Part I
Review on correlation functions of the XXZ spin chain

- (1) H. Bethe(1930): Exact solutions of the one-dimensional Heisenberg model (XXX spin chain)
- (2) C.N. Yang and C.P. Yang (1966): the ground state of the XXZ spin chain
- (3) L.D. Faddeev and L. Takhtajan (1979): Algebraic Bethe-ansatz solution of the XXZ spin chain

(1) and (2): Coordinate Bethe ansatz (波動関数を仮定して固有状態を導く)

(3) Algebraic Bethe ansatz（固有状態の「生成演算子」を用いる）

異方的１次元ハイゼンベルグ模型(ＸＸＺ鎖)の定義

The Hamiltonian of the spin-1/2 XXZ spin chain under P. B. C. (the Periodic Boundary Conditions)

$$\mathcal{H}_{\rm XXZ} = \frac{1}{2} \sum_{j=1}^{L} \left(\sigma_j^X \sigma_{j+1}^X + \sigma_j^Y \sigma_{j+1}^Y + \Delta \sigma_j^Z \sigma_{j+1}^Z \right) .$$

Here σ_j^a $(a = X, Y, Z)$ are Pauli matrices on the jth site. We define q by

$$\Delta = (q + q^{-1})/2 \qquad (q = \exp \eta)$$

Quantum phase transitions at $\Delta = \pm 1$:

For $-1 < \Delta \le 1$, $\mathcal{H}_{\rm XXZ}$ is gapless. ($\Delta = \cos \zeta$ by $\boldsymbol{q = e^{i\zeta}}$, $0 \le \zeta < \pi$.)

Low excited spectrum is consistent with **CFT with $c = 1$**

For $\Delta > 1$ or $\Delta < -1$, it is gapful. ($\Delta = \pm \cosh \zeta$ by $q = e^{-\zeta}$, $0 < \zeta$.)

ＸＸＺ鎖に対する相関関数の多重積分表示の研究の流れ

- ●ｑ頂点演算子の方法（無限系、外部磁場ゼロ）
  M. Jimbo, K. Miki, T. Miwa and A. Nakayashiki,
  Phys. Lett. **A 168** (1992) 256–263.

- ●ｑＫＺ方程式（差分方程式）を解いて相関関数を求める方法
  M. Jimbo and T. Miwa, J. Phys. A: Math. Gen. **29** (1996) 2923-2958.

- ●代数的ベーテ仮設の方法（有限系で求めて無限極限：有限磁場）
  N. Kitanine, J.M. Maillet and V. Terras, Nucl. Phys. B **567** [FS] (2000) 554–582.

cf. Exact form factors of the sine-Gordon model (F. Smirnov, 1980's)

Emptiness Formation Probability (EFP)

Let us consider unit matrices $e^{a,b}$ for $a, b = 0, 1$.

$$e_j^{1,1} = \frac{1}{2}(1 - \sigma_j^z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} .$$

For the XXZ spin chain (massless regime) we have

$$\tau(m) \equiv \langle e_1^{1,1} \cdots e_m^{1,1} \rangle$$

$$= (-1)^m \left(-\frac{\pi}{\zeta} \right)^{m(m+1)/2} \int_{-\infty}^{\infty} \frac{d\lambda_1}{2\pi} \cdots \int_{-\infty}^{\infty} \frac{d\lambda_m}{2\pi} \prod_{a>b} \frac{\sinh \frac{\pi}{\zeta}(\lambda_a - \lambda_b)}{\sinh(\lambda_a - \lambda_b - i\zeta)}$$

$$\times \prod_{j=1}^{m} \frac{\sinh^{j-1}(\lambda_j - i\zeta/2) \sinh^{m-j}(\lambda_j + i\zeta/2)}{\cosh^m \frac{\pi}{\zeta}\lambda_j}$$

cf. N. Kitanine et al., NPB **567**, 554 (2000).

- ●代数的ベーテ仮説法の方法 II（有限系 $\to$ 無限系：有限温度の相関関数）

F. Göhmann, A. Klümper and A. Seel,
J. Phys. A: Math. Gen. Vol. 37 (2004) 7625-7651.

量子転送行列(quantum transfer matrix) の最大固有値を用いて、有限温度相関関数の多重積分表示を導いた。

Cf. 鈴木・Trotter 変換の応用

- ● 相関関数の多重積分表示を級数展開して数値評価($\zeta(n)$:ゼータ関数)

H. Boos and V.E. Korepin, J. Phys. A **34**, 5311 (2001).

J. Sato, M. Shiroishi, and M. Takahashi, Nucl. Phys. B **729**, 441 (2005).

Cf. M. Takahashi, J. Phys. C: Solid State Phys. **10**, 1289 (1976) ハバード模型の強結合極限で$\langle \sigma_1^z \sigma_3^z \rangle$を導いた。

J. Sato et al, PRL**106**, 257201 (2011) による低温展開の一部

$$\langle \sigma_1^z \sigma_2^z \rangle = \frac{1}{3} - \frac{4}{3}\ln 2 + \frac{1}{36}(T/J)^2$$

$$\langle \sigma_1^z \sigma_3^z \rangle = \frac{1}{3} - \frac{16}{3}\ln 2 + 3\zeta(3) + \left(\frac{1}{9} - \frac{\pi^2}{72}\right)(T/J)^2$$

- ● 相関関数の長距離漸近極限の導出（代数的ベーテ仮説）

N. Kitanine, K.K. Kozlowski, J.M. Maillet, A.N. Slavnov and V. Terras, JSTAT (2009) P04003

$$\langle \sigma_1^z \sigma_{m+1}^z \rangle = \langle \sigma^z \rangle^2 - \frac{2\mathcal{Z}^2}{\pi^2 m^2} + 2|F_\sigma|^2 \cdot \frac{\cos(2k_F m)}{m^2 \mathcal{Z}^2} + \cdots .$$

F_σ はσ^z の形状因子（form factor)

$$F_\sigma = \langle M|\sigma^z|\Psi_g\rangle$$

$|\Psi_g\rangle$：基底状態

$|M\rangle$：ウムクラップ過程に対応する励起状態

（両端のフェルミ点にホールと粒子）

$\mathcal{Z}$: the dressed charge

形状因子（form factor)：演算子を二つの固有状態で挟んだ行列要素

• ● 相関関数の長距離漸近極限の導出：形状因子展開法

N. Kitanine, K.K. Kozlowski, J.M. Maillet, A.N. Slavnov and V. Terras, JSTAT (2011) P12010

$$\langle \sigma_1^z \sigma_{m+1}^z \rangle = \sum_{n:\, all\ Bethe\ states} \langle \sigma_1^z | n \rangle \langle n | \sigma_{m+1}^z \rangle$$
$$= \sum_{n:\, all\ Bethe\ states} \langle \sigma_1^z | n \rangle \langle n | \sigma_1^z \rangle \, m^{-2\Delta(n)}$$

$\langle \sigma_1^z | n \rangle$, $\langle n | \sigma_{m+1}^z \rangle$: 形状因子 (form factors)

フェルミ面近傍の粒子・正孔励起を中間状態n：ＣＦＴと対応。

１Ｄボース気体の動的密度相関関数$\langle \rho(x,t)\rho(0,0) \rangle$ の漸近展開

N. Kitanine et al., arXiv:1206.2630

Introduction to the lgebraic Bethe ansatz

We define the R-matrix by

$$R_{12}(\lambda_1, \lambda_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{[1,2]}$$

$$b(u) = \sinh u / \sinh(u+\eta), \quad c(u) = \sinh \eta / \sinh(u+\eta) \quad (1)$$

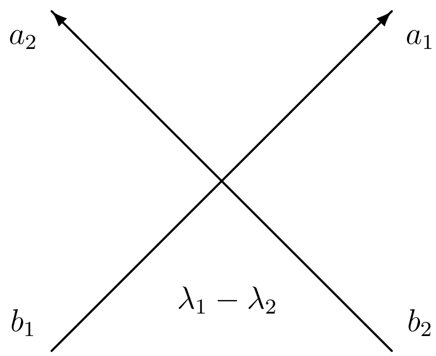
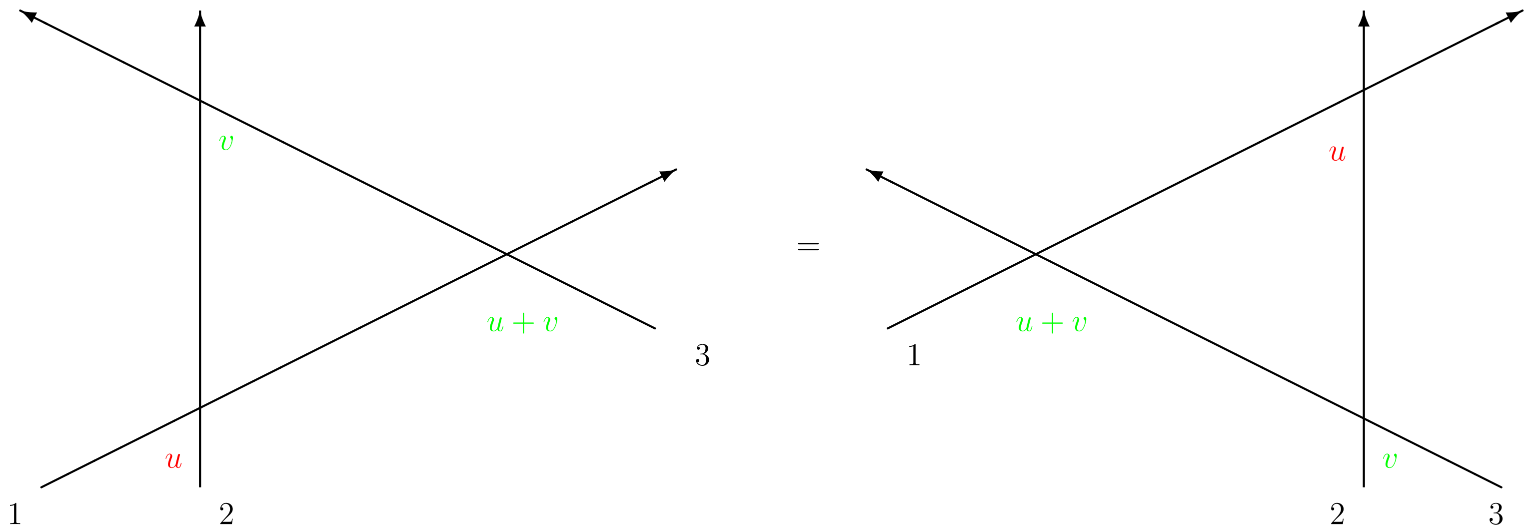


Figure 1: $R(\lambda_1, \lambda_2)^{a_1, a_2}_{b_1, b_2}$

Here $u = \lambda_1 - \lambda_2$ and $q = \exp \eta$.

The Yang-Baxter equations:



The R-matrices satisfy the Yang-Baxter equations.

$$R_{2,\,3}(v)R_{1,\,3}(u+v)R_{1,\,2}(u) = R_{1,\,2}(u)R_{1,\,3}(u+v)R_{2,\,3}(v)$$

Spectral parameter u is expressed by the angle between lines 1 and 2, where the interstion corresponds to $R_{1,\,2}(u)$.

モノドロミー行列の定義

We introduce the monodromy matrix $T_{0,12\cdots L}(\lambda)$:

$$T_{0,12\cdots L}(\lambda) = R_{0L}(\lambda, w_L) R_{0L-1}(\lambda, w_{L-1}) \cdots R_{02}(\lambda, w_2) R_{01}(\lambda, w_1) \,.$$

Here $w_1, w_2, \ldots, w_L$ are *inhomogeneity parameters.*

Figure 2: Matrix element of the monodromy matrix $(T_{\alpha,\beta})^{a_1,\ldots,a_L}_{b_1,\ldots,b_L}$.

The operator-valued matrix element of the monodromy matrix give the creation and annihilation operators

$$T_{0,12\cdots L}(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}_{[0]} .$$

The transfer matrix, $t(u)$, is given by the trace of the monodromy matrix with respect to the 0th space:

$$\begin{aligned} t(u) &= \mathrm{tr}_0 \left(T_{0,12\cdots L}(u) \right) \\ &= A(u) + D(u) . \end{aligned} \tag{2}$$

The logarithmic derivative of the transfer matrix gives the XXZ Hamiltonian:

$$\mathcal{H}_{XXZ} = \frac{d}{du} \log t(u) \,|_{u=0}$$

Thus, the transfer matrix and the Hamiltonian share the eigenvectors.

Let $|0\rangle$ be the vacuum vector with all spins being up.

$$|0\rangle = |\uparrow\uparrow\cdots\uparrow\rangle$$

The Bethe vector

$$\prod_{k=1}^{n} B(\lambda_k)|0\rangle = B(\lambda_1)\cdots B(\lambda_n)|0\rangle$$

becomes an eigenvector of the transfer matrix if rapidities $\lambda_1, \ldots, \lambda_n$ satisfy the Bethe ansatz equations:

$$\left(\frac{\sinh(\lambda_j + \eta/2)}{\sinh(\lambda_j - \eta/2)}\right) = \prod_{k=1;\, k\neq j}^{n} \frac{\sinh(\lambda_j - \lambda_k + \eta)}{\sinh(\lambda_j - \lambda_k - \eta)}, \quad (j = 1, 2, \ldots, n)$$

Review: algebraic BA derivation of the multiple-integral representation of the spin-1/2 XXZ correlation functions

- ●Quantum Inverse Scattering Problem (QISP) 「量子逆散乱問題」:

  Local spin-1/2 operators expressed by A, B, C, D (spin-1/2)

  局所演算子を基本演算子ＡＢＣＤの積や和で表すこと

QISP formula

$$x_n = \prod_{j=1}^{n-1} t(w_j) \mathrm{tr}_0(x_0 T_{0,\,12\cdots L}(w_n)) \prod_{j=1}^{n} t(w_j)^{-1}$$

For example, we have

$$\sigma_n^- = \prod_{j=1}^{n-1} (A(w_j) + D(w_j)) \ \cdot \ B(w_n) \ \cdot \ \prod_{j=1}^{n} (A(w_j) + D(w_j))^{-1}$$

- ● Scalar products of the BA:

Suppose that $\{\mu_j\}$ or $\{\lambda_j\}$ are Bethe roots,

(i) the Gaudin-Korepin formula for the Bethe ansatz norm

$$\langle 0|C(\lambda_1)\cdots C(\lambda_M)\, B(\lambda_1)\cdots B(\lambda_M)|0\rangle = \det \Phi^{'}$$

$\Phi^{'}$: the Gaudin matrix

(ii) Slavnov's formula:

$$\langle 0|C(\mu_1)\cdots C(\mu_M)\, B(\lambda_1)\cdots B(\lambda_M)|0\rangle = \det \Psi^{'}$$

Thus, we obtain the expectation values of local operators by calculating the ratio for Bethe roots $\{\lambda_j\}$ and arbitrary parameters $\{\mu_j\}$:

$$\frac{\langle 0|C(\mu_1)\cdots C(\mu_M)\, B(\lambda_1)\cdots B(\lambda_M)|0\rangle}{\langle 0|C(\lambda_1)\cdots C(\lambda_M)\, B(\lambda_1)\cdots B(\lambda_M)|0\rangle} = \det(\Psi^{'}/\Phi^{'})$$

- ● Integral equations for the matrix elements of Ψ'/Φ':

  To evaluate the matrix elements of $\left(\Psi'/\Phi'\right)$ we solve integral equations for them (cf. Izergin)

The thermodynamic property of the ground state is taken into accout by the density of the roots of the Bethe-ansatz equations.

Commutation relations $(b_{j\beta} = b(\lambda_j - \lambda_\beta)\ c_{j\beta} = c(\lambda_j - \lambda_\beta))$

$$\langle 0| \prod_{k=1}^{n} C(\mu_k) \cdot D(\mu_0) = \sum_{\alpha=0}^{n} d_\alpha c_{\alpha 0} \prod_{j=0; j\neq\alpha} b_{\alpha j}^{-1} \cdot \langle 0| \prod_{k=1; k\neq\alpha}^{n} C(\mu_k)$$

$$\langle 0| \prod_{k=1}^{n} C(\mu_k) \cdot A(\mu_0) = \sum_{\alpha=0}^{n} a_\alpha c_{0\alpha} \prod_{j=0; j\neq\alpha} b_{j\alpha}^{-1} \cdot \langle 0| \prod_{k=1; k\neq\alpha}^{n} C(\mu_k)$$

$$\langle 0| \prod_{k=1}^{n} C(\mu_k) \cdot B(\mu_0) = \sum_{\alpha=0}^{n} d_\alpha c_{\alpha 0} \prod_{j=0; j\neq\alpha} b_{\alpha j}^{-1}$$

$$\times \sum_{\beta=0; \beta\neq\alpha}^{n} a_\beta c_{0\beta} \prod_{j=0; j\neq\alpha,\beta} b_{j\beta}^{-1} \cdot \langle 0| \prod_{k=1; k\neq\alpha,\beta}^{n} C(\mu_k)$$

Here $a_\alpha = a(\lambda_\alpha)$ and $d_\alpha = d(\lambda_\alpha)$ are defined by

$$A(\lambda)|0\rangle = a(\lambda)|0\rangle, \quad D(\lambda)|0\rangle = d(\lambda)|0\rangle.$$

Examples of multiple integrals (T.D. and C. Matsui, NPB(2010)).

For $s=1$ and $m=1$ ($w_1^{(2)}=\xi_1$, $w_2^{(2)}=\xi_1-\eta$), we have

$$\langle E_1^{11\,(2+)} \rangle = \langle \psi_g^{(2+)} | E_1^{11\,(2+)} | \psi_g^{(2+)} \rangle / \langle \psi_g^{(2+)} | \psi_g^{(2+)} \rangle$$

$$= \quad 2 \left(\int_{-\infty+i\epsilon}^{\infty+i\epsilon} + \int_{-\infty-i\zeta+i\epsilon}^{\infty-i\zeta+i\epsilon} \right) d\lambda_1 \left(\int_{-\infty-i\epsilon}^{\infty-i\epsilon} + \int_{-\infty-i\zeta-i\epsilon}^{\infty-i\zeta-i\epsilon} \right) d\lambda_2$$
$$\times Q(\lambda_1, \lambda_2) \, \mathrm{det} S(\lambda_1, \lambda_2) \tag{3}$$

$$Q(\lambda_1, \lambda_2) = (-1) \frac{\varphi(\lambda_2 - w_2^{(2)}) \varphi(\lambda_1 - w_1^{(2)} - \eta)}{\varphi(\lambda_2 - \lambda_1 + \eta + \epsilon_{2,1}) \varphi(\eta)} \tag{4}$$

and matrix $S(\lambda_1, \lambda_2)$ is given by

$$\begin{pmatrix} \rho(\lambda_1 - w_1^{(2)} + \eta/2) \delta(\alpha(\lambda_1), 1) & \rho(\lambda_1 - w_2^{(2)} + \eta/2) \delta(\alpha(\lambda_1), 2) \\ \rho(\lambda_2 - w_1^{(2)} + \eta/2) \delta(\alpha(\lambda_2), 1) & \rho(\lambda_2 - w_2^{(2)} + \eta/2) \delta(\alpha(\lambda_2), 2) \end{pmatrix} . \tag{5}$$

Evaluating the integrals for the spin-1 one-point function

(T. D. and J. Sato, SIGMA(2011))

Evaluating the multiple integrals explicitly, we have obtained all the one-point function for the integrable spin-1 XXZ chain as

$$
\begin{aligned}
E^{2,\,2\,(2\,p)}\rangle = \langle E^{0,\,0\,(2\,p)}\rangle &= \frac{\zeta - \sin\zeta\cos\zeta}{2\zeta\sin^2\zeta}, \\
\langle E^{1,\,1\,(2\,p)}\rangle &= \frac{\cos\zeta(\sin\zeta - \zeta\cos\zeta)}{\zeta\sin^2\zeta}.
\end{aligned} \tag{6}
$$

In particular, we have

$$
\langle E^{22}\rangle = \langle E^{00}\rangle\,. \tag{7}
$$

Through the direct evaluation of the multiple integrals we confirm the identity: $\langle E^{22}\rangle + \langle E^{11}\rangle + \langle E^{00}\rangle = 1$.

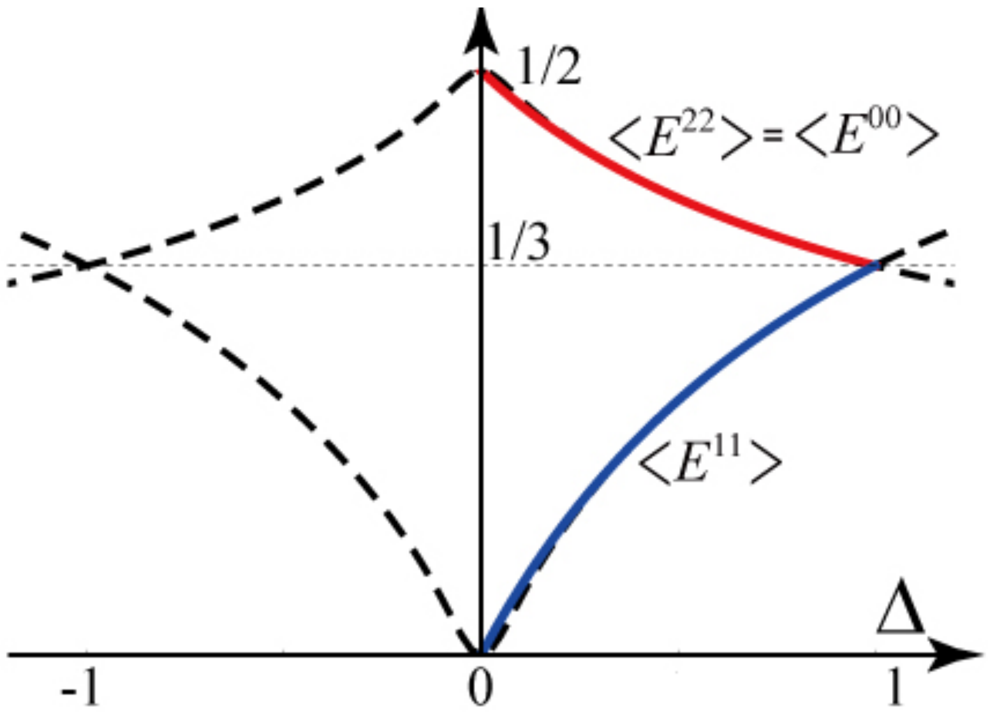


Figure 3: Comparison with the exact numerical diagonalization. The red and blue lines represent analytical results obtained by the multiple integrals for $\langle E^{22}\rangle = \langle E^{00}\rangle$ and $\langle E^{11}\rangle$, respectively. The black dotted lines represent those obtained by exact diagonalization with the system size $N_s = 8$.