くりこみ群法の基礎
と
その非平衡物理への応用

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Introduction

--- 非平衡、スケールの階層、平均化(粗視化)、漸近解析 ---

- (非平衡)物理学:系のダイナミクスを少数の変数のダイナミクスに還元(縮約)する理論
- 例

(i) BBGKY(Bogoliubov-Born-Green-Kirkwood-Yvon) hierarchy

for s-body distribution function (s=1,2, ,,,,),

equivqlent to Liouville eq. (time-reversible) ➔

Boltzmann (kinetic) equation with one-body distribution function (time-irreversible)

Bogoliubov (1945, 1946): hierarchy of relaxation times of s-body distribution functions. F_s (s>1)の緩和時間は短い。F_1のslaving variable. 粗視化した時間では、F_s (s>1)はすべてF_1の関数。

Ref. N.N. Bogoliubov, in “Studies in Statistical Mechanics”, (J. de Boer and G. E. Uhlenbeck, Eds.) vol2, (North-Holland, 1962)

Introduction Continued

Separation of scales in the time evolution of a rarefied gas

Liouville → Boltzmann → Fluid dyn.

Hamiltonian (BBGKY hierarchy) | One-body dist. fn. | T,n, u^μ

Slower dynamics

Fewer d.o.f

Hydrodynamics is the effective dynamics with fewer variables of the kinetic (Boltzmann) equation in the infrared refime.

ii) Boltzmann to hydrodynamics (Hilbert, Chapman-Enskog,Bogoliubov)

After some time, the one-body distribution function is asymptotically well described by local temperature T(x), density n(x), and the flow velocity ***u*** **,** ***i.e., the hydrodynamic variables.***

iii) Langevin as a kinetic equation to Fokker-Planck equation

R. L. Stratonovich, "Topics in the Theory of Random Noise", vol.1,2, (Gordon and Breach, NY, 1963),
C. W. Gardiner, "Handbook of Stochastic Methods for Physics,,," (Springer, 1985),
H. Risken," The Fokker-Planck Equation " (Springer, 1989),
N. G. van Kampen, "Stocastic Processes in Physics and Chemistry", (North-Holland, 1992)

iv) Critical dynamics: the order parmaeter becomes a soft (slow) mode, in particlular, around a 2nd-ordr critical point; TDGL etc

One needs to construct a method to
(1) extract slow variables and the governing equations of them, and in some cases,
(2) make a coarse graining of time, leading to time-irreversible equations.

(This is a mission physicists should do. (Hilbert)

粗視化:

H. Moriによる粗視化された時間微分 の定義 (H. Mori, 1956, 1858, 1959)
輸送係数を定義する時間微分は微視的時間微分の「平均」である:

$$\frac{\delta}{\delta t}\langle F\rangle(t) \equiv \frac{1}{\tau}\{\langle F\rangle(t+\tau) - \langle F\rangle(t)\} = \frac{1}{\tau}\int_0^\tau ds \frac{d}{ds}\langle F\rangle(t+s)$$

τ 微視的時間と巨視的時間(緩和時間)の中間の時間.

初期条件の設定: 不変(吸引)多様体の存在

Boltzmann:分子混沌仮説 $\longrightarrow$ 2体相関のない状況の分布関数を初期値に取る。

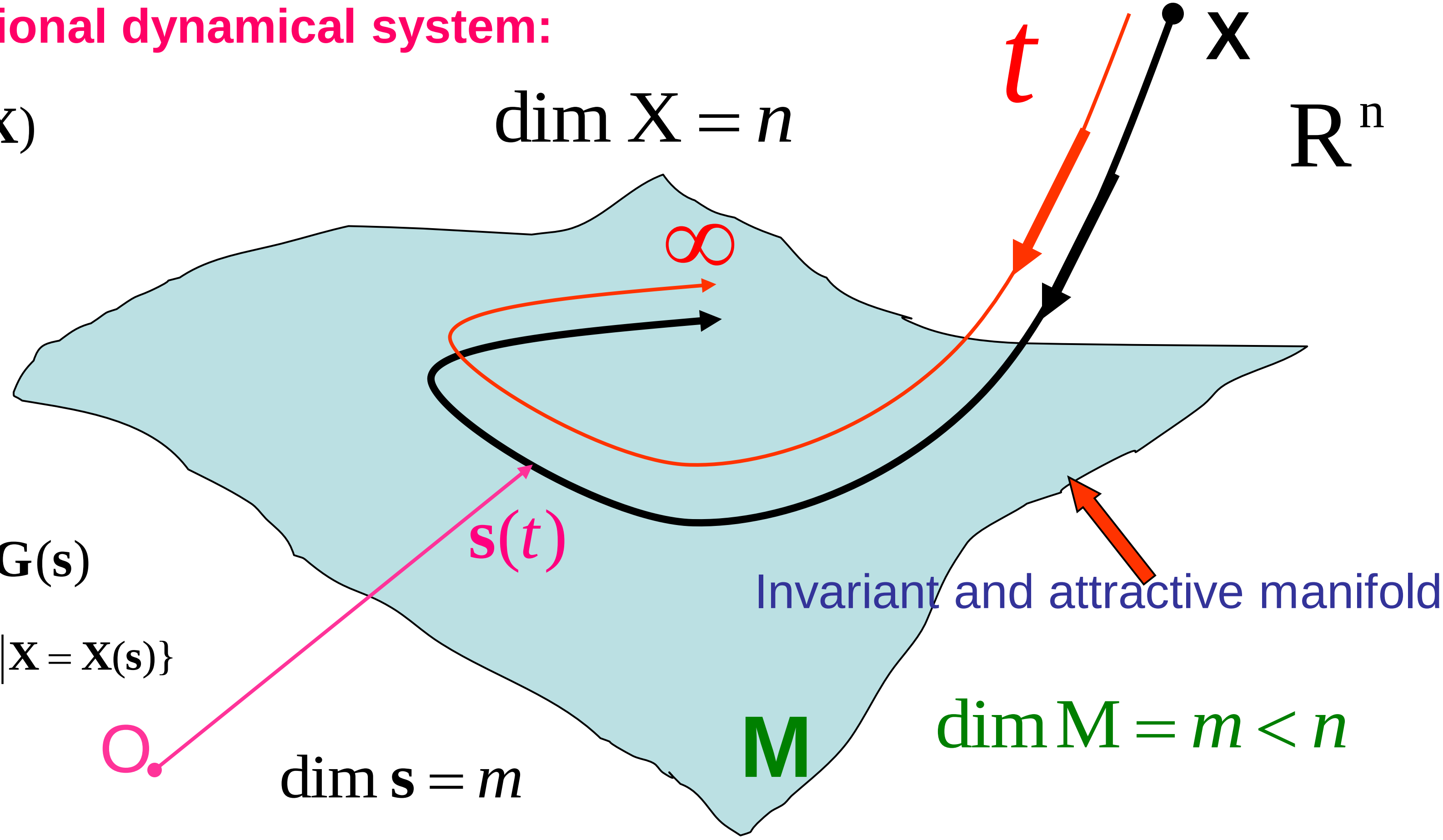
Bogoliubov (1946), 久保ら「統計物理学」(旧 物理学の基礎、岩波)
J.L. Lebowitz, Physica A 194 (1993),1.
川崎恭治「非平衡と相転移」(浅倉 2000), 第7章

Geometrical image of reduction of dynamics

n-dimensional dynamical system:

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X})$$

$$\begin{cases} \dfrac{d\mathbf{s}}{dt} = \mathbf{G}(\mathbf{s}) \\ \mathrm{M} = \{\mathbf{X} \,|\, \mathbf{X} = \mathbf{X}(\mathbf{s})\} \end{cases}$$



eg.

In Field theory, $\mathbf{X} = (g_1, g_2, \ldots, g_n) \equiv \boldsymbol{g} \to \boldsymbol{s} = (s_1, s_2, \ldots, s_m)$

renormalizable $\Longleftrightarrow$ $\exists$ **Invariant manifold M** dim M $= m < n \leq \infty$

The RG/flow equation

$$\Gamma(\phi, \boldsymbol{g}(\Lambda), \Lambda) = \Gamma(\phi, \boldsymbol{g}(\Lambda'), \Lambda').$$

If we take the limit $\Lambda' \to \Lambda$, we have

$$\frac{d\Gamma(\phi, \boldsymbol{g}(\Lambda), \Lambda)}{d\Lambda} = 0,$$

which is the RG/flow equation.

$$\frac{\partial \Gamma}{\partial \boldsymbol{g}} \cdot \frac{d\boldsymbol{g}}{d\Lambda} = -\frac{\partial \Gamma}{\partial \Lambda}.$$

The yet unknown function $\mathbf{g}$ is solved exactly and inserted into Γ, which then becomes valid in a global domain of the energy scale.

The merits of the Renormalization Group/Flow eq:

Owing to the very non-perturbative nature, the RG has at least two merits:

(A) **Resummation of the perturbation series**

Applying the RG equation of Gell-Mann-Low type to perturbative calculations up to first lowest orders, a resummation in the infinite order of diagrams of some kind can be achieved. That is, the RG method gives a powerful resummation method.

(B) **Construction of infrared effective actions**

The RG of Wilson type provides us with a systematic method for constructing low-energy effective actions which are asymptotically valid in the low-frequency and long-wave length limit.

Perturbative Approach:

For dynamical systems:

$$\frac{dX}{dt} = F(X, t),$$

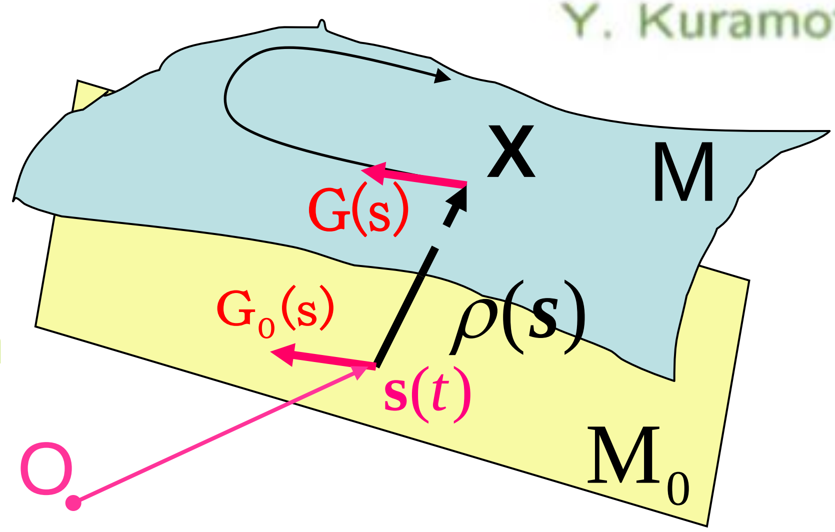
$$\downarrow$$

$$\frac{ds}{dt} = G(s), \quad \text{, reduced dynamics on M}$$

$$X = R(s); \quad \text{, representation of M}$$

Y.Kuramoto('89)

Y. Kuramoto ('89)



Perturbative reduction of dynamics

$$\frac{dX}{dt} = F(X, t),$$

$$\downarrow$$

$$\frac{ds}{dt} = G_0(s) + \gamma(s), \quad \text{, reduced dynamics on M}$$

$$X = R_0(s) + \rho(s); \quad \text{, the invariant manifold M}$$

Krylov-Bogoliubov-Mitropolosky

Geometrical image of perturbative reduction of dynamics

The RG may provide a powerful method for the reduction of dynamics;

The purpose of this talk:

(1) Show that the RG as developed by Illinois group can be reformulated to give a powerful and systematic method for the reduction of dynamics with a coarse graining; construction of the (coarse-grained) attractive slow manifold and extraction of the reduced dynamics on it

(2) Apply the method to somere examples in nonequilibrium problems including a derivation of the relativistic fluid dynamics from the Boltzmann equation.

(3) An emphasis put on the relation to the classical theory of envelopes; the resummed solution obtained through the RG is the envelope of the set of solutions given in the perturbation theory.

ゼロモードと永年項

例　$$\ddot{x} + \epsilon\dot{x} + x = 0,$$

$$x(t) = A(t)\sin\phi(t), \qquad A(t) = \bar{A}\exp(-\epsilon t/2), \quad \phi(t) = \omega t + \bar{\theta}$$

$$(\omega \equiv \sqrt{1 - \epsilon^2/4})$$

素朴な摂動展開

$$x = x_0 + \epsilon x_1 + \epsilon x_2 + \cdots$$
$$x_0 = A\sin(t + \theta)$$

$$\mathcal{L}x_1 \equiv \ddot{x}_1 + x_1 = -\dot{x}_0 = -A\cos(t + \theta)$$

固有振動による強制振動➔共鳴

L　のゼロモード

$$x_1 = -A/2 \cdot t\sin(t + \theta)$$

永年項！

2次の摂動解

$$x(t) = A\sin(t+\theta) - \epsilon\frac{A}{2}t\sin(t+\theta) + \epsilon^2\frac{A}{8}\{t^2\sin(t+\theta) - t\cos(t+\theta)\}$$

振幅増大！
摂動の破綻

総和法の必要性とゆっくりした運動へのくりこみ

$$A(t) = \bar{A}\exp(-\epsilon t/2), \quad \phi(t) = \omega t + \bar{\theta}$$

$$\dot{A} = -\epsilon/2 \cdot A$$

References for RG method

L.Y.Chen, N. Goldenfeld and Y.Oono,
PRL.72('95),376; Phys. Rev. E54 ('96),376.
Discovery that allowing secular terms appear, and then applying RG-like eq gives `all' basic equations of exiting asymptotic methods

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- 八田佳孝、素粒子論研究 104-4 (2002), 75

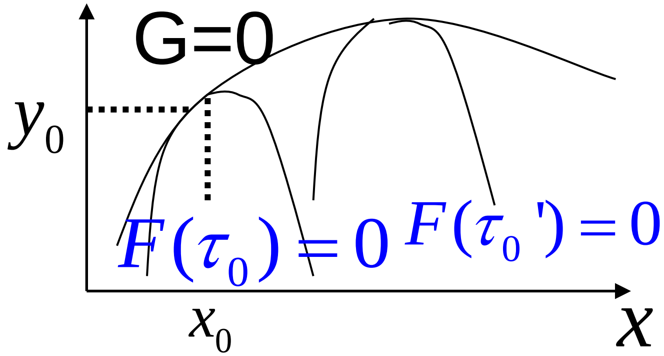
A geometrical interpretation: T.K. ('95)

construction of the envelope of the perturbative solutions

Let $\{C_\tau\}_\tau$ be a family of curves parametrized by τ in the x-y plane;

$$C_\tau : F(x, y, \tau, \mathbf{C}(\tau)) = 0 \;\textbf{?}$$

E: The envelope of C_τ $G(x, y) = 0.$



$$F_{\tau_0}(x_0, y_0, \tau_0, C(\tau_0)) \equiv \frac{\partial F(x_0, y_0, \tau_0)}{\partial \tau_0} + \frac{\partial C}{\partial \tau_0}\frac{\partial F(x_0, y_0, \tau_0, C(\tau_0))}{\partial C} = 0.$$

The envelop equation: $dF / d\tau_0 = 0$ ⟺ RG eq.

the solution is inserted to F with the condition

$\tau_0 = x_0$ ← the tangent point

⇒ $G(x, y) = F(x, y, \mathbf{C}(x))$

A simple example: resummation and extracting slowdynamics

T.K. ('95)

$$\frac{d^2x}{dt^2} + \epsilon\frac{dx}{dt} + x = 0,$$

the dumped oscillator!

$$x(t) = \bar{A}\exp(-\frac{\epsilon}{2}t)\sin(\sqrt{1-\frac{\epsilon^2}{4}}t + \bar{\theta}),$$

$$x(t, t_0) = x_0(t, t_0) + \epsilon x_1(t, t_0) + \epsilon^2 x_2(t, t_0) + \ldots,$$

$$\ddot{x}_0 + x_0 = 0, \qquad \ddot{x}_{n+1} + x_{n+1} = -\dot{x}_n.$$

$$x(t_0, t_0) = W(t_0).$$

$$W(t_0) = W_0(t_0) + \epsilon W_1(t_0) + \epsilon^2 W_2(t_0) + \ldots,$$

$$x_0(t, t_0) = A(t_0)\sin(t + \theta(t_0)), \quad W_0(t_0) = x_0(t_0, t_0) = A(t_0)\sin(t_0 + \theta(t_0)).$$

$$x_1(t, t_0) = -\frac{A}{2}\cdot(t - t_0)\sin(t + \theta), \qquad W_1(t_0) = 0$$

a secular term appears, invalidating P.T.

$$x_2(t) = \frac{A}{8}\{(t-t_0)^2\sin(t+\theta) - (t-t_0)\cos(t+\theta)\}, \qquad W_2(t_0) = 0$$

Secular terms appear again!

Collecting the terms, we have

$$\boxed{\begin{aligned} x(t,t_0) &= A\sin(t+\theta) - \epsilon\frac{A}{2}(t-t_0)\sin(t+\theta) \\ &\quad + \epsilon^2\frac{A}{8}\{(t-t_0)^2\sin(t+\theta) - (t-t_0)\cos(t+\theta)\} \end{aligned}}$$

With I.C.: $W(t_0) = W_0(t_0) = A(t_0)\sin(t_0 + \theta(t_0))$

; parameterized by the functions,

$$A(t_0), \phi(t_0) \equiv t_0 + \theta(t_0)$$

Let us try to construct the envelope function of the set of locally divergent functions, Parameterized by t_0 !

The secular terms invalidate the pert. theroy, like the log-divergence in QFT!

$\{C_{t_0}\}_{t_0}$: $\{x(t,t_0)\}_{t_0}$ $\qquad x_E(t) = x(t,t) = W(t).$

$$\frac{dx(t,t_0)}{dt_0} = 0; \quad t_0 = t. \quad \Longrightarrow \quad A(t_0) \text{ and } \theta(t_0)$$

$$\frac{dA}{dt_0} + \epsilon A = 0, \quad \frac{d\theta}{dt_0} + \frac{\epsilon^2}{8} = 0,$$

$$A(t_0) = \bar{A}\mathrm{e}^{-\epsilon t_0/2}, \quad \theta(t_0) = -\frac{\epsilon^2}{8}t_0 + \bar{\theta},$$

Extracted the amplitude and phase equations, separately!

$$x_E(t) = x(t,t) = W_0(t) = \bar{A}\exp(-\frac{\epsilon}{2}t)\sin((1-\frac{\epsilon^2}{8})t + \bar{\theta}),$$

$$\sqrt{1-\epsilon^2/4} = 1 - \epsilon^2/8 + O(\epsilon^4)$$

The envelop function $x_E(t) = W_0(t)$ an approximate but

global solution in contrast to the pertubative solutions

which have secular terms and valid only in local domains.

c.f. Chen et al ('95)

Notice also the resummed nature!

非線形振動子

Van der Pol eq.: $$\frac{d^2x}{dt^2} + x = \epsilon(1 - x^2)\frac{dx}{dt}.$$

Krylov-Bogoliubov-Mitropolsky の方法

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$$

$$\ddot{x}_0 + x_0 = 0$$

$$x_0(t) = A_0 \cos(t + \theta_0) \equiv u_0(A_0, \phi); \quad (\phi \equiv t + \theta_0)$$

KBM Ansatz:
$$\begin{aligned} x(t) &= u(A, \phi) = u_0(A, \phi) + \rho(A, \phi), \\ \rho(A, \phi) &= \epsilon\rho_1(A, \phi) + \epsilon^2\rho_2(A, \phi) + \cdots, \end{aligned}$$

高次の展開項の一部は 0 次解 u_0 の振幅と位相にくりこむ

その他にくりこめない項 ρ があることを想定 *0* 次解である基準振動を含んではならない。

QEDのパウリ項:に対応

ERGに一般化したものはPolchinsky の定理

$$\frac{dA}{dt} = F(A), \quad \frac{d\phi}{dt} = 1 + \Omega(A).$$

$$F(A) = \epsilon F_1(A) + \epsilon^2 F_2 + \cdots, \quad \Omega(A) = \epsilon\Omega_1(A) + \epsilon^2\Omega_2 + \cdots.$$

$$\frac{\partial^2 \rho_1}{\partial \phi^2} + \rho_1 = (2F_1 - A + \frac{A^3}{4}) \sin\phi + 2A\Omega_1 \cos\phi + \frac{A^3}{4} \sin 3\phi \equiv B_1(A, \phi).$$

可解条件(くりこみ条件): Fredholm の交代定理

B_1がゼロモードを含まない(永年項が出ない)ことを課す。

$$(u_0, B_1) = 0, \quad (\frac{du_0}{d\phi}, B_1) = 0. \quad (f, g) = \int_0^{2\pi} d\phi\, f(\phi) g(\phi).$$

$$\Downarrow$$

$$F_1 = A/2 \cdot (1 - A^2/4),\ \Omega_1 = 0$$

Reduced dynamics

$$\frac{dA}{dt} = \epsilon \frac{A}{2}(1 - \frac{A^2}{4}), \quad \frac{d\phi}{dt} = 1,$$ with a limit cycle with a radius 2!

不変多様体のゆがみ $\rho_1 = -A^3/32 \cdot \sin 3\phi$

不変多様体 $x(t) = A(t) \cos\phi(t) - \epsilon \frac{A^3(t)}{32} \sin 3\phi(t)$

くりこみ群法による扱い

$$\ddot{x} + x = \epsilon(1 - x^2)\dot{x}.$$

Let $\tilde{x}(t;\, t_0)$ be a local solution around $t \sim \forall t_0$

Expand: $\tilde{x}(t;\, t_0) = \tilde{x}_0(t;\, t_0) + \epsilon\, \tilde{x}_1(t;\, t_0) + \epsilon^2\, \tilde{x}_2(t;\, t_0) + \cdots.$

I.C. $W(t_0) \equiv \tilde{x}(t_0;\, t_0) = x(t_0)$;supposed to be an exact sol.

$$W(t_0) = W_0(t_0) + \epsilon\, W_1(t_0) + \epsilon^2\, W_2(t_0) + \cdots.$$

$O(\varepsilon^0)$

$$\mathcal{L}\tilde{x}_0 \equiv \left[\frac{d^2}{dt^2} + 1\right]\tilde{x}_0 = 0, \qquad \tilde{x}_0(t;\, t_0) = A(t_0)\, \cos(t + \theta(t_0)).$$

$O(\varepsilon^1)$

3倍角の公式

$$\mathcal{L}\tilde{x}_1 = -A\left(1 - \frac{A^2}{4}\right)\sin\phi(t) + \frac{A^3}{4}\sin 3\,\phi(t),$$

$$\phi(t) = t + \theta_0(t_0).$$

$$\tilde{x}_1(t;t_0) = (t-t_0)\,\frac{A}{2}\left(1-\frac{A^2}{4}\right)\cos\phi(t) - \frac{A^3}{32}\,\sin 3\,\phi(t).$$

素朴な摂動論;永年項の出現を許す:ただし、初期時間でゼロになるように解を書き下す。

RG/Envelope equation:

$$\left.\frac{\mathrm{d}\tilde{x}}{\mathrm{d}t_0}\right|_{t_0=t} = 0, \qquad \tilde{x} = \tilde{x}_0 + \epsilon\,\tilde{x}_1$$

$$\Downarrow$$

$$\dot{A} = \epsilon\,\frac{A}{2}\left(1-\frac{A^2}{4}\right),$$
$$\dot{\phi} = 1.$$

The same reduced dynamics as KBM derived

Extracted the asymptotic dynamics that is a slow dynamics.

An approximate but globally valid sol. as the envelope;

$$x_{\mathrm{E}}(t) \equiv \tilde{x}(t;\,t) = W(t) = A(t)\,\cos(t+\theta_0) - \epsilon\,\frac{A^3(t)}{32}\,\sin(3\,t + 3\,\theta_0),$$

非摂動多様体

摂動による多様体の歪み

(Mysterious) **KBM ansatzが自然に導出された！！**
非摂動(ゼロ次)解の積分定数が不変多様体の自然な座標を与える。

A foundation of the RG method a la ERG. T.K. (1998); Ei, Fujii and T.K.('00)

Let us take the following n-dimensional equation;

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} = \boldsymbol{F}(\boldsymbol{X}, t), \tag{B·11}$$

where n may be infinity. Let $\boldsymbol{X}(t) = \boldsymbol{W}(t)$ be an yet unknown exact solution to Eq.(B·11), and we try to solve the equation with the initial condition at $t = \forall t_0$;

$$\boldsymbol{X}(t = t_0) = \boldsymbol{W}(t_0). \tag{B·12}$$

Then, the solution may be written as $\boldsymbol{X}(t; t_0, \boldsymbol{W}(t_0))$.

$$\boldsymbol{X}(t; t_0, \boldsymbol{W}(t_0)) = \boldsymbol{X}(t; t_0', \boldsymbol{W}(t_0')).$$

Taking the limit $t_0' \to t_0$, we have

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t_0} = \frac{\partial \boldsymbol{X}}{\partial t_0} + \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{W}}\frac{\mathrm{d}\boldsymbol{W}}{\mathrm{d}t_0} = \boldsymbol{0}.$$

Pert. Theory:

$\boldsymbol{X}(t; t_0, \boldsymbol{W}(t_0))$ and $\boldsymbol{X}(t; t_0', \boldsymbol{W}(t_0'))$ may be valid only for $t \sim t_0$ and $t \sim t_0'$,

$$t_0 < t < t_0' \;\; (\text{or } t_0' < t < t_0)$$

$$\left.\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t_0}\right|_{t_0=t} = \left.\frac{\partial \boldsymbol{X}}{\partial t_0}\right|_{t_0=t} + \left.\frac{\partial \boldsymbol{X}}{\partial \boldsymbol{W}}\frac{\mathrm{d}\boldsymbol{W}}{\mathrm{d}t_0}\right|_{t_0=t} = \boldsymbol{0},$$ with $t_0 = t$ **RG equation!**

Let $\boldsymbol{X}(t;t_0)$ is an approximate solution to Eq.(B·11) around $t \sim t_0$;

$$\frac{\mathrm{d}\boldsymbol{X}(t;t_0)}{\mathrm{d}t} \simeq \boldsymbol{F}(\boldsymbol{X}(t;t_0),t).$$

Then, we have

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{W}(t)}{\mathrm{d}t} &= \left.\frac{\partial \boldsymbol{X}(t;t_0)}{\partial t}\right|_{t_0=t} + \left.\frac{\partial \boldsymbol{X}(t;t_0)}{\partial t_0}\right|_{t_0=t} \\ &= \left.\frac{\partial \boldsymbol{X}(t;t_0)}{\partial t}\right|_{t_0=t} \\ &\simeq \boldsymbol{F}(\boldsymbol{X}(t;t_0),\, t)|_{t_0=t}, \\ &= \boldsymbol{F}(\boldsymbol{W}(t),t),\end{aligned}$$

showing that our envelope function satisfies the original equation (B.11) in the global domain uniformly.

Generic example with zero modes

S.Ei, K. Fujii & T.K.('00)

$$\partial_t \mathbf{u} = A\mathbf{u} + \epsilon \mathbf{F}(\mathbf{u}), \quad |\epsilon| < 1$$

$$\mathbf{u}(t;t_0) = \mathbf{u}_0(t;t_0) + \epsilon \mathbf{u}_1(t;t_0) + \epsilon^2 \mathbf{u}_2(t;t_0) + \cdots$$

$$\begin{aligned}\mathbf{W}(t_0) &= \mathbf{W}_0(t_0) + \epsilon \mathbf{W}_1(t_0) + \epsilon^2 \mathbf{W}_2(t_0) + \cdots, \\ &= \mathbf{W}_0(t_0) + \boldsymbol{\rho}(t_0),\end{aligned}$$

$$\begin{aligned}(\partial_t - A)\mathbf{u}_0 &= 0, \\ (\partial_t - A)\mathbf{u}_1 &= \mathbf{F}(\mathbf{u}_0), \\ (\partial_t - A)\mathbf{u}_2 &= \mathbf{F}'(\mathbf{u}_0)\mathbf{u}_1,\end{aligned}$$

$$(\mathbf{F}'(\mathbf{u}_0)\mathbf{u}_1)_i = \sum_{j=1}^{n} \left\{\partial(F'(\mathbf{u}_0))_i / \partial(u_0)_j\right\}(u_1)_j$$

When A has semi-simple zero eigenvalues.

$$A\mathrm{U}_i = 0, \quad (i = 1, 2, \ldots, m).$$

We suppose that other eigenvalues have negative real parts;

$$A\mathrm{U}_\alpha = \lambda_\alpha \mathrm{U}_\alpha, \quad (\alpha = m+1, m+2, \cdots, n),$$

where $\mathrm{Re}\lambda_\alpha < 0$. One may assume without loss of generality that U_i's and U_α's are linearly independent.

The adjoint operator $A^\dagger$ has the same eigenvalues as A has;

$$A^\dagger \tilde{\mathrm{U}}_i = 0, \quad (i = 1, 2, \ldots, m),$$
$$A^\dagger \tilde{\mathrm{U}}_\alpha = \lambda^*_\alpha \tilde{\mathrm{U}}_\alpha, \quad (\alpha = m+1, m+2, \cdots, n).$$

Def. P **the projection onto the kernel** $\ker A$

$$P + Q = 1$$

Since we are interested in the asymptotic state as $t \to \infty$, we may assume that the lowest-order initial value belongs to $\ker A$:

$$\mathbf{W}_0(t_0) = \sum_{i=1}^{m} C_i(t_0)\mathbf{U}_i = \mathbf{W}_0[\boldsymbol{C}]. \quad \Longleftrightarrow \quad \mathrm{M}_0$$

$$\mathbf{u}_0(t;t_0) \ = \ \mathrm{e}^{(t-t_0)A}\mathbf{W}_0(t_0) \ = \ \textstyle\sum_{i=1}^{m} C_i(t_0)\mathbf{U}_i.$$

Parameterized with m variables,
Instead of n!

$$\boldsymbol{C} = {}^t(C_1, C_2, \cdots, C_m)$$

$$\mathbf{u}_1(t;t_0) = \mathrm{e}^{(t-t_0)A}[\mathbf{W}_1(t_0) + A^{-1}Q\mathbf{F}(\mathbf{W}_0(t_0))] \\ +(t-t_0)P\mathbf{F}(\mathbf{W}_0(t_0)) - A^{-1}Q\mathbf{F}(\mathbf{W}_0(t_0)).$$

The would-be rapidly changing terms can be eliminated by the choice;

$$\mathbf{W}_1(t_0) = -A^{-1}Q\mathbf{F}(\mathbf{W}_0(t_0)), \qquad P\mathbf{W}_1(t_0) = 0$$

Then, the secular term appears only the P space;

$$\mathbf{u}_1(t;t_0) = (t-t_0)P\mathbf{F} - A^{-1}Q\mathbf{F}$$

a deformation of the manifold ρ

Deformed (invariant) slow manifold: $M_1 = \{\mathbf{u} | \mathbf{u} = \mathbf{W}_0 - \epsilon A^{-1} Q \mathbf{F}(\mathbf{W}_0)\}$

$$\mathbf{u}(t; t_0) = \mathbf{W}_0 + \epsilon\{(t - t_0) P\mathbf{F} - A^{-1} Q \mathbf{F}\}$$

A set of locally divergent functions parameterized by t_0 !

The RG/E equation $\partial \mathbf{u} / \partial t_0 \big|_{t_0 = t} = \mathbf{0}$ gives the envelope, which is globally valid:

$$\dot{\mathbf{W}}_0(t) = \epsilon P \mathbf{F}(\mathbf{W}_0(t)),$$

which is reduced to an m-dimensional coupled equation,

$$\dot{C}_i(t) = \epsilon \langle \tilde{\mathbf{U}}_i, \mathbf{F}(\mathbf{W}_0[C]) \rangle, \quad (i = 1, 2, \cdots, m).$$

The global solution (the invariant manifod):

$$\mathbf{u}(t) = \mathbf{u}(t; t_0 = t) = \sum_{i=1}^{m} C_i(t) \mathbf{U}_i - \epsilon A^{-1} Q \mathbf{F}(\mathbf{W}_0[C]).$$

We have derived the invariant manifold and the **slow dynamics** on the manifold by the RG method.

Extension; (a) A Is not semi-simple. (2) Higher orders.

(Ei,Fujii and T.K. Ann.Phys.('00))

Layered pulse dynamics for TDGL and NLS.

The RG/E equation $\partial \mathbf{u} / \partial t_0 \big|_{t_0 = t} = \mathbf{0}$

gives the envelope, which is globally valid:

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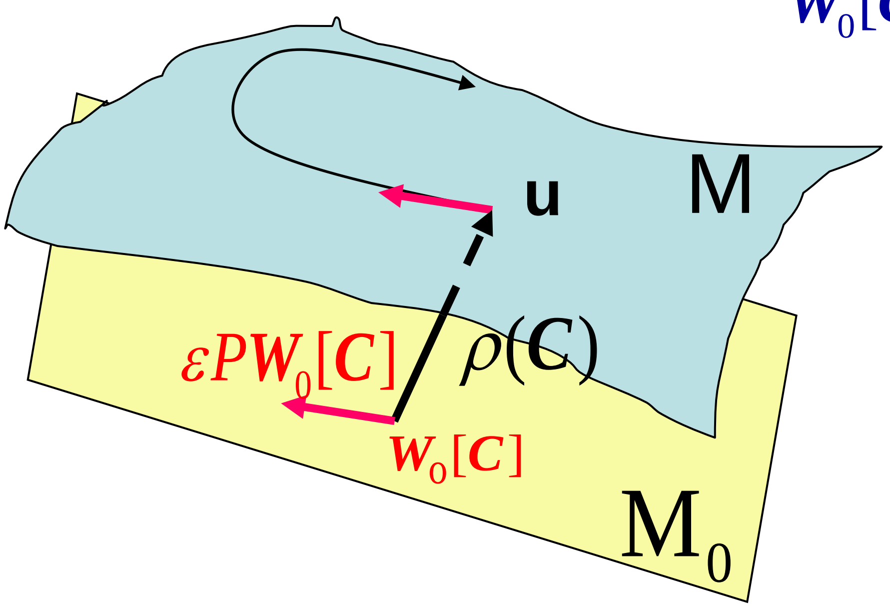
The global solution (the invariant manifod):

$$\mathbf{u}(t) = \mathbf{u}(t; t_0 = t) = \underbrace{\sum_{i=1}^{m} C_i(t) \mathbf{U}_i}_{W_0[C]} \underbrace{- \epsilon A^{-1} Q \mathbf{F}(\mathbf{W}_0[C])}_{\rho(C)}.$$



c.f. Polchinski theorem
in renormalization theory
In QFT.

We have derived the invariant manifold and the **slow dynamics** on the manifold by the RG method.

Extension;

a) A Is not semi-simple.(ジョルダン細胞を持つ場合） b) Higher orders.

c) PD equations;

Layered pulse dynamics for TDGL and Non-lin.Schroedinger.

d) Reduction of stocastic equation with several variables

S. Ei, K. Fujii and T.K. , Ann.Phys.('00)
Y. Hatta and T.K. Ann. Phys. (2002)

非摂動線形演算子がジョルダン細胞を持つ場合の例

KdV 方程式にしたがうソリトン-ソリトン相互作用

$$\partial t u + 6u\partial x u + \partial_x^3 u = 0, \qquad \partial_t u = F[u], \quad F[u] \equiv c\partial_z - 6u\partial_z u - \partial_z^3 u.$$

$$u(x,\ t) = \frac{c}{2}\mathrm{sech}^2\Big[\frac{\sqrt{c}(x-ct)}{2}\Big] \equiv \varphi(x-ct,\ c)$$

時刻 t_0 において、中心がそれぞれ $z_i(t_0)$ $(i = i,\ 2)$ にあるとする。

$$u(z,t) = \varphi(z - z_1(t_0); c) + \varphi(z - z_2(t_0); c) + v(z,t)$$

線形演算子 $A = F'[\varphi] = c\partial_z - \partial_z^2 - 6(\partial_z\phi + \varphi\partial_z)$

Aはジョルダン細胞を持つ： $AU_1 = 0 \quad AU_2 = U_1 \qquad U_1 = \partial_z\varphi, \quad U_2 = -\partial_c\varphi$

RG法による(new) reduced dynamics(Ei, Fujii and T.K.):

$$\dot{z}_1 = -C_1(t) - 28c\mathrm{e}^{-\sqrt{c}(z_2-z_1)}, \quad \dot{C}_1 = 16c^{5/2}\mathrm{e}^{-\sqrt{c}(z_2-z_1)}$$

C_1を消去して、

$$\ddot{z}_1 = -16c^{5/2}\mathrm{e}^{-\sqrt{c}(z_2-z_1)}, \quad \ddot{z}_2 = 16c^{5/2}\mathrm{e}^{-\sqrt{c}(z_2-z_1)}$$

(Ei-Ohta, 1994)

Hydrodynamic limit of Boltzmann equation

---RG method---

The separation of scales in the relativistic heavy-ion collisions

Liouville → Boltzmann → Fluid dyn.

Hamiltonian

Navier−Stokes eq.

Slower dynamics

on the basis of the RG method; Chen-Goldenfeld-Oono('95),T.K.('95)

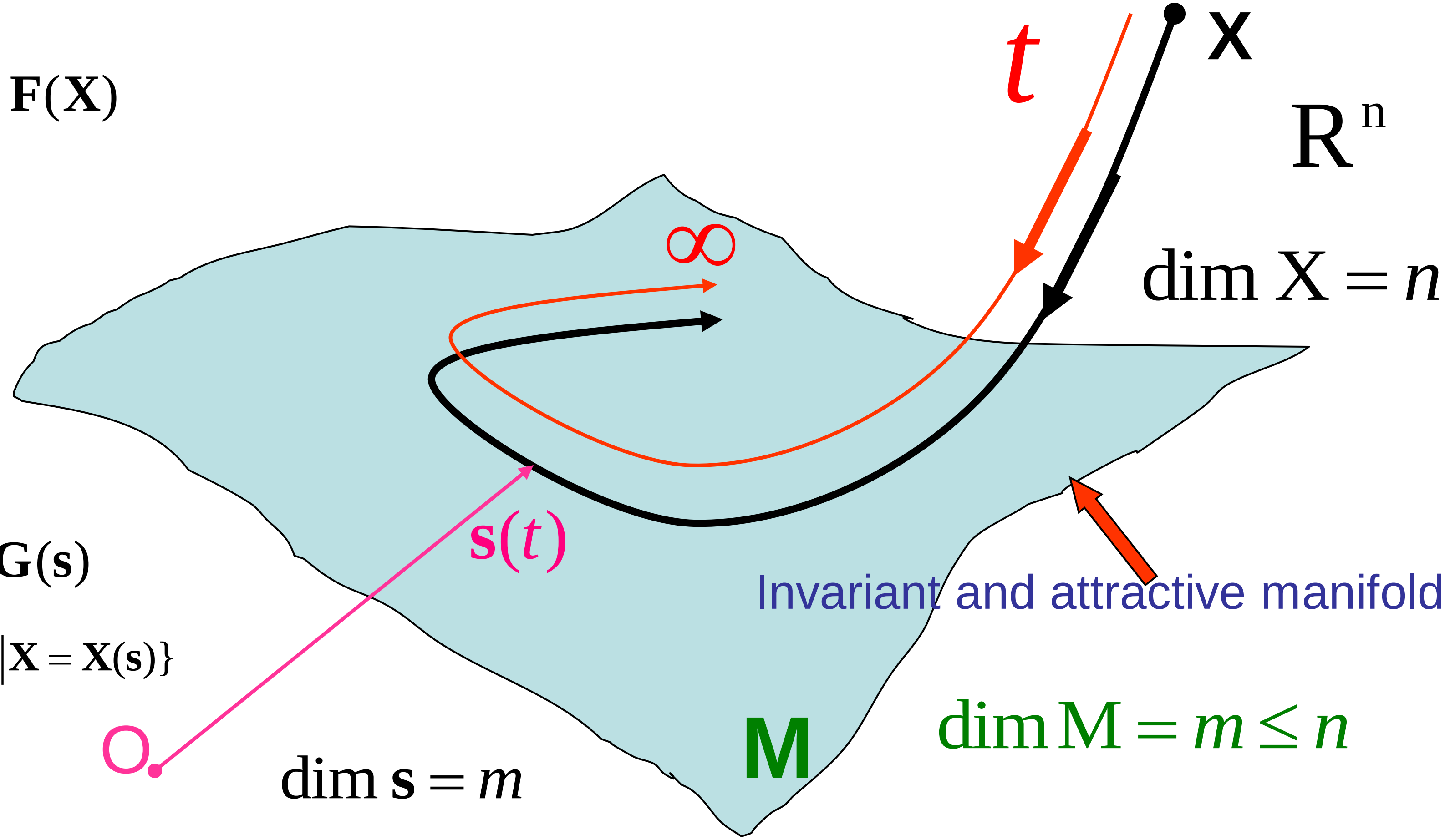
C.f. Y. Hatta and T.K. ('02) , K.Tsumura and TK ('05)

Hydrodynamics is the effective dynamics of the kinetic (Boltzmann) equation in the infrared refime.

Geometrical image of reduction of dynamics

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X})$$

$$\left\{\begin{array}{l} \dfrac{d\mathbf{s}}{dt} = \mathbf{G}(\mathbf{s}) \\ \mathrm{M}=\{\mathbf{X} \,|\, \mathbf{X} = \mathbf{X}(\mathbf{s})\} \end{array}\right.$$



eg.

$\mathbf{X} = f(\mathbf{r}, \mathbf{p})$; **distribution function in the phase space (infinite dimensions)**

$\mathbf{s} = \{u^{\mu}, T, n\}$; **the hydrodinamic quantities (5 dimensions), conserved quantities.**

Relativistic Boltzmann equation

$$p^\mu \partial_\mu f_p(x) = C[f]_p(x),$$

Collision integral: $C[f]_p(x) \equiv \frac{1}{2!}\sum_{p_1}\frac{1}{p_1^0}\sum_{p_2}\frac{1}{p_2^0}\sum_{p_3}\frac{1}{p_3^0}\,\omega(p\,,\,p_1|p_2\,,\,p_3)\Big(f_{p_2}(x)\,f_{p_3}(x) - f_p(x)\,f_{p_1}(x)\Big),$

Symm. property of the transition probability:

$$\omega(p\,,\,p_1|p_2\,,\,p_3) = \omega(p_2\,,\,p_3|p\,,\,p_1) = \omega(p_1\,,\,p|p_3\,,\,p_2) = \omega(p_3\,,\,p_2|p_1\,,\,p) \quad \text{--- (1)}$$

Energy-mom. conservation; $\omega(p\,,\,p_1|p_2\,,\,p_3) \propto \delta^4(p+p_1-p_2-p_3)$ --- (2)

Owing to (1),

$$\sum_p \frac{1}{p^0}\varphi_p(x)\,C[f]_p(x) = \frac{1}{2!}\sum_p\frac{1}{p^0}\sum_{p_1}\frac{1}{p_1^0}\sum_{p_2}\frac{1}{p_2^0}\sum_{p_3}\frac{1}{p_3^0}\frac{1}{4}\Big[\omega(p\,,\,p_1|p_2\,,\,p_3)\Big(\varphi_p(x)+\varphi_{p_1}(x)-\varphi_{p_2}(x)-\varphi_{p_3}(x)\Big)\times\Big(f_{p_2}(x)\,f_{p_3}(x)-f_p(x)\,f_{p_1}(x)\Big)\Big]. \quad (3)$$

Collision Invariant $\varphi_p(x)$ **:** $\displaystyle\sum_p \frac{1}{p^0}\varphi_p(x)\,C[f]_p(x) = 0.$

Eq.'s (3) and (2) tell us that

the general form of a collision invariant; $\varphi_p(x) = \alpha(x) + p^\mu\,\beta_\mu(x),$

which can be x-dependent!

Local equilibrium distribution

The entropy current: $$S^\mu(x) \equiv -\sum_p \frac{1}{p^0} p^\mu f_p(x) (\ln f_p(x) - 1)$$

$$\partial_\mu S^\mu(x) = -\sum_p \frac{1}{p^0} C[f]_p(x) \ln f_p(x).$$

Conservation of entropy $\longrightarrow$ $\ln f_p(x) = \alpha(x) + p^\mu \beta_\mu(x),$

$$f_p(x) = \frac{1}{(2\pi)^3} \exp\left[\frac{\mu(x) - p^\mu u_\mu(x)}{T(x)}\right] \equiv f_p^{\mathrm{eq}}(x)$$

i.e., **the local equilibrium distribution fn;**

(Maxwell-Juettner dist. fn.)

Remark:

Owing to the energy-momentum conservation,
the collision integral also vanishes for the local equilibrium distribution fn.;

$$C[f_p^{eq}](x) = 0.$$

Typical hydrodynamic equations for a viscous fluid
--- Choice of the frame and ambiguities in the form ---

Fluid dynamics = a system of balance equations

$$\partial_\mu T^{\mu\nu} = 0, \qquad \partial_\mu N^\mu = 0.$$

energy−momentum : $T^{\mu\nu}$ number : N^μ

$$T^{\mu\nu} \equiv \epsilon u^\mu u^\nu - p \Delta^{\mu\nu} + \delta T^{\mu\nu} \qquad N^\mu \equiv n u^\mu + \delta N^\mu$$

Dissipative part

Eckart eq. no dissipation in the number flow; ⇨ Describing the flow of matter

$$\delta T^{\mu\nu} = u^\mu T \lambda \left(\frac{1}{T} \nabla^\nu T - D u^\nu \right) + u^\nu T \lambda \left(\frac{1}{T} \nabla^\mu T - D u^\mu \right)$$
$$+ 2\eta \frac{1}{2} \left(\nabla^\mu u^\nu + \nabla^\nu u^\mu - \frac{2}{3} \Delta^{\mu\nu} \nabla \cdot u \right) + \zeta \Delta^{\mu\nu} \nabla \cdot u$$
$$\delta N^\mu = 0.$$

with $D \equiv u^\mu \partial_\mu \quad \nabla^\mu \equiv \Delta^{\mu\nu} \partial_\nu$

$\Delta_p^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu \equiv \Delta^{\mu\nu}$,

--- Involving time-like derivative ---.

Landau−Lifshits no dissipation in energy flow ⇨ describing the energy flow.

$$\delta T^{\mu\nu} = 2\eta \frac{1}{2} \left(\nabla^\mu u^\nu + \nabla^\nu u^\mu - \frac{2}{3} \Delta^{\mu\nu} \nabla \cdot u \right) + \zeta \Delta^{\mu\nu} \nabla \cdot u,$$
$$\delta N^\mu = -\lambda \frac{nT}{\epsilon + p} \left(\frac{1}{T} \nabla^\mu T - \frac{1}{\epsilon + p} \nabla^\mu p \right)$$

⇔

$\delta T^{\mu\nu} u_\nu = 0$, No dissipative energy-density nor energy-flow

$u_\mu \delta N^\mu = 0$ No dissipative particle density

--- Involving only space-like derivatives ---

with transport coefficients:

ς ; Bulk viscocity, η; Shear viscocity

λ ;Heat conductivity

Previous attempts to derive the dissipative hydrodynamics as a reduction of the dynamics

N.G. van Kampen, J. Stat. Phys. 46(1987), 709

unique but non-covariant form and hence not Landau either Eckart!

Cf. Chapman-Enskog method to derive Landau and Eckart eq.'s; see, eg, de Groot et al ('80)

Here,

In the covariant formalism,
in a unified way and systematically
derive dissipative rel. hydrodynamics at once!

Derivation of the relativistic hydrodynamic equation from the rel. Boltzmann eq. --- an RG-reduction of the dynamics

K. Tsumura, T.K. K. Ohnishi; Phys. Lett. B646 (2007) 134-140

c.f. Non-rel. Y.Hatta and T.K., Ann. Phys. 298 ('02), 24; T.K. and K. Tsumura, J.Phys. A:39 (2006), 8089

Ansatz of the origin of the dissipation= the spatial inhomogeneity,
leading to Navier-Stokes in the non-rel. case .

$\boldsymbol{a}_p^\mu$ would become a macro flow-velocity ⟵ **Coarse graining of space-time**

$\boldsymbol{a}_p^\mu$ may not be u^μ

$$\tau \equiv \boldsymbol{a}_p^\mu x_\mu, \quad \sigma^\mu \equiv \left(g^{\mu\nu} - \frac{\boldsymbol{a}_p^\mu \boldsymbol{a}_p^\nu}{\boldsymbol{a}_p^2} \right) x_\nu \equiv \boldsymbol{\Delta}_p^{\mu\nu} x_\nu \qquad x^\mu \Longrightarrow \tau \quad \sigma^\mu$$

$$\frac{\partial}{\partial \tau} = \frac{1}{\boldsymbol{a}_p^2} \boldsymbol{a}_p^\mu \partial_\mu \equiv \boldsymbol{D},$$ time-like derivative

$$\boldsymbol{\Delta}_p^{\mu\nu} \frac{\partial}{\partial \sigma^\nu} = \boldsymbol{\Delta}_p^{\mu\nu} \partial_\nu \equiv \boldsymbol{\nabla}^\mu$$ space-like derivative

Rewrite the Boltzmann equation as,

$$\Longrightarrow \quad \frac{\partial}{\partial \tau} f_p(\tau, \sigma) = \frac{1}{p \cdot \boldsymbol{a}_p} C[f]_p(\tau, \sigma) - \underbrace{\frac{1}{p \cdot \boldsymbol{a}_p} p \cdot \boldsymbol{\nabla} f_p(\tau, \sigma)}_{\text{perturbation}}$$

Only spatial inhomogeneity leads to dissipation.

RG gives a resummed distribution function, from which $T^{\mu\nu}$ and N^μ are obtained.

Solution by the perturbation theory

0th
$$\frac{\partial}{\partial\tau}\tilde{f}_p^{(0)} = \frac{1}{p\cdot \boldsymbol{a}_p} C[f]_p \Big|_{f=\tilde{f}^{(0)}}$$

"slow" $\frac{\partial}{\partial\tau}\tilde{f}_p^{(0)} = 0 \Longrightarrow \frac{1}{p\cdot \boldsymbol{a}_p} C[f]_p \Big|_{f=\tilde{f}^{(0)}} = 0$

$$\Longrightarrow \tilde{f}_p^{(0)}(\tau\,,\,\sigma\,;\,\tau_0) = (2\pi)^{-3}\exp\left[\frac{\mu(\sigma\,;\,\tau_0) - p^\mu u_\mu(\sigma\,;\,\tau_0)}{T(\sigma\,;\,\tau_0)}\right] \equiv f_p^{\mathrm{eq}}(\sigma\,;\,\tau_0)$$

$$\Longrightarrow \tilde{f}^{(0)}(\tau) = f^{\mathrm{eq}}$$

written in terms of the hydrodynamic variables.
Asymptotically, the solution can be written solely in terms of the hydrodynamic variables.

- Five conserved quantities $T(\sigma\,;\,\tau_0),\ \mu(\sigma\,;\,\tau_0),\ u_\mu(\sigma\,;\,\tau_0)$

reduced degrees of freedom $\boxed{m = 5}$ $\boxed{u^\mu(\sigma\,;\,\tau_0)\,u_\mu(\sigma\,;\,\tau_0) = 1}$

- 0th invariant manifold $f_p^{(0)}(\tau_0\,,\,\sigma) = f_p^{\mathrm{eq}}(\sigma\,;\,\tau_0)$

$$\Longrightarrow f^{(0)}(\tau_0) = f^{\mathrm{eq}}$$

Local equilibrium

1st

$$\frac{\partial}{\partial \tau}\tilde{f}_p^{(1)} = \sum_q A_{pq}\,\tilde{f}_q^{(1)} + F_p$$

Evolution op.：

$$A_{pq} \equiv \frac{1}{p\cdot \boldsymbol{a}_p}\frac{\partial}{\partial f_q}C[f]_p\Big|_{f=f^{\mathrm{eq}}}$$

inhomogeneous：

$$F_p \equiv -\frac{1}{p\cdot \boldsymbol{a}_p}\,p\cdot \boldsymbol{\nabla} f_p^{\mathrm{eq}}$$

Collision operator

$$L_{pq} \equiv f_p^{\mathrm{eq}\text{-}1}\,A_{pq}\,f_q^{\mathrm{eq}}$$

$$L_{pq} = -\frac{1}{p\cdot \boldsymbol{a}_p}\frac{1}{2!}\sum_{p_1}\frac{1}{p_1^0}\sum_{p_2}\frac{1}{p_2^0}\sum_{p_3}\frac{1}{p_3^0}\,\omega(p\,,p_1|p_2\,,p_3)\,f_{p_1}^{\mathrm{eq}}\left(\delta_{pq}+\delta_{p_1q}-\delta_{p_2q}-\delta_{p_3q}\right)$$

The lin. op. L has good properties:

Def. **inner product:** $\langle\,\varphi\,,\,\psi\,\rangle \equiv \sum_p \frac{1}{p^0}(p\cdot \boldsymbol{a}_p)\,f_p^{\mathrm{eq}}\,\varphi_p\,\psi_p$

1. $\langle\,\varphi\,,\,L\,\psi\,\rangle = \langle\,L\,\varphi\,,\,\psi\,\rangle$ — Self-adjoint

2. $\langle\,\varphi\,,\,L\,\varphi\,\rangle \leq 0$ for all φ — Semi-negative definite

3. $L\,\varphi_0^{\alpha} = 0 \Longrightarrow \varphi_{0p}^{\alpha} = \begin{cases} p^{\mu} & \alpha=\mu, \\ m & \alpha = 4 \end{cases}$

L has 5 zero modes、other eigenvalues are negative.

1. Proof of self-adjointness

$$
\begin{aligned}
\langle \varphi \,,\, L\,\psi \rangle &= \sum_{pq} \frac{1}{p^0} (p \cdot \boldsymbol{a}_p)\, f_p^{\rm eq}\, \varphi_p\, L_{pq}\, \psi_q \\
&= -\frac{1}{4}\frac{1}{2!} \sum_{p} \frac{1}{p^0} \sum_{p_1} \frac{1}{p_1^0} \sum_{p_2} \frac{1}{p_2^0} \sum_{p_3} \frac{1}{p_3^0}\, \omega(p \,,\, p_1 | p_2 \,,\, p_3) \\
&\qquad\qquad f_p^{\rm eq}\, f_{p_1}^{\rm eq} \Big(\varphi_p + \varphi_{p_1} - \varphi_{p_2} - \varphi_{p_3}\Big) \Big(\psi_p + \psi_{p_1} - \psi_{p_2} - \psi_{p_3}\Big) \\
&= \langle L\,\varphi \,,\, \psi \rangle .
\end{aligned}
$$

2. Semi-negativeness of the L

$$
\begin{aligned}
\langle \varphi \,,\, L\,\varphi \rangle &= -\frac{1}{4}\frac{1}{2!} \sum_{p} \frac{1}{p^0} \sum_{p_1} \frac{1}{p_1^0} \sum_{p_2} \frac{1}{p_2^0} \sum_{p_3} \frac{1}{p_3^0}\, \omega(p \,,\, p_1 | p_2 \,,\, p_3)\, f_p^{\rm eq}\, f_{p_1}^{\rm eq} \Big(\varphi_p + \varphi_{p_1} - \varphi_{p_2} - \varphi_{p_3}\Big)^2 \\
&\leq 0 \text{ for all } \varphi
\end{aligned}
$$

3.Zero modes

$$
\varphi_{0p}^{\alpha} = \begin{cases} p^{\mu} & \alpha = \mu \qquad \text{en-mom.} \\ m & \alpha = 4 \qquad \text{Particle \#} \end{cases}
$$

$$
\varphi_p + \varphi_{p_1} = \varphi_{p_2} + \varphi_{p_3}
$$

Collision invariants!
or conserved quantities.

Def. Projection operators:

$$\begin{cases} \left[P\,\psi\right]_p \equiv \sum_{\alpha\beta} \varphi_{0p}^{\alpha}\,\eta_{\alpha\beta}^{-1}\,\langle\,\varphi_0^{\beta}\,,\,\psi\,\rangle, \\ Q \equiv 1 - P. \end{cases}$$

metric

$$\eta^{\alpha\beta} \equiv \langle\,\varphi_0^{\alpha}\,,\,\varphi_0^{\beta}\,\rangle$$

$$\eta_{\alpha\beta}^{-1}\ ;\ \sum_{\gamma} \eta^{\alpha\gamma}\,\eta_{\gamma\beta}^{-1} = \delta_{\beta}^{\alpha}$$

$$\frac{\partial}{\partial\tau}\tilde{f}^{(1)} = A\,\tilde{f}^{(1)} + F$$

$$\Rightarrow \tilde{f}^{(1)}(\tau) = e^{(\tau-\tau_0)A}\left\{\underline{f^{(1)}(\tau_0)} + \underline{A^{-1}\,\bar{Q}\,F}\right\} + (\tau-\tau_0)\,\bar{P}\,F - A^{-1}\,\bar{Q}\,F.$$

The initial value yet not determined

fast motion to be avoided

$$\bar{P} \equiv f^{\mathrm{eq}}\,P\,f^{\mathrm{eq}\,-1},\qquad \bar{Q} \equiv f^{\mathrm{eq}}\,Q\,f^{\mathrm{eq}\,-1}.$$

$$f_{pq}^{\mathrm{eq}} \equiv f_p^{\mathrm{eq}}\,\delta_{pq}$$

$$\Rightarrow \tilde{f}^{(1)}(\tau) = (\tau-\tau_0)\,\bar{P}\,F - A^{-1}\,\bar{Q}\,F$$

eliminated by the choice

Modification of the manifold： $f^{(1)}(\tau_0) = -A^{-1}\,\bar{Q}\,F$

Second order solutions

$$\frac{\partial}{\partial \tau}\tilde{f}^{(2)} = A\,\tilde{f}^{(2)} + I$$

with $I_p \equiv \frac{1}{p\cdot \boldsymbol{a}_p}\, p\cdot \boldsymbol{\nabla}\left[A^{-1}\,\bar{Q}\,F\right]_p$

$$\tilde{f}^{(2)}(\tau) = e^{(\tau-\tau_0)A}\left\{\underline{f^{(2)}(\tau_0)} + A^{-1}\,\bar{Q}\,I\right\} + (\tau-\tau_0)\,\bar{P}\,I - A^{-1}\,\bar{Q}\,I$$

The initial value not yet determined

fast motion

$$\tilde{f}^{(2)}(\tau) = (\tau-\tau_0)\,\bar{P}\,I - A^{-1}\,\bar{Q}\,I.$$

eliminated by the choice

- Modification of the invariant manifold in the 2^{nd} order; $f^{(2)}(\tau_0) = -A^{-1}\,\bar{Q}\,I,$

Application of RG/E equation to derive slow dynamics

Collecting all the terms, we have;

- Invariant manifold (hydro dynamical coordinates) as the initial value:

$$f(\tau_0) = f^{\mathrm{eq}} + \varepsilon\Big(- A^{-1}\,\bar{Q}\,F\Big) + \varepsilon^2\Big(- A^{-1}\,\bar{Q}\,I\Big) + O(\varepsilon^3),$$

- The perturbative solution with secular terms:

$$\tilde{f}(\tau) = f^{\mathrm{eq}} + \varepsilon\Big(\underline{(\tau - \tau_0)}\,\bar{P}\,F - A^{-1}\,\bar{Q}\,F\Big) + \varepsilon^2\Big(\underline{(\tau - \tau_0)}\,\bar{P}\,I - A^{-1}\,\bar{Q}\,I\Big) + O(\varepsilon^3).$$

RG/E equation

$$\left.\frac{\mathrm{d}}{\mathrm{d}\tau_0}\tilde{f}_p(\tau\,,\,\sigma\,;\,\tau_0)\right|_{\tau_0=\tau} = 0.$$

The meaning of $\tau_0 = \tau$ $\Rightarrow$ found to be the coarse graining condition

The novel feature in the relativistic case;

Choice of the flow $\boldsymbol{a}_p^{\mu}$ $\quad \boldsymbol{a}_p^{\mu} = u^{\mu}$ Unique! (Tsumura, T.K. (2012))

$$\partial_\mu J^{\mu\alpha}_{\mathrm{hydro}} = 0,$$

$$J^{\mu\alpha}_{\mathrm{hydro}} \equiv \sum_p \frac{1}{p^0} p^\mu \varphi^\alpha_{0p} \left\{ f^{\mathrm{eq}}_p - [A^{-1} \bar{Q} F]_p \right\} = J^{\mu\alpha}_0 + \Delta J^{\mu\alpha},$$

$$J^{\mu\alpha}_0 \equiv \sum_p \tfrac{1}{p^0} p^\mu \varphi^\alpha_{0p} f^{\mathrm{eq}}_p$$

$$\Delta J^{\mu\alpha} \equiv -\sum_p \tfrac{1}{p^0} p^\mu \varphi^\alpha_{0p} [A^{-1} \bar{Q} F]_p$$

produce the dissipative terms!

The distribution function;

$$f(\tau_0) = f^{\mathrm{eq}} - A^{-1} \bar{Q} F - A^{-2} \bar{Q} H - A^{-1} \bar{Q} I$$

Notice that the distribution function as the solution is represented solely by the hydrodynamic quantities!

Evaluation of the dissipative term

thermal forces

$$\left[A^{-1}\,\bar{Q}\,F\right]_p = f_p^{\rm eq} \sum_q L_{pq}^{-1} \frac{1}{q\cdot\theta_q}\,\frac{1}{T}\left(\Pi_q\,X_\theta + Q_q^\mu\,X_{\theta\mu} + \Pi_q^{\mu\nu}\,X_{\mu\nu}\right)$$

The microscopic representation of the dissipative flow:

bulk pressure $\Pi_p \equiv \left(\frac{4}{3}-\gamma\right)(p\cdot u)^2 + \{(\gamma-1)T\,\hat{h} - \gamma T\}\,(p\cdot u) - \frac{1}{3}m^2$

heat flow $Q_p^\mu \equiv -\{(p\cdot u) - T\,\hat{h}\}\,\Delta^{\mu\nu}\,p_\nu$

stress tensor $\Pi_p^{\mu\nu} \equiv \frac{1}{2}\left(\Delta^{\mu\rho}\,\Delta^{\nu\sigma} + \Delta^{\mu\sigma}\,\Delta^{\nu\rho} - \frac{2}{3}\Delta^{\mu\nu}\,\Delta^{\rho\sigma}\right)p_\rho\,p_\sigma$

- $\hat{h} \equiv \dfrac{\epsilon + nT}{nT}$ Reduced enthalpy
- $n \equiv (2\pi)^{-3}\,4\pi\,m^3\,e^{\frac{\mu}{T}}\left[z^{-1}\,K_2(z)\right]$
- $\gamma \equiv 1 + \left[z^2 + 3\,h - (h-1)^2\right]^{-1}$ $= c_P\,/\,c_V$ Ratio of specific heats
- $\epsilon \equiv m\,n\left[\dfrac{K_3(z)}{K_2(z)} - z^{-1}\right]$
- $z \equiv \dfrac{m}{T}$

de Groot et al

All quantities are functions of the hydrodynamic variables, T, μ and u^μ

$$\boldsymbol{a}_p^{\mu} = u^{\mu}$$

$$\partial_\mu J^{\mu\alpha}_{\mathrm{hydro.}} = 0$$

$$\boxed{p \equiv nT}$$

$$\Delta J^{\mu\alpha} = \begin{cases} -\zeta\,\Delta^{\mu\nu}\,X + 2\,\eta\,X^{\mu\nu} & \alpha = \nu \\ -T\,\lambda\,z\,\hat{h}^{-1}\,X^{\mu} & \alpha = 4. \end{cases}$$

$\Rightarrow$ satisfies the Landau constraints

$$u_\mu u_\nu \delta T^{\mu\nu} = 0, u_\mu \Delta_{\sigma\nu} \delta T^{\mu\nu} = 0$$

$$u_\mu \delta N^{\mu} = 0$$

$$\begin{aligned} X &\equiv -\nabla_\mu u^\mu, \\ X_\mu &\equiv \nabla_\mu \ln T - \hat{h}^{-1}\,\nabla_\mu \ln(n\,T), \\ X_{\mu\nu} &\equiv \frac{1}{2}\left(\Delta_{\mu\rho}\,\Delta_{\nu\sigma} + \Delta_{\mu\sigma}\,\Delta_{\nu\rho} - \frac{2}{3}\,\Delta_{\mu\nu}\,\Delta_{\rho\sigma}\right)\nabla^\rho\,u^\sigma. \end{aligned}$$

$$T^{\mu\nu} = \epsilon\,u^\mu\,u^\nu - (p + \zeta\,X)\,\Delta^{\mu\nu} + 2\,\eta\,X^{\mu\nu}$$

$$N^{\mu} = n\,u^{\mu} - \lambda\,\frac{n\,T}{\epsilon + p}\,X^{\mu}.$$

Landau frame and Landau eq.!

with the microscopic expressions for the transport coefficients;

Bulk viscosity $$\zeta \equiv -\frac{1}{T}\sum_{pq}\frac{1}{p^0}f_p^{\mathrm{eq}}\,\Pi_p\,\mathcal{L}_{pq}^{-1}\,\Pi_q$$

Heat conductivity $$\lambda \equiv -\frac{1}{3}\frac{1}{T^2}\sum_{pq}\frac{1}{p^0}f_p^{\mathrm{eq}}\,Q_p^{\mu}\,\mathcal{L}_{pq}^{-1}\,Q_{\mu q},$$

Shear viscosity $$\eta \equiv -\frac{1}{10}\frac{1}{T}\sum_{pq}\frac{1}{p^0}f_p^{\mathrm{eq}}\,\Pi_p^{\mu\nu}\,\mathcal{L}_{pq}^{-1}\,\Pi_{\mu\nu q}$$

$$\mathcal{L}_{pq} \equiv (p\cdot\theta_p)\,L_{pq}$$ ⟸ θ_p -independent

c.f. $$L_{pq} = -\frac{1}{p\cdot a_p}\frac{1}{2!}\sum_{p_1}\frac{1}{p_1^0}\sum_{p_2}\frac{1}{p_2^0}\sum_{p_3}\frac{1}{p_3^0}\,\omega(p\,,\,p_1|p_2\,,\,p_3)\,f_{p_1}^{\mathrm{eq}}\left(\delta_{pq}+\delta_{p_1q}-\delta_{p_2q}-\delta_{p_3q}\right)$$

$$(a_p{}^{\mu} = \theta_p{}^{\mu})$$

In a Kubo-type form;

$$\zeta \equiv \frac{1}{T}\int_0^{\infty}\mathrm{d}s\,\langle\,\Pi(0)\,,\,\Pi(s)\,\rangle_{\mathrm{eq}},$$

$$\lambda \equiv -\frac{1}{3}\frac{1}{T^2}\int_0^{\infty}\mathrm{d}s\,\langle\,Q^{\mu}(0)\,,\,Q_{\mu}(s)\,\rangle_{\mathrm{eq}},$$

$$\eta \equiv \frac{1}{10}\frac{1}{T}\int_0^{\infty}\mathrm{d}s\,\langle\,\Pi^{\mu\nu}(0)\,,\,\Pi_{\mu\nu}(s)\,\rangle_{\mathrm{eq}}.$$

$$\left[\Pi(s)\right]_p \equiv \sum_q\left[\mathrm{e}^{s\,\mathcal{L}}\right]_{pq}\Pi_q$$

$$\langle\,\varphi\,,\,\psi\,\rangle_{\mathrm{eq}} \equiv \sum_p\frac{1}{p^0}f_p^{\mathrm{eq}}\,\varphi_p\,\psi_p$$

C.f. Bulk viscosity may play a role in determining the acceleration of the expansion of the universe, and hence the dark energy!

Brief Summary and concluding remarks

(1) The RG v.s. the envelop equation
(2) The RG eq. gives the reduction of dynamics and the invariant manifold.
(3) The RG eq. was applied to reduce the Boltzmann eq. to the fluid dynamics in the limit of a small space variation.

Other applications:
a. the elimination of the rapid variable from Focker-Planck eq.
b. Derivation of Boltzmann eq. from Liouvill eq.
c. Derivation of the slow dynamics around bifurcations
and so on.

See for the details,
Y.Hatta and T.K., Ann. Phys. 298(2002),24