Black brane hydrodynamics & membrane paradigm

Mukund Rangamani

Durham University
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R. Emparan, V. Hubeny, MR (wip)

Black Holes as Fluid Membranes

- An invitation to the black hole membrane paradigm
- The old membrane paradigm
- Detour: relativistic fluid dynamics
- Blackfolds: effective dynamics of black branes
- From blackfolds to the membrane paradigm
- Summary & open questions

The fluid/gravity correspondence

- The fluid/gravity correspondence establishes a correspondence between Einstein's equations with a negative cc and those of relativistic conformal fluids.

Einstein's eqn with negative cc

$$E_{MN} = R_{MN} - \frac{1}{2} G_{MN} R - \frac{d(d-1)}{2} G_{MN} = 0$$

$$\updownarrow$$

Relativistic ideal fluid equations

$$(\rho + P)\, \nabla_\mu u^\mu + u^\mu \nabla_\mu \rho = 0$$

$$P_\alpha{}^\mu \nabla_\mu P + (\rho + P)\, P_{\nu\alpha}\, u^\mu\, \nabla_\mu u^\nu = 0$$

Bhattacharyya, Hubeny, Minwalla, MR (2007)

The fluid/gravity correspondence

- The fluid/gravity correspondence establishes a correspondence between Einstein's equations with a negative cc and those of relativistic conformal fluids.
- Given any solution to the hydrodynamic equations, one can construct, in a gradient expansion, an approximate *inhomogeneous, dynamical black hole* solution in an asymptotically AdS spacetime.
- The construction heuristically can be viewed as patching together planar AdS black holes of different temperatures with slow variation between patches.
- The fluid in question lives on the timelike boundary of AdS spacetime, and as is familiar, holographically encodes the entire dynamics of the bulk spacetime geometry.

The black hole membrane paradigm

- Connections between gravity and fluids originated in the *black hole membrane paradigm.*

T. Damour; K. Thorne, R. Price (1970s)

- The membrane paradigm associates a dynamical membrane with electro-mechanical properties to the black hole.
- In particular, it does away with the interior of the black hole; matter falling into the black hole instead interacts with the membrane.
- Membrane dynamics, obtained by projecting Einstein's equations onto a null hypersurface, has formal similarities with the non-relativistic Navier-Stokes dynamics.

C. Eling, Y. Oz (2008)

- More recently, using a gradient expansion in the near-horizon Rindler region, the membrane dynamics has been shown to correspond to an incompressible Navier-Stokes system.

Bredberg, Lysov, Keeler, Strominger (2010-11)
Compere, McFadden, Skenderis, Taylor (2011-12)
Eling, Meyer, Oz (2012)

The moving membrane

- What is the connection between the membrane dynamics and the fluid on the boundary, when we consider AdS black holes?
- Given the connection between the radial position in AdS and the energy scale in the field theory, is it possible that the membrane dynamics is obtained as some RG flow of the boundary relativistic fluid?
- A-priori this sounds strange, given that hydrodynamics is already a long-wavelength effective description, but one can visualize the non-relativistic dynamics as the ultra-low-energy (or very late time) description of the relativistic system.
- In any event, with our modern viewpoint on connections between gravity and hydrodynamics, which in fact transcend the AdS/CFT regime, it is apposite to revisit the membrane paradigm.

The black hole membrane paradigm

- What precisely is the definition of the black hole membrane paradigm?

- How do we derive the dynamics of the membrane?

The membrane paradigm: motivations

- Surface dynamics associated with the event horizon should capture the dynamics of the internal states of the black hole.

- What is a good description of such surface dynamics? One should be able to derive it from Einstein's equations.

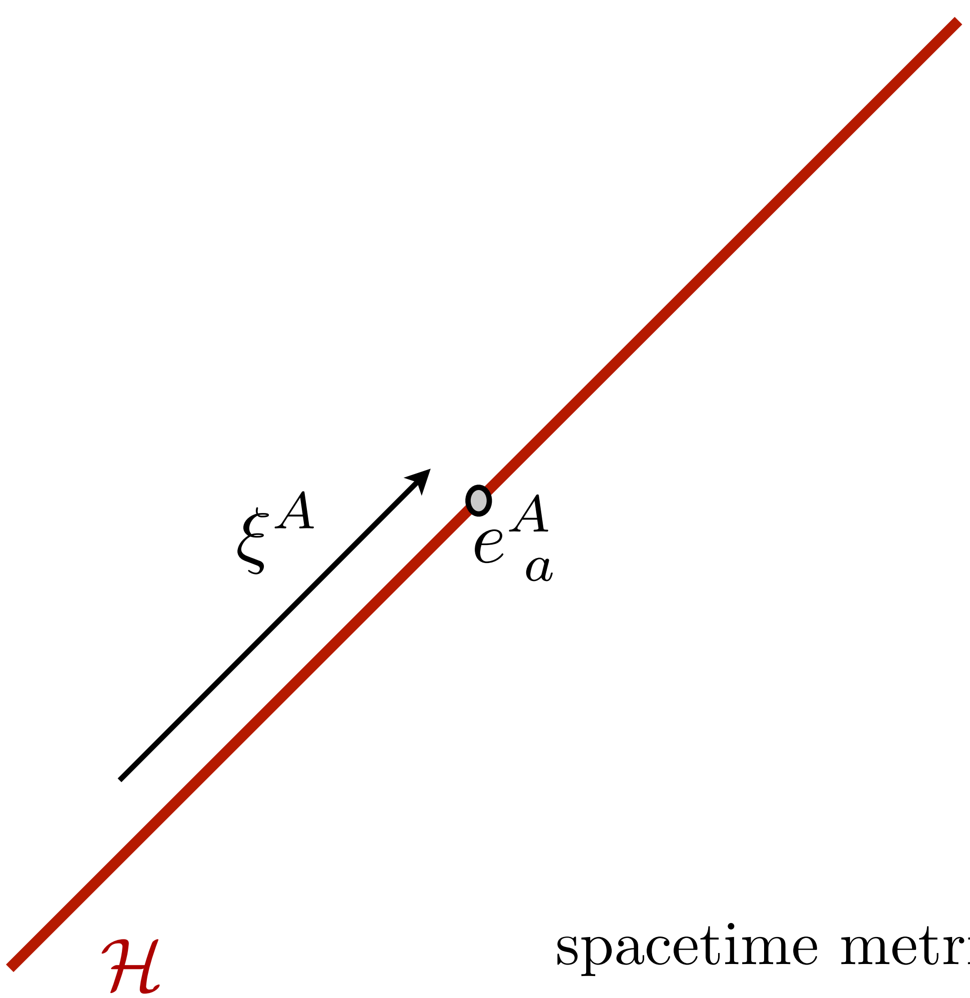
- The membrane paradigm for black holes was invented to understand some of these aspects and demystify the characteristics of the black hole and to describe the associated physics as one would for "ordinary bodies".

- *Claim*: the internal dynamics of a black hole can be modeled effectively as a membrane with electromechanical properties. The dynamics of Einstein's equations allows determination of the response of the black hole to external disturbances.

Damour (1978); Price, Thorne (1986)

The membrane paradigm: derivation

- Dynamical equations: project Einstein's equations along the event horizon which is a null hypersurface, using the the generator of the event horizon, (which is a null vector in the spacetime).



Projecting Einstein's equations

(i). "Hamiltonian constraint:" $R_{AB}\,\xi^A\,\xi^B$

(ii). "Momentum constraint:" $R_{AB}\,\xi^A\,e^B$

spacetime metric: $\mathcal{G}_{AB}$

induced metric: $g_{\mu\nu}$

metric on spatial sections: γ_{ab}

The membrane paradigm

- Dynamical equations: project Einstein's equations along the event horizon which is a null hypersurface, using the the generator of the event horizon, (which is a null vector in the spacetime).

- Surface dynamics controlled by the extrinsic curvature induced on the horizon which provides a measure of gravitational energy momentum.

- The diffeomorphism symmetry inherent in Einstein's equations implies the conservation for the "gravitational energy-momentum" obtained via such a projection as an identity.

$$T_{\mu\nu} \sim K_{\mu\nu} - K\, g_{\mu\nu} + \text{counter-terms}\,[g_{\mu\nu}]$$

- The equations associated with the membrane paradigm are these conservation equations; these can be written in a form that is tantalizingly similar to fluid dynamical equations albeit of a peculiar kind.

Kinematics: variables for the paradigm

- From the horizon generator we can determine the extrinsic curvature of the horizon; introduce a basis on spatial sections with vectors e_μ

$$\nabla_\mu \xi = -K^\nu_\mu \, e_\nu$$

- Components of this extrinsic curvature are decomposed based on their transformations of the spatial rotation group.

$$\text{expansion:} \qquad \theta = -K^a_a \ ,$$

$$\text{shear:} \qquad \sigma_{ab} = -\gamma_{ac} \, K^c{}_b + \frac{1}{2} \gamma_{ab} \, \theta$$

$$\text{vorticity vector:} \qquad \Omega_a = -K^\xi_a$$

- The surface gravity can also be recovered from the extrinsic curvature by looking at the component along the generator:

$$\text{surface gravity:} \qquad \kappa = -K^\xi_\xi \ ,$$

Membrane dynamics

- The equations on a given spatial section of the horizon are interpreted as fluid dynamical equations and those along the generator as a gravitational analog of the Clausius equation (relating entropy production to heat).

$$D_t p_a = -\nabla_a \left(\frac{\kappa}{8\pi}\right) + \frac{1}{16\pi}\,\nabla_b \sigma_a{}^b - \frac{1}{16\pi}\,\nabla_a \theta - \xi^\mu\, T^{\text{matter}}_{\mu a}$$

Equation bears similarity to a hydrodynamic equation, with

$$P = \frac{\kappa}{8\pi}\,, \qquad \eta = \frac{1}{16\pi}\,, \qquad \zeta = -\frac{1}{16\pi}$$

$$\boxed{\eta/s = \frac{1}{4\pi}}$$

The equation along the generators is the famous Raychaudhuri equation:

$$D_t s - \frac{1}{\kappa}\,D_t^2 s = \frac{1}{T}\left(2\,\frac{1}{16\pi}\,\sigma_{ab}\,\sigma^{ab} - \frac{1}{16\pi}\,\theta^2 + \text{horizon-momenta}^2\right)$$

Damour (1978); Price, Thorne (1986)

Inadequacies of the membrane paradigm

- The equations for the most part are essentially kinematical: conservation of the horizon stress tensor is guaranteed a-priori.
- The derivation is predicated on the equation of motion of gravity being solved and one subsequently focuses on the projection of the full dynamics.
- The negative value of the bulk viscosity, usually attributed to the teleological nature of the event horizon, indicates a serious pathology of the horizon fluid.
- Analysis of linearized fluctuations about spherically symmetric black holes (such as asymptotically flat Schwarzschild) does not show any evidence of hydrodynamic behaviour.
- More critically, hydrodynamics is a low energy effective theory. At no stage in the derivation of the membrane paradigm is one focussing on the low energy excitations of the black hole.

Relativistic hydrodynamics

- Hydrodynamics is an IR effective field theory, valid when systems attain local but not global thermal equilibrium.
- We require that deviations away from equilibrium are long-wavelength in nature, i.e., we allow fluctuations that occur at scales larger than the typical mean free path of the theory. $\ell_m \ll L \qquad t_m \ll t$
- This allows for a gradient expansion: higher derivative operators are suppressed by powers of our expansion parameter ℓ_m/L .
- The dynamical content of fluid dynamics is just conservation. The energy momentum tensor and charge currents if any should be covariantly conserved.

$$\nabla_\mu T^{\mu\nu} = 0 \ , \qquad \nabla_\mu J^\mu = 0$$

- Conservation alone does not make for a good dynamical system since there are more dof than equations, but things simplify in the long-wavelength limit.

Relativistic hydrodynamics

- In the long-wavelength limit (+ local equilibrium) the dynamical dof are reduced, to local charge densities, local temperature and a (normalized) velocity field which indicates direction of flow of energy flux.

$$T^{\mu\nu}(x) = [P(x) + \rho(x)]\, u^\mu\, u^\nu + P(x)\, g^{\mu\nu} + \Pi^{\mu\nu}(x)$$

$$J^\mu_I = q_I\, u^\mu + J^\mu_{I,\mathrm{diss}}$$

- The definition of the velocity field can be fixed by a choice of fluid frame; typically one chooses the velocity to be the timelike eigenvector of the energy -momentum tensor (defining thus the Landau frame).

- Further specification of the fluid requires constitutive relations which require the operators which characterize the dissipative tensors.

- In addition, a fluid also has an entropy current, which satisfies the 2nd law.

$$\mathcal{J}^\mu_S = s\, u^\mu + \mathcal{J}^\mu_{S,\mathrm{diss}}\ , \qquad \nabla_\mu \mathcal{J}^\mu_S \geq 0$$

Relativistic hydrodynamics

- The dissipative parts of stress-tensor and charge currents can be expanded out in a basis of on-shell inequivalent operators built from the dynamical variables and their derivatives.

- From the velocity field we can for instance define:

$$
\begin{aligned}
\theta &= \nabla_\mu u^\mu = P^{\mu\nu}\,\nabla_\mu u_\nu \\
a^\mu &= u^\nu\,\nabla_\nu u^\mu \equiv \mathscr{D} u^\mu \\
\sigma^{\mu\nu} &= \nabla^{(\mu} u^{\nu)} + u^{(\mu}\,a^{\nu)} - \frac{1}{d-1}\,\theta\,P^{\mu\nu} = P^{\mu\alpha}\,P^{\nu\beta}\,\nabla_{(\alpha} u_{\beta)} - \frac{1}{d-1}\,\theta\,P^{\mu\nu} \\
\omega^{\nu\mu} &= \nabla^{[\mu} u^{\nu]} + u^{[\mu}\,a^{\nu]} = P^{\mu\alpha}\,P^{\nu\beta}\,\nabla_{[\alpha} u_{\beta]}\ .
\end{aligned}
$$

- At first order, upon using the conservation of ideal fluid to eliminate themal gradients, we have

$$
\Pi^{\mu\nu}_{(1)} = -2\,\eta\,\sigma^{\mu\nu} - \zeta\,\theta\,P^{\mu\nu}
$$

- Second law requires that $\eta \geq 0\ , \qquad \zeta \geq 0$

Blackfolds: an effective theory of black branes

- The discussion of hydrodynamics makes it clear that one way to approach the membrane paradigm is to view it as an effective theory of black objects.

- The natural language for such a discussion is provided within the framework of blackfolds.

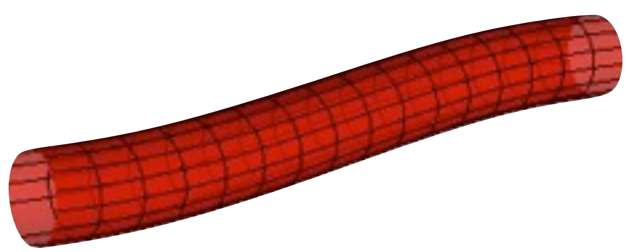
  Emparan, Harmark, Nairchos, Obers (2009)

- However to proceed, we need a useful way to characterize the low energy degrees of freedom.

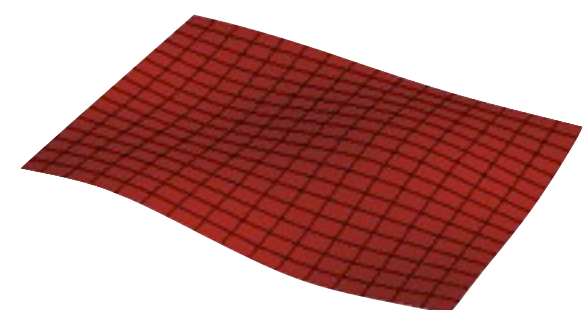
- Requirement: ability to separate the IR dofs (associated with the horizon) from the UV dof (associated with asymptopia).

From black holes to black branes

- The black holes in 3+1 dimensions are spherical (Hawking's topology theorem) and inherently have only one scale.
- In higher dimensions, one can have extended horizons, with multiple scales.



black string



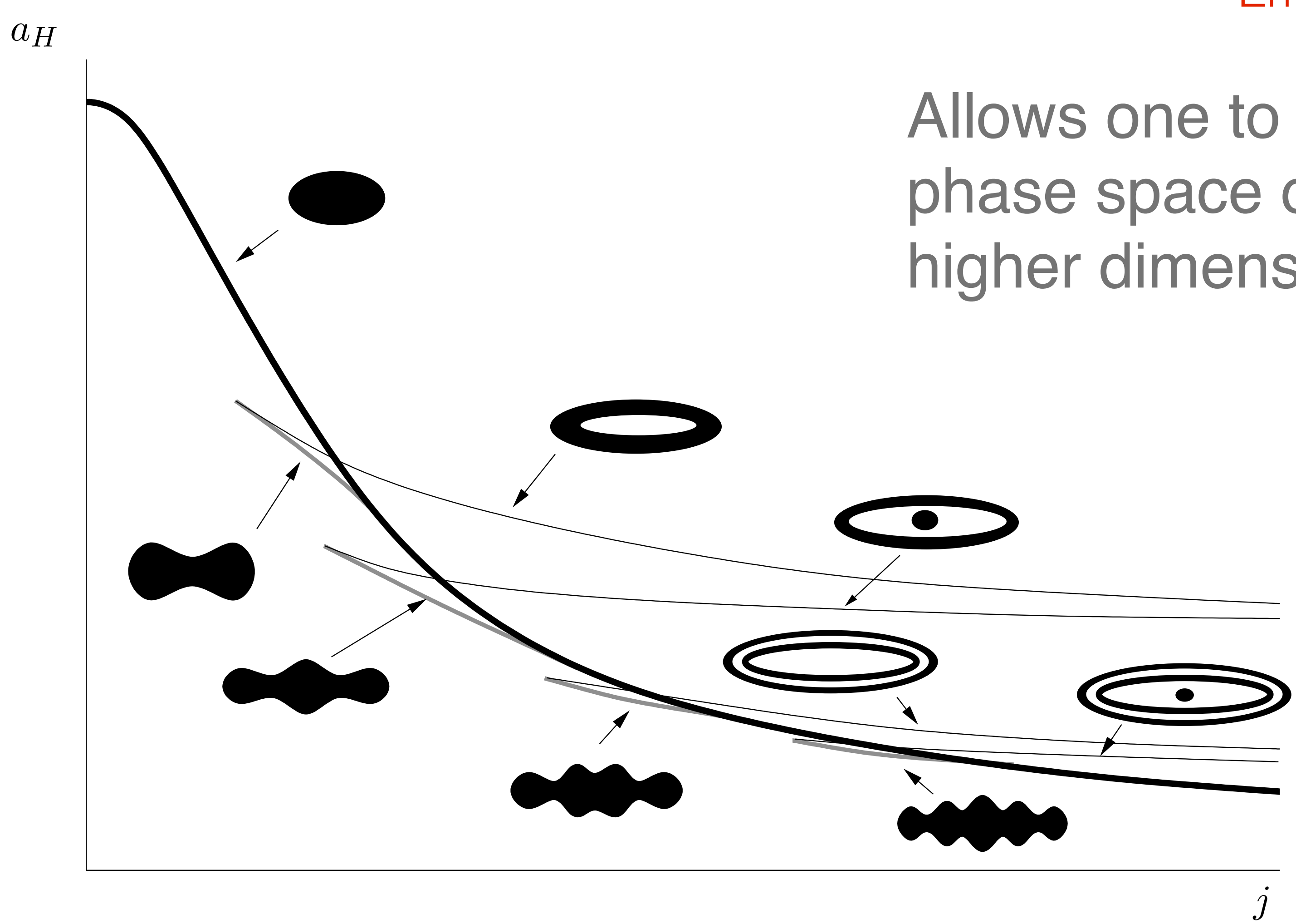
black brane

- Separation of scales allows us to investigate the behaviour of infra-red physics of black holes systematically.
- Historically, this was done in the context of black holes in a universe with a negative cosmological constant first: *fluid/gravity correspondence.*
- More generally the *blackfold* approach allows one to investigate this physics and moreover allows construction of approximate black hole solutions in higher dimensions.

Blackfolds

- A world-volume effective field theory for the dynamics of black branes.
- Gives approximate black hole solutions to Einstein's equations when the horizons in question admit two widely separated scales.

Emparan, Harmark, Nairchos, Obers (2009)

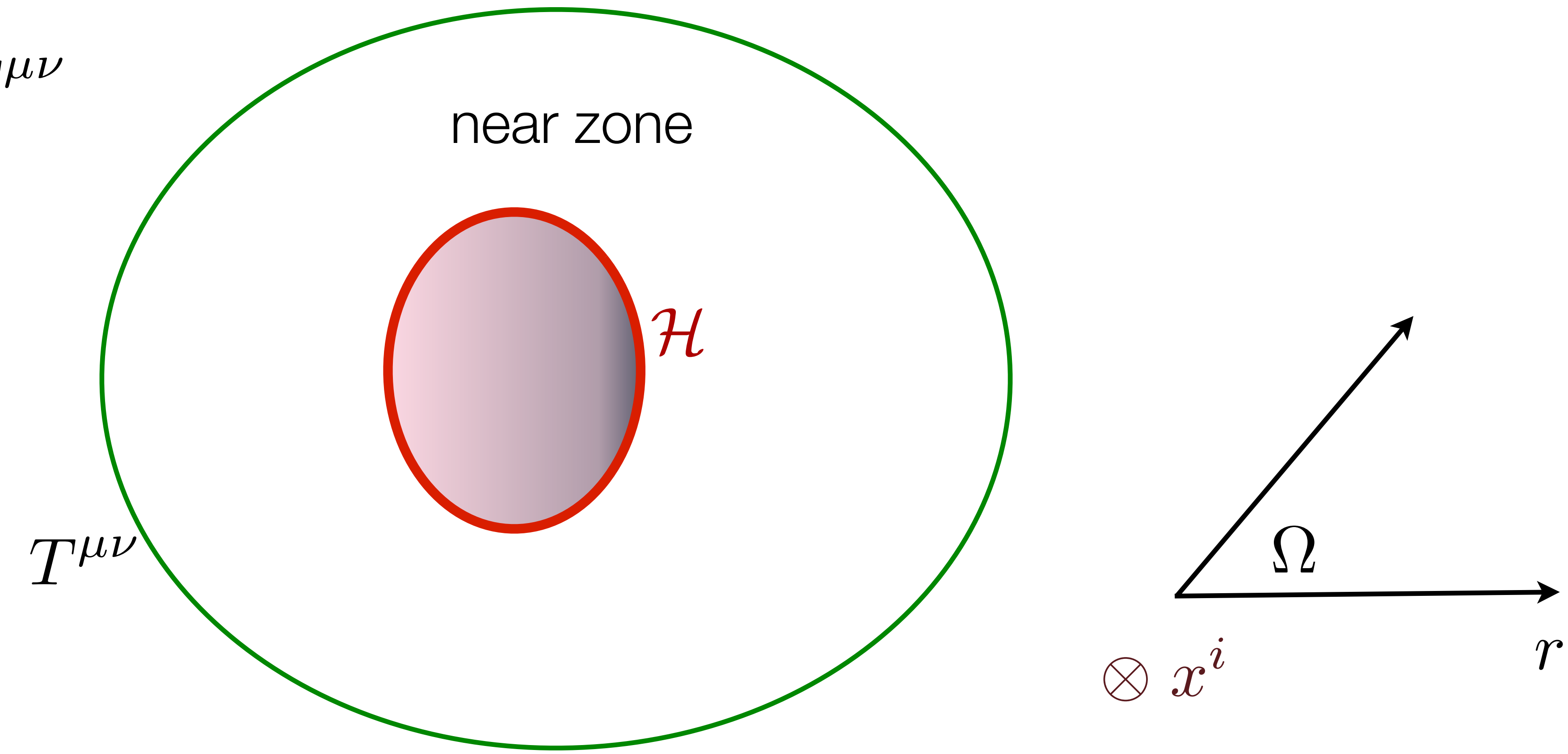


Allows one to explore the vast classical phase space of gravitational solutions in higher dimensions.

Emparan, Figueras (2010)

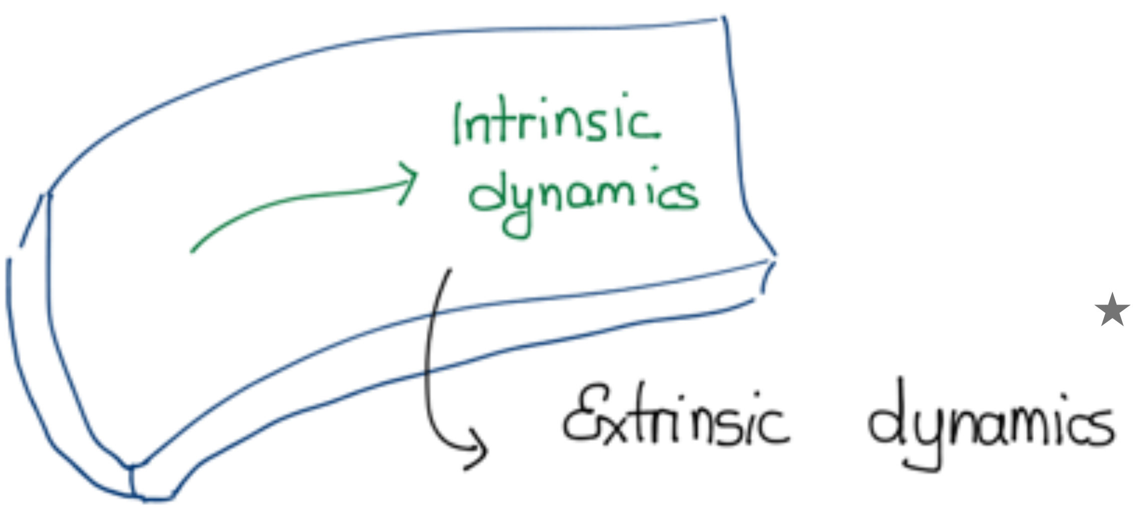
Blackfolds: qualitative picture

Far zone sourced by $T^{\mu\nu}$



- Dynamics of black branes naturally splits into two
  - Intrinsic dynamics: dynamics along the world-volume, essentially given by the conservation of the brane stress tensor.
  - Extrinsic dynamics: which describes how the brane bends in response to the ambient curvature.

Blackfolds: low energy dynamics



⋆ Intrinsic: conservation of the induced stress tensor along the horizon directions.

$$\nabla_\mu T^{\mu\nu} = 0$$

⋆ Extrinsic: minimization of the stress induced by the extrinsic curvature of the brane.

$$K_{\mu\nu}{}^{\rho}\, T^{\mu\nu} = 0$$

❖ In the long wavelength limit (momenta along the world-volume)

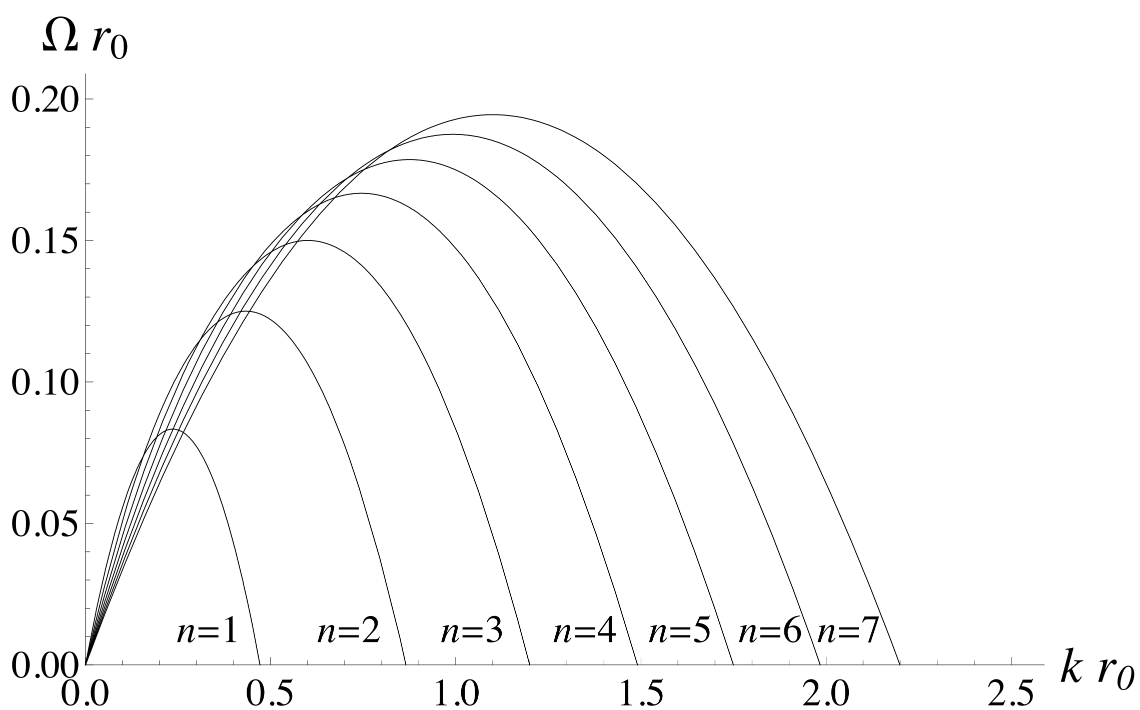
⋆ Intrinsic: Black branes behave like fluids under strains along the horizon.

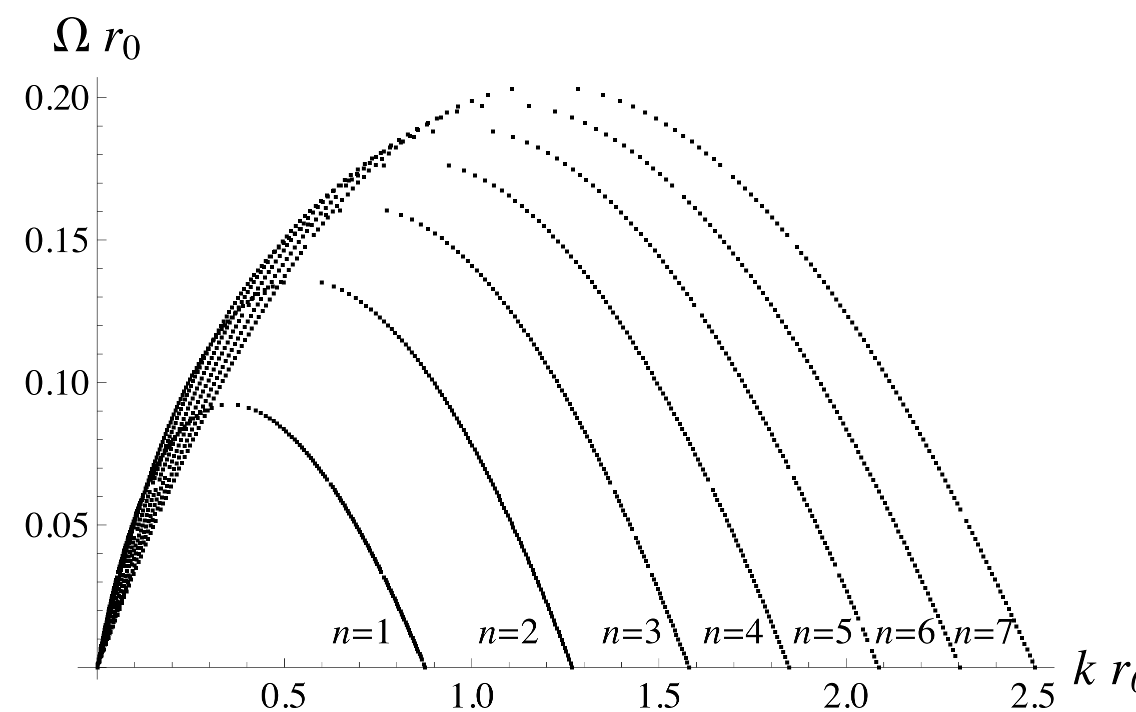
⋆ Extrinsic: Dynamics is that of elastic solids for strains normal to the horizon.

Camps, Emparan (2012).

Predictions of blackfolds

- Black branes are known to be unstable to long-wavelength fluctuations.
- This feature should be visible in the effective field theory approach and indeed the intrinsic dynamics of the branes i.e., fluid dynamics carries a clear signature of this instability.

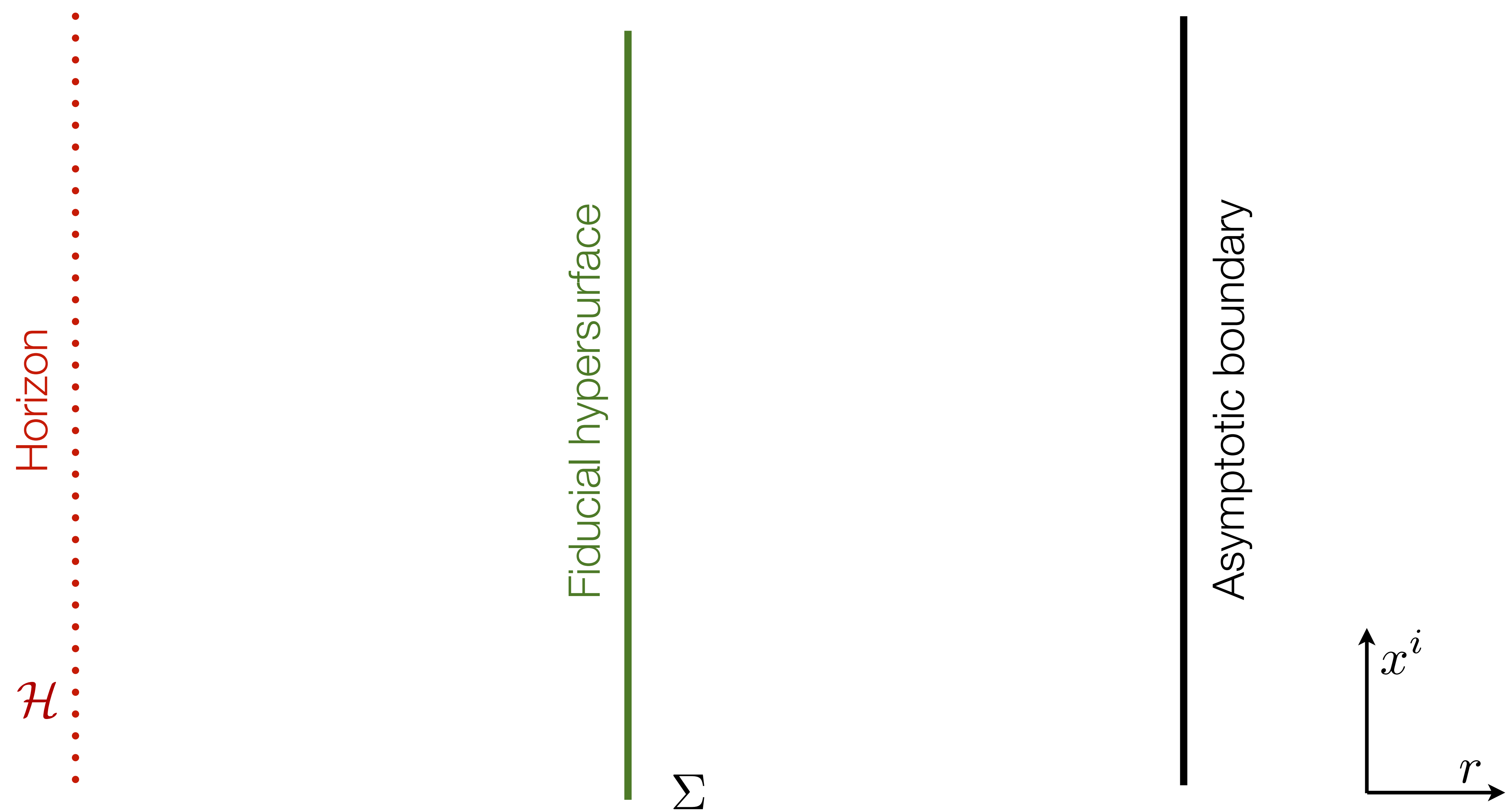




Camps, Emparan, Haddad (2010).

The membrane boundary conditions

- A-priori there are two distinct ways to view the dynamics on a fiducial timelike hypersurface, which we view as a regulated membrane.



The membrane boundary conditions

- A-priori there are two distinct ways to view the dynamics on a fiducial timelike hypersurface, which we view as a regulated membrane.

Dirichlet problem

- ⋆ Fix the metric data (background) on the hypersurface and solve gravity eoms subject to regularity in the interior.
- ⋆ The stress tensor on the surface is induced a la Brown-York once we solve for the geometry in the interior.
- ⋆ This stress tensor encodes the dynamics of the degrees of freedom on the surface.

Induced problem

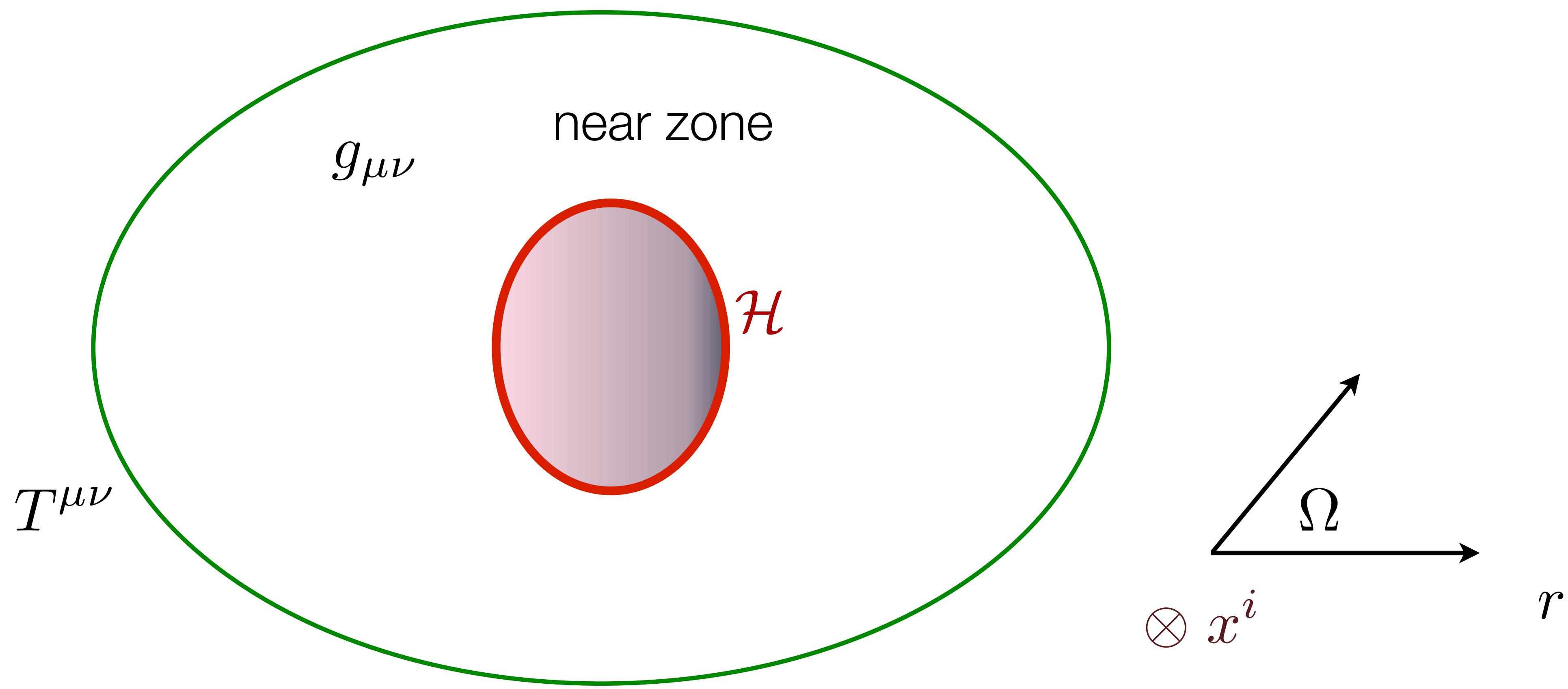
- ⋆ Project the bulk metric solved with asymptotic boundary conditions + regularity onto the hypersurface.
- ⋆ The stress tensor likewise is constructed from the extrinsic curvature.
- ⋆ There is a non-trivial correlation between the stress-tensor and the induced metric: “multi-trace” boundary condition in AdS/CFT parlance.

Dirichlet versus induced problems

- The Dirichlet boundary condition allows us to identify the dynamical degrees of freedom since we get to prescribe the background.
- The induced problem on the other hand requires a specific the boundary condition the surface so that the desired asymptotics is attained.
- The induced problem on the horizon is precisely the construction of the membrane paradigm:
  - The induced geometry and the dynamical degrees of freedom (spatial components of the horizon generators) are correlated precisely to achieve the desired asymptotics.

- Strategy: Use the Dirichlet problem to isolate the dynamical degrees of freedom & subsequently engineer the appropriate induced boundary condition to ensure asymptotics.

Dirichlet problem: branes in a box



- We focus on the near-zone of the black object, isolating the brane in a box.
- By prescribing rigid boundary conditions on the hypersurface we fix the geometry of our box and ask how the black hole responds to this confinement.
- This allows us to decipher the dynamical degrees of freedom on the surface.

Branes in a box: implications

- Putting a black hole/brane in a box has consequences on its dynamics.
- Given the new boundary conditions the black hole has to adapt itself and in particular its horizon needs to adjust appropriately. Indeed one can have multiple black hole solutions in the box.
- One important consequence is the change in the nature of the black hole thermodynamics:
  - Small black holes: Don’t see the box and have thermodynamics of an asymptotically flat space solution, including the negative specific heat.
  - Large black holes: Are sensitive to the box and associated boundary conditions. They can come to equilibrium and have positive specific heat.

Natural way to implement a covariant gravitational box: put the black hole in a universe with a negative cosmological constant.

Hydrodynamic limit of the Dirichlet problem

- Einstein's equations can be solved in the near-zone in a certain long-wavelength approximation (focus on large black branes).

- Solution is parameterized by variables that characterize the low energy intrinsic dynamics of the hypersurface on which we impose the boundary conditions.

- This dynamics is the aforementioned fluid dynamics: one finds the hypersurface is described by a relativistic fluid with explicit constitutive relations determining its transport properties.

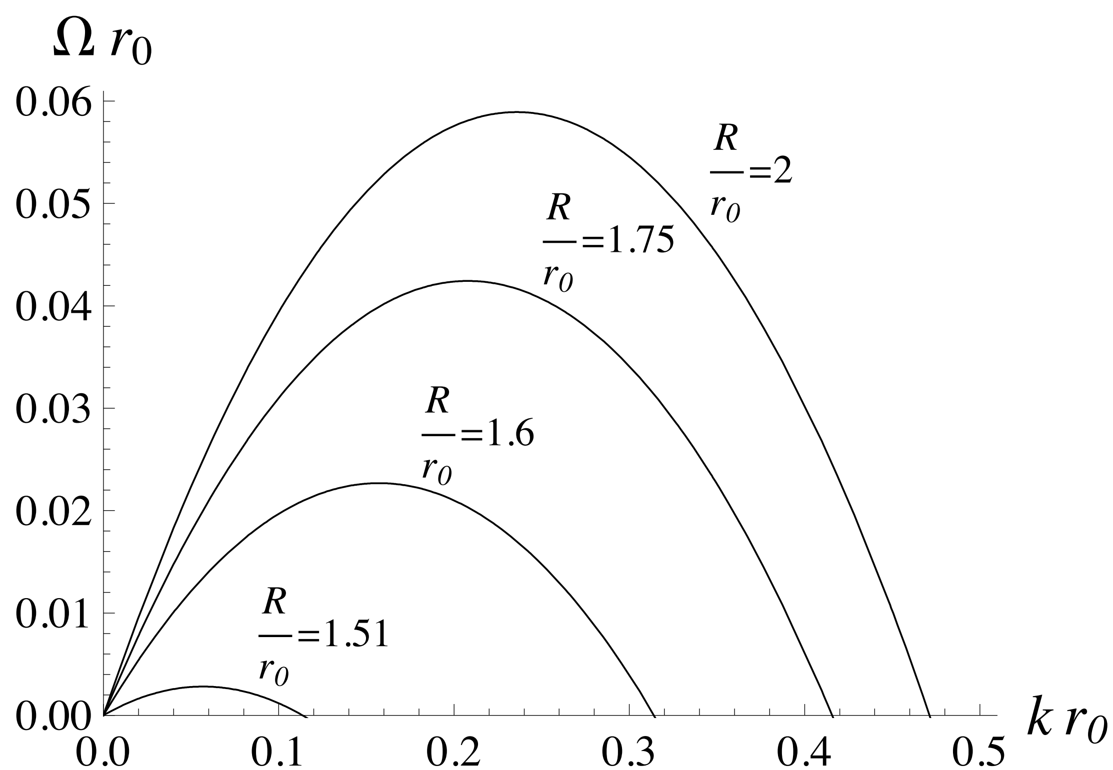
- E.g., AdS black holes in a box: transport is unchanged but equation of state changes as a function of the hypersurface location.

Brattan, Camps, Loganayagam, MR (2011)

- To leading order the longitudinal and transverse modes are decoupled; the coupling between the dof occurs at higher orders.

Dirichlet hydrodynamics II

- The fluid dynamical behaviour being sensitive to the boundary conditions displays this for e.g., via a modification of the spectrum of small amplitude fluctuations.



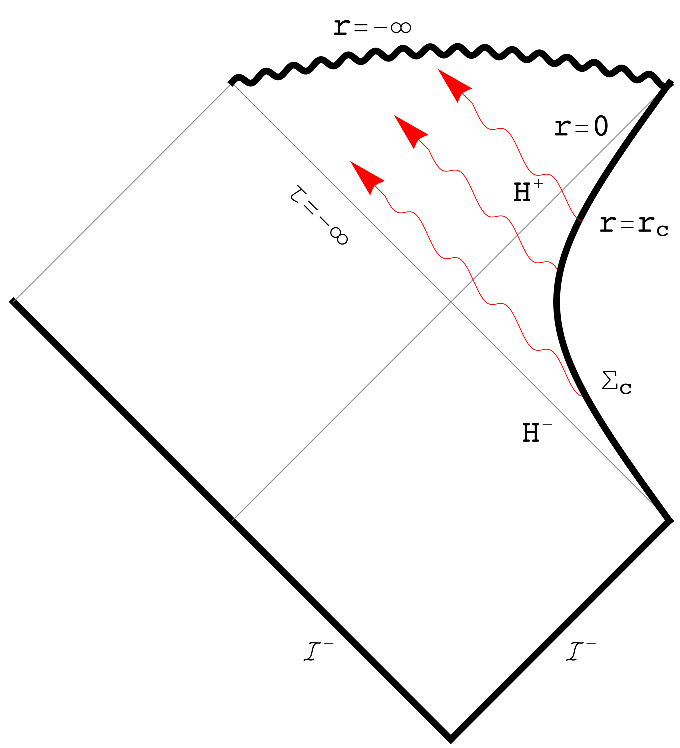
- One sees the disappearance of the GL mode and the speed of sound monotonically increases as we move the hypersurface close to the horizon.

Emparan, Martinez (2012)

‘Membrane’ limit of the Dirichlet problem

- Near-horizon limit: enclose the black hole/brane with a box that hugs the horizon (cf., stretched horizon).

- The dynamics can be studied by working in the Rindler geometry which is the effective geometry in the sliver of spacetime between the horizon and the Dirichlet surface.



$$\partial_i v^i = 0 \ , \qquad \partial_i p + \partial_t v_i + v^j\, \partial_j v_i - \nu\, \partial^2 v_i = 0$$

- The low energy physics of the Rindler region is universal to all branes and is simply the *incompressible Navier-Stokes dynamics.*

Bredberg, Keeler, Lysov, Strominger (2011)

Charged branes: Decoupling

- One can study more interesting systems, by considering black branes carrying charges. e.g., D-branes in string theory.
- Charges imply that one has a new scale in the problem, which one can use to tune the temperature of the black hole (independently of its size).
- In the near-extremal (vanishing temperature) limit one encounters an infinite throat: the proper distance to the horizon diverges.
- The dynamics in the throat in the low energy limit decouples from the asymptotic dof: this is the basis of the famous AdS/CFT correspondence.
- Of course, in string theory one can identify the decoupled theory which describes the physics of the throat region.

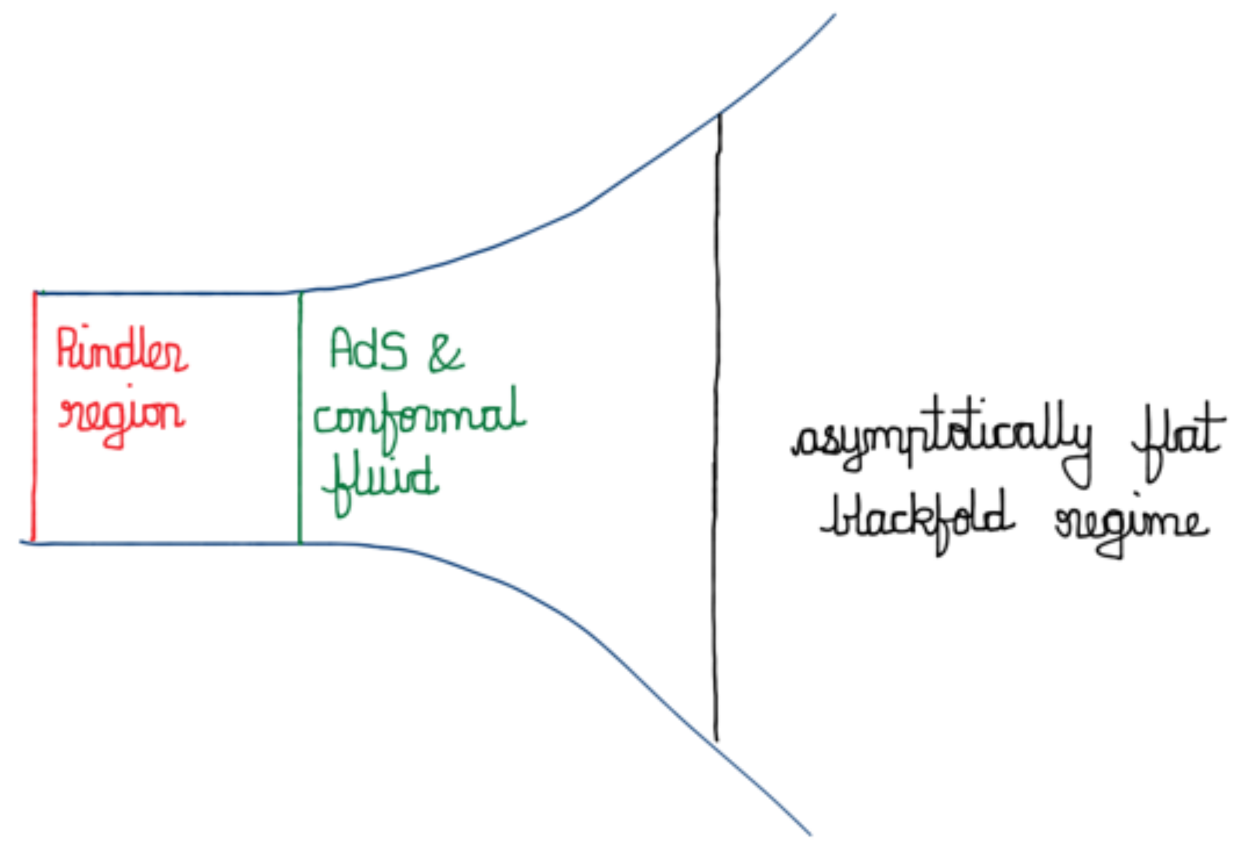
D-branes and membranes

- In classical gravity one can explore the hypersurface dynamics in various regimes of charged black branes.

- Asymptotic region: Blackfold fluid with features described earlier.

- Throat region: Conformal fluid dual to AdS geometries. Low energy limit of a QFT.

- Rindler region: incompressible fluid

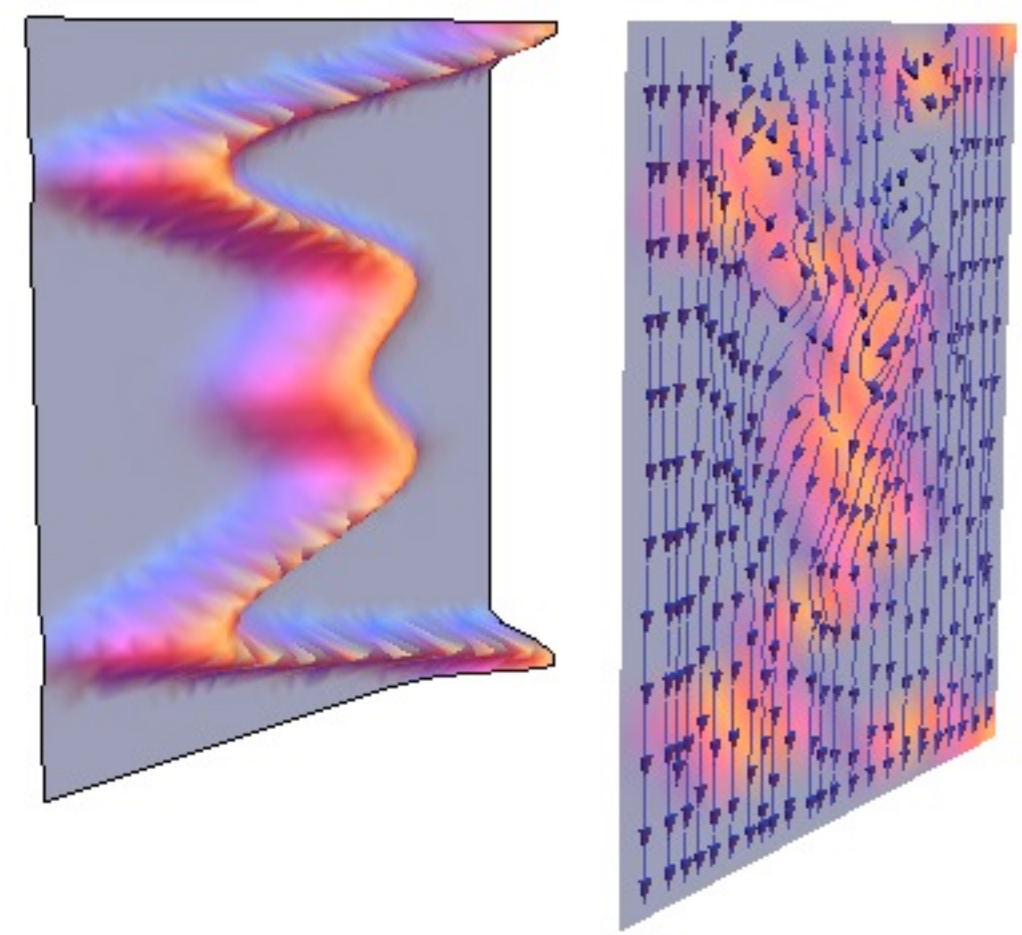


- Monitor the variation of transport properties of the fluid across the regimes: transport coefficients are pretty much determined by the throat dynamics.

Emparan, Hubeny, MR (wip) Brattan, Camps, Loganayagam, MR (2011)

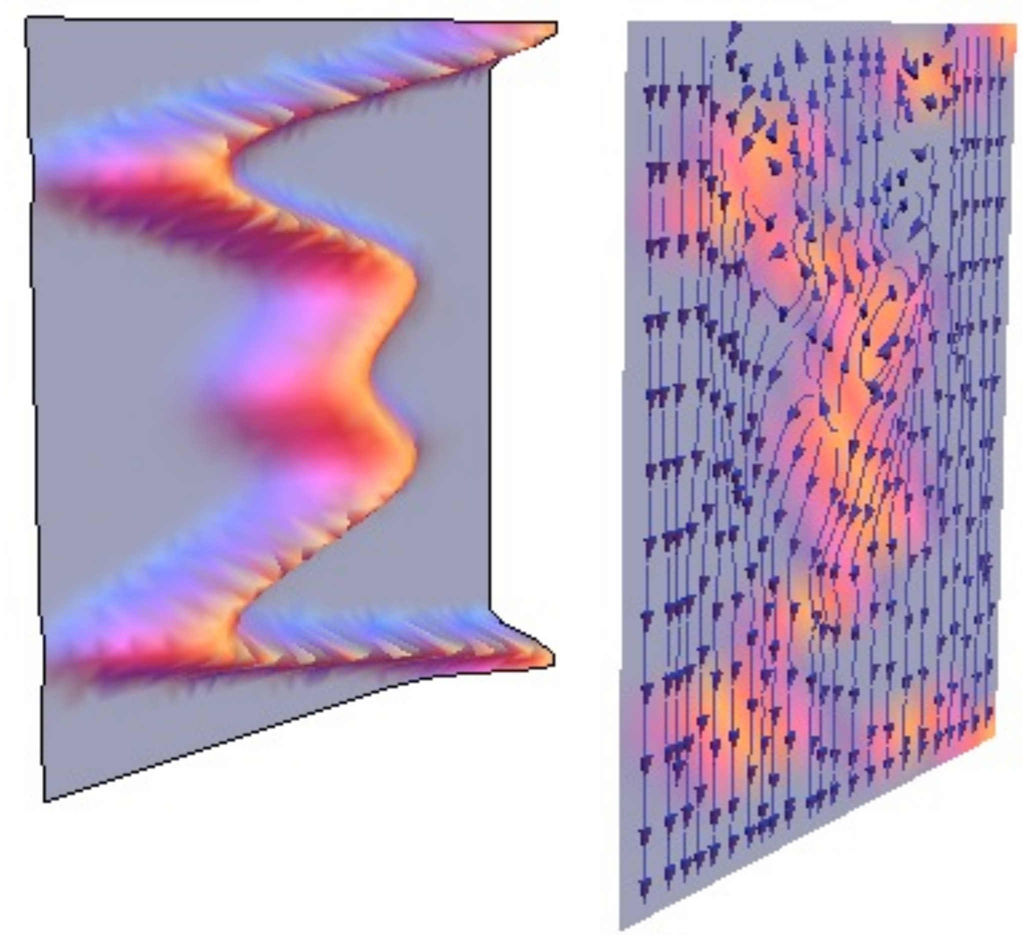
Black branes as lumps of fluid

- Black branes really behave as lumps of fluid in the low energy limit.
- In the fluid/gravity correspondence, the fluid lives at the end of the universe, on the asymptotic boundary of the spacetime where the black hole resides.
- Here the fluid is a hologram, honestly capturing all the low energy physics of the entire geometry.



Black branes as lumps of fluid

- More generally, the blackfold approach allows us to isolate the fluid regime associated with a black brane. (Dirichlet problem)
- Based on where one chooses to put the hypersurface demarcating the near and far zones, one obtains different constitutive relations.
- However, there is an universality of the very near horizon region: this Rindler region is described by a nearly ideal, non-relativistic, incompressible viscous fluid.



The paradigmatic membrane

- The membrane of the paradigm is obtained by fine tuning the boundary condition on the hypersurface (multi-trace deformation).
- The physical consequence of such a boundary condition on the horizon should be given by the equations for evolution of the horizon generator.
- Thus the Damour-Navier-Stokes equations which describe the evolution of the horizon geometry as a function of the affine parameter arise from the fluid dynamics of the Dirichlet membrane, upon allowing the background to become ‘dynamical’.
- In holographic RG parlance, the Damour membrane is the effective dynamics obtained after integrating out the UV modes; this induces the multi-trace boundary conditions.

A hierarchy of effective field theories

Blackfolds

$\cup$

Fluid/Gravity

$\cup$

Membrane paradigm