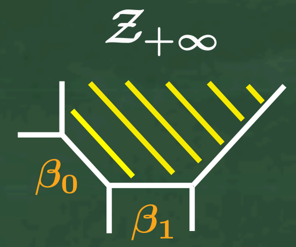
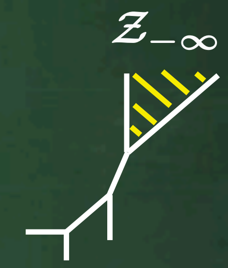
Results of wall-crossings





$$\mathcal{Z} = \sum_{n,m_0,m_1} \Omega(\mathrm{D4} + m^i\mathrm{D2}_{(i)} + n\mathrm{D0})\, q^n {Q_0}^{m^0} {Q_1}^{m^1}$$

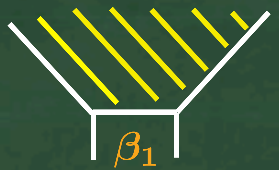
<u>Result of wall-crossing formula</u>

$$\mathcal{Z}_{+\infty} = \mathcal{Z}_{-\infty} \prod_{n=0}^{\infty} (1 - q^n Q_0)(1 - q^n Q_0 Q_1)$$

$$\times \prod_{m=1}^{\infty} (1 - q^n {Q_0}^{-1})(1 - q^m Q_0^{-1} Q_1^{-1})$$

$$\prod_{n=1}^{\infty} \frac{1}{1 - q^n}$$

Results of wall-crossings



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$$\times \prod_{m=1}^{\infty} (1 - q^n {Q_0}^{-1})(1 - q^m Q_0^{-1} Q_1^{-1})$$

$$\prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

omit β_0

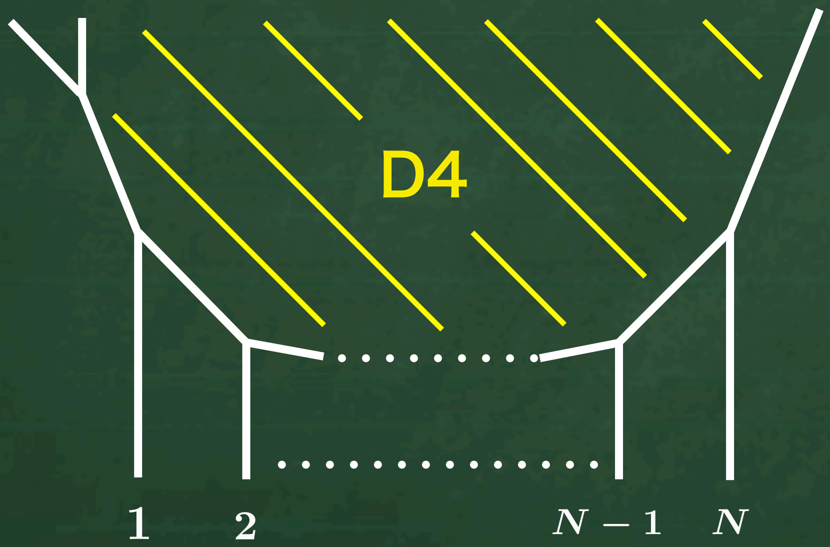
$$\mathcal{Z}_{+\infty}|_{Q_0-\text{independent}} = \prod_{n=1}^{\infty} \left(\frac{1}{1-q^n}\right)^3 \sum_{m=-\infty}^{\infty} q^{m^2} {Q_1}^m = \frac{q^{1/8}}{\eta(q)^2} \chi_0^{\widehat{\mathrm{su}}(2)_1}(q, Q_1)$$

We can generalize it to A_{N-1} ALE space!!

Generalization

Multiple flops

We now have multiplet flop transitions.



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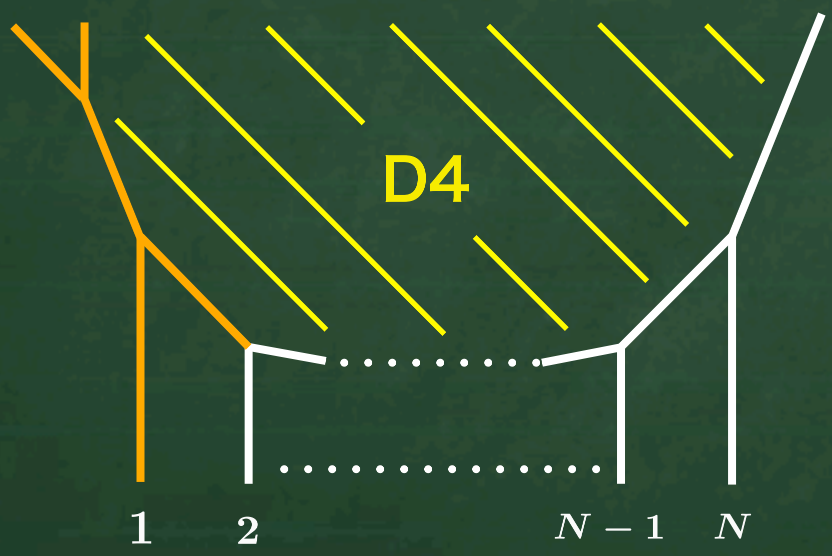

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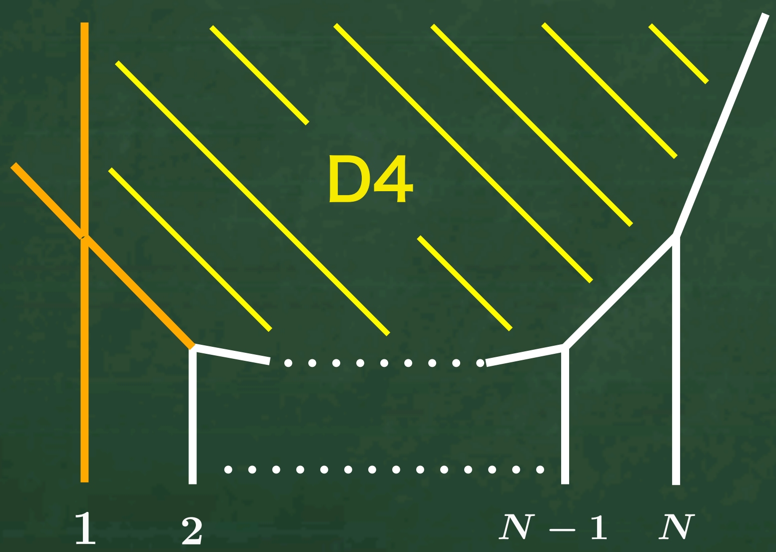

Generalization

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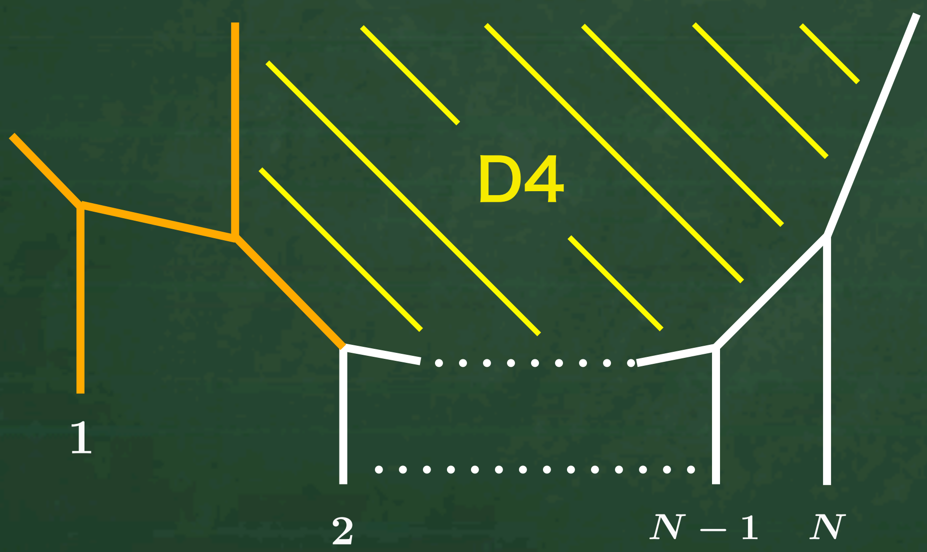

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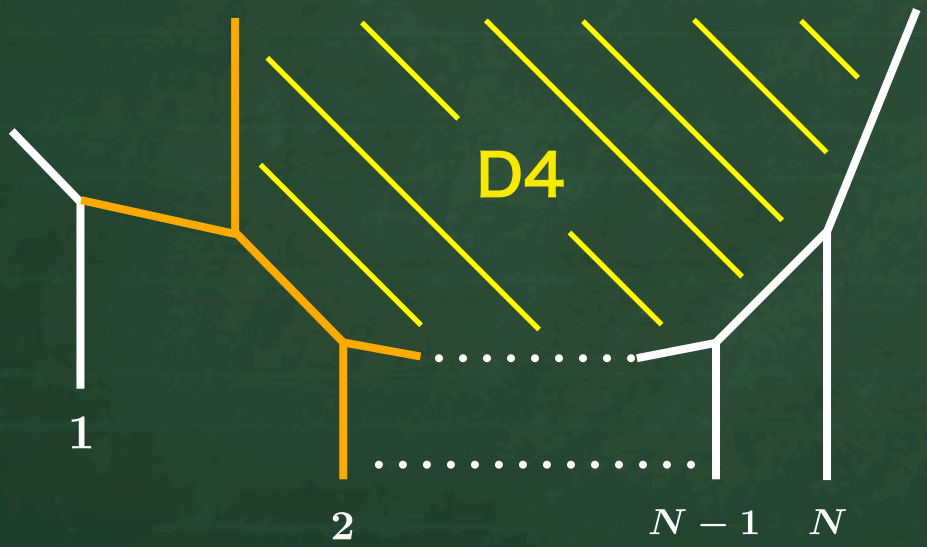

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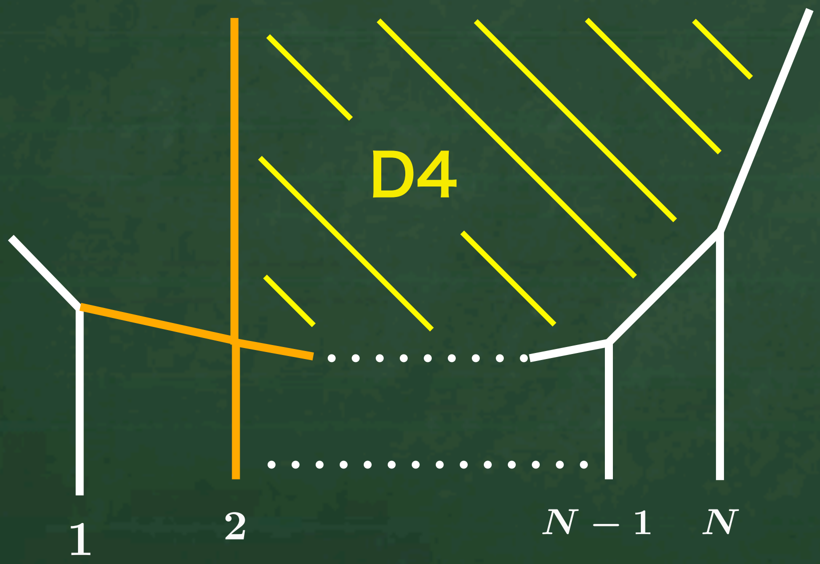

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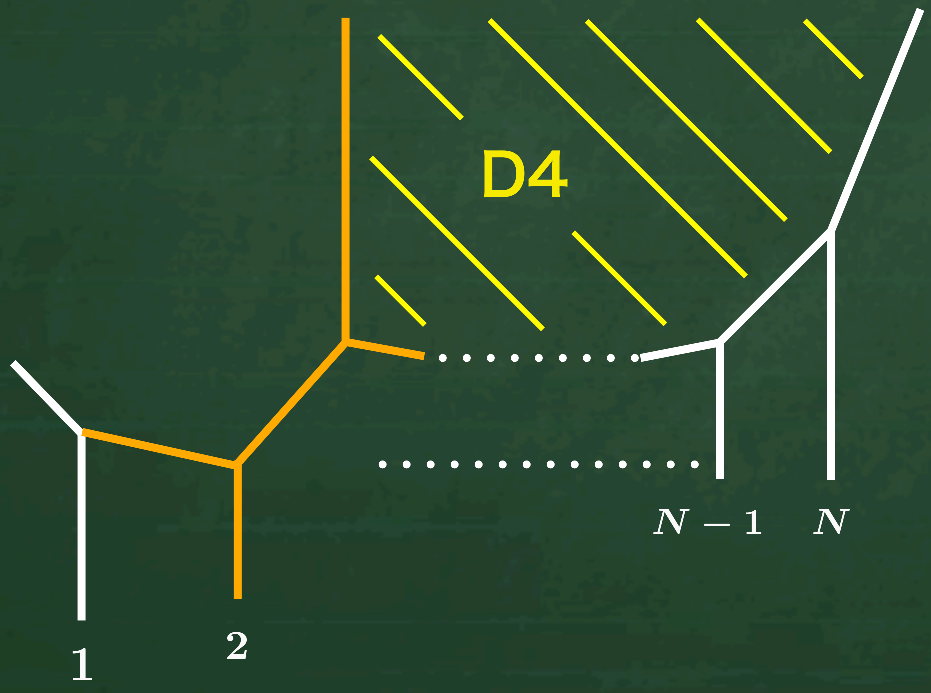

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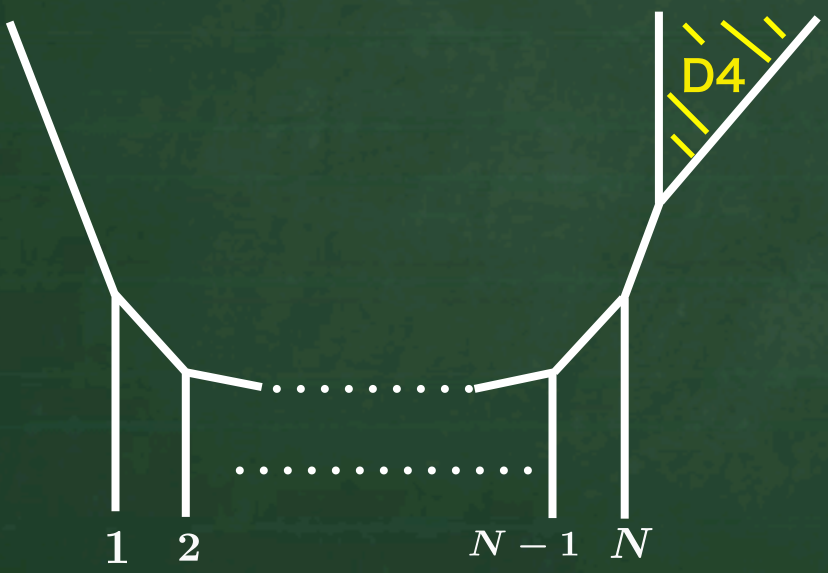

Instantons on $\mathbb{C}^2$

Generalization

Affine SU(N) character from wall-crossings

$$\mathcal{Z}_{-\infty}(q,Q) = \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

wall-crossings

$$\mathcal{Z}_{-\infty}(q,Q) \longleftrightarrow \mathcal{Z}_{+\infty}(q,Q)$$

From the wall-crossing formula, we find

$$\mathcal{Z}_{+\infty}(q,Q) = \mathcal{Z}_{-\infty}(q,Q) \prod_{k=0}^{N-1} \left[\prod_{n=0}^{\infty} (1 - q^n Q_0 \cdots Q_k) \prod_{m=1}^{\infty} (1 - q^m Q_0^{-1} \cdots Q_k^{-1}) \right]$$

$$= \prod_{\ell=1}^{\infty} \left(\frac{1}{1-q^{\ell}} \right)^{N+1} \sum_{n_0,\cdots,n_{N-1} \in \mathbb{Z}} q^{\sum_{i=0}^{N-1} \frac{n_i(n_i-1)}{2}} \prod_{k=0}^{N-1} (-Q_0 \cdots Q_k)^{n_k}$$

Generalization

$$\mathcal{Z}_{+\infty}(q,Q) = \prod_{\ell=1}^{\infty}\left(\frac{1}{1-q^{\ell}}\right)^{N+1} \sum_{n_0,\cdots,n_{N-1}\in\mathbb{Z}} q^{\sum_{i=0}^{N-1}\frac{n_i(n_i-1)}{2}} \prod_{k=0}^{N-1}(-Q_0\cdots Q_k)^{n_k}$$

After removing dummy cycle...

$$\mathcal{Z}_{+\infty}(q,Q)|_{Q_0-\mathrm{independent}} = \prod_{n=1}^{\infty}\left(\frac{1}{1-q^{n}}\right)^{2} \chi_0^{\widehat{su}(N)_1}(q,Q)$$

where

$$\chi_0^{\widehat{su}(N)_1}(q,Q) = \frac{1}{\eta(q)^{N-1}} \sum_{n_1,\cdots,n_{N-1}\in\mathbb{Z}} q^{\sum_{i=1}^{N-1} n_i^2 - \sum_{i=1}^{N-2} n_i n_{i+1}} \prod_{j=1}^{N-1} {Q_j}^{n_j}$$

Affine SU(N) character is correctly reproduced by wall-crossings!

Summary

- We study the relation between ALE instantons and wall-crossings of D4-D2-D0 on CY3.
- By adding a "dummy" cycle, the wall-crossings interpolate the instanton partition functions on ALE space and $\mathbb{C}^2$.
- Our analysis relies on the wall-crossing formula of d=4, N=2 theories.

Future work

- The parameter r of $\chi_r^{\widehat{su}(N)_1}$ is left for future work.
- Generalizations to other instanton partition functions on Vafa-Witten theories are to be studied.

That‘s all for my presentation.

Thank you very much.