Hidden symmetry of correlation functions and amplitudes in $\mathcal{N}=4$ SYM

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Observables in $\mathcal{N}=4$ SYM

✔ Natural observables in a (conformal) gauge theory:

- ✗ Scattering amplitudes: $A_n(p_i) = \langle p_1, p_2, \ldots, p_n | S | 0 \rangle$
- ✗ Correlation functions: $G_n(x_i) = \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\ldots\mathcal{O}(x_n)\rangle$
- ✗ (Light-like) Wilson loops: $W(C) = \langle \mathrm{tr}\, P \exp\left(i \oint_C dx \cdot A(x)\right)\rangle$

✔ Carry different/supplementary information about $\mathcal{N}=4$ SYM:

$G_n =$ off-shell (anomalous dimensions, structure constants of OPE)

$A_n =$ on-shell (S-matrix)

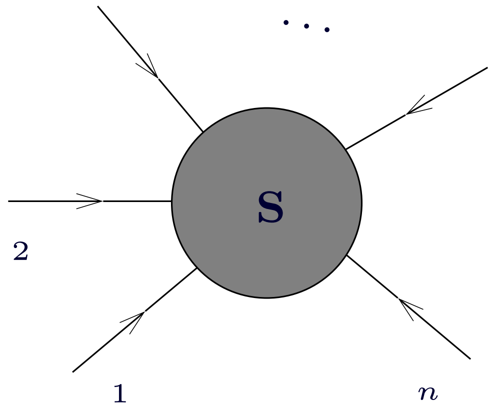
✔ Are related to each other in planar $\mathcal{N}=4$ SYM

✔ Have a new hidden symmetry (ultimately related to integrability of $\mathcal{N}=4$ SYM)

✔ Allows us to predict Correlators/Amplitudes/Wilson loops at higher loops without any Feynman graph calculations!

Gluon amplitudes in $\mathcal{N}=4$ SYM

✔ On-shell matrix elements of $S-$matrix:

$\mathcal{A}_n =$ 

- Quantum numbers of scattered gluons:

Color:	$a_i = 1, \dots, N_c^2 - 1$
Light-like momenta:	$(p_i^\mu)^2 = 0$
Polarization state (helicity):	$h_i = \pm 1$

✔ Color-ordered planar gluon amplitudes:

$$\mathcal{A}_n = \mathrm{tr}\left[T^{a_1} T^{a_2} \dots T^{a_n}\right] A_n^{h_1,h_2,\dots,h_n}(p_1, p_2, \dots, p_n) + \text{[Bose symmetry]}$$

✗ Supersymmetry relations:

$$A^{++\dots+} = A^{-+\dots+} = 0\,, \qquad A^{(\mathrm{MHV})} = A_n^{--+\dots+}\,, \qquad A^{(\mathrm{next-MHV})} = A_n^{---+\dots+}\,, \quad \dots$$

✗ The $n=4$ and $n=5$ planar gluon amplitudes are all MHV

✗ Loop corrections to all MHV amplitudes are described by a single scalar function

$$A_n^{\mathrm{MHV}}(p_i) = A_n^{(\mathrm{tree})}(p_i) M_n^{\mathrm{MHV}}(\{s_{ij}\}; a)$$

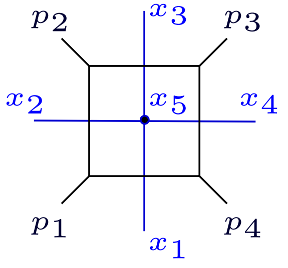
Scattering amplitudes/Wilson loop duality

- Four-gluon amplitude in $\mathcal{N} = 4$ SYM at weak coupling

$$A_4/A_4^{(\text{tree})} = 1 + a\, I^{(1)}(s,t) + O(a^2)\,, \qquad a = \frac{g^2 N_c}{8\pi^2}$$

Scalar box in dim.regularization (for IR divergences) with $D = 4 - 2\epsilon$

$$I^{(1)}(s,t) \;=$$



$$\sim \int \frac{d^D x_5\, x_{13}^2 x_{24}^2}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}$$

Dual variables $p_i = x_i - x_{i+1}, \qquad p_i^2 = x_{i,i+1}^2 = 0$

- Light-like rectangular Wilson loop $C_4 = [x_1, x_2] \cup [x_2, x_3] \cup [x_3, x_4] \cup [x_4, x_1]$

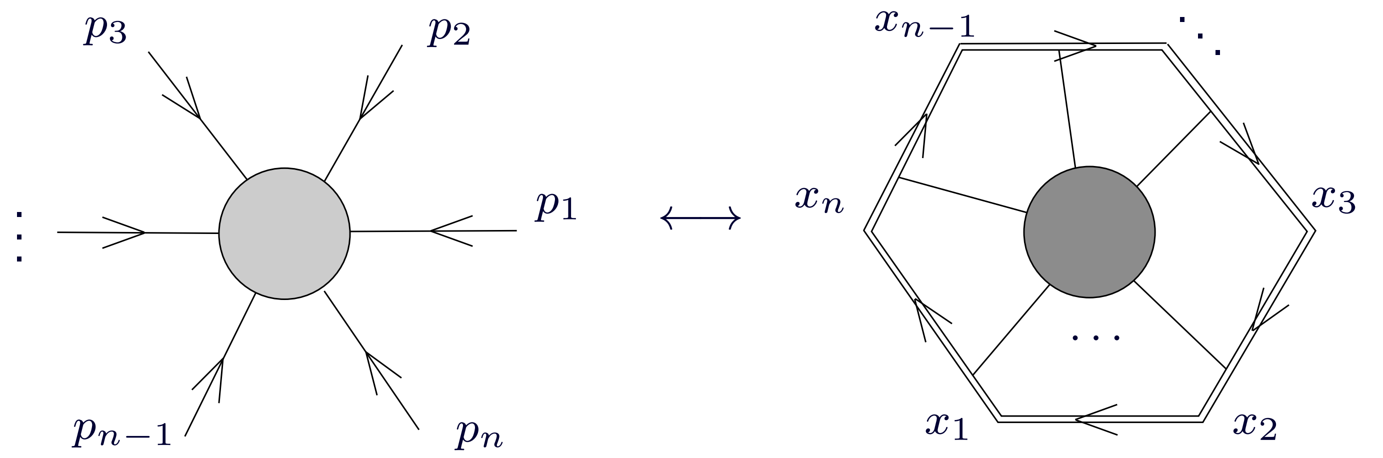
$$\begin{aligned} W[C_4] &= 1 + \frac{1}{2}(ig)^2 \int_{C_4} dx^\mu \int_{C_4} dy^\nu\, D_{\mu\nu}(x-y) + O(g^4) \\ &= 1 + a\, I^{(1)}(s,t) + O(a^2) \end{aligned}$$

- Duality relation in planar $\mathcal{N} = 4$ SYM:

$$\ln\left(A_4/A_4^{(\text{tree})}\right) = \ln W[C_4]$$

dual conformal symmetry!

MHV scattering amplitudes/Wilson loop duality



MHV amplitudes are dual to light-like Wilson loops

$$\ln \mathcal{A}_n^{(\mathrm{MHV})} \sim \ln W(C_n) + O(1/N_c^2)\,, \qquad C_n = p_1 \cup \cdots \cup p_n \text{ = light-like } n\text{–gon}$$

- ✔ At strong coupling, agrees with the AdS/CFT prediction for $n = 4$
- ✔ At weak coupling, agrees with two-loop calculation for $n \geq 4$
- ✔ Recent progress:
  - ✗ Analytical expressions at weak coupling
  - ✗ Strong coupling prediction for $n > 5$
  - ✗ Rich mathematical structure at strong coupling (integrability, Y-system, TBA)

Correlation functions in $\mathcal{N}=4$ SYM

- Two-point functions:

$$\langle O_1(x_1)O_2(x_2)\rangle = \frac{\delta_{12}}{(x_{12}^2)^{\Delta_1(a)}}\,, \qquad x_{12}\equiv x_1-x_2$$

Scaling dimensions $\Delta_i=\Delta_i(a)$ are known in planar $\mathcal{N}=4$ SYM from integrability

- Three-point functions:

$$\langle O(x_1)O_2(x_2)O_3(x_3)\rangle = \frac{C_{123}(a)}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}x_{23}^{\Delta_2+\Delta_3-\Delta_1}x_{31}^{\Delta_3+\Delta_1-\Delta_2}}$$

C_{123} are structrure constants of the OPE

- Four-point functions:

$$\langle O(x_1)O(x_2)O(x_3)O(x_4)\rangle = \frac{1}{(x_{12}^2x_{23}^2x_{34}^2x_{41}^2)^{\Delta/2}}\mathcal{F}(u,v;a)$$

$\mathcal{F}(u,v;a)$ are (complicated) conformal invariant functions of

$$u=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}\,, \qquad v=\frac{x_{23}^2x_{41}^2}{x_{13}^2x_{24}^2}$$

Correlation functions of 1/2 BPS operators

- Protected superconformal operators made from scalars ϕ^I

$$\mathcal{O}(x) = \mathrm{Tr}(ZZ)\,, \qquad \tilde{\mathcal{O}}(x) = \mathrm{Tr}(\bar{Z}\bar{Z})\,, \qquad Z = \phi^1 + i\phi^2$$

  Two- and three-point correlation functions do not receive quantum corrections

- Simplest 4-point correlation function

$$G_4 = \langle \mathcal{O}(x_1)\tilde{\mathcal{O}}(x_2)\mathcal{O}(x_3)\tilde{\mathcal{O}}(x_4)\rangle = \frac{1}{x_{12}^2 x_{23}^2 x_{34}^2 x_{41}^2}\left[1 + 2a\, x_{13}^2 x_{24}^2 g(1,2,3,4) + O(a^2)\right]$$

  One-loop 'cross' integral

$$g(1,2,3,4) = \frac{1}{4\pi^2}\int \frac{d^4x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}\,, \qquad (x_{12}^2, x_{23}^2, x_{34}^2, x_{41}^2 \neq 0)$$

- Involves *the same* integral as 4-gluon amplitude at one loop but for *different* kinematics: on-shell $x_{i,i+1}^2 = 0$ for A_4 and off-shell $x_{i,i+1}^2 \neq 0$ for G_4

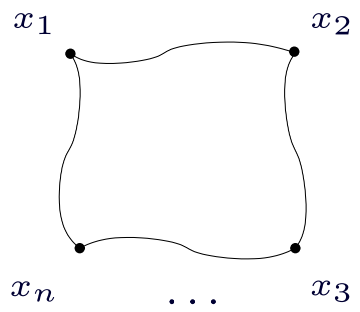
- Novel limit: let all the neighboring points be light-like separated at the same time

$$x_{i,i+1}^2 \;\to\; 0\,, \qquad x_i \neq x_{i+1}\,, \qquad (i = 1, \ldots, n)$$

Correlation functions on the light cone

- First-quantized description (random walk of a scalar particle)

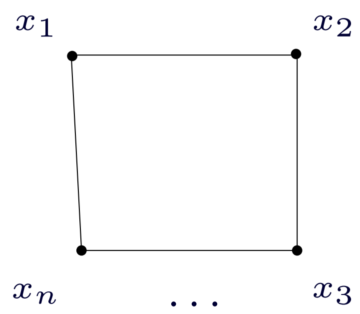
$$G_n \to \sum_C \mathrm{e}^{-iL(C)} \, \langle 0 | \mathrm{Tr}_{\mathrm{adj}} \, \mathrm{P} \, \mathrm{e}^{i \oint_C dx^\mu A_\mu(x)} | 0 \rangle \, , \qquad C =$$



Interaction with gauge field $\to$ path-ordered exponential in the *adjoint* of the $SU(N_c)$

- In the light-cone limit, $x^2_{i,i+1} \to 0$, the sum is dominated by the saddle point

$$G_n \to G_n^{(0)} \times \langle 0 | \mathrm{Tr}_{\mathrm{adj}} \, \mathrm{P} \, \mathrm{e}^{i \oint_{C_n} dx^\mu A_\mu(x)} | 0 \rangle \, , \qquad C_n =$$



$C_n =$ classical trajectory of an infinitely fast particle interacting with a slowly varying gauge field

- All-loop result (valid in any gauge theory!)

$$\lim_{x^2_{i,i+1} \to 0} \left(G_n / G_n^{(0)} \right) = W_{\mathrm{adj}}[C_n] = \left(W[C_n] \right)^2 + O(1/N_c^2)$$

Dualities

Wilson loops/Scattering amplitudes duality

$$\ln\left(A_n^{\mathrm{MHV}}/A_n^{(\mathrm{tree})}\right) = \ln\left(W[C_n]\right) + O(1/N_c^2)$$

Correlation functions/Wilson loops duality

$$\lim_{x_{i,i+1}^2\to 0} \ln\left(G_n/G_n^{(0)}\right) = 2\ln\left(W[C_n]\right) + O(1/N_c^2)$$

... leading to Correlation functions/Scattering amplitudes duality

$$\lim_{x_{i,i+1}^2\to 0} \ln\left(G_n/G_n^{(0)}\right) = 2\ln\left(A_n^{\mathrm{MHV}}/A_n^{(\mathrm{tree})}\right) + O(1/N_c^2)$$

- ✔ Has been verified to two loops for $n = 4,\ 5,\ 6$
- ✔ The dualities can be extended to generic non-MHV amplitudes:

  Super-Amplitude $\leftrightarrow$ *Super-Wilson loop* $\leftrightarrow$ *Super-correlation function*

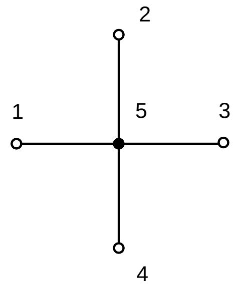
- ✔ The correlation functions are related to Wilson loops/Amplitudes in the light-cone limit $x_{i,i+1}^2 \to 0$

  What are symmetries of the correlation functions for arbitrary $x_{i,i+1}^2$?

A hidden symmetry of the correlation functions

Examine one-loop correction to the correlator

$$G_4^{(1)} \sim \int \frac{d^4 x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} \quad =$$



The corresponding integrand

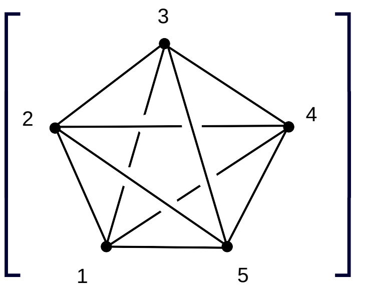
$$[G_4^{(1)}]_{\text{Integrand}} \sim \frac{1}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}$$

The r.h.s. has S_4 permutation symmetry w.r.t. exchange of the external points $1, 2, 3, 4$

Equivalent form of writing

$$[G_4^{(1)}]_{\text{Integrand}} \sim x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2 \times \left[\prod_{i<j} \frac{1}{x_{ij}^2} \right]$$

$$= x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2 \times$$



The second factor in the r.h.s. has the complete S_5 permutation symmetry!

- ✔ Explicit two-loop calculation: [Eden,Schubert,Sokatchev'00],[Bianchi,Kovacs,Rossi,Stanev'00]

$$G^{(2)} = h(1,2;3,4) + h(3,4;1,2) + h(2,3;1,4) + h(1,4;2,3)$$
$$+ h(1,3;2,4) + h(2,4;1,3) + \frac{1}{2}\left(x_{12}^2 x_{34}^2 + x_{13}^2 x_{24}^2 + x_{14}^2 x_{23}^2\right)[g(1,2,3,4)]^2$$

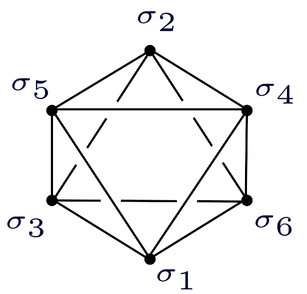
$h(1,2;3,4)-$ 'double' scalar box integral

- ✔ Go to a common denominator

$$G_4^{(2)} = x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2 \int d^4x_5 d^4x_6 \, f^{(2)}(x_1, \ldots, x_6)$$

- ✔ 7 integrals in $G_4^{(2)}$ are described by a single $f-$function

$$f^{(2)}(x_1, \ldots, x_6) = \frac{1}{48} \sum_{\sigma \in S_6} \frac{x^2_{\sigma_1\sigma_2} x^2_{\sigma_3\sigma_4} x^2_{\sigma_5\sigma_6}}{\prod_{1 \le i < j \le 6} x_{ij}^2} \quad =$$



Has the complete S_6 permutation symmetry !

- ✔ Integrand of the correlator has the complete permutation symmetry exchanging the external and integration points ... Where does it come from?

$\mathcal{N}=4$ stress-tensor supermultiplet

$$G_4 = \langle \mathcal{O}(x_1,y_1)\dots\mathcal{O}(x_4,y_4)\rangle = \sum_{l=0}^{\infty} a^l\, G_4^{(\ell)}(1,2,3,4)$$

Half-BPS operators made of the six scalars (complex null vector, $y^2 \equiv y_I y_I = 0$)

$$\mathcal{O}(x,y) = y_I\, y_J\, \mathcal{O}^{IJ}_{\mathbf{20'}}(x) = y_I\, y_J\ \mathrm{tr}\left[\Phi^I \Phi^J(x)\right]$$

The lowest-weight state of the $\mathcal{N}=4$ stress-tensor (chiral) supermultiplet

$$\mathcal{T}(x,\rho,y) = \exp\left(\rho^a_\alpha\, Q^\alpha_a\right)\mathcal{O}(x,y) = \mathcal{O}(x,y) + \ldots + (\rho)^4 \mathcal{L}_{\mathcal{N}=4}(x)$$

The on-shell action of the $\mathcal{N}=4$ theory

$$S_{\mathcal{N}=4} = \int d^4x \int d^4\rho\ \mathcal{T}(x,\rho,y)$$

Compute loop corrections using the method of Lagrangian insertions:

$$\begin{aligned} a\frac{\partial}{\partial a} G_4 &= a\frac{\partial}{\partial a}\int D\Phi\, \mathrm{e}^{-\frac{1}{a}S_{\mathcal{N}=4}[\Phi]}\, \mathcal{O}(x_1,y_1)\dots\mathcal{O}(x_4,y_4) \\ &= \int d^4x_5\, \langle \mathcal{O}(x_1,y_1)\dots\mathcal{O}(x_4,y_4)\mathcal{L}_{\mathcal{N}=4}(x_5)\rangle \end{aligned}$$

The loop correction is determined by integrated 5-point correlation function with insertions of the $\mathcal{N}=4$ SYM action

Method of Lagrangian insertions

The $\ell-$loop correction – (integrated) tree-level correlation function with ℓ insertions of $\mathcal{L}_{\mathcal{N}=4}$

$$G_4^{(\ell)}(1,2,3,4) = \int d^4x_5 \ldots \int d^4x_{4+\ell} \, \langle \mathcal{O}(x_1,y_1) \ldots \mathcal{O}(x_4,y_4) \mathcal{L}(x_5) \ldots \mathcal{L}(x_{4+\ell}) \rangle^{(0)}$$

The operators $\mathcal{O}(x,y)$ and $\mathcal{L}(x)$ belong to the same supermultiplet!

$$G_4^{(\ell)}(1,2,3,4) \ = \int d^4x_5 ... d^4x_{4+\ell} \left(\int d^4\rho_5 \ldots d^4\rho_{4+\ell} \, \langle \mathcal{T}(1) ... \mathcal{T}(4) \mathcal{T}(5) ... \mathcal{T}(4+\ell) \rangle^{(0)} \right)$$

The *integrand* of the loop corrections to the four-point correlation function

$$\left[G_4^{(\ell)}(1,2,3,4) \right]_{\rm Integrand} = \int d^4\rho_5 \ldots d^4\rho_{4+\ell} \, \langle \mathcal{T}(1) ... \mathcal{T}(4) \mathcal{T}(5) ... \mathcal{T}(4+\ell) \rangle^{(0)}$$

The correlation function $\langle \mathcal{T}(1) ... \mathcal{T}(1) \mathcal{T}(5) ... \mathcal{T}(4+\ell) \rangle$ is symmetric under exchange of points:

- ✔ Integrand reveals a new permutation $S_{4+\ell}$ symmetry involving all the $(4+\ell)$ points
- ✔ The OPE leads to powerful restrictions on the form of the integrand of the correlation function
- ✔ This information is sufficient to unambiguously fix the form of the integrand to all loops

All-loop integrand

Loop corrections to 4-point correlator

$$G_4^{(\ell)}(1,2,3,4) = x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2 \int d^4x_5 \dots d^4x_{4+\ell}\, f^{(\ell)}(x_1,\dots,x_{4+\ell})$$

✔ General form of $f^{(\ell)}$ for arbitrary ℓ:

$$f^{(\ell)}(x_1 \dots, x_{4+\ell}) = \frac{P^{(\ell)}(x_1,\dots x_{4+\ell})}{\prod_{1\le i<j\le 4+\ell} x_{ij}^2}$$

Can be deduced from the OPE analysis of the tree-level correlator

✔ The polynomial $P^{(\ell)}$ should satisfy the conditions:

- ✗ be invariant under $S_{4+\ell}$ permutations of $x_1, ..., x_{4+\ell}$
- ✗ have a uniform conformal weight $(1-\ell)$ at each point, both external and internal

$$P^{(\ell)}(x_i^{-1}) = \prod_{i=1}^{4+\ell} (x_i^2)^{-\ell+1} P^{(\ell)}(x_i)$$

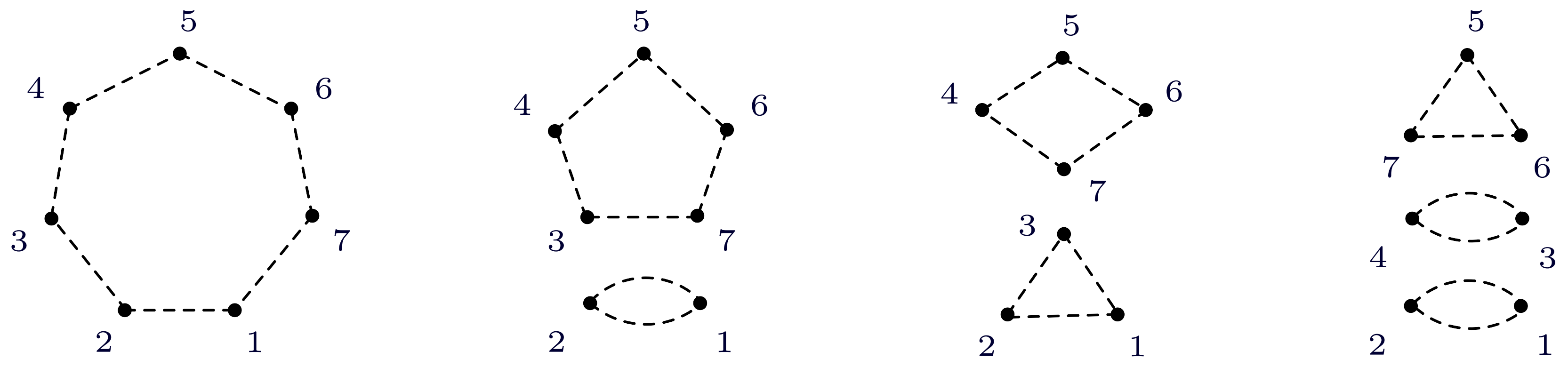
✔ Graph theory solution:

$P^{(\ell)} \mapsto$ ***Multi-graph with*** $(4+\ell)$ ***vertices of degree*** $(\ell-1)$

Warm up exercise: three loops

$$P^{(3)} \mapsto \textbf{\textit{Multi-graph with } } 7 \textbf{\textit{ vertices of degree }} 2$$

There are only 4 independent possibilities for $P^{(3)}$:



(a) (b) (c) (d)

$$P_a^{(3)} = x_{12}^2 x_{23}^2 x_{34}^2 x_{45}^2 x_{56}^2 x_{67}^2 x_{71}^2 \; + \; S_7 \text{ permutations} \,, \qquad P_{b,c,d}^{(3)} = \ldots$$

$P^{(3)}$ is a *linear combination of four terms*

$$P^{(3)} = c_a P_a^{(3)} + c_b P_b^{(3)} + c_c P_c^{(3)} + c_d P_d^{(3)}$$

The expansion coefficients are uniquely fixed by the duality with amplitudes for $x_{i,i+1}^2 \to 0$ and/or consistency with the OPE in the short distance limit $x_i \to x_j$

$$c_a = 1 \,, \qquad c_b = c_c = c_d = 0$$

All-loop integrand II

- The all-loop integrand of 4-point correlator possesses a complete symmetry under the exchange of the four external and all internal (integration) points
- The symmetry is exact for *arbitrary* gauge group $SU(N_c)$ (no need for the planar limit)
- The symmetry alone allowes us to construct 6-loop integrand of the correlation function (without doing Feynman diagram calculation!)
- In the light-cone limit, the scattering amplitude/correlator duality predicts the integrand for 4-gluon amplitude
- In the short-distance limit, the OPE leads to analytical result for the Konishi anomalous dimension at 5 loops

Conclusions + open questions

- ✔ $\mathcal{N}=4$ SYM has more symmetries as one might expect (integrability?)

- ✔ Correlators/Wilson loops/Scattering amplitudes are related to each other in $\mathcal{N}=4$ SYM:

$$\begin{tikzcd}
G_n(x_i) \arrow[rr, leftrightarrow, "x^2_{i,i+1}\to 0"] \arrow[dr, leftrightarrow, "p_i=x_{i,i+1}"'] & & W[C_n] \arrow[dl, leftrightarrow, "C_n=p_1\cup\ldots\cup p_n"] \\
& A_n^{\mathrm{MHV}}(p_i) &
\end{tikzcd}$$

- ✔ Scattering amplitudes in planar $\mathcal{N}=4$ SYM possess *conventional* + *dual* superconformal symmetries

  What is manifestation of these symmetries for the correlation functions ?

- ✔ Correlation functions have permutation symmetry for an arbitrary gauge group $SU(N_c)$:

  Do scattering amplitudes have an additional symmetry beyond the planar limit ?