General Black Hole solutions in D=5 SUGRA

To be published in PRD

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Contents

- Kaluza-Klein (K-K) black holes
  - K-K black holes in D=5 Einstein gravity
  - K-K black holes in D=5 minimal supergravity
  - Squashed K-K black holes
  - Caged black holes
- Electric-magnetic SL(2,R) duality in D=5 minimal supergravity
- New black hole solutions
- Summary & future works

K-K black holes in D=5 Einstein theory

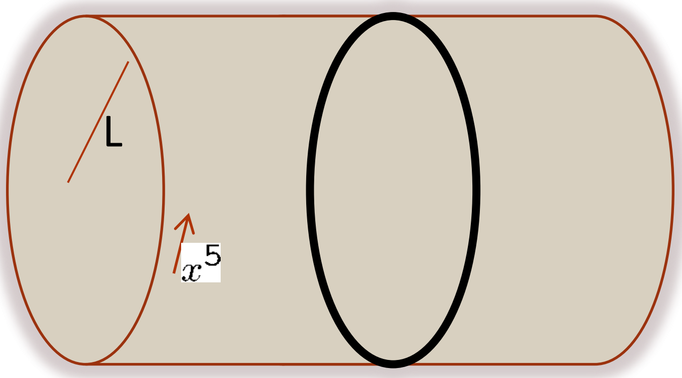
Black holes in Kaluza-Klein theory

- Simplest ''trivial'' example

Black string (Schwarzschild string = Schwarzschild bh × S^1)

$$ds^2 = \underbrace{-\left(1-\frac{m}{r}\right)dt^2+\left(1-\frac{m}{r}\right)^{-1}dr^2+r^2d\Omega^2_{S^2}}_{\text{Schwarzschild bh}}+(dx^5)^2$$

$0 \leq x^5 < 2\pi L$



- Horizon ≃ $S^2 \times S^1$
- Asymptotics : D=4 Minkowski × S^1

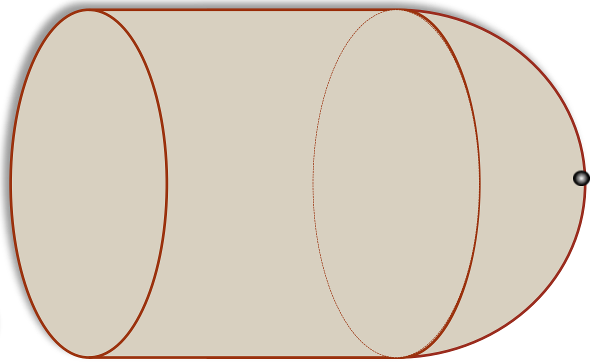
$$ds^2 \underset{r\to\infty}{\simeq} -dt^2+dr^2+r^2d\Omega^2_{S^2}+(dx^5)^2$$

Black holes in Kaluza-Klein theory

- “Non-trivial” example

  Squashed black holes

$$ds^2 = \rho^{-1} ds^2_{4DBH} + \rho^2 (dx^5 + w_i dx^i)^2$$



- Horizon ≃ S³

$$ds^2 \underset{r^2 \to M}{\simeq} -\left(1 - \frac{M}{r^2}\right) dt^2 + \left(1 - \frac{M}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2_{S^3}$$

Like D=5 Schwarzschild bh

- Asymptotics : D=4 Minkowski × S¹

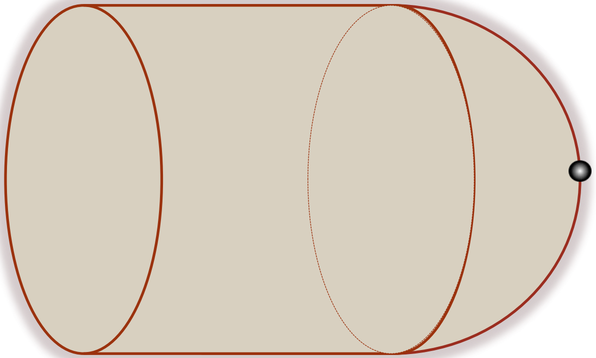
$$ds^2 \underset{r \to \infty}{\simeq} -dt^2 + dr^2 + r^2 d\Omega^2_{S^2} + (dx^5)^2$$

Spherical black holes with a compact dimension in D=5 Einstein & minimal supergravity

D=5 black hole solutions	M	q_e	J_5	J_4	q_m
Dobiash-Maison 82	Yes		Yes		
Rasheed 95	Yes		Yes	Yes	
Gaiotto-Strominger-Yin 06	Yes		Yes		
Elvang-Emparan-Mateos-Reall 05	Yes		Yes		Yes
Ishihara-Matsuno 06	Yes	Yes			
Nakagawa-Ishihara-Matsuno-Tomizawa 08	Yes	Yes	Yes		
Tomizawa-Yasui-Morisawa 08	Yes	Yes		Yes	
Tomizawa-Ishihara-Matsuno-Nakagawa 08	Yes	Yes	Yes		Yes

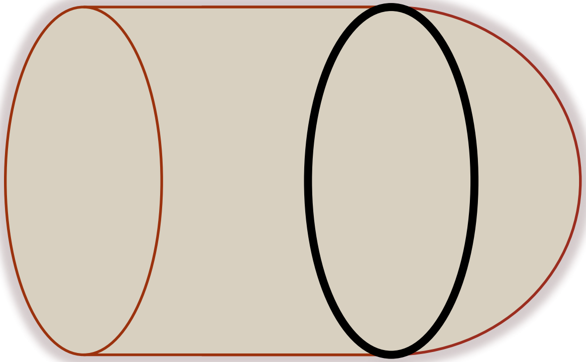
Various configuration of compactified black holes

Black hole on a nut



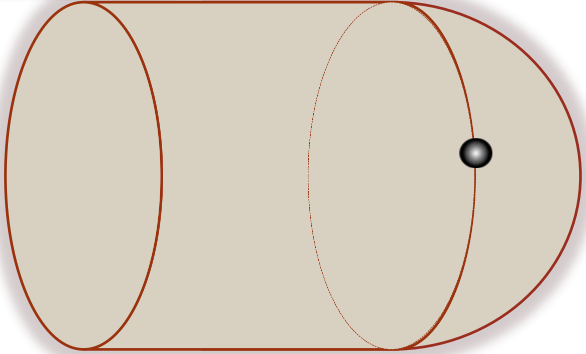
(Ishihara-Matsuno 05, Nakagawa-Ishihara-Matsuno-Tomizawa 08,
Tomizawa-Yasui-Morisawa 08, Tomizawa-Ishibashi 08,
Tomizawa-Ishishara-Nakagawa-Matsuno 08, Tomizawa 10, Elvang-Emparan-Reall ･･･)

Nutty black ring



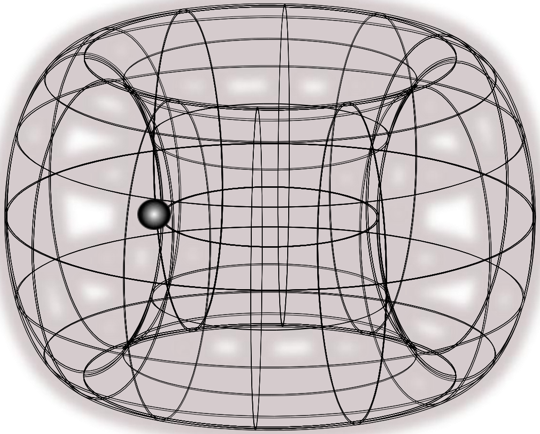
(Bena-Kraus-Warner 05, Ford-Giusto-Peet-Saxena 08,
Camps-Emparan-Figueras-Guito-Saxena 09･･･)

Caged black hole



(Myers 87, Maeda-Ohta-Tanabe 06)

Black hole on a bubble



(Elvang-Horowiz 06, Tomizawa-Iguchi-Mishima 07,
Iguchi-Mishima-Tomizawa 07)

D=5 Kaluza-Klein theory

- D=5 metric:

$$ds^2 = e^{\frac{4\sigma}{\sqrt{3}}}(dx^5 + B_\mu dx^\mu)^2 + e^{-\frac{2\sigma}{\sqrt{3}}} g^4_{\mu\nu} dx^\mu dx^\nu$$

$$B_\mu = B_\mu(x^\mu),\ \sigma = \sigma(x^\mu),\ g^4_{\mu\nu} = g^4_{\mu\nu}(x^\mu)$$

$$0 \leq x^5 < 2\pi L$$

- D=5 Einstein-Hilbert action:

$$\mathcal{L} = \frac{1}{16\pi G}\int d^5x\sqrt{-g}R$$

dilaton

U(1) gauge field

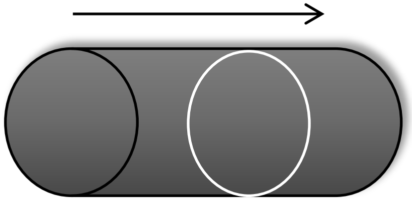
$$\Rightarrow \quad \mathcal{L} = \int d^4x\sqrt{-g^4}\left[R^4 - 2(\partial\sigma)^2 + \frac{1}{4}e^{2\sqrt{3}\sigma}F^2\right]$$

dimensional reduction

$$(F = dB)$$

Einstein-Maxwell-dilaton system

How to find charged K-K black holes



e.g. Schwarzschild string = Schwarzschild bh × S¹

$$ds^2 = -\left(1-\frac{m}{r}\right)dt^2+\left(1-\frac{m}{r}\right)^{-1}dr^2+r^2d\Omega^2_{S2}+(dx^5)^2$$

Lorentz boost along the fifth dimension:

$$t \to \cosh\alpha\, t+\sinh\alpha\, x^5,\ x^5 \to \sinh\alpha\, t+\cosh\alpha\, x^5$$

- Boosted Schwarzschild string

$$ds^2 = \left(1+\frac{m\sinh^2\alpha}{r}\right)^{-1}(dx^5+A_t dt)^2+\left(1+\frac{m\sinh^2\alpha}{r}\right)^{-\frac{1}{2}}ds^2_{(4)}$$

- D=4 dimensionally reduced metric:

$$ds^2_{(4)} = -\frac{r-m}{\sqrt{r^2+(\cosh^2\alpha-1)mr}}dt^2+\frac{\sqrt{r^2+(\cosh^2\alpha-1)mr}}{r-m}dr^2+r\sqrt{r^2+(\cosh^2\alpha-1)mr}d\Omega^2_{S2}$$

- Gauge potential for Maxwell field:

$$A_t = -\frac{2m\cosh\alpha\sinh\alpha}{r+m\sinh^2\alpha}$$

$$Q(\text{Electric charge}) = m\cosh\alpha\sinh\alpha$$

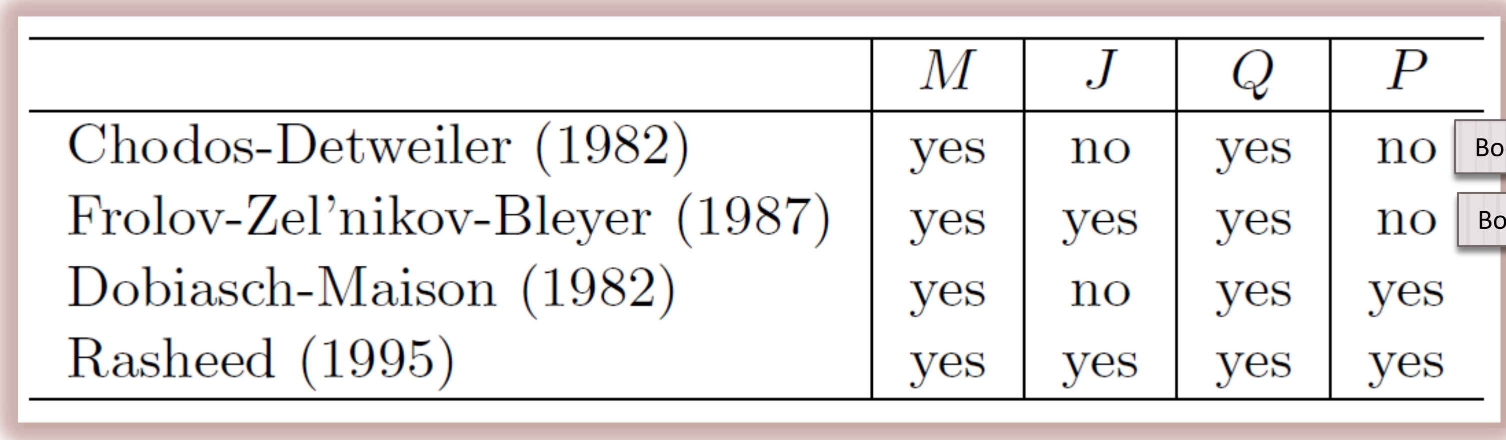
Remarks:

- The D=5 metric is a vacuum solution in D=5 Einstein equation but the D=4 metric is no longer a solution to the D=4 Einstein equation.
- From the D=4 point of view, this yields a nontrivial electric charge (K-K electric charge).
- The spatial twist of the fifth dimension and the D=4 metric can yield a magnetic charge (K-K magnetic monopole charge).

Classification of Kaluza-Klein black holes in D=5 Einstein theory

- In D=5 Kaluza-Klein theory, an asymptotically flat, stationary regular (dimensionally reduced D=4) black hole is specified by 4-charges:
    - M (mass)
    - J (angular momentum)
    - Q (K-K electric charge)
    - P (K-K magnetic monopole charge)

- Classification of ``known” Kaluza-Klein black holes in D=5 Einstein theory



	M	J	Q	P
Chodos-Detweiler (1982)	yes	no	yes	no
Frolov-Zel'nikov-Bleyer (1987)	yes	yes	yes	no
Dobiasch-Maison (1982)	yes	no	yes	yes
Rasheed (1995)	yes	yes	yes	yes

Boosted schwarzschild string

Boosted Kerr string

Rasheed solutions (Rasheed 95)

- The Rasheed solutions are the most general Kaluza-Klein rotating dyonic black hole solutions (with 4-parameters) in D=5 Einstein theory
- Metric in D=5:

$$\boxed{ds^2 = \frac{B}{A}(dx^5 + B_\mu dx^\mu)^2 + \sqrt{\frac{A}{B}}ds^2_{(4)},}$$

  - Dimensionally reduced D=4 metric:

$$ds^2_{(4)} = -\frac{f^2}{\sqrt{AB}}(dt + \omega^0{}_\phi d\phi)^2 + \frac{\sqrt{AB}}{\Delta}dr^2 + \sqrt{AB}d\theta^2 + \frac{\sqrt{AB}\Delta}{f^2}\sin^2\theta d\phi^2$$

  - Kaluza-Klein U(1) gauge field:

$$B_\mu dx^\mu = \frac{C}{B}dt + \left(\omega^5{}_\phi + \frac{C}{B}\omega^0{}_\phi\right)d\phi,$$

$$A = \left(r - \frac{\Sigma}{\sqrt{3}}\right)^2 - \frac{2P^2\Sigma}{\Sigma - \sqrt{3}M} + a^2\cos^2\theta + \frac{2JPQ\cos\theta}{(M + \Sigma/\sqrt{3})^2 - Q^2},$$

$$B = \left(r + \frac{\Sigma}{\sqrt{3}}\right)^2 - \frac{2Q^2\Sigma}{\Sigma + \sqrt{3}M} + a^2\cos^2\theta - \frac{2JPQ\cos\theta}{(M - \Sigma/\sqrt{3})^2 - P^2},$$

$$C = 2Q(r - \Sigma/\sqrt{3}) - \frac{2PJ\cos\theta(M + \Sigma/\sqrt{3})}{(M - \Sigma/\sqrt{3})^2 - P^2},$$

$$\omega^0{}_\phi = \frac{2J\sin^2\theta}{f^2}\left[r - M + \frac{(M^2 + \Sigma^2 - P^2 - Q^2)(M + \Sigma/\sqrt{3})}{(M + \Sigma/\sqrt{3})^2 - Q^2}\right],$$

$$\omega^5{}_\phi = \frac{2P\Delta\cos\theta}{f^2} - \frac{2QJ\sin^2\theta[r(M - \Sigma/\sqrt{3}) + M\Sigma/\sqrt{3} + \Sigma^2 - P^2 - Q^2]}{f^2[(M + \Sigma/\sqrt{3})^2 - Q^2]},$$

$$\Delta = r^2 - 2Mr + P^2 + Q^2 - \Sigma^2 + a^2,$$

$$f^2 = r^2 - 2Mr + P^2 + Q^2 - \Sigma^2 + a^2\cos^2\theta,$$

Scalar charge

- The solutions have five parameters, (M,J,Q,P,Σ), but all of these are not independent because of the relation: $\frac{Q^2}{\Sigma + \sqrt{3}M} + \frac{P^2}{\Sigma - \sqrt{3}M} = \frac{2\Sigma}{3},$

K-K black holes in D=5 minimal supergravity

K-K reduction in D=5 minimal SUGRA (Chamseddine and Nicolai 80)

- Lagrangian :

$$\boxed{\mathcal{L} = E^{(5)} \left(R - \frac{1}{4} F_{MN} F^{MN} \right) - \frac{1}{12\sqrt{3}} \epsilon^{MNPQR} F_{MN} F_{PQ} A_R}$$

  - D=5 Metric:

$$ds^2 = \rho^2 (dx^5 + B_\mu dx^\mu)^2 + \rho^{-1} ds^2_{(4)}$$
$$(\mu = 0, 1, 2, 3)$$

$$E_M^{(5)A} = \left(\begin{array}{c|c} \rho^{-1/2} E_\mu^{(4)\alpha} & \rho B_\mu \\ \hline 0 & \rho \end{array} \right)$$

  - Maxwell field:

$$A = A_\mu dx^\mu + A_5 dx^5$$

  - K-K reduction to D=4:

K-K U(1) gauge field   Maxwell U(1) gauge field

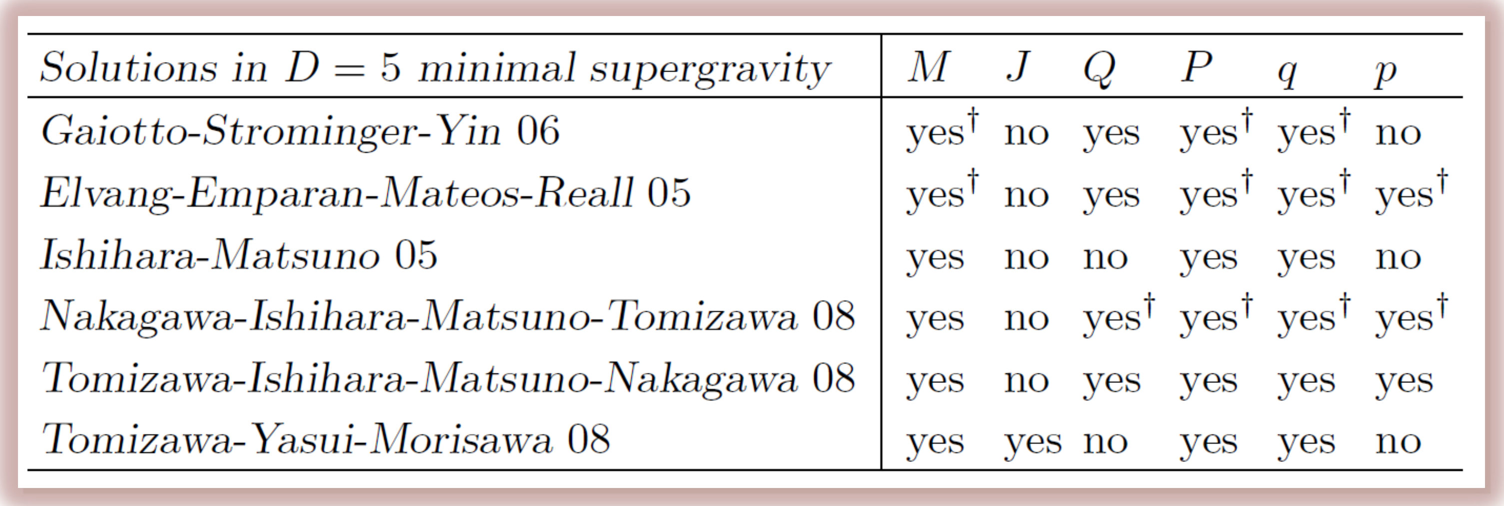
$$\mathcal{L} = E^{(4)} \left(R^{(4)} - \frac{3}{2} (\partial \ln \rho)^2 - \frac{1}{4} \rho^3 B_{\mu\nu} B^{\mu\nu} - \frac{1}{2} \rho^{-2} (\partial A_5)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) - \frac{1}{4\sqrt{3}} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} A_5$$

dilaton   axion   CS term

⇨ **Einstein-Maxwells-dilaton-axion system**

Classification of Kaluza-Klein black holes in D=5 minimal SUGRA

- In D=5 Kaluza-Klein theory, an asymptotically flat, stationary regular (dimensionally reduced D=4) black hole is specified by 6-charges:
  - M (mass)
  - J (angular momentum)
  - Q (K-K electric charge)
  - P (K-K magnetic monopole charge)
  - q (electric charge of Maxwell field)
  - p (magnetic momopole charge of Maxwell field)
- Classification of ``known" Kaluza-Klein black holes in D=5 Minimal SUGRA



Solutions in $D = 5$ minimal supergravity	M	J	Q	P	q	p
Gaiotto-Strominger-Yin 06	yes†	no	yes	yes†	yes†	no
Elvang-Emparan-Mateos-Reall 05	yes†	no	yes	yes†	yes†	yes†
Ishihara-Matsuno 05	yes	no	no	yes	yes	no
Nakagawa-Ishihara-Matsuno-Tomizawa 08	yes	no	yes†	yes†	yes†	yes†
Tomizawa-Ishihara-Matsuno-Nakagawa 08	yes	no	yes	yes	yes	yes
Tomizawa-Yasui-Morisawa 08	yes	yes	no	yes	yes	no

- So far, the most general (non-BPS) solutions having six ``*independent"* charges (though expected to exist) have not been found

Squashing

Squashed black holes

- The “Squashing” deforms a class of cohomogeneity-1 non-compactified solutions into a class of cohomogeneity-1 compactified solutions

- E.g. D=5 Schwarzschild black holes with asymptotic flatness

$$ds^2 = -\left(1-\frac{m}{r^2}\right)dt^2+\left(1-\frac{m}{r^2}\right)^{-1}dr^2+\frac{r^2}{4}[\sigma_1^2+\sigma_2^2+\sigma_3^2]$$

(σ1,σ2,σ3) : SU(2)-invariant 1-form:

$$\sigma_1 = \cos\psi d\theta+\sin\psi\sin\theta d\phi, \sigma_2 = -\sin\psi d\theta+\cos\psi\sin\theta d\phi, \sigma_3 = d\psi+\cos\theta d\phi$$

- S^3 can be regarded as S^1 Fiber bundle over S^2 (Hofp bundle)

$$\sigma_1^2+\sigma_2^2+\sigma_3^2 = 4d\Omega_{S^3}^2$$

$$\sigma_1^2+\sigma_2^2 = d\Omega_{S^2}^2 = d\theta^2+\sin^2\theta d\phi^2, \quad (0\le\theta<\pi,\ 0\le\phi<2\pi,\ 0\le\psi<4\pi)$$

- Squashed Schwarzschild black holes

$$ds^2 = -\left(1-\frac{m}{r^2}\right)dt^2 + \left(1-\frac{m}{r^2}\right)^{-1} k(r)^2 dr^2 + \frac{r^2}{4}[k(r)(\sigma_1^2+\sigma_2^2)+\sigma_3^2]$$

  - Squashing function:

$$k(r) := \frac{(c^2-m)c^2}{(c^2-r^2)^2} \to \infty \ (r \to c)$$

  - Coordinate range:

$$0 < r \leq c$$

(r→c : infinity)

  - Proper length:

$$\int^r \sqrt{g_{rr}}dr \to \infty \ (r \to c)$$

  - Asymptotics: asymptotically Kaluza-Klein

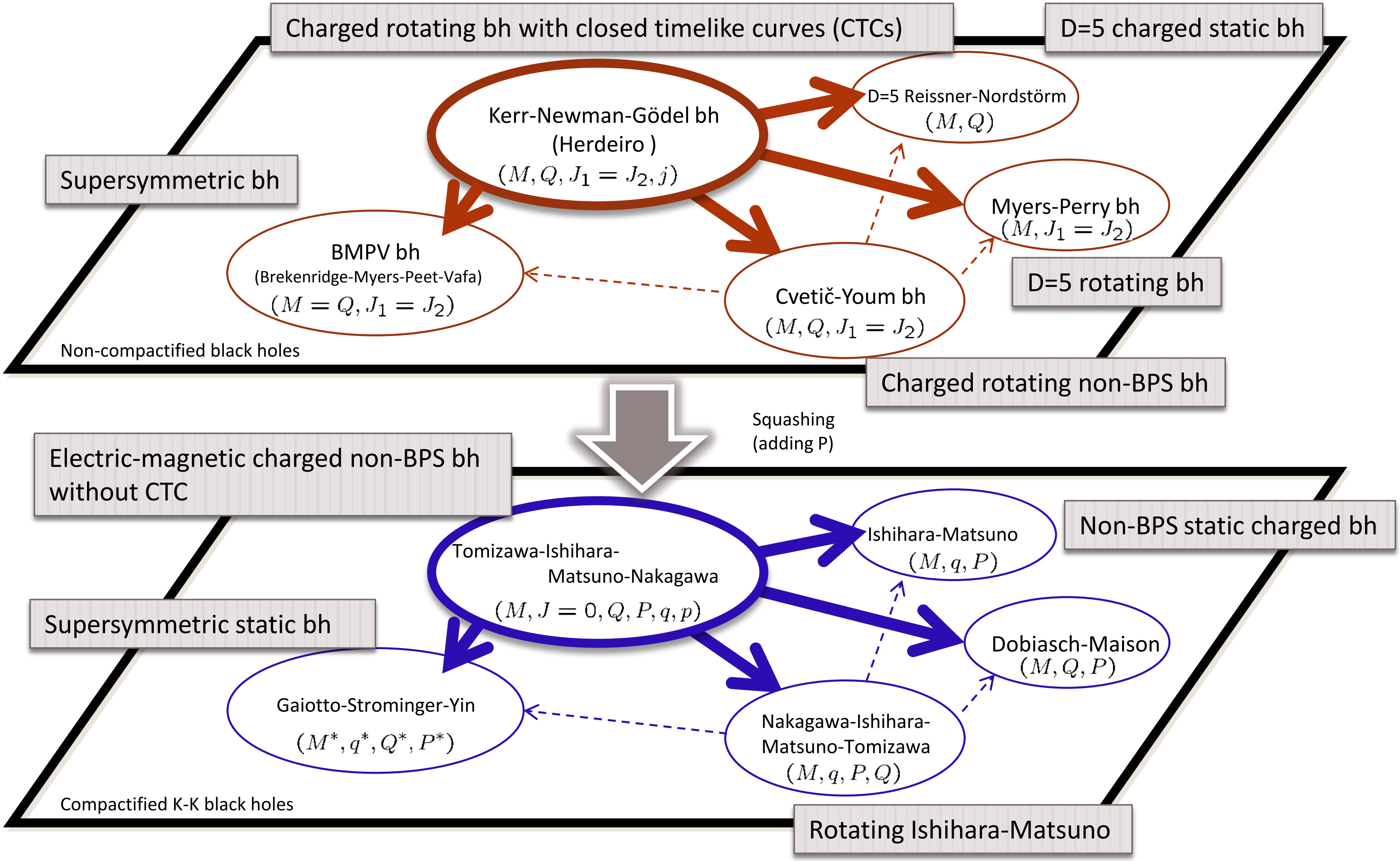
$$ds^2 \simeq -dt^2 + d\rho^2 + \rho^2 d\Omega_{S^2}^2 + \frac{c^2}{4}(d\psi + \cos\theta d\phi)^2$$

$$\rho := \frac{c^2}{4(c^2-r^2)} \to \infty \ (r \to c)$$

Squashed solutions

Solutions in D=5 Einstein/supergravity	***Squashed solutions (Regular, causal)***
D=5 Minkowski	GPS monopole
D=5 Myers-Perry bh with equal angular momenta	Dobiash-Maison 82
D=5 Reissner-Nordström bh	Ishihara-Matsuno 05
D=5 Cvetič-Youm bh with equal charges	Nakagawa-Ishihara-Matsuno-Tomizawa 08
Gödel universe	Rotating GPS monopole 08
Kerr-Newman-Gödel bh	Tomizawa-Ishihara-Matsuno-Nakagawa 08
Charged Gödel bh	Tomizawa-Ishibashi 08
D=5 Cvetič-Youm bh	Tomizawa 10

Relations between squashed solutions



Charged rotating bh with closed timelike curves (CTCs)

D=5 charged static bh

Kerr-Newman-Gödel bh
(Herdeiro)
$(M, Q, J_1 = J_2, j)$

D=5 Reissner-Nordstörm
(M, Q)

Supersymmetric bh

Myers-Perry bh
$(M, J_1 = J_2)$

BMPV bh
(Brekenridge-Myers-Peet-Vafa)
$(M = Q, J_1 = J_2)$

D=5 rotating bh

Cvetič-Youm bh
$(M, Q, J_1 = J_2)$

Non-compactified black holes

Charged rotating non-BPS bh

Squashing
(adding P)

Electric-magnetic charged non-BPS bh
without CTC

Tomizawa-Ishihara-
Matsuno-Nakagawa
$(M, J = 0, Q, P, q, p)$

Ishihara-Matsuno
(M, q, P)

Non-BPS static charged bh

Supersymmetric static bh

Dobiasch-Maison
(M, Q, P)

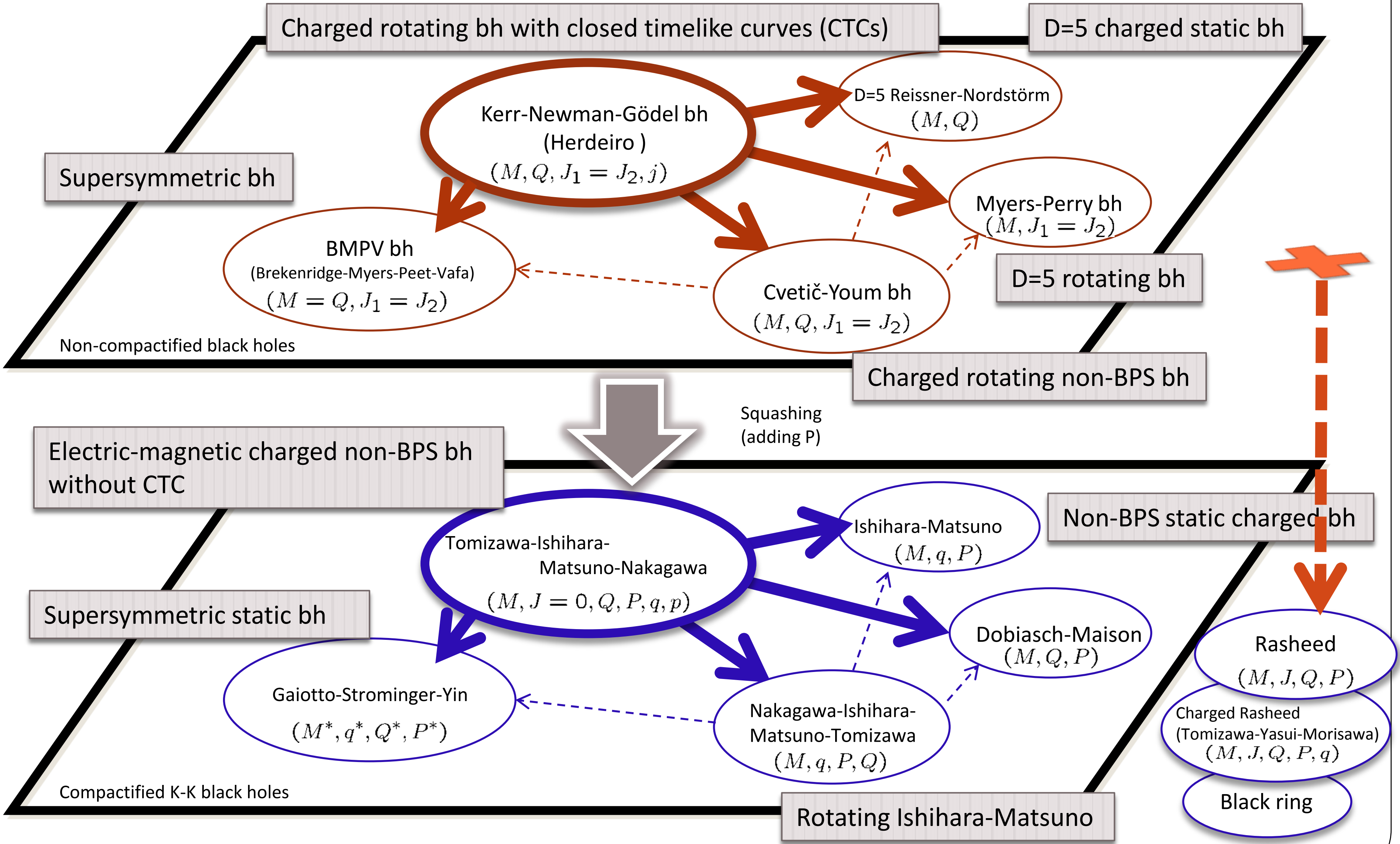
Gaiotto-Strominger-Yin
(M^*, q^*, Q^*, P^*)

Nakagawa-Ishihara-
Matsuno-Tomizawa
(M, q, P, Q)

Compactified K-K black holes

Rotating Ishihara-Matsuno

Relations between squashed solutions



Charged rotating bh with closed timelike curves (CTCs)

D=5 charged static bh

Supersymmetric bh

Kerr-Newman-Gödel bh (Herdeiro) $(M, Q, J_1 = J_2, j)$

D=5 Reissner-Nordstörm (M, Q)

Myers-Perry bh $(M, J_1 = J_2)$

BMPV bh (Brekenridge-Myers-Peet-Vafa) $(M = Q, J_1 = J_2)$

Cvetič-Youm bh $(M, Q, J_1 = J_2)$

D=5 rotating bh

Non-compactified black holes

Charged rotating non-BPS bh

Squashing (adding P)

Electric-magnetic charged non-BPS bh without CTC

Non-BPS static charged bh

Tomizawa-Ishihara-Matsuno-Nakagawa $(M, J = 0, Q, P, q, p)$

Ishihara-Matsuno (M, q, P)

Supersymmetric static bh

Dobiasch-Maison (M, Q, P)

Gaiotto-Strominger-Yin (M^*, q^*, Q^*, P^*)

Nakagawa-Ishihara-Matsuno-Tomizawa (M, q, P, Q)

Rasheed (M, J, Q, P)

Charged Rasheed (Tomizawa-Yasui-Morisawa) (M, J, Q, P, q)

Black ring

Compactified K-K black holes

Rotating Ishihara-Matsuno

Caged black holes

Caged black holes

- HD Majumdar-Papapetrou (MP) multi-black holes (Myers 95)

Extreme black hole solutions in Einstein-Maxwell theory

$$\left\{\begin{array}{l} ds^2 = -H^2 dt^2 + H^{-1} ds^2_{(4)} \\ \\ A = \frac{\sqrt{3}}{2}\frac{1}{H} \end{array}\right.$$

D=4 flat metric:

$$ds^2_{(4)} = dx^2 + dy^2 + dz^2 + dw^2$$

$$\triangle_{(4)} H = 0 \quad \Rightarrow \quad H = 1 + \sum_i \frac{Q_i}{|\vec{r} - \vec{r_i}|^2}$$

Caged black holes

- HD Majumdar-Papapetrou (MP) multi-black holes (Myers 87)

Extreme black hole solutions in Einstein-Maxwell theory

$$\begin{cases} ds^2 = -H^2 dt^2 + H^{-1} ds^2_{(4)} \\ A = \dfrac{\sqrt{3}}{2}\dfrac{1}{H} \end{cases}$$

D=4 flat metric:

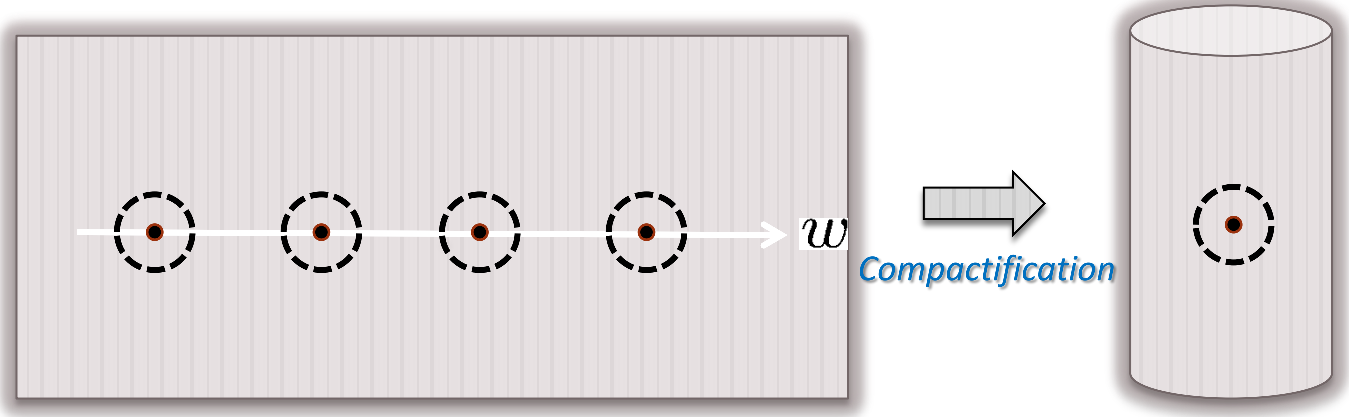
$$ds^2_{(4)} = dx^2 + dy^2 + dz^2 + dw^2$$

$$\triangle_{(4)} H = 0 \quad \Rightarrow \quad H = 1 + \sum_i \frac{Q_i}{|\vec{r} - \vec{r}_i|^2}$$

- Caged black holes (Myers 87) :

MP black holes with the same separation in w-direction

$$\vec{r}_i = (0, 0, 0, 2\pi L n_i) \ \ (n_i \in Z)$$



- Rotational caged black holes (Maeda-Ohta-Tanabe 06)

Multi-BMPV black holes

$$ds^2 = -H^2(dt+\omega)^2 + H^{-1} ds^2_{(4)}$$

$$A = \frac{\sqrt{3}}{2}\frac{H^{-1}}{r^2}(dt+\omega)$$

$$\omega = \frac{J}{2r^2}(\sin^2\theta d\phi^2 + \cos^2\theta d\psi^2)$$

$$H = 1 + \sum_i \frac{Q_i}{|\vec{r}-\vec{r_i}|^2}$$

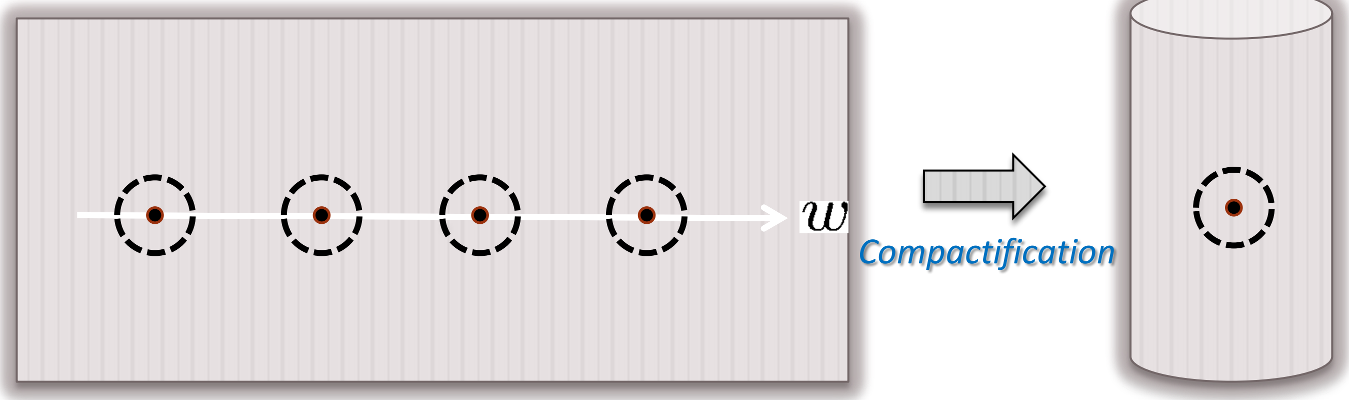
D=4 flat metric:

$$ds^2_{(4)} = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2)$$

Caged black holes :

Multi-BMPV black holes with the same separation in w-direction

$$\vec{r_i} = (0,0,0,2\pi L n_i) \ (n_i \in Z)$$



- Smoothness of the MP metric (Welch 95, Candlish-Reall 07)
    - D=4 ⇒ analytic (Hartle-Hawking 72)
    - D=5 ⇒ C^2 but not C^3
    - D>5 ⇒ C^1 but not C^2 ⇒ curvature singularity
- Smoothness of the multi-BMPV metric (Candlish 10)
    - D=5 ⇒ C^1 but not C^2 ⇒ curvature singularity

- D>5 caged MP black holes & D=5 caged multi-BMPV black holes: regularity (unknown) ?

Electric-magnetic SL(2,R) duality in D=5 minimal SUGRA

Electric-Magnetic duality invariance in electrodynamics

- Sourceless Maxwell equation:
  - Bianchi identity:

$$\mathrm{rot}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad [dF = 0]$$
$$\mathrm{div}\mathbf{B} = 0$$

  - Field equations:

$$\mathrm{rot}\mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} \quad [d * F = 0]$$
$$\mathrm{div}\mathbf{E} = 0$$

⇒ invariant under the Hodge duality transformation:

$$\begin{cases} \mathbf{E} \to -\mathbf{B} \\ \mathbf{B} \to \mathbf{E} \end{cases}$$

$$[F_{\mu\nu} \to (*F)_{\mu\nu}]$$

- D=4 Maxwell-Chern-Simons theory coupled with an axion and a dilaton admits the more general SL(2,R)-duality invariance (Gibbons-Rasheed 95, 96)

- In D=5 minimal SUGRA, the dimensionally reduced D=4 theory (Maxwell+Maxwell+Chern-Simons theory coupled with an axion and a dilaton) has the SL(2,R)-duality invariance (Mizoguchi-Ohta 98)

Electric-Magnetic duality invariance in electrodynamics

- Sourceless Maxwell equation:
  - Bianchi identity:

$$\mathrm{rot}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad [dF = 0]$$
$$\mathrm{div}\mathbf{B} = 0$$

  - Field equations:

$$\mathrm{rot}\mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} \quad [d * F = 0]$$
$$\mathrm{div}\mathbf{E} = 0$$

⇒ In general, Maxwell eqs are invariant under SO(2)-duality rotation:

$$\begin{cases} \mathbf{E} \to \cos\alpha\ \mathbf{E} - \sin\alpha\ \mathbf{B} \\ \mathbf{B} \to \sin\alpha\ \mathbf{E} + \cos\alpha\ \mathbf{B} \end{cases}$$

$$[F_{\mu\nu} \to \cos\alpha\ F_{\mu\nu} + \sin\alpha\ (*F)_{\mu\nu}]$$

- D=4 Maxwell-Chern-Simons theory coupled with an axion and a dilaton admits the more general SL(2,R)-duality invariance (Gibbons-Rasheed 95, 96)

- In D=5 minimal SUGRA, the dimensionally reduced D=4 theory (Maxwell+Maxwell+Chern-Simons theory coupled with an axion and a dilaton) has the SL(2,R)-duality invariance (Mizoguchi-Ohta 98)

K-K reduction in D=5 minimal SUGRA

- **Lagrangian:**

$$\mathcal{L} = E^{(5)}\left(R - \frac{1}{4}F_{MN}F^{MN}\right) - \frac{1}{12\sqrt{3}}\epsilon^{MNPQR}F_{MN}F_{PQ}A_R$$

- **D=5 Metric:**

$$ds^2 = \rho^2(dx^5 + B_\mu dx^\mu)^2 + \rho^{-1}ds^2_{(4)}$$

$$(\mu = 0, 1, 2, 3)$$

$$E_M^{(5)A} = \left(\begin{array}{c|c} \rho^{-1/2}E_\mu^{(4)\alpha} & \rho B_\mu \\ \hline 0 & \rho \end{array}\right)$$

- **Maxwell field:**

$$A = A_\mu dx^\mu + A_5 dx^5$$

- **K-K reduction to D=4:**

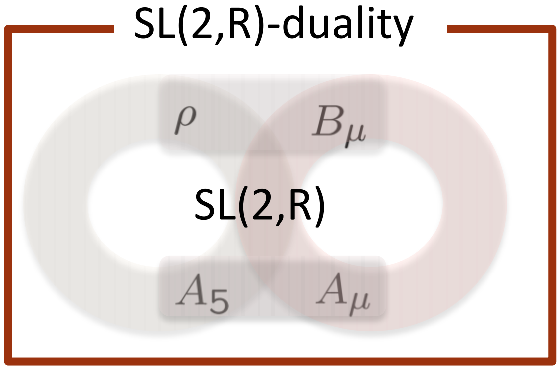
$$\mathcal{L} = E^{(4)}\left(R^{(4)} - \frac{3}{2}\underbrace{(\partial \ln \rho)^2}_{\text{dilaton}} - \frac{1}{4}\rho^3 \overbrace{B_{\mu\nu}B^{\mu\nu}}^{\text{K-K U(1) gauge field}} - \frac{1}{2}\rho^{-2}\underbrace{(\partial A_5)^2}_{\text{axion}} - \frac{1}{4}\overbrace{F_{\mu\nu}F^{\mu\nu}}^{\text{Maxwell U(1) gauge field}}\right) - \underbrace{\frac{1}{4\sqrt{3}}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}A_5}_{\text{CS term}}$$

⇨ **Einstein-Maxwells-dilaton-axion system**

SL(2,R)-duality invariance in D=5 minimal SUGRA (Mizoguchi-Ohta 98)

- Lagrangian for vectors & scalar fields:

$$\mathcal{L}_{\mathcal{V}}+\mathcal{L}_{\mathcal{S}} = \sqrt{|g^{(4)}|}\left(\frac{1}{4}\mathcal{G}^T_{\mu\nu}\mathcal{H}^{\mu\nu} + \frac{3}{40}\mathrm{Tr}\partial_\mu\mathcal{R}^{-1}\partial^\mu\mathcal{R}\right)$$


SL(2,R)-duality

- Fields:

$$\mathcal{G}_{\mu\nu} := \begin{pmatrix} \tilde{A}_{\mu\nu} \\ B_{\mu\nu} \end{pmatrix}$$

$$\partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu$$

$$\partial_\mu B_\nu - \partial_\nu B_\mu$$

$$\mathcal{H}_{\mu\nu} := m * \mathcal{G}_{\mu\nu} - a\mathcal{G}_{\mu\nu}$$

$$m := \frac{1}{\rho+4A_5^2/3}\begin{pmatrix} 1 & -\frac{A_5^2}{\sqrt{3}\rho} \\ -\frac{A_5^2}{\sqrt{3}\rho} & \frac{A_5^3}{3\sqrt{3}\rho^3}(1+\frac{A_5^4}{3\rho^2+4A_5^2\rho^2}) \end{pmatrix}$$

$$a := \frac{2\rho}{\rho+4A_5^2/3}\begin{pmatrix} \frac{2A_5}{\sqrt{3}\rho} & A_5(1+\frac{2A_5^2}{3\rho^2}) \\ A_5(1+\frac{2A_5^2}{3\rho^2}) & \frac{2A_5^3}{3\sqrt{3}}(1+\frac{A_5^2}{3\rho^2}) \end{pmatrix}$$

$$\mathcal{R} = \begin{pmatrix} \rho^{-1} & -\frac{1}{\sqrt{3}}\rho^{-1}A_5 \\ -\frac{1}{\sqrt{3}}\rho^{-1}A_5 & \frac{1}{3}\rho^{-1}A_5^2+\rho \end{pmatrix}$$

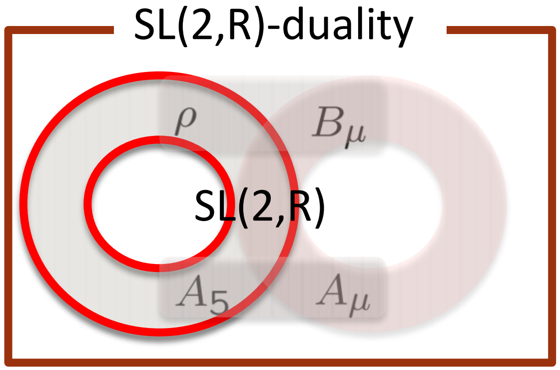
- EOM + Bianchi id:

$$d\mathcal{H}_{\mu\nu} = 0 \quad d\mathcal{G}_{\mu\nu} = 0$$

- Duality invariance: EOM has SL(2,R)-duality invariance

$$\mathcal{R} \to \Lambda^t\mathcal{R}\Lambda \qquad \begin{pmatrix} \mathcal{G}_{\mu\nu} \\ \mathcal{H}_{\mu\nu} \end{pmatrix} \to \Lambda^{-1}\begin{pmatrix} \mathcal{G}_{\mu\nu} \\ \mathcal{H}_{\mu\nu} \end{pmatrix} \quad \text{For } \Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathrm{R})$$

Transformation for axion and dilaton (Mizoguchi-Tomizawa 11)



- SL(2,R) generators (Chevalley generators):

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- General non-trivial transformation:

$$\Lambda = e^{-\alpha E} e^{-\beta F}$$

- SL(2,R)-duality transformation from a seed into a new solution:

$$(\rho^{New})^{-1} \qquad -\frac{1}{\sqrt{3}}(\rho^{New})^{-1} A_5^{New}$$

$$\mathcal{R} = \begin{pmatrix} \rho^{-1} & -\frac{1}{\sqrt{3}}\rho^{-1} A_5 \\ -\frac{1}{\sqrt{3}}\rho^{-1} A_5 & \frac{1}{3}\rho^{-1} A_5^2 + \rho \end{pmatrix}$$

Field for a seed

$$\rightarrow \mathcal{R}' = (e^{-\alpha E} e^{-\beta F})^t \mathcal{R} e^{-\alpha E} e^{-\beta F} = \begin{pmatrix} \rho^{-1} + 2\alpha\beta\rho^{-1} + \alpha^2\beta^2\rho^{-1} + \beta^2\rho & -\alpha\rho^{-1} - \alpha^2\beta\rho^{-1} - \beta\rho \\ -\alpha\rho^{-1} - \alpha^2\beta\rho^{-1} - \beta\rho & \alpha^2\rho^{-1} + \rho \end{pmatrix}$$

Field for a new solution

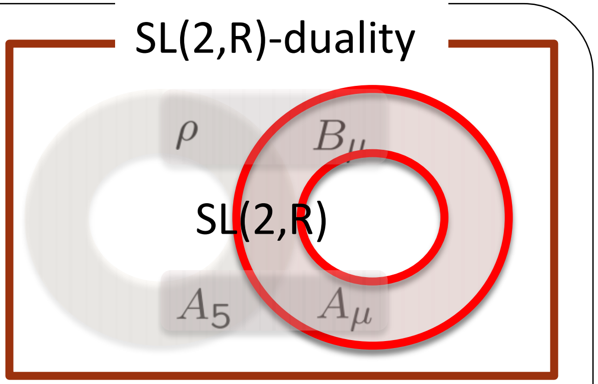
- New fields (in particular when a seed is a vacuum solution):

$$\rho^{new} = \frac{\rho}{(1+\alpha\beta)^2 + \beta^2\rho^2},$$

$$A_5^{new} = \sqrt{3}\,\frac{\alpha(1+\alpha\beta) + \beta\rho^2}{(1+\alpha\beta)^2 + \beta^2\rho^2}.$$

Transformation for U(1) gauge fields

(Mizoguchi-Tomizawa 11)



- SL(2,R) generators (4-representation):

$$E = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

- SL(2,R)-transformation from a seed into a new solution:

$$\mathcal{F}_{\mu\nu} = \begin{pmatrix} \tilde{A}_{\mu\nu} = 0 \\ B_{\mu\nu} \\ -F'_{\mu\nu} = 0 \\ \rho^3(*B)_{\mu\nu} \end{pmatrix}$$

Field for a seed

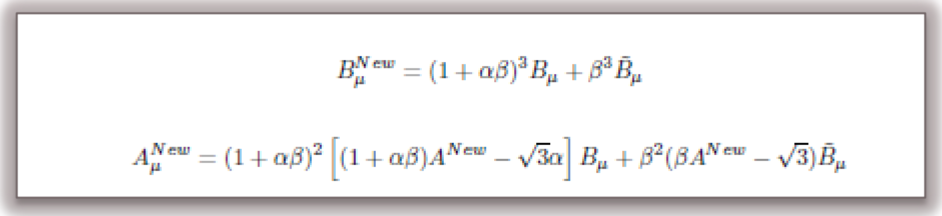
$B^{New}_{\mu\nu}$

$$\to \mathcal{F}'_{\mu\nu} = e^{\beta F} e^{\alpha E} \mathcal{F}_{\mu\nu} = \begin{pmatrix} \sqrt{3}\alpha^2(1+\alpha\beta)B_{\mu\nu} + \sqrt{3}\beta\rho^3(*B)_{\mu\nu} \\ (1+\alpha\beta)^3 B_{\mu\nu} + \beta^3\rho^3(*B)_{\mu\nu} \\ \sqrt{3}\alpha(1+\alpha\beta)^2 B_{\mu\nu} + \sqrt{3}\beta^2\rho^3(*B)_{\mu\nu} \\ \alpha^3 B_{\mu\nu} + \rho^3(*B)_{\mu\nu} \end{pmatrix}$$

Field for a new solution

$$-F'^{New}_{\mu\nu} = -(dA'^{New})_{\mu\nu}$$
$$(A'^{New}_\mu = A^{New}_\mu - A^{New}_5 B^{New}_\mu)$$

- New vector fields:



$$B^{New}_\mu = (1+\alpha\beta)^3 B_\mu + \beta^3 \bar{B}_\mu$$

$$A^{New}_\mu = (1+\alpha\beta)^2 \left[(1+\alpha\beta)A^{New} - \sqrt{3}\alpha\right] B_\mu + \beta^2(\beta A^{New} - \sqrt{3})\bar{B}_\mu$$

Our solutions

Application to Rasheed solutions (Mizoguchi-Tomizawa 11)

- Metric (in D=5) for new solutions:

$$ds^2 = \frac{AB}{(A(1+\alpha\beta)^2+\beta^2 B)^2}\left[dx^5+(1+\alpha\beta)^3 B_\mu dx^\mu+\beta^3\tilde{B}_\mu dx^\mu\right]^2+\frac{A(1+\alpha\beta)^2+\beta^2 B}{\sqrt{AB}}ds^2_{(4)},$$

  - The dimensionally reduced D=4 metric is

$$ds^2_{(4)} = -\frac{f^2}{\sqrt{AB}}(dt+\omega^0{}_\phi d\phi)^2+\frac{\sqrt{AB}}{\Delta}dr^2+\sqrt{AB}d\theta^2+\frac{\sqrt{AB}\Delta}{f^2}\sin^2\theta d\phi^2$$

  - Kaluza-Klein gauge field is given by

$$B^{new}_\mu = (1+\alpha\beta)^3 B^{Rasheed}_\mu+\beta^3\tilde{B}_\mu$$

with

$$\begin{aligned}\tilde{B}_\mu dx^\mu &= \frac{2P(r+\Sigma/\sqrt{3})+2JQ\frac{(M-\Sigma/\sqrt{3})}{(M+\Sigma/\sqrt{3})^2-Q^2}\cos\theta}{A}dt \\ &-\left(2Q\cos\theta+\frac{2JP\frac{(M+\Sigma/\sqrt{3})}{(M-\Sigma/\sqrt{3})^2-P^2}r+2JP\frac{P^2-Q^2+\Sigma(\sqrt{3}M-\Sigma/3)}{(M-\Sigma/\sqrt{3})^2-P^2}+2a^2Q\cos\theta}{A}\sin^2\theta\right)d\phi\end{aligned}$$

- Gauge potential for new solutions:

$$\mathcal{A}^{new} = (1+\alpha\beta)^2\left[(1+\alpha\beta)A^{new}_5-\sqrt{3}\alpha\right]B_\mu dx^\mu+\beta^2(\beta A^{new}_5-\sqrt{3})\tilde{B}_\mu dx^\mu+A^{new}_5 dx^5$$

With the axion:

$$A^{new}_5 = \frac{\sqrt{3}(\alpha+\alpha^2\beta+\beta B/A)}{(1+\alpha\beta)^2+\beta^2 B/A}$$

Horizon topology

$$ds^2 = \frac{AB}{(A(1+\alpha\beta)^2+\beta^2 B)^2}\left[dx^5 + (1+\alpha\beta)^3 B_\mu dx^\mu + \beta^3 \tilde{B}_\mu dx^\mu\right]^2 + \frac{A(1+\alpha\beta)^2+\beta^2 B}{\sqrt{AB}} ds^2_{(4)},$$

$$ds^2_{(4)} = -\frac{f^2}{\sqrt{AB}}(dt+\omega^0{}_\phi d\phi)^2 + \frac{\sqrt{AB}}{\Delta}dr^2 + \sqrt{AB}d\theta^2 + \frac{\sqrt{AB}\Delta}{f^2}\sin^2\theta d\phi^2$$

- Horizon locations : $\Delta(r):=r^2-2Mr+P^2+Q^2-\Sigma^2+a^2=0$
  - From the D=4 point of view, each t,r=constant surface turns out to be S^2 because of Gauss-Bonnet theorem:
  - Therefore, from the D=5 point of view, each t,r=constant surface can be regarded as a U(1) principal fiber bundle over S^2 base space

Horizon topology

$$ds^2 = \frac{AB}{(A(1+\alpha\beta)^2+\beta^2 B)^2}\left[dx^5 + (1+\alpha\beta)^3 B_\mu dx^\mu + \beta^3 \tilde{B}_\mu dx^\mu\right]^2 + \frac{A(1+\alpha\beta)^2+\beta^2 B}{\sqrt{AB}} ds^2_{(4)},$$

$$ds^2_{(4)} = -\frac{f^2}{\sqrt{AB}}(dt + \omega^0{}_\phi d\phi)^2 + \frac{\sqrt{AB}}{\Delta} dr^2 + \sqrt{AB} d\theta^2 + \frac{\sqrt{AB}\Delta}{f^2}\sin^2\theta d\phi^2$$

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  - Therefore, from the D=5 point of view, each t,r=constant surface can be regarded as a U(1) principal fiber bundle over a S^2 base space
- The 1st Chern-number:

$$c_1(\mathcal{B}) = -\frac{1}{\Delta x^5}\int_{S^2} \mathcal{B},$$

  - Curvature: $\mathcal{B} = \frac{1}{2}(\partial_\mu B_\nu^{new} - \partial_\nu B_\mu^{new}) dx^\mu \wedge dx^\nu$
  - Periodicity: $\Delta x^5 = 4\pi(2\tilde{\gamma}^3 P - 2\tilde{\beta}^3 Q)$

0	**$S^1 \times S^2$**
±1	S^3
others	L(n;1)

Horizon topology

$$ds^2 = \frac{AB}{(A(1+\alpha\beta)^2+\beta^2 B)^2}\left[dx^5 + (1+\alpha\beta)^3 B_\mu dx^\mu + \beta^3 \tilde{B}_\mu dx^\mu\right]^2 + \frac{A(1+\alpha\beta)^2+\beta^2 B}{\sqrt{AB}} ds^2_{(4)},$$

$$ds^2_{(4)} = -\frac{f^2}{\sqrt{AB}}(dt + \omega^0{}_\phi d\phi)^2 + \frac{\sqrt{AB}}{\Delta}dr^2 + \sqrt{AB}d\theta^2 + \frac{\sqrt{AB}\Delta}{f^2}\sin^2\theta d\phi^2$$

- Horizon locations : $\Delta(r) := r^2 - 2Mr + P^2 + Q^2 - \Sigma^2 + a^2 = 0$
  - From the D=4 point of view, each t,r=constant surface turns out to be S^2 because of Gauss-Bonnet theorem:
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others	L(n;1)

  - Curvature: $\mathcal{B} = \frac{1}{2}(\partial_\mu B^{new}_\nu - \partial_\nu B^{new}_\mu)dx^\mu \wedge dx^\nu$
  - Periodicity: $\Delta x^5 = 4\pi(2\tilde{\gamma}^3 P - 2\tilde{\beta}^3 Q)$
- $c_1(\mathcal{B})| = 1$ ⇒ the spatial cross section of the horizon ≈ S^3

Asymptotics

- Asymptotic behavior @ r=∞

Metric:

$$ds^2 \simeq \left(-1+\overbrace{\frac{-\frac{4\beta^2\Sigma}{\sqrt{3}N}+2M+\frac{2\Sigma}{\sqrt{3}}}{N^{-\frac{1}{2}}r}}^{M^{new}}\right)dt^2 \overbrace{+2\frac{2P\beta^2+2QN_2}{Nr}}^{Q^{new}}dtdx^5$$

$$+4\frac{2(PN_2-Q\beta^3)(QN_2+P\beta^3)N^{-2}\cos\theta-\overset{J^{new}}{JN}\sin^2\theta}{r}dtd\phi$$

$$+\left(1+\frac{4[(1+\alpha\beta)^2-\beta^2]\Sigma}{\sqrt{3}N^{\frac{1}{2}}r}\right)(dx^5)^2+r^2\sin^2\theta d\phi^2\overbrace{+4\frac{(PN_2-Q\beta^3)}{N}}^{P^{new}}dx^5d\phi+dr^2+r^2d\theta^2,$$

Gauge potential:

$$A \simeq \underbrace{\frac{2\sqrt{3}\beta(1+\alpha\beta)[Q(1+\alpha\beta)-P\beta]}{Nr}}_{q_e^{new}}dt+\sqrt{3}(\alpha+\beta+\alpha^2\beta)dx^5+\underbrace{\frac{2\sqrt{3}\beta(1+\alpha\beta)[P(1+\alpha\beta)+Q\beta]\cos\theta}{N}}_{q_m^{new}}d\phi,$$

where

$N := (1+\alpha\beta)^2+\beta^2$ and $N_2 := (1+\alpha\beta)^3$.

- Our solutions seem to have 6 independent parameters

Parameter independence

Physical parameters:

- Mass & angular momentum:

$$M = -\frac{1}{8\pi}\int_{S^2_\infty} *d\xi_t, \;\; J = \frac{1}{16\pi}\int_{S^2_\infty} *d\xi_\phi.$$

- K-K electric/magnetic charge:

$$Q^{new} \equiv \frac{1}{8\pi}\int_{S^2} \rho^3 * \mathcal{B} = \tilde{\beta}^3 P + \tilde{\gamma}^3 Q, \;\; P^{new} \equiv \frac{1}{8\pi}\int_{S^2} \mathcal{B}^{new} = \tilde{\gamma}^3 P - \tilde{\beta}^3 Q,$$

- Maxwell electric/magnetic charge:

$$q^{new} \equiv \frac{1}{8\pi}\int_{S^2} \left(\rho * \mathcal{F}^{(4)} - \frac{2}{\sqrt{3}} A_5 \mathcal{F}\right) = \tilde{\beta}\tilde{\gamma}(\tilde{\gamma}Q - \tilde{\beta}P), \;\; p^{new} \equiv \frac{1}{8\pi}\int_{S^2} \left(\mathcal{F}^{(4)} - A_5 \mathcal{B}\right) = \tilde{\beta}\tilde{\gamma}(\tilde{\beta}Q + \tilde{\gamma}P)$$

Jacobian:

$$\left|\frac{\partial(Q^{new}, P^{new}, q^{new}, p^{new})}{\partial(Q, P, \alpha, \beta)}\right| = 0.$$

Remarks:

- Our solutions have six-charges (M,J,Q,P,q,p)
- However, four of these are not independent but related by a constraint

D=5 black hole solutions	M	q	Q	J	P	p
Chodos-Detweiler 82	*Yes*		*Yes*			
Dobiash-Maison 82	*Yes*		*Yes*		*Yes*	
Rasheed 95	*Yes*		*Yes*	*Yes*	*Yes*	
Frolov-Zel'nikov-Bleyer 87	*Yes*		*Yes*	*Yes*		
Gaiotto-Strominger-Yin 06	*Yes*	*Yes*	*Yes*		*Yes*	
Elvang-Emparan-Mateos-Reall 05	*Yes*	*Yes*	*Yes*		*Yes*	*Yes*
Ishihara-Matsuno 06	*Yes*	*Yes*			*Yes*	
Nakagawa-Ishihara-Matsuno-Tomizawa 08	*Yes*	*Yes*	*Yes*		*Yes*	
Tomizawa-Yasui-Morisawa 08	*Yes*	*Yes*		*Yes*	*Yes*	
Tomizawa-Ishihara-Matsuno-Nakagawa 08	*Yes*	*Yes*	*Yes*		*Yes*	*Yes*
Tomizawa-Mizoguchi 11	*Yes*	*Yes*	*Yes*	*Yes*	*Yes*	*Yes*

Summary

- We have obtained new K-K black hole solutions in D=5 minimal supergravity, by using SL(2,R) symmetry of dimensionally reduced D=4 space.
- Our solutions can be regarded as dyonic rotating black holes in D=4 Einstein-Maxwell+Maxwell with coupled dilaton-axion system
- Charges: (M,J,P,Q,q,p) with C(P,Q,q,p)=0

Future works

- How to find the most general K-K black holes with independent six charges ?
  - Flip+SL(2,R)-duality transformation (Imazato-Mizoguchi-Tomizawa: in progress)
    Use a timelike Killing vector for SL(2,R)-duality transformation
- Applications to other interesting solutions ?
  - Caged black holes (Mizoguchi-Tomizawa: in progress)
  - Dipole rings with a supersymmetric limit
  - Less symmetric solutions
  - ⋯
- Applications to other theories ?
- Thermodynamics ?
- Stability ?