A master solution of the Yang-Baxter equation and classical discrete integrable equations.

Vladimir Bazhanov

(in collaboration with Sergey Sergeev & Vladimir Mangazeev)

Australian National University

Mathematical Statistical Mechanics, Kyoto University,
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- Lattice models of statistical mechanics and field theory,
  - Quantum Yang-Baxter equation. Star-triangle relation.
  - low-temperature (quasi-classical) limit and its relation to classical mechanics.
- New "master" solution to the star-triangle relation (STR) contains
  - all previously known solutions to STR
    - Ising & Kashiwara-Miwa models
    - Fateev-Zamolodchikov & chiral Potts models
  - elliptic gamma-functions & Spiridonov's elliptic beta integral
- Low-temperature (quasi-classical) limit of the "master solution".
  - relation to the Adler-Bobenko-Suris classical non-linear integrable equations on quadrilateral graphs,
  - new integrable models of statistical mechanics where the Boltzmann weights are determined by classical integrable equations (Q_4).

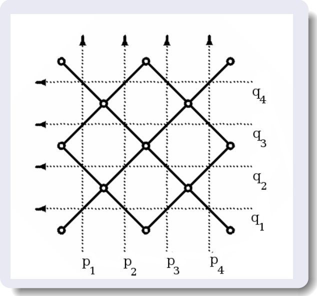
Space of solutions to the Yang-Baxter equation

- YBE is an overdetermined system of algebraic equations. Its general solution is unknown even in the simplest cases.
- Known solutions, various methods:
  Onsager, McGuire, Yang, Baxter, ... (over 65 different authors; native languages: Russian 26, Japanese 15, English 9, German 4, French 4, ..., Norwegian 1.)
- Algorithmic recipes (Drinfeld,Jimbo) Universal R-matrix for quantized (affine) Lie algebras, or quantum groups.
- 3D-generalization: tetrahedron equation, Zamolodchikov (1980) followed by Baxter, Bazhanov, Kashaev, Korepanov, Mangazeev, Maillet-Nijhoff, Sergeev, Stroganov,...
- New result (VB-Mangazeev-Sergeev): 3D integrable model with POSITIVE Boltzmann weights

Local "spins": $\sigma_i \in$ (set of values), $\quad \sigma_i \in \mathbb{R}$

$$Z = \sum_{\{spins\}} e^{-E(\sigma)/T},$$

$$E(\{\sigma\}) = \sum_{(ij)\in edges} \epsilon(\sigma_i, \sigma_j),$$



Boltzmann weights

$$W(\sigma_i, \sigma_j) = e^{-\epsilon(\sigma_i,\sigma_j)/T}$$

$$Z = \sum_{\{spins\}} \prod_{(ij)\in edges} W(\sigma_i, \sigma_j).$$

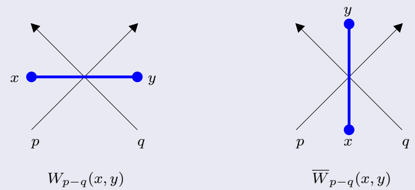
The problem: calculate partition function when number of edges is infinite,

$$\log Z \ = \ -Nf/T + O(\sqrt{N}), \qquad N \to \infty$$

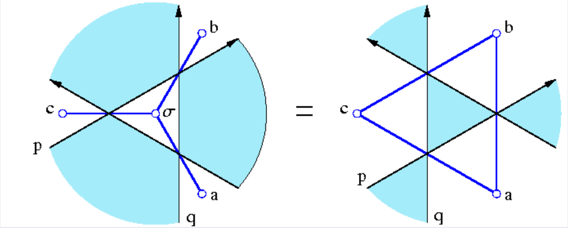
Solvable analytically if the Boltzmann weights satisfy the Yang-Baxter equation

Two types of Boltzmann weights, depending on the arrangement of rapidity line wrt the edge

$W_{p-q}(x,y)$ and $\overline{W}_{p-q}(x,y)$.



Simplest form of the Yang-Baxter equation: the *star-triangle relation*



$$\sum_{\sigma} \overline{W}_{p-q}(\sigma,b)\, W_{p-r}(c,\sigma)\, \overline{W}_{q-r}(a,\sigma) \;=\; W_{p-q}(c,a)\overline{W}_{p-r}(a,b)\, W_{q-r}(c,b)\,.$$

In general, weights $\overline{W}$ are related to W via

$$\overline{W}_{p-q}(x,y) \;=\; \sqrt{S(x)S(y)}W_{\eta-p+q}(x,y)\;,$$

where $S(x)$ are one-"spin" weights and η is the non-zero crossing parameter (value of an open angle).

In most cases the Boltzmann weights W are symmetric,

$$W_{p-q}(x,y) = W_{p-q}(y,x)\;.$$

Let for shortness

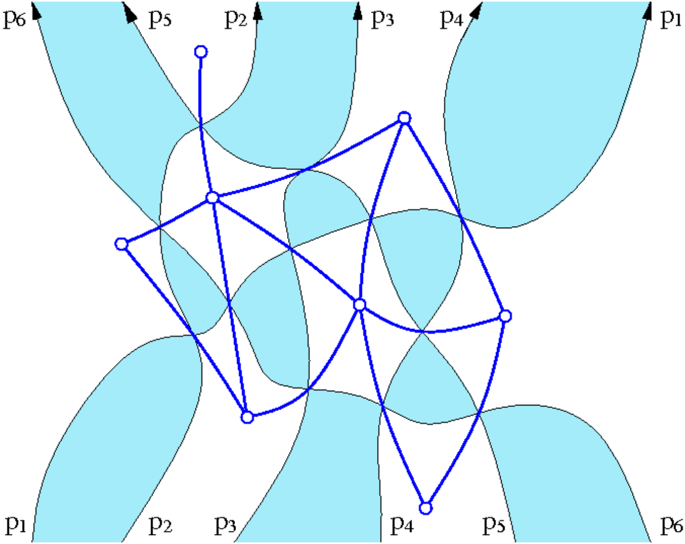
$$p-q=\alpha_1\;,\quad q-r=\alpha_3\;.$$

The star-triangle relation takes the form (assume continuous spins)

$$\int dx_0\, S(x_0)\, W_{\eta-\alpha_1}(x_1,x_0)W_{\alpha_1+\alpha_3}(x_2,x_0)W_{\eta-\alpha_3}(x_3,x_0)$$

$$=\; W_{\alpha_1}(x_2,x_3)W_{\eta-\alpha_1-\alpha_3}(x_1,x_3)W_{\alpha_3}(x_1,x_2)$$

Planar graph G, where $\mathcal{L}$ is the medial graph



Partition function

$$Z = \int \prod_{(ij)} W_{\alpha_{ij}}(x_i, x_j) \ \prod_m S(x_m)\ dx_m, \quad \alpha_{ij} = \begin{cases} p - q, & 1^{\text{st}}\text{-type} \\ \eta - p + q, & 2^{\text{nd}}\text{-type} \end{cases}$$

Assume, there is a temperature-like parameter ε, such for $\varepsilon \to 0$

$$W_\alpha(x,y) \ = \ e^{-\Lambda_\alpha(x,y)/\varepsilon + \mathcal{O}(1)}, \quad S(x) \ = \ \varepsilon^{-1/2} e^{-C(x)/\varepsilon + \mathcal{O}(1)}$$

$$\log Z = -\frac{1}{\varepsilon}\,\mathcal{E}(x^{(cl)}) + O(1), \qquad \mathcal{E}(x) = \sum_{(ij)} \Lambda_{\alpha_{ij}}(x_i, x_j) + \sum_m C(x_m)$$

and the variables $x^{(cl)} = \{\boldsymbol{x}_1^{(cl)}, x_2^{(cl)}, \ldots\}$ solve the variational equations

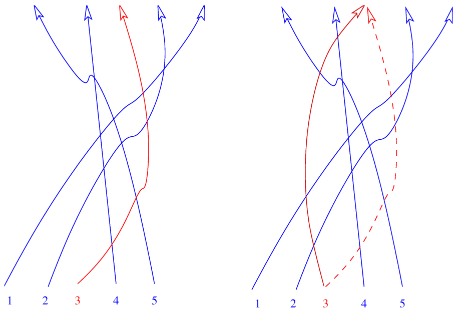
$$\frac{\partial \mathcal{E}(x)}{\partial x_j}\Big|_{x = x^{(cl)}} = 0$$

Can one obtain in this way the Q4 system of Adler-Bobenko-Suris, 2003?

$$\Lambda_\alpha(x,y) = -\mathsf{i}\int_0^{x-y} d\xi \log \frac{\vartheta_4((\xi - \mathsf{i}\alpha)\,|\,\tau)}{\vartheta_4(\xi + \mathsf{i}\alpha\,|\,\tau)} - \mathsf{i}\int_{\pi/2}^{x+y} d\xi \log \frac{\vartheta_4(\xi - \mathsf{i}\alpha\,|\,\tau)}{\vartheta_4(\xi + \mathsf{i}\alpha\,|\,\tau)}$$

$$\mathcal{C}(x) \ = \ 2\Big(|x| - \frac{\pi}{2}\Big)^2. \qquad |x| < \frac{\pi}{2} \tag{1}$$

Partition function depends only on the boundary data (i.e., on values of boundary spins and values of rapidities) but not on details of the lattice inside.



Baxter's factorization theorem (1979)

$$\log Z = -\frac{1}{T}\sum_{<ij>} f(\alpha_{ij}) + O(\sqrt{N})$$

$$\int \varepsilon^{-1/2} dx_0 \exp\Big\{ -\frac{\mathcal{E}_\star(x_0)}{\varepsilon} + \mathcal{O}(1) \Big\} \;=\; \exp\Big\{ -\frac{\mathcal{E}_\triangle}{\varepsilon} + \mathcal{O}(1) \Big\}$$

where

$$\mathcal{E}_\star = \Lambda_{\eta-\alpha_1}(x_0, x_1) + \Lambda_{\alpha_1+\alpha_3}(x_0, x_2) + \Lambda_{\eta-\alpha_3}(x_0, x_3) + C(x_0)\ ,$$

$$\mathcal{E}_\triangle = \Lambda_{\alpha_1}(x_2, x_3) + \Lambda_{\eta-\alpha_1-\alpha_3}(x_1, x_2) + \Lambda_{\alpha_3}(x_1, x_2)$$

the STR implies

$$\mathcal{E}_\star \;=\; \mathcal{E}_\triangle$$

at the stationary point

$$\frac{\partial \mathcal{E}_\star}{\partial x_0} \;=\; 0$$

Any solution of STR, admitting low-temperature expansion, leads to classical discrete integrable system, whose action is invariant under star-triangle moves

Chiral Potts and Kashiwara-Miwa models

N-state chiral Potts model (Albertini, McCoy et al'87, Baxter-Perk-AuYang'87)

$$W_{pq}(a,b) = \left(\frac{\mu_p}{\mu_q}\right)^{(a-b)} \prod_{k=1}^{a-b} \frac{y_q - \omega^k x_p}{y_p - \omega^k x_q}$$

$\omega^N = 1$, and (x_p, y_p, μ_p) is a point on genus ≥ 1 algebraic curve

- Positive Boltzmann weights. Reduces to Ising model for $N = 2$.
- Contains Z_N model (Fateev-Zamolodchikov'82)
- R-matrix

$$R_{ab}^{cd} = W_{pq}(a,b)\overline{W}_{pq}(b,c)W_{pq}(d,c)\overline{W}_{pq}(a,d)$$

intertwines two cyclic representations of $U_q(\widehat{sl}(2))$ (VB-Stroganov'90)

Chiral Potts and Kashiwara-Miwa models

N-state model with broken Z_N symmetry (Kashiwara-Miwa'86)

$$W_\theta(a,b) = r_\theta(a-b)\, t_\theta(a+b)$$

$$r_\theta(n) = \prod_{k=1}^{n} \frac{\vartheta_1\big(\frac{\pi}{N}(k-\frac{1}{2}) - \frac{\theta}{2N}\big)}{\vartheta_1\big(\frac{\pi}{N}(k-\frac{1}{2}) + \frac{\theta}{2N}\big)}, \quad t_\theta(n) = \prod_{k=1}^{n} \frac{\vartheta_4\big(\frac{\pi}{N}(k-\frac{1}{2}) - \frac{\theta}{2N}\big)}{\vartheta_4\big(\frac{\pi}{N}(k-\frac{1}{2}) + \frac{\theta}{2N}\big)},$$

- Reduces to Ising model for $N = 2$.
- In the trig. case reduces to Z_N model (Fateev-Zamolodchikov'82)
- The correponding R-matrix intertwines two (special) cyclic representations of Sklyanin algebra (Hasegawa-Yamada'90)

Chiral Potts and Kashiwara-Miwa models

Is there a generalised KM-model corresponding to the most general cyclic representations of the Sklyanin algebra? (VB-Stroganov,90 unpublished)

$$W_\theta(a,b) = r_\theta(a-b,\alpha-\beta)\,t_\theta(a+b,\alpha+\beta)$$

$$r_\theta(n,\phi) \;=\; \left[\frac{\mathcal{N}(\theta+\phi)}{\mathcal{N}(\theta-\phi)}\right]^{n/N} \prod_{k=1}^{n} \frac{\vartheta_1\left(\frac{\pi}{N}(k-\frac{1}{2}) - \frac{1}{2N}(\theta-\phi)\right)}{\vartheta_1\left(\frac{\pi}{N}(k-\frac{1}{2}) + \frac{1}{2N}(\theta+\phi)\right)}$$

What is the meaning of the additional parameters?

Elliptic gamma-function

$$\frac{\Gamma(x+1)}{\Gamma(x)} = x\,, \quad \frac{\Gamma_{\text{trig}}(x+\delta)}{\Gamma_{\text{trig}}(x)} \sim \sinh(x)\,, \quad \frac{\Gamma_{\text{ell}}(x+\delta)}{\Gamma_{\text{ell}}(x)} \sim \vartheta_1(x|\tau)$$

Let q,p be the temperature-like parameters (elliptic nomes)

$$\mathsf{q} \;=\; e^{\mathsf{i}\pi\tau'}\;, \quad \mathsf{p} = e^{\mathsf{i}\pi\tau} \quad \mathrm{Im}(\tau,\tau') > 0\;.$$

The crossing parameter $\eta > 0$ is given by

$$e^{-2\eta} \;=\; \mathsf{pq}\;, \quad \mathsf{i}\eta \;=\; \frac{1}{2}\pi(\tau+\tau')\;.$$

In what follows, we consider the primary physical regimes

$$\eta > 0\;, \quad \mathsf{p},\mathsf{q} \in \mathbb{R} \quad \text{or} \quad \mathsf{p}^* = \mathsf{q}\;.$$

The elliptic gamma-function is defined by

$$\Phi(z) \;=\; \prod_{j,k=0}^{\infty} \frac{1 - e^{2\mathsf{i}z}\mathsf{q}^{2j+1}\mathsf{p}^{2k+1}}{1 - e^{-2\mathsf{i}z}\mathsf{q}^{2j+1}\mathsf{p}^{2k+1}} \;=\; \exp\left\{\sum_{n\neq 0} \frac{e^{-2\mathsf{i}zn}}{k(\mathsf{q}^n - \mathsf{q}^{-n})(\mathsf{p}^n - \mathsf{p}^{-n})}\right\}\;.$$

- $\Phi(z)$ is π-periodic,

$$\Phi(z+\pi) = \Phi(z)\ ,$$

- $\log \Phi$ is odd,

$$\Phi(z)\Phi(-z) = 1\ ,$$

- Zeros and poles:

$$\text{Zeros of } \Phi(z) = \{-\mathsf{i}\eta - j\pi\tau - k\pi\tau' \mod \pi\ , \quad j,k \geq 0\}\ ,$$

$$\text{Poles of } \Phi(z) = \{+\mathsf{i}\eta + j\pi\tau + k\pi\tau' \mod \pi\ , \quad j,k \geq 0\}\ ,$$

Exponential formula for $\Phi(z)$ is valid in the strip

$$-\eta < \mathrm{Im}(z) < \eta\ .$$

- Diference property:

$$\frac{\Phi(z-\frac{\pi\tau'}{2})}{\Phi(z+\frac{\pi\tau'}{2})} = \prod_{n=0}^{\infty}(1-e^{2\mathsf{i}z}\mathsf{p}^{2n+1})(1-e^{-2\mathsf{i}z}\mathsf{p}^{2n+1}) \sim \vartheta_4(z\,|\,\tau)\ ,$$

and similarly with $\tau \leftrightarrows \tau'$.

Boltzmann weights

Define the weights $\mathbb{W}$ and $\overline{\mathbb{W}}$ by

$$\mathbb{W}_\alpha(x,y) \;=\; \kappa(\alpha)^{-1}\,\frac{\Phi(x-y+\mathsf{i}\alpha)}{\Phi(x-y-\mathsf{i}\alpha)}\,\frac{\Phi(x+y+\mathsf{i}\alpha)}{\Phi(x+y-\mathsf{i}\alpha)}$$

and

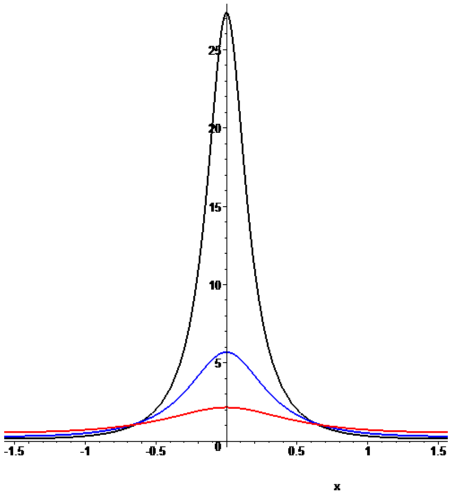
$$\overline{\mathbb{W}}_\alpha(x,y) = \sqrt{\mathbb{S}(x)\mathbb{S}(y)}\,\mathbb{W}_{\eta-\alpha}(x,y)\;, \quad \mathbb{S}(x) \;=\; \frac{e^{\eta/2}}{2\pi}\vartheta_1(2x\,|\,\tau)\vartheta_1(2x\,|\,\tau')\;.$$

Normalization factor (partition function per edge – exact solution) $\kappa(\alpha)$ is given by

$$\kappa(\alpha) \;=\; \exp\left\{\sum_{n\neq 0}\frac{e^{4\alpha n}}{n(\mathsf{p}^n-\mathsf{p}^{-n})(\mathsf{q}^n-\mathsf{q}^{-n})(\mathsf{p}^n\mathsf{q}^n+\mathsf{p}^{-n}\mathsf{q}^{-n})}\right\}\;.$$

It satisfies

$$\frac{\kappa(\eta-\alpha)}{\kappa(\alpha)} \;=\; \Phi(\mathsf{i}\eta-2\mathsf{i}\alpha)\;, \quad \kappa(\alpha)\kappa(-\alpha)=1\;.$$

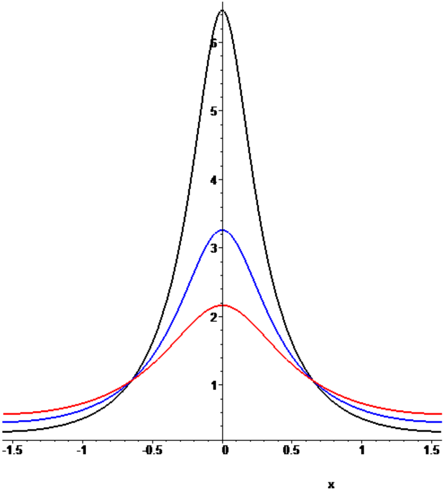


Plot of the real π-periodic function

$$R_\alpha(x) = \frac{\Phi(x + \mathsf{i}\alpha)}{\Phi(x - \mathsf{i}\alpha)}$$

for $\mathsf{p} = \mathsf{q} = \frac{1}{2}$ and

- red: $\alpha = \frac{\eta}{4}$
- blue: $\alpha = \frac{\eta}{2}$
- black: $\alpha = \frac{3\eta}{4}$



Plot of the real π-periodic function

$$R_\alpha(x) = \frac{\Phi(x + \mathsf{i}\alpha)}{\Phi(x - \mathsf{i}\alpha)}$$

for $\alpha = \eta/4$ and

- red: $\mathsf{p} = \mathsf{q} = 0.5$
- blue: $\mathsf{p} = \mathsf{q} = 0.6$
- black: $\mathsf{p} = \mathsf{q} = 0.7$

- The weights $\mathbb{W}_\alpha(x,y)$ and $\overline{\mathbb{W}}_\alpha(x,y)$ are real positive for

$$x, y \in \mathbb{R} \quad \text{and} \quad 0 < \alpha < \eta$$

- The weights are symmetric and π-periodic,

$$\mathbb{W}_\alpha(x,y) = \mathbb{W}_\alpha(y,x) = \mathbb{W}_\alpha(-x,y) = \mathbb{W}_\alpha(x+\pi,y) = \ldots\ .$$

- Difference properties of the weights:

$$\frac{\mathbb{W}_\alpha(x-\frac{\pi\tau'}{2},y)}{\mathbb{W}_\alpha(x+\frac{\pi\tau'}{2},y)} = \frac{\vartheta_4(x-y+\mathsf{i}\alpha\,|\,\tau)\,\vartheta_4(x+y+\mathsf{i}\alpha\,|\,\tau)}{\vartheta_4(x-y-\mathsf{i}\alpha\,|\,\tau)\,\vartheta_4(x+y-\mathsf{i}\alpha\,|\,\tau)}$$

and similarly with $\tau \leftrightarrows \tau'$.

As a mathematical identity the star-triangle relation for this solution is equivalent to Spiridonov's celebrated elliptic beta integral (2001).

This identity lies in the basis of the theory of elliptic hypergeometric functions.

Its connection with the Yang-Baxter equation (star-triangle relation) was not hitherto known

"Trigonometric" limit.

$$\tau = \mathsf{ib}/R\ , \quad \tau' = \mathsf{ib}^{-1}/R\ , \quad R \to \infty$$

Gamma-function with small argument

$$\Phi(\frac{\pi}{R}\sigma) \ \to\ \varphi(\sigma) = \ \exp\left\{\frac{1}{4}\int_{\mathrm{pv}}\frac{dw}{w}\,\frac{e^{-2\mathsf{i}\sigma w}}{\sinh{(\mathsf{b}w)}\sinh{(w/\mathsf{b})}}\right\}$$

Gamma-function with big argument

$$\Phi(\frac{\pi}{R}\sigma + \mathrm{const}) \ \to\ 1\ , \quad \mathrm{const} = \mathcal{O}(R^0)\ .$$

Two regimes of the star-triangle equation:

$$x_j = \mathrm{const} + \frac{\pi}{R}\sigma_j \quad \text{and} \quad x_j = \frac{\pi}{R}\sigma_j\ .$$

We consider the low-temperature limit outside the primary physical regime:

$$\mathsf{p}^2 = \boldsymbol{e}^{2\mathsf{i}\pi\tau} \quad \text{and} \quad \mathsf{q}^2 = e^{-T/N^2}\omega\,, \quad \omega = e^{2\pi\mathsf{i}/N}\,, \quad T \to 0\,.$$

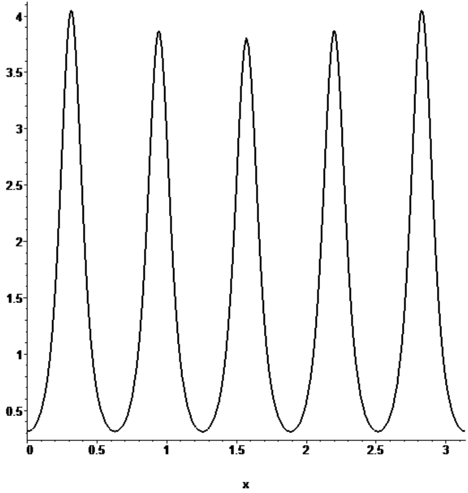
Asymptotic of $\mathbb{W}$: the low-T expansion

$$\mathbb{W}_\alpha(x,y) = \exp\left\{-\frac{\Lambda_\alpha(x,y)}{T}\right\} \cdot W_\alpha(x,y) \cdot (1 + \mathcal{O}(T))$$

where the Lagrangian density $\Lambda_\alpha(x,y)$ is $\frac{\pi}{N}$ periodic in x and y while the finite part $W_\alpha(x,y)$ is π-periodic.
Asymptotic of the partition function:

$$\mathcal{Z} = \underset{0\le x_m\le\pi}{\int\ldots\int} \exp\left\{-\frac{\mathcal{E}(\{x\})}{T} + \mathcal{O}(1) + \mathcal{O}(T)\right\} \prod_m \frac{dx_m}{\sqrt{T}}\,, \quad T \to 0\,,$$

where $\mathcal{E}(\{x\})$ is an action for a classical discrete integrable system. The ground state of the system is highly degenerate due to π/N periodicity.



Plot of

$$\text{abs}\left(\frac{\Phi(x+\mathsf{i}\alpha)}{\Phi(x-\mathsf{i}\alpha)}\right)$$

with $\mathsf{p} = \frac{1}{2}$ and

$$\mathsf{q} = 0.99 \cdot e^{\mathsf{i}\pi/5} \ .$$

The peaks are at

$$x = \frac{\pi}{N}(n + \tfrac{1}{2}),$$

Expression for the Lagrangian density:

$$\Lambda_\alpha(x,y) = 2\mathsf{i}N\int_0^{x-y} d\xi \log\frac{\vartheta_3(N(\xi-\mathsf{i}\alpha)\,|\,N\tau)}{\vartheta_3(N(\xi+\mathsf{i}\alpha)\,|\,N\tau)} + 2\mathsf{i}N\int_{\pi/2N}^{x+y} d\xi \log\frac{\vartheta_3(N(\xi-\mathsf{i}\alpha)\,|\,N\tau)}{\vartheta_3(N(\xi+\mathsf{i}\alpha)\,|\,N\tau)}$$

$$\begin{aligned}\Lambda_{\eta-\alpha}(x,y) &= \frac{\pi^2}{2} - (Nx)^2 - (Ny)^2 \\ &+2\mathsf{i}N\int_0^{x-y} d\xi \log\frac{\vartheta_1\big(N(\mathsf{i}\alpha+\xi)\,|\,N\tau\big)}{\vartheta_1\big(N(\mathsf{i}\alpha-\xi)\,|\,N\tau\big)} + 2\mathsf{i}N\int_{\pi/2N}^{x+y} d\xi \log\frac{\vartheta_1\big(N(\mathsf{i}\alpha+\xi)\,|\,N\tau\big)}{\vartheta_1\big(N(\mathsf{i}\alpha-\xi)\,|\,N\tau\big)}\ .\end{aligned} \tag{2}$$

$$\mathcal{C}(x) \;=\; 2\Big(x-\frac{\pi}{2}\Big)^2 . \qquad 0<x<\frac{\pi}{N} \tag{3}$$

Energy for the regular square lattice

$$\mathcal{E}(X) \;=\; \sum_{(ij)}\Lambda(\alpha\,|\,x_i,x_j) + \sum_{(kl)}\Lambda(\eta-\alpha\,|\,x_k,x_l) + \sum_m \mathcal{C}(x_m)\ , \tag{4}$$

Variational equations (Adler-Bobenko-Suris Q_4 eqns.)

$$\frac{\partial\mathcal{E}(X)}{\partial x_i} = 0, \quad \Rightarrow \quad \Psi_3\big(x,x_r\big)\Psi_3(x,x_\ell) = \Psi_1\big(x,x_u\big)\Psi_1(x,x_d)\ ,$$

$$\Psi_j(x,y) \;=\; \frac{\vartheta_j\big(N(x-y+\mathsf{i}\alpha)\,|\,N\tau\big)\ \vartheta_j\big(N(x+y+\mathsf{i}\alpha)\,|\,N\tau\big)}{\vartheta_j\big(N(x-y-\mathsf{i}\alpha)\,|\,N\tau\big)\ \vartheta_j\big(N(x+y-\mathsf{i}\alpha)\,|\,N\tau\big)}\ , \qquad j=1,2,3,4.$$

Zeroth order

Due to $\frac{\pi}{N}$-periodicity of the leading term, we introduce the discrete spin variables n_j,

$$x_j = \xi_j + \frac{\pi}{N} n_j \ , \quad 0 < \mathrm{Re}(\xi_j) < \frac{\pi}{2N} \ , \quad n_j \in \mathbb{Z}_N$$

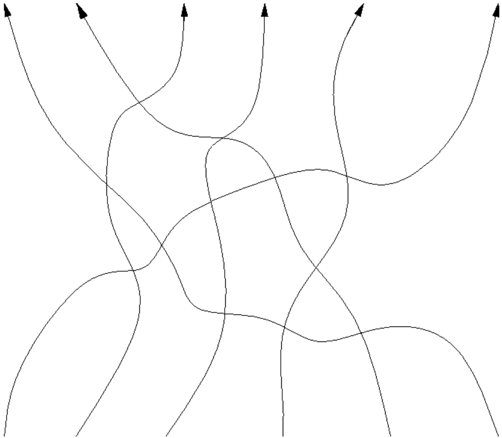
where parameter ξ_0 is the solution of the variational equation (in general: parameters ξ_j are solution of classical integrable equations). Canceling then the T^{-1} term, we come to the most general discrete-spin star-triangle equation:

$$\sum_{n_0 \in \mathbb{Z}_N} \overline{W}_{pq}(x_0, x_1) W_{pr}(x_0, x_2) \overline{W}_{qr}(x_0, x_3) \\ = \mathcal{R}_{pqr} W_{pq}(x_2, x_3) \overline{W}_{pr}(x_1, x_3) W_{qr}(x_1, x_2)$$

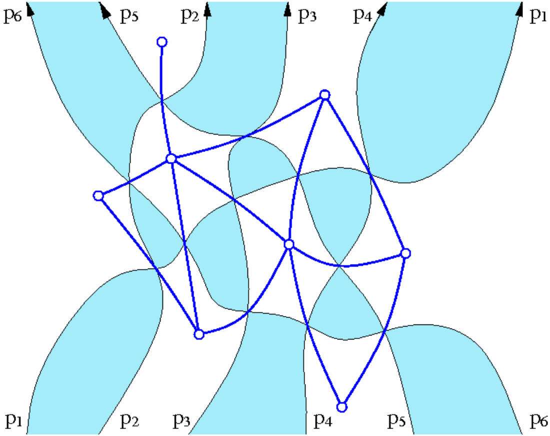
Note: we consider the star-triangle equation in the orders T^{-1} and T^0, however it is satisfied in all orders of T-expansion.

Hybrid model

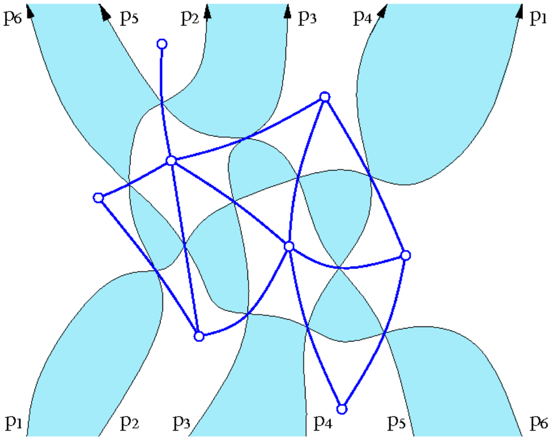
$$\mathcal{Z} = \underbrace{\int \ldots \int}_{0 \leq x_m \leq \pi} \exp\left\{ -\frac{\mathcal{E}(\{x\})}{T} + \mathcal{O}(1) + \mathcal{O}(T) \right\} \prod_m \frac{dx_m}{\sqrt{T}}\,, \quad T \to 0\,,$$



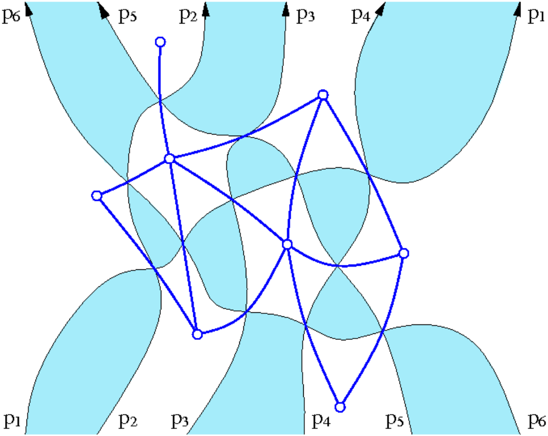
I. Rapidity lattice



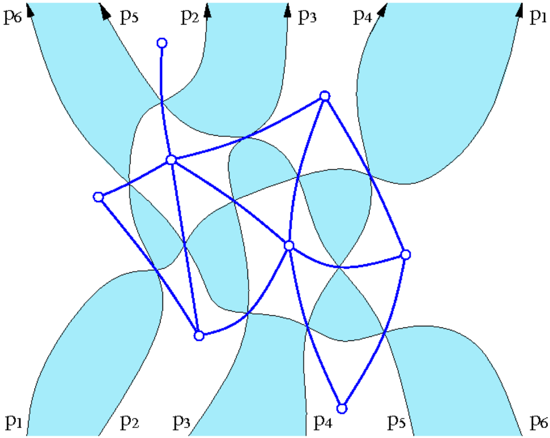
II. Bipartite graph, to each site assign a pair (ξ_j, n_j), where ξ_j are continuous and $n_j \in \mathbb{Z}_N$.



III. Fix all boundary variables (ξ_i, n_i).



IV. Solve classical integrable variational equations for the parameters ξ_j in the bulk (Dirichlet problem for the *Adler-Bobenko-Suris* system)



V. All discrete-spin Boltzmann weights W and $\overline{W}$ entering the partition function are now defined, the lattice statistical mechanics begins.

Asymptotics of the partition function:

$$\log \mathcal{Z} = -\frac{\mathcal{E}(\{\xi^{(cl)}\})}{T} + \log \mathcal{Z}_0 + \mathcal{O}(T) \ ,$$

where $\{\xi^{(cl)}\}$ denote the stationary point of the classical action,

$$\frac{\partial \mathcal{E}(\{\xi\})}{\partial \xi_m}\Big|_{\{\xi\}=\{\xi^{(cl)}\}} \ = \ 0 \ ,$$

and $\mathcal{Z}_0 = \mathcal{Z}_0(\{\xi^{(cl)}\})$ is the partition function for the discrete-spin system.

Yang-Baxter equation

$$R_{12}\,R_{13}\,R_{23} = R_{23}\,R_{13}\,R_{12} \tag{5}$$

Tetrahedron equation

$$R_{123}R_{145}R_{246}R_{356} = R_{356}R_{246}R_{145}R_{123}\ , \tag{6}$$

where R_{123} acts in a product of three oscillator Fock spaces, $n = 0, 1, 2\ldots$

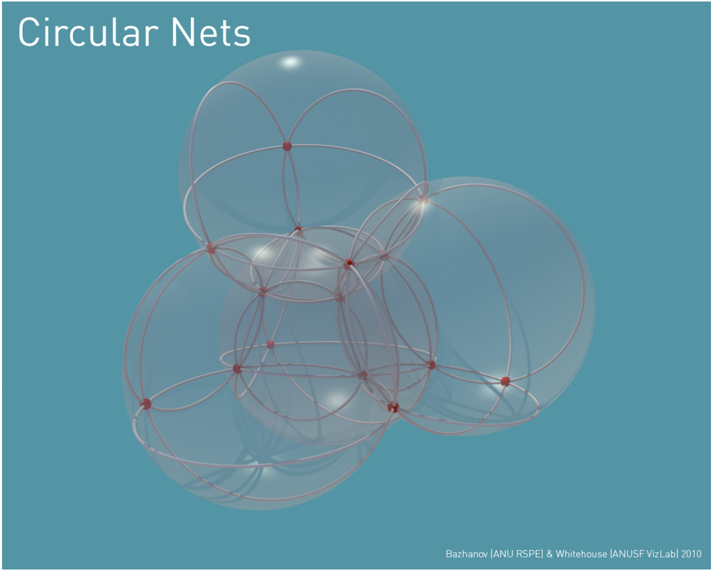
$$R_{n_1,n_2,n_3}^{n_1',n_2',n_3'} = \delta_{n_1+n_2,n_1'+n_2'}\,\delta_{n_2+n_3,n_2'+n_3'}\,q^{n_2(n_2+1)-(n_2-n_1')(n_2-n_3')}$$

$$\times\phi_1^{n_1}\phi_2^{n_2}\phi_3^{n_3}\phi_4^{n_1'}\sum_{r=0}^{n_2}\frac{(q^{-2n_1'};q^2)_{n_3-r}}{(q^2;q^2)_{n_3-r}}\frac{(q^{2+2n_3};q^2)_r}{(q^2;q^2)_r}q^{-2r(n_2+n_1'+1)}$$

For $0<q<1$ all nonzero matrix elements of R are positive. Layer-to-layer transfer matrix of the size $M\times N$, possesses rank-size duality for $U_q(\widehat{sl}_N)$ and $U_q(\widehat{sl}_M)$

$$\mathbf{T}(\{\phi\}) = \bigoplus_{\mu} T_M^{\widehat{sl}_N}(\mu) = \bigoplus_{\nu} T_N^{\widehat{sl}_M}(\nu)$$

Quasiclassical limit leads to 3D circular nets (Bobenko, Konopelchenko-Schief)



- We presented a new solution to the star-triangle equation expressed in terms of elliptic Gamma-functions
- This solution involves two temperature-like parameters (elliptic nomes p and q)
- This solution contains as specials cases all previously known solutions of the star-triangle equation both with discrete and continuous spin variables
- When one elliptic nome tends to a root of unity, $\mathsf{q}^2 \to e^{2\pi \mathsf{i}/N}$, we obtain a hybrid of a classical non-linear integrable system and a solvable model of statistical mechanics. In particular, it contains the chiral Potts and Kashiwara-Miwa models. This is analogous to the background field quantization in Quantum Field Theory.
- Connection to superconformal indices and electric-magnetic dualities (Dolan-Osborn, Spiridonov-Vartanov)

THANK YOU

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