Simple analytical results for quenches in 1D quantum gases

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Kyoto, August 2013

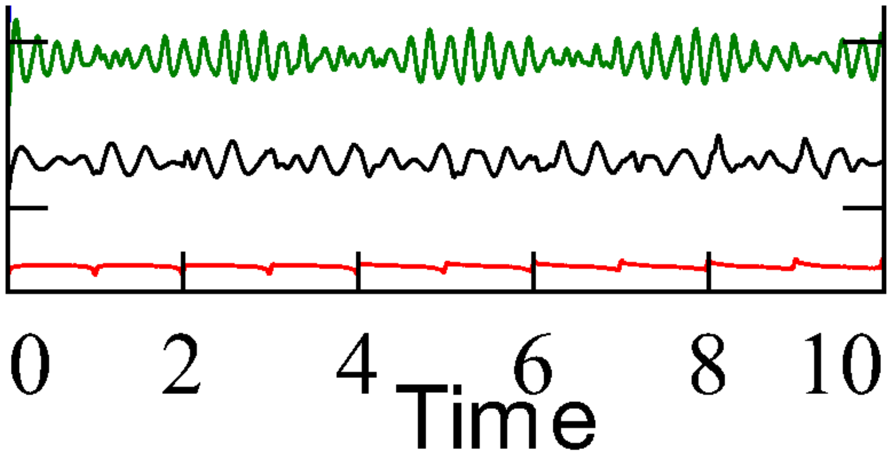
Based on:

M. Collura, S. Sotiriadis, P. Calabrese, Phys. Rev. Lett. 110, 245301 (2013) + arXiv:1306.5604.
M. Kormos, M. Collura, P. Calabrese, ArXiv:1307.2142 + unpublished

Earlier works with **J. Cardy, F. Essler, and M. Fagotti**

Quantum quench dynamics

- A many-body quantum system is prepared in the ground-state of H_0, *i.e.* $|\Psi_0\rangle$
- At $t=0$, $H_0 \Rightarrow H$, *i.e.* an Hamiltonian parameter is quenched
- For $t>0$, it evolves unitarily: $|\Psi(t)\rangle = e^{-iHt}|\Psi_0\rangle$
- No contact with "external" world
- How can we describe the dynamics?



von Neumann in 1929 posed the question [1003.2133]

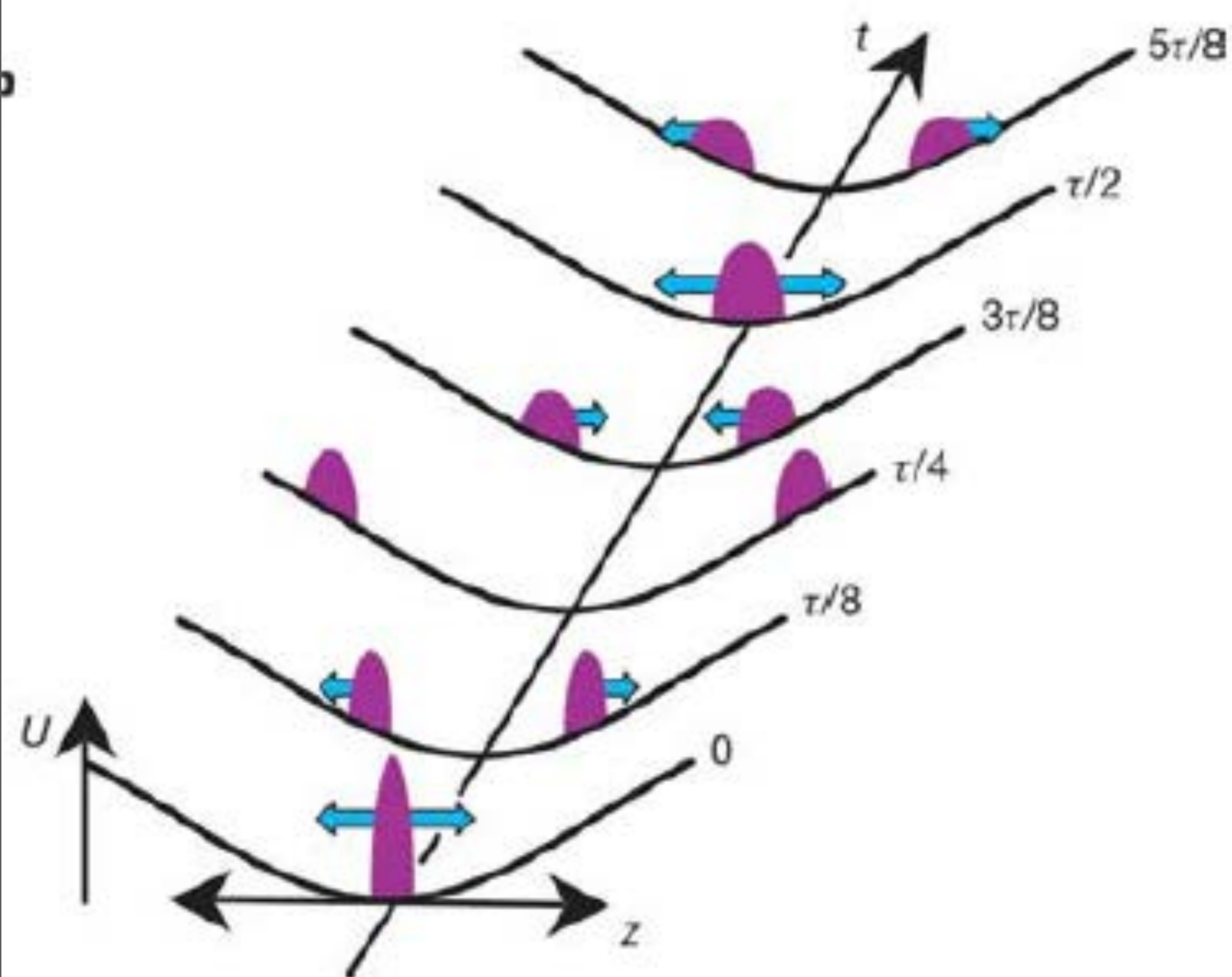
It stayed a purely academic question: for condensed matter systems the coupling to the environment is unavoidable

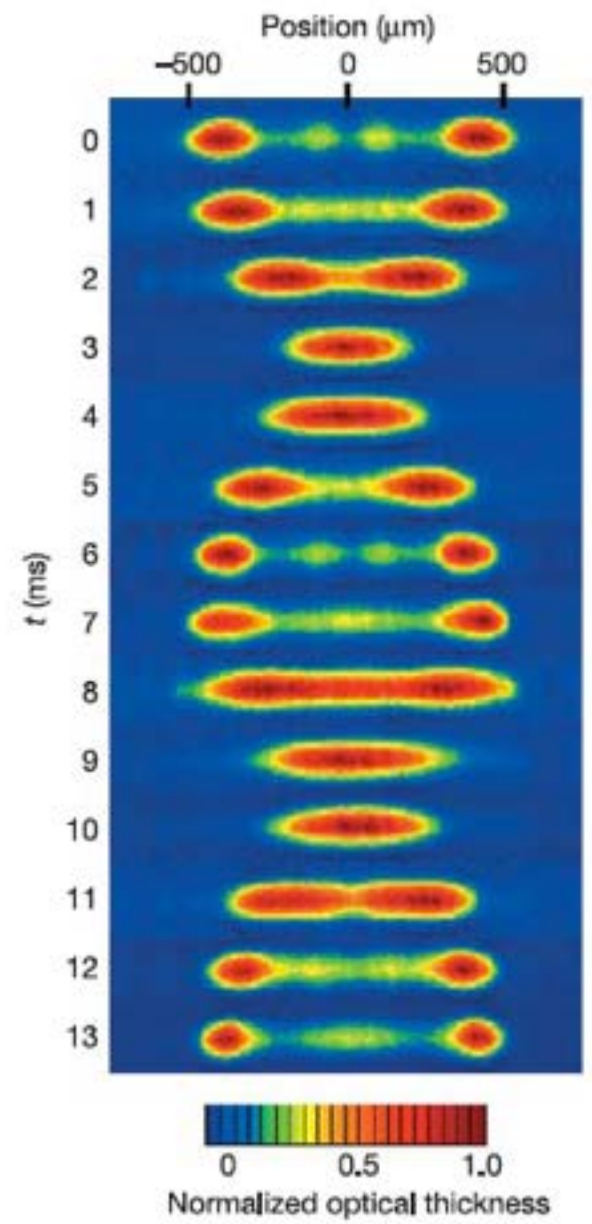
Not anymore in cold atoms!

Quantum Newton cradle

T. Kinoshita, T. Wenger and D.S. Weiss, Nature 440, 900 (2006)

few hundreds ^{87}Rb atoms in a 1D trap

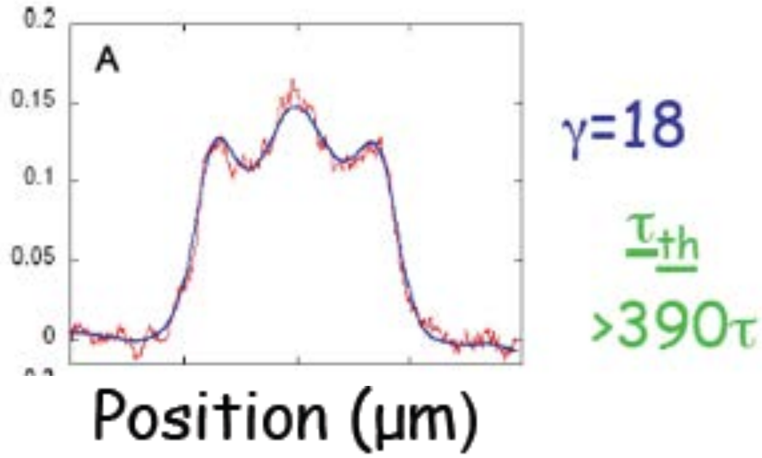


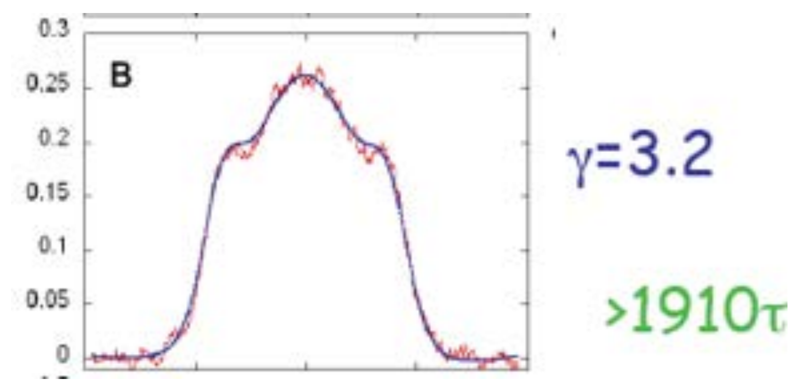


Essentially unitary time evolution

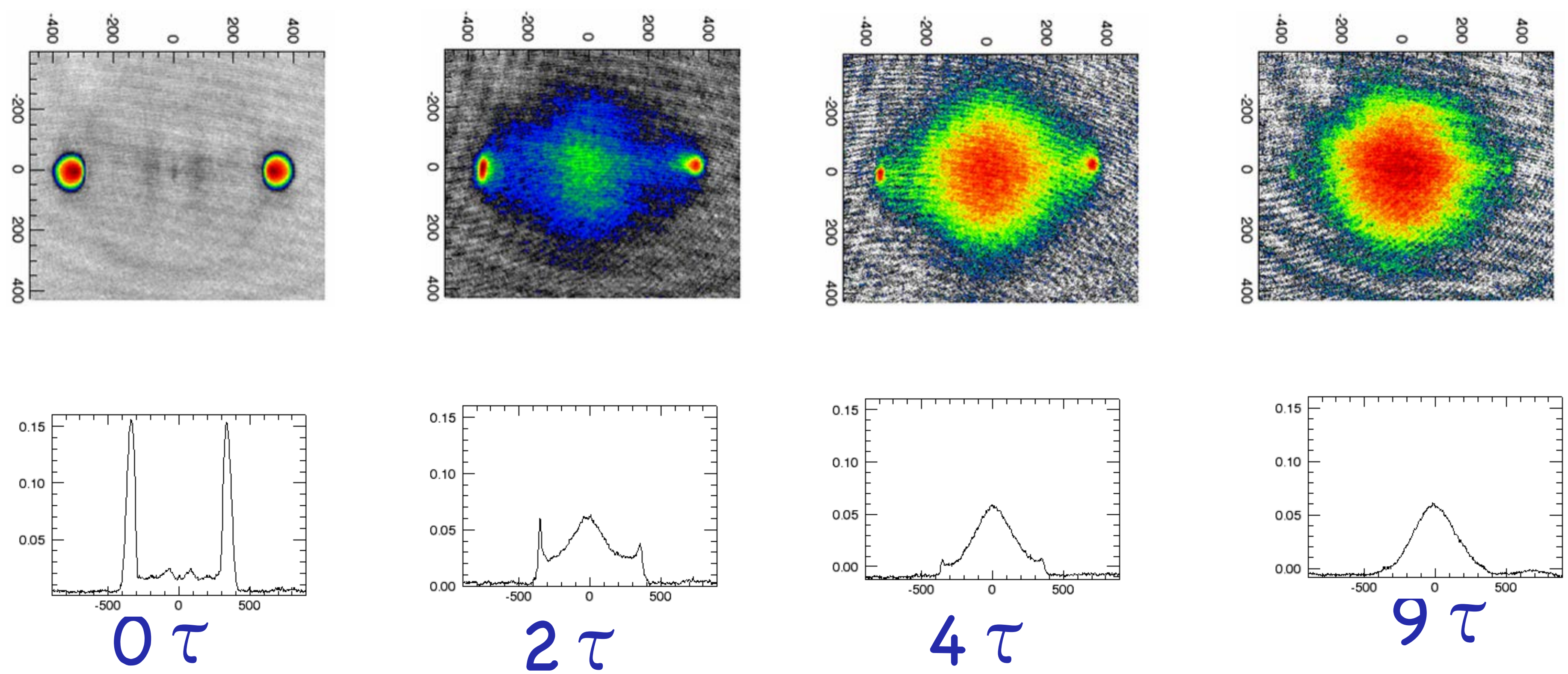
Can a steady state be attained? Surprisingly, YES

- 1D system relaxes slowly in time, to a non-thermal distribution





- 2D and 3D systems relax quickly and thermalize:



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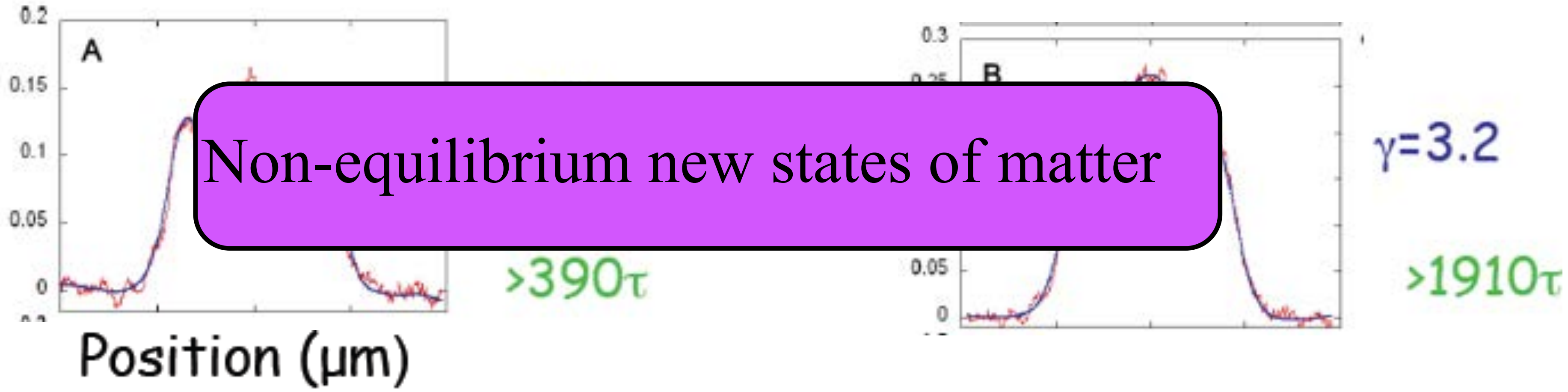

Non-equilibrium new states of matter

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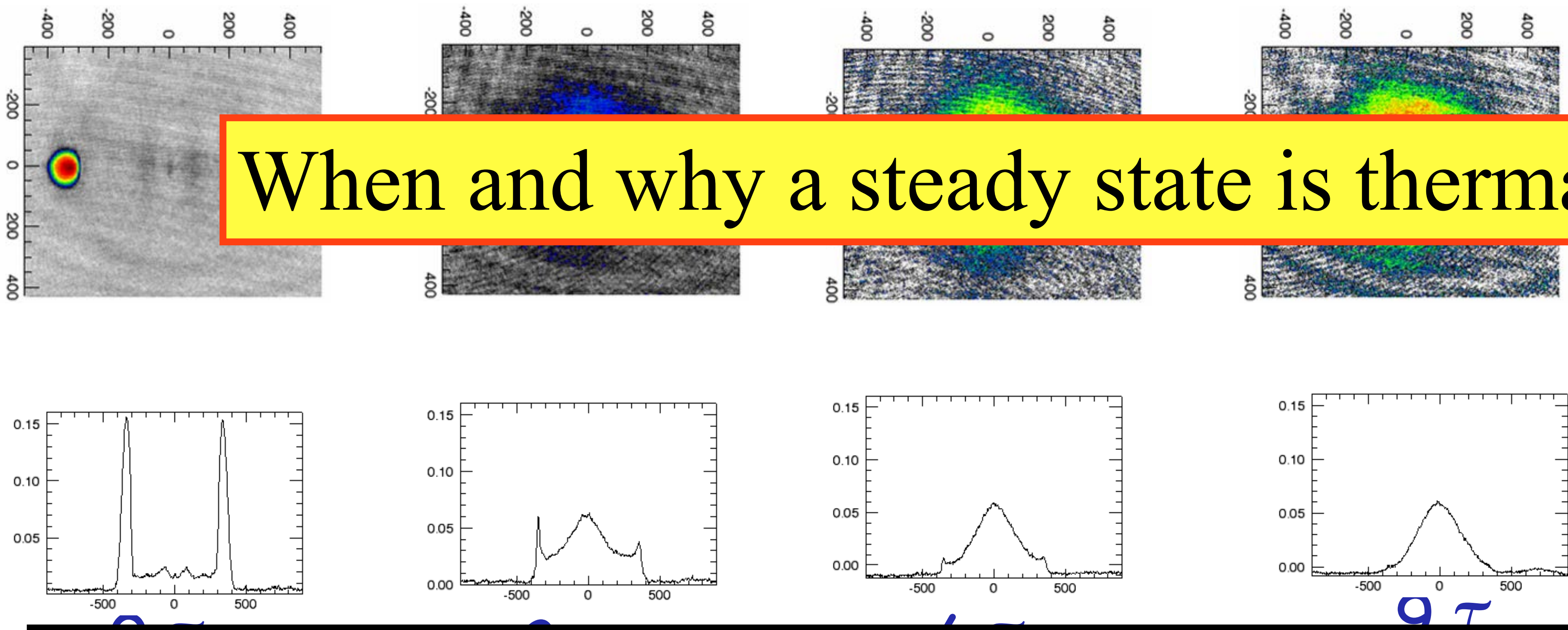
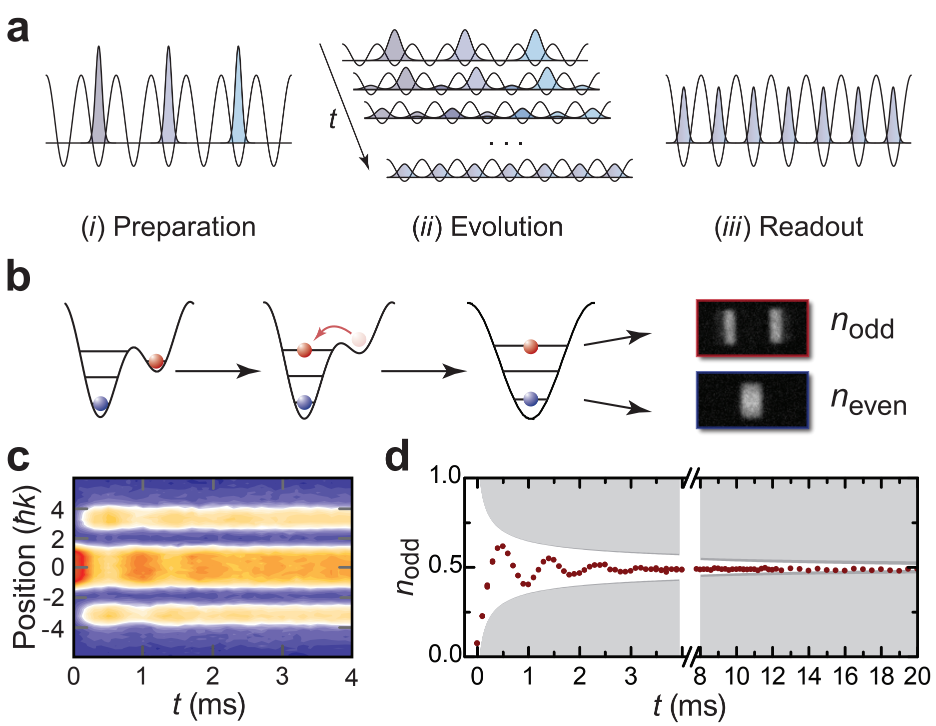


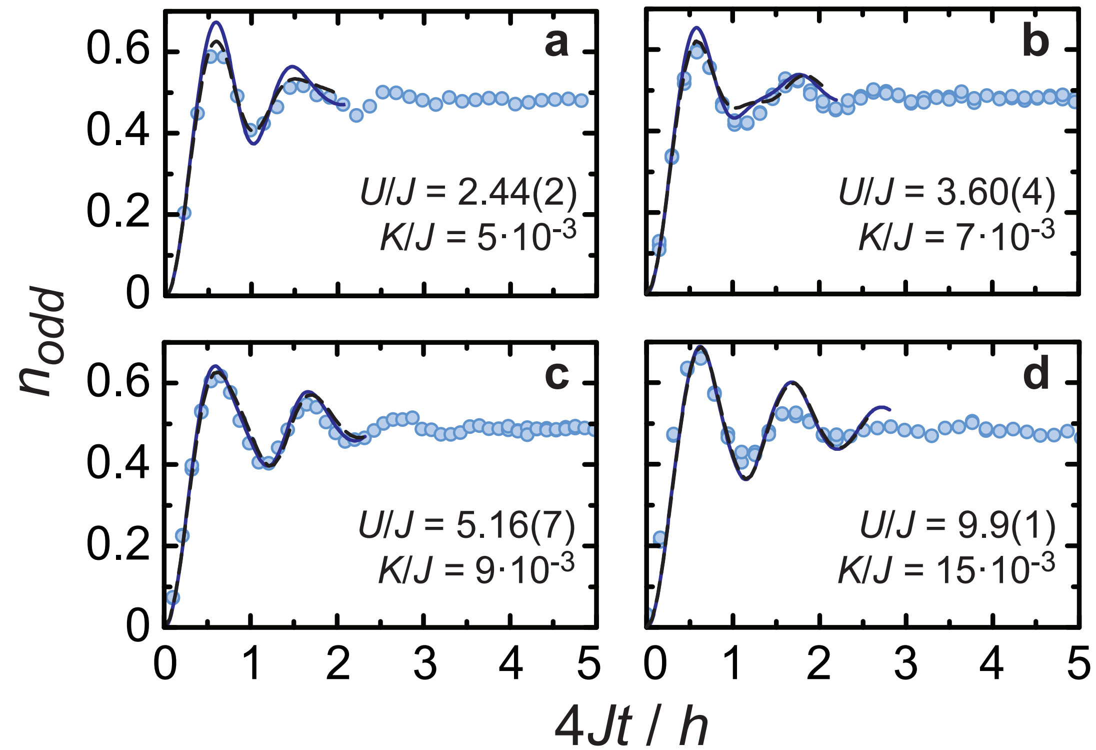
When and why a steady state is thermal??

The 1D case is special because the system is almost integrable

Probing relaxation

S Trotzky et al, Nature Phys. 8, 325 (2012)





- Numerical DMRG and experiment agree perfectly
- The stationary state looks thermal

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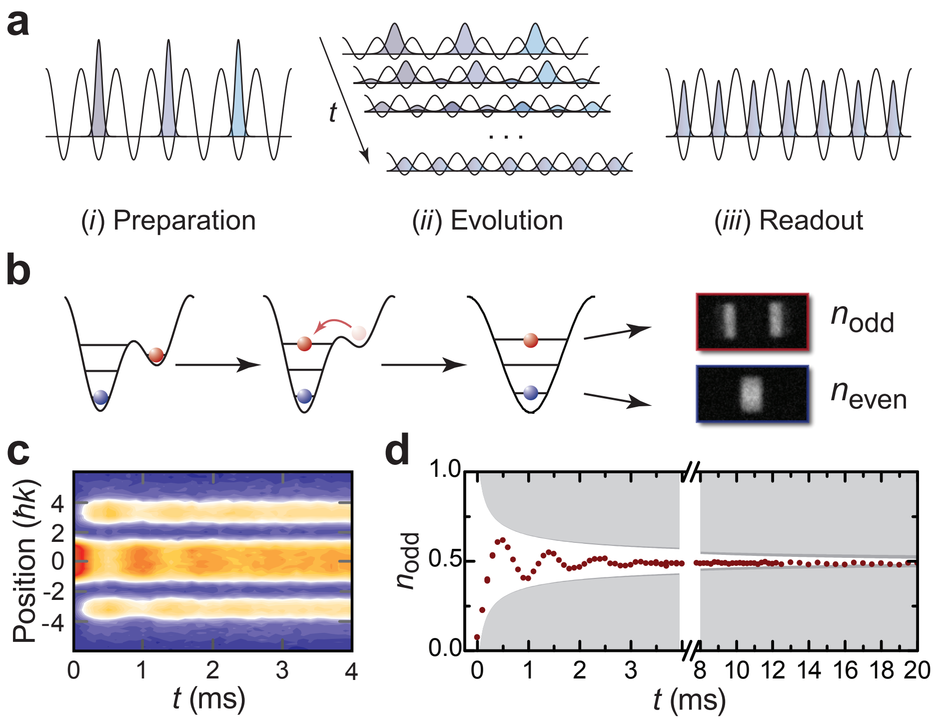

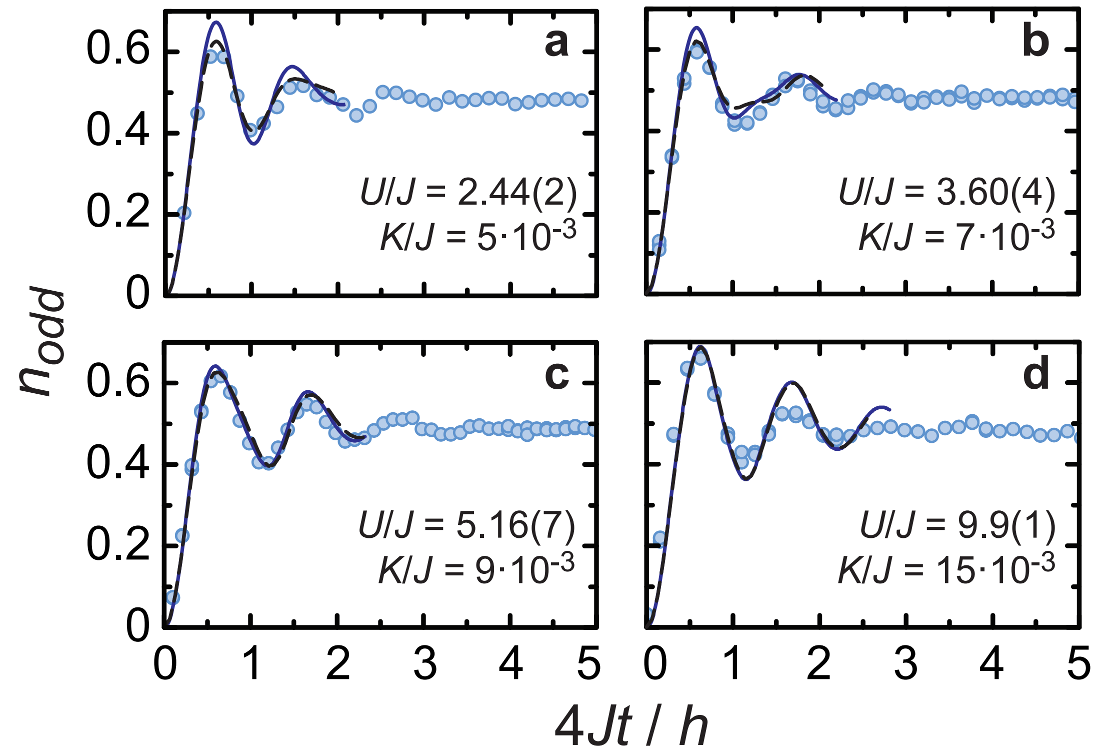

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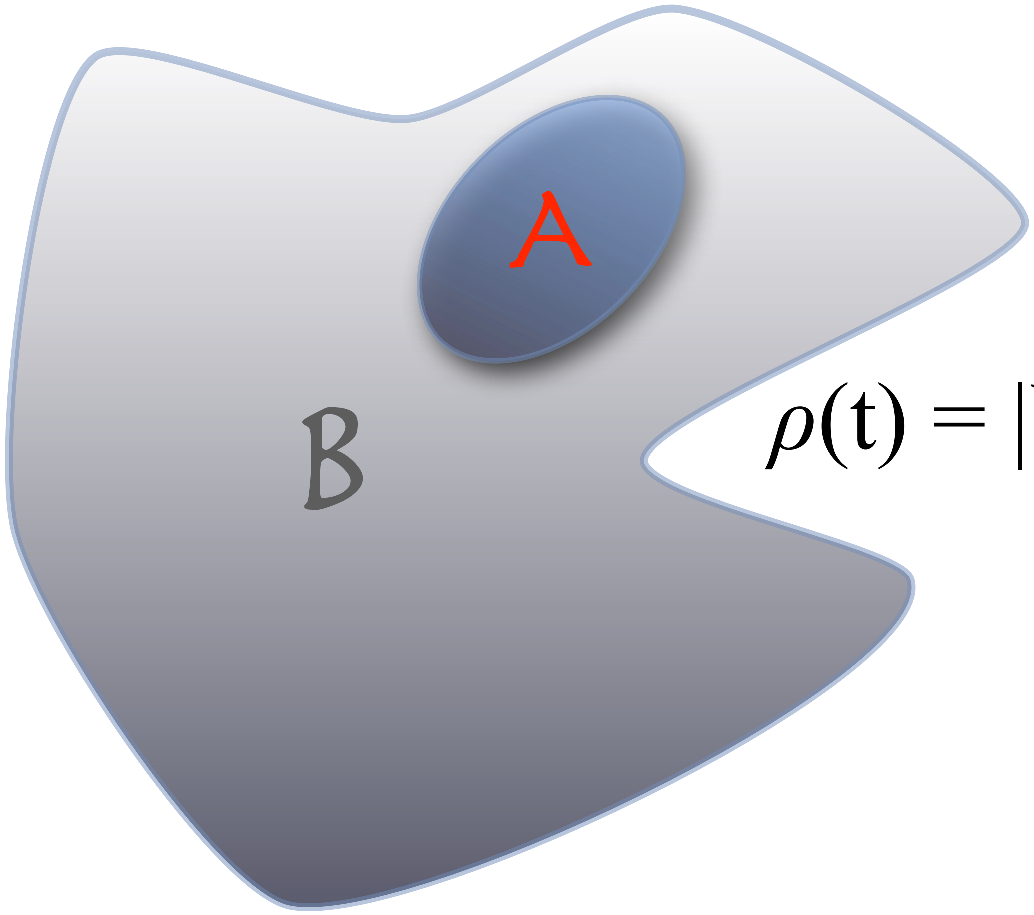
Common Belief: - Generic systems "thermalizes" — Deutsch '91, Srednicki '95

- Integrable systems are different — Rigol et al '07

But the system is always in a pure state!

Reduced density matrix



$|\Psi(t)\rangle$ time dependent pure state

$\rho(t) = |\Psi(t)\rangle\langle\Psi(t)|$ density matrix of A∪B (Infinite)

Reduced density matrix: $\rho_A(t) = \mathrm{Tr}_B\, \rho(t)$

The expectation values of all local observables in A are

$$\langle\Psi(t)|O_A(x)|\Psi(t)\rangle = \mathrm{Tr}[\rho_A(t)\, O_A(x)]$$

Stationary state: If for any finite subsystem A it exists the limit

$$\lim_{t\to\infty} \rho_A(t) = \rho_A(\infty)$$

Thermalization

Consider the Gibbs ensemble for the whole system A∪B

$\rho_T = e^{-H/T_{eff}}/Z$ with $\langle\Psi_0|\, H\,|\Psi_0\rangle = \mathrm{Tr}[\rho_T\, H]$

T_{eff} "is" the energy in the initial state: no free parameter!!

Reduced density matrix for subsystem A: $\rho_{A,T} = \mathrm{Tr}_B\, \rho_T$

The system thermalizes if for any finite subsystem A

$$\rho_{A,T} = \rho_A(\infty)$$

The infinite part B of the system "acts as an heat bath for A"

Generalized Gibbs Ensemble

[Rigol et al 2007]

What about integrable systems?

I_m is a complete set of local (in space) integrals of motion

$[I_m, I_n]=0 \quad [I_m, H]=0 \qquad I_m=\sum_x O_m(x)$

The GGE density matrix is

$\rho_{GGE}= e^{-\sum \lambda_m I_m}/Z$ with λ_m fixed by $\langle\Psi_0| I_m |\Psi_0\rangle = \mathrm{Tr}[\rho_{GGE} I_m]$

Again no free parameter!!

Reduced density matrix for subsystem A: $\rho_{A,GGE}=\mathrm{Tr}_B\, \rho_{GGE}$

The system is described by GGE if for any finite subsystem A

$$\rho_{A,GGE} = \rho_A(\infty)$$

[Barthel-Schollwock '08]
[Cramer, Eisert, et al '08] + ...
[PC, Essler, Fagotti '12]

B is not a standard heat bath for A:
infinite information on the initial state is retained!

Global quenches:

❶ extensive energy

❷ translational invariant

Local quenches

❶ little energy, localized

❷ non-translational invariant

Quantum quenches

Inhomogeneous quenches

❶ extensive energy

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How to attack the problem:

❶ Purely numerically (tDMRG, exact diagonalization)

❷ “approximate theories”, (CFT, Luttinger, RG...)

❸ Exploiting integrability

❹ Solving “free theories”

Quantum quenches in “free” theories

- Mass quenches in (lattice) field theories

PC-Cardy ’07, Barthel-Schollwock ’08, Cramer, Eisert, et al ’08, Sotiriadis et al ’09.....

- Luttinger model quartic term quench

Cazalilla ’06, Cazalilla-Iucci ’09, Mitra-Giamarchi ’10....

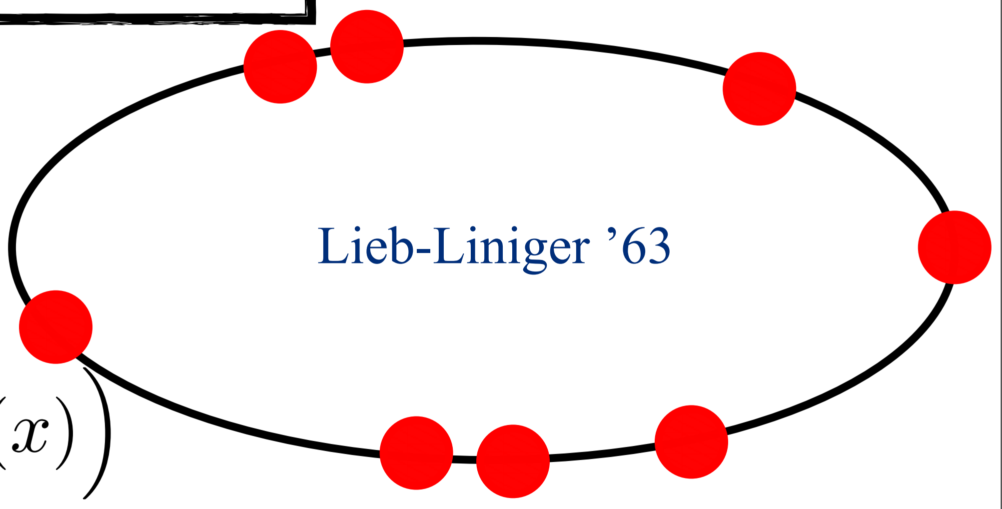
- Transverse field quench in Ising/XY model

Barouch-McCoy ’70, Igloi-Rieger ’00-13, Sengupta et al ’04, Rossini et al. ’10, PC, Essler, Fagotti ’11-13........

- Few more.....

All of them rely on a linear mapping between pre- and post-quench mode operators

Global quenches in Lieb-Liniger



$$
\begin{aligned}
H &= -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + c \sum_{i \neq j} \delta(x_i - x_j) \\
&= \int_0^L dx \left(\partial_x \hat{\phi}^\dagger(x) \partial_x \hat{\phi}(x) + c\, \hat{\phi}^\dagger(x) \hat{\phi}^\dagger(x) \hat{\phi}(x) \hat{\phi}(x) \right)
\end{aligned}
$$

paradigmatic Bethe ansatz solvable model with infinitely many local conserved charges

Most general global quench: $c_0 \rightarrow c$

In the TD limit, beyond present knowledge, both time-evolution and GGE

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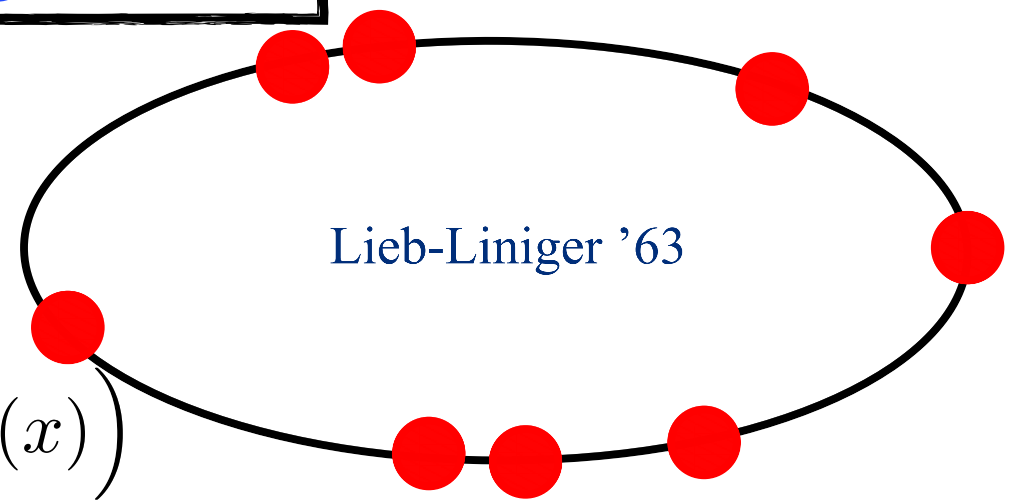

paradigmatic Bethe ansatz solvable model with infinitely many local conserved charges

Most general global quench: $c_0 \rightarrow c$

In the TD limit, beyond present knowledge, both time-evolution and GGE

"Easier" global quench: $c_0 = 0 \rightarrow c$

Simple initial state: $|\psi_0(N)\rangle = \frac{1}{\sqrt{N!}}\hat{\xi}_0^N|0\rangle$ $\qquad \hat{\phi}(x) = \frac{1}{\sqrt{L}}\sum_q \mathrm{e}^{iqx}\hat{\xi}_q$

❶ Very difficult to address the time evolution

❷ GGE construction: the expectation values of local charges diverges

[firstly pointed out by JS Caux now in Kormos et al 1305.7202, problem bypassed by q-boson regularization]

The easiest global quench: $c=0 \longrightarrow c=\infty$ (BEC $\longrightarrow$ TG)

[studied numerically by Gritsev et al. 2010]

$$H = \int dx\, \partial_x \hat{\phi}^\dagger \partial_x(x) \hat{\phi}(x) \quad \xrightarrow{\text{quench}} \quad H = \int dx\, \partial_x \hat{\Phi}^\dagger \partial_x(x) \hat{\Phi}(x)$$

canonical bosons → hard-core bosons

$$[\hat{\Phi}(x), \hat{\Phi}(y)] = [\hat{\Phi}(x), \hat{\Phi}^\dagger(y)] = 0 \qquad x \neq y,$$

$$[\hat{\Phi}(x)]^2 = [\hat{\Phi}^\dagger(x)]^2 = 0\,, \quad \{\hat{\Phi}(x), \hat{\Phi}^\dagger(x)\} = 1$$

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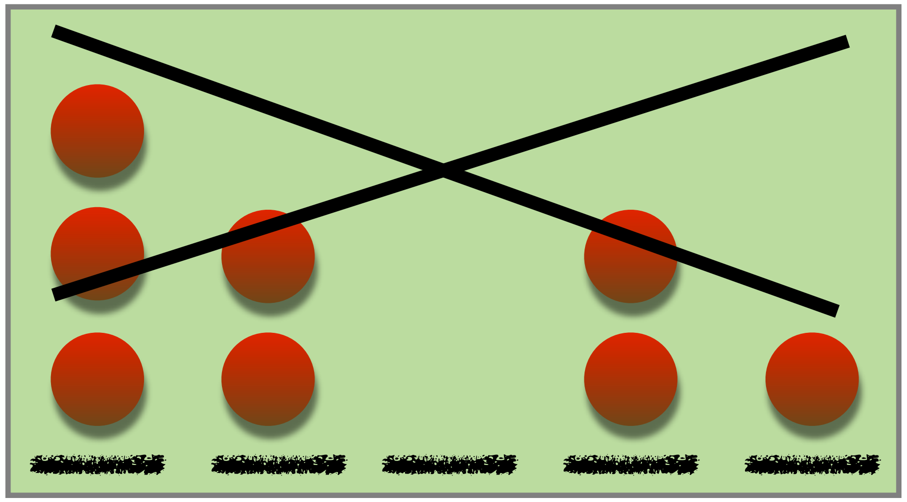

It is a non-linear transformation in the eigenmodes:

$$\hat{\Phi}^{(\dagger)}(x) = P_x \hat{\phi}^{(\dagger)}(x) P_x \qquad P_x = |0\rangle\langle 0|_x + |1\rangle\langle 1|_x$$

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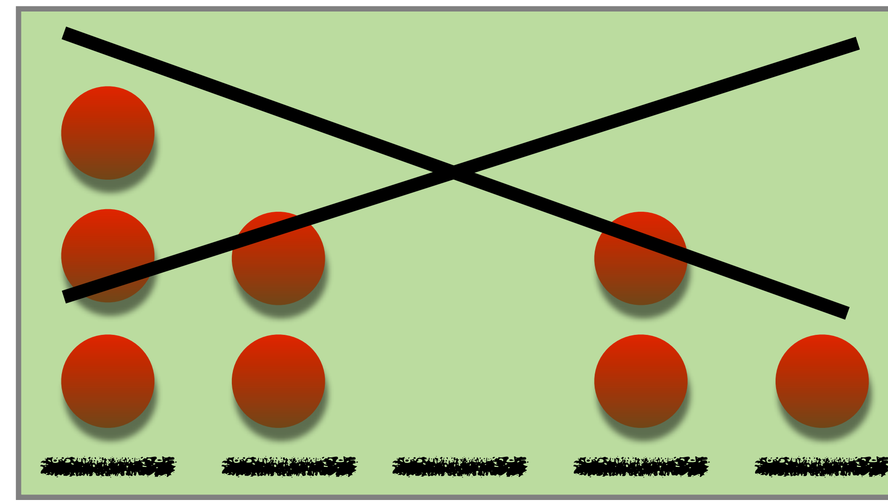

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Diagonalization of the post-quench Hamiltonian:

JW: $\hat{\Psi}(x) = \exp\left\{ i\pi \int_0^x dz \hat{\Phi}^\dagger(z)\hat{\Phi}(z) \right\} \hat{\Phi}(x)$

$$H = \int dx\, \partial_x \hat{\Psi}^\dagger(x)\, \partial_x \hat{\Psi}(x)$$

free fermions

Fourier: $\hat{\eta}_k = \int_0^L dx \frac{e^{-ikx}}{\sqrt{L}} \hat{\Psi}(x)$

$$H = \sum_{k=-\infty}^{\infty} k^2 \hat{\eta}_k^\dagger \hat{\eta}_k$$

❶ Non-local charges: $\hat{n}_k = \hat{\eta}_k^\dagger \hat{\eta}_k$

❷ Local charges: $\hat{I}_j = \int dx\, \hat{\Psi}^\dagger(x)(-i)^j \frac{\partial^j}{\partial x^j} \hat{\Psi}(x) = \sum_k k^j \hat{n}_k$ Linear relation I_j vs n_k

The two GGEs are equivalent: $\sum \gamma_j I_j = \sum \lambda_k n_k$

Two-point fermionic correlation

$\langle \hat{\Psi}^\dagger(x)\hat{\Psi}(y)\rangle$ does not depend on time because Fourier transform of n_k

$$\langle \hat{\Psi}^\dagger(x)\hat{\Psi}(y)\rangle = \sum_{j=0}^{\infty} \frac{(-2)^j}{j!} \int_x^y dz_1 \cdots \int_x^y dz_j \langle \hat{\Phi}^\dagger(x)\hat{\Phi}^\dagger(z_1)\cdots\hat{\Phi}^\dagger(z_j)\hat{\Phi}(z_j)\cdots\hat{\Phi}(z_1)\hat{\Phi}(y)\rangle$$

expansion of JW + normal ordering

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How to get this?

We known for canonical bosons:

$$\langle \hat{\phi}^\dagger(x)\hat{\phi}^\dagger(z_1)\cdots\hat{\phi}^\dagger(z_j)\hat{\phi}(z_j)\cdots\hat{\phi}(z_1)\hat{\phi}(y)\rangle = \frac{1}{L^{j+1}}\langle N|(\hat{\xi}_0^\dagger)^{j+1}(\hat{\xi}_0)^{j+1}|N\rangle = \frac{1}{L^{j+1}}\frac{N!}{(N-j-1)!}$$

A carefully lattice regularization shows that canonical and HC bosons "are the same", because in the TD limit ${}_N\langle \mathrm{BEC}|a_l^\dagger a_l|\mathrm{BEC}\rangle_N \approx \nu e^{-\nu}$ with $\nu=N/M$, M lattice sites and LL is $\nu\rightarrow 0$

$$\langle \hat{\Psi}^\dagger(x)\hat{\Psi}(y)\rangle = \frac{N}{L}\sum_{j=0}^{\infty}\frac{[-2|x-y|/L]^j}{j!}\frac{(N-1)!}{(N-j-1)!} = n\left(1-\frac{2n|x-y|}{N}\right)^{N-1} \xrightarrow{N\to\infty} n\mathrm{e}^{-2n|x-y|}$$

Fourier transform gives $n_k = \dfrac{4n^2}{k^2+4n^2}$ and hence the GGE

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The GGE bosonic correlation is given by Wick theorem $\langle \hat{\phi}^\dagger(x)\hat{\phi}(y)\rangle_{\mathrm{GGE}} = ne^{-2n|x-y|}$

Important: $\hat{I}_j = \int \frac{dk}{2\pi}k^j n_k = \int \frac{dk}{2\pi}k^j \frac{4n^2}{k^2+4n^2}$ diverges for $j\neq 0$, but no problem for n_k GGE

Dynamical density-density correlation function

By definition we have:

$$\langle \hat\rho(x_1,t_1)\hat\rho(x_2,t_2)\rangle = \frac{1}{L^2}\sum_{k_1,k_2,k_3,k_4} \mathrm{e}^{-i(k_1-k_2)x_1-i(k_3-k_4)x_2}\mathrm{e}^{i(k_1^2-k_2^2)t_1}\mathrm{e}^{i(k_3^2-k_4^2)t_2}\langle\psi_0|\hat\eta^\dagger_{k_1}\hat\eta_{k_2}\hat\eta^\dagger_{k_3}\hat\eta_{k_4}|\psi_0\rangle$$

4-pt function non trivial because Wick theorem holds in usual form only for t=∞ (and t=0).
To get it let's go back to real space:

$$\langle\psi_0|\hat\eta^\dagger_{k_1}\hat\eta_{k_2}\hat\eta^\dagger_{k_3}\hat\eta_{k_4}|\psi_0\rangle = \frac{1}{L^2}\int_0^L dz_1dz_2dz_3dz_4\mathrm{e}^{i(k_1z_1-k_2z_2+k_3z_3-k_4z_4)}\langle\psi_0|\hat\Psi^\dagger(z_1)\hat\Psi(z_2)\hat\Psi^\dagger(z_3)\hat\Psi(z_4)|\psi_0\rangle$$

In a nutshell: expand the string, treat hc boson as canonical bosons, sum up the 24 terms...

$$\langle\hat\Psi^\dagger(z_1)\hat\Psi(z_2)\hat\Psi^\dagger(z_3)\hat\Psi(z_4)\rangle = \delta(z_2-z_3)n\mathrm{e}^{-2n|z_4-z_1|} + \sum_{\mathcal{P}}\theta(z_{\mathcal{P}})\sigma_{\mathcal{P}}\, n^2\mathrm{e}^{-2n(z_{\mathcal{P}_4}-z_{\mathcal{P}_3}+z_{\mathcal{P}_2}-z_{\mathcal{P}_1})}$$

in the integral this "anomalous" term is fundamental!

Dynamical density-density correlation function

Plugging in the integral

the rest is Wick...

$$\langle\psi_0|\hat{\eta}^\dagger_{k_1}\hat{\eta}_{k_2}\hat{\eta}^\dagger_{k_3}\hat{\eta}_{k_4}|\psi_0\rangle = n(k_1)\delta_{k_2,k_3}\delta_{k_1,k_4} + (\delta_{k_1,k_2}\delta_{k_3,k_4} - \delta_{k_2,k_3}\delta_{k_1,k_4})n(k_1)n(k_3) + \delta_{k_1,-k_3}\delta_{k_2,-k_4}\frac{k_1k_2}{4n^2}n(k_1)n(k_2)$$

Summing over momenta

$$\begin{aligned}\langle\hat{\rho}(x,t)\hat{\rho}(x+\Delta x,t+\Delta t)\rangle &= \frac{1+i\,\mathrm{sgn}(\Delta t)}{2\sqrt{2\pi|\Delta t|}}\mathrm{e}^{-i\frac{\Delta x^2}{4\Delta t}}\int\frac{dk}{2\pi}e^{ik\Delta x-ik^2\Delta t}n(k)+\\ & n^2-\left|\int\frac{dk}{2\pi}e^{ik\Delta x-ik^2\Delta t}n(k)\right|^2+\left|\frac{1}{2n}\int\frac{dk}{2\pi}\mathrm{e}^{ik\Delta x+ik^2(2t+\Delta t)}kn(k)\right|^2\end{aligned}$$

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$$n^2 - \left|\int\frac{dk}{2\pi}e^{ik\Delta x-ik^2\Delta t}n(k)\right|^2 + \left|\frac{1}{2n}\int\frac{dk}{2\pi}\mathrm{e}^{ik\Delta x+ik^2(2t+\Delta t)}kn(k)\right|^2$$

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Summing over momenta

$$\langle\hat{\rho}(x,t)\hat{\rho}(x+\Delta x,t+\Delta t)\rangle = \frac{1+i\,\mathrm{sgn}(\Delta t)}{2\sqrt{2\pi|\Delta t|}}\mathrm{e}^{-i\frac{\Delta x^2}{4\Delta t}}\int\frac{dk}{2\pi}e^{ik\Delta x-ik^2\Delta t}n(k) + n^2 - \left|\int\frac{dk}{2\pi}e^{ik\Delta x-ik^2\Delta t}n(k)\right|^2 + \left|\frac{1}{2n}\int\frac{dk}{2\pi}\mathrm{e}^{ik\Delta x+ik^2(2t+\Delta t)}kn(k)\right|^2$$

Dynamical density-density correlation function

Plugging in the integral

the rest is Wick...

$$\langle\psi_0|\hat\eta^\dagger_{k_1}\hat\eta_{k_2}\hat\eta^\dagger_{k_3}\hat\eta_{k_4}|\psi_0\rangle = n(k_1)\delta_{k_2,k_3}\delta_{k_1,k_4} + (\delta_{k_1,k_2}\delta_{k_3,k_4} - \delta_{k_2,k_3}\delta_{k_1,k_4})n(k_1)n(k_3) + \delta_{k_1,-k_3}\delta_{k_2,-k_4}\frac{k_1k_2}{4n^2}n(k_1)n(k_2)$$

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$$\langle \psi_0 | \hat{\eta}^\dagger_{k_1} \hat{\eta}_{k_2} \hat{\eta}^\dagger_{k_3} \hat{\eta}_{k_4} | \psi_0 \rangle = n(k_1) \delta_{k_2,k_3} \delta_{k_1,k_4} + (\delta_{k_1,k_2} \delta_{k_3,k_4} - \delta_{k_2,k_3} \delta_{k_1,k_4}) n(k_1) n(k_3) + \delta_{k_1,-k_3} \delta_{k_2,-k_4} \frac{k_1 k_2}{4n^2} n(k_1) n(k_2)$$

Summing over momenta

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Features:

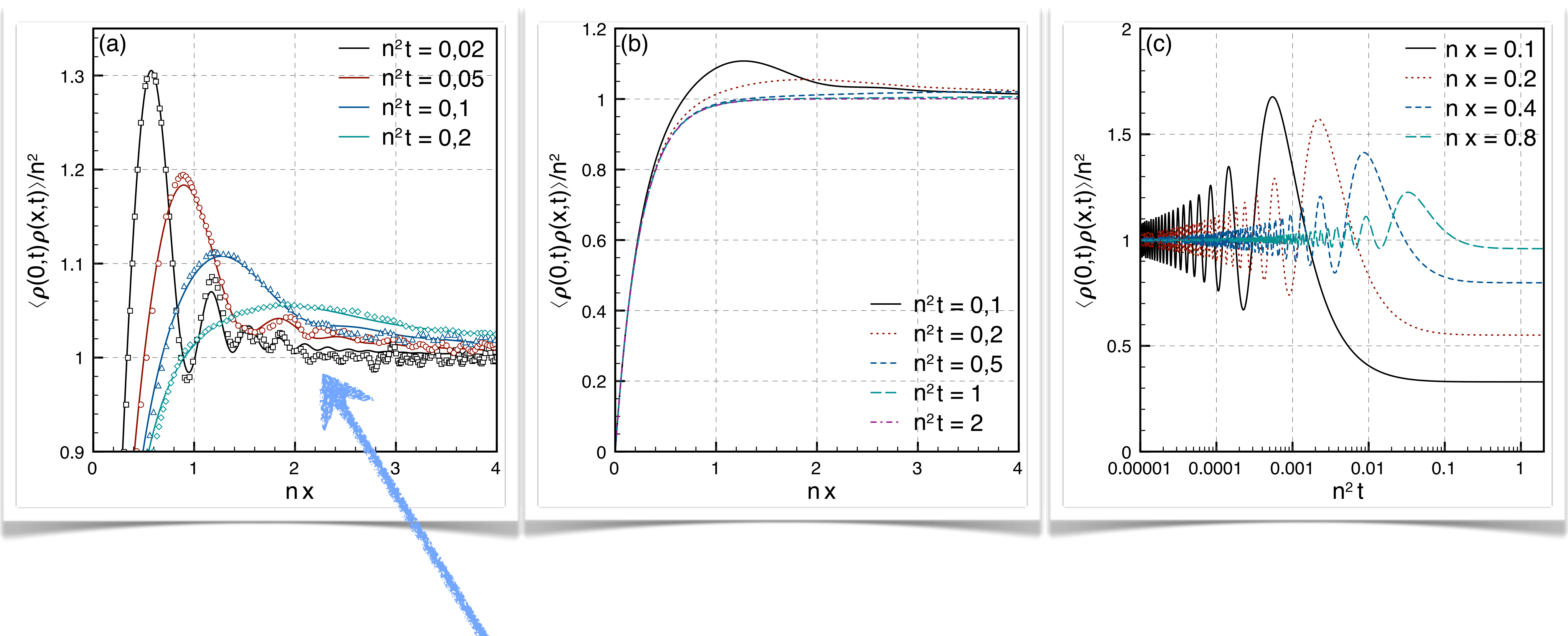
❶ Only the last term depend on t

❷ Wick, i.e. GGE, gives the rest, hence for $t \to \infty$ GGE is valid

❸ auto-correlation ($\Delta x=0$) is time-independent [numerically noticed in Gritsev et al]

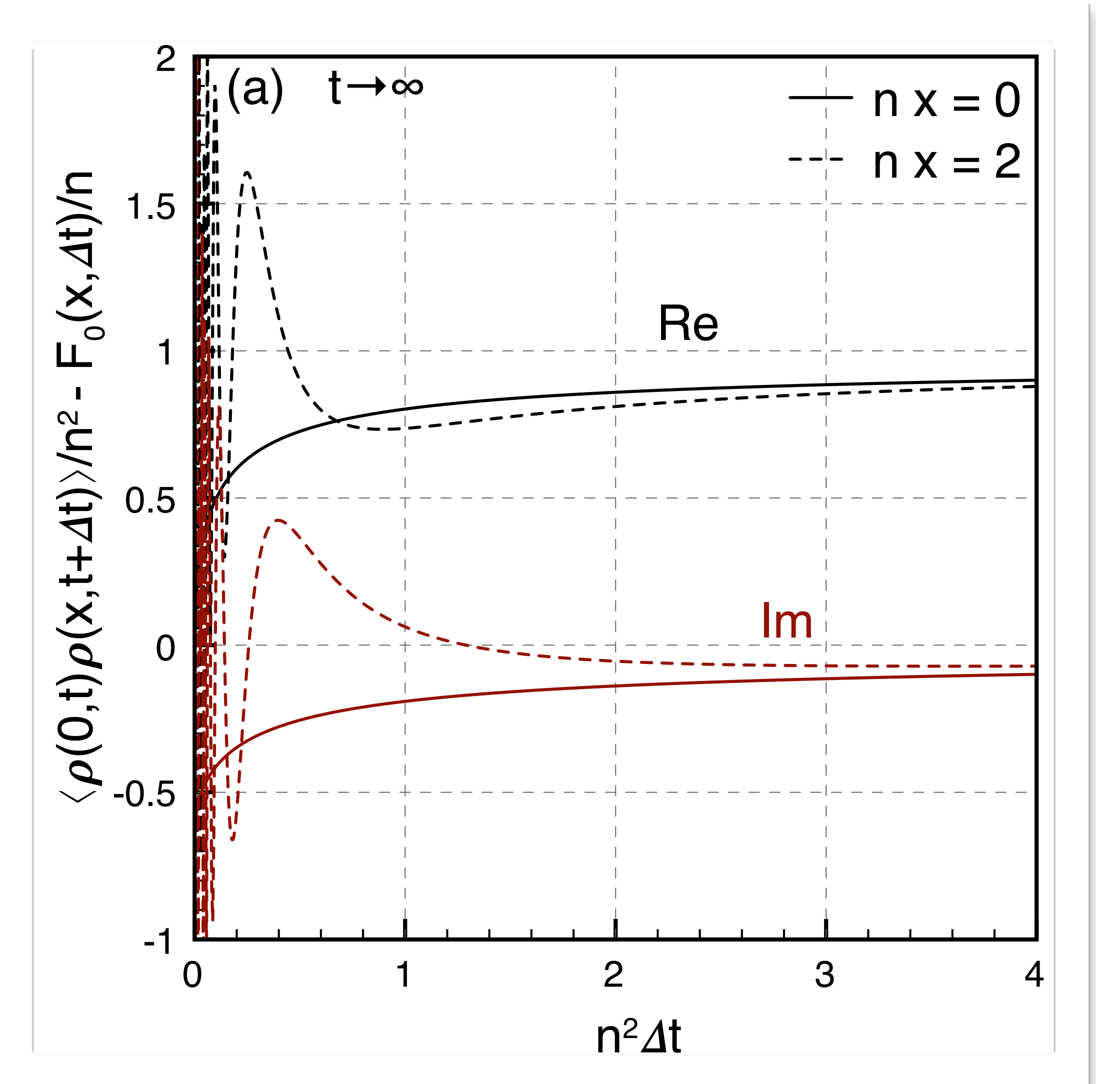
Equal time density correlation

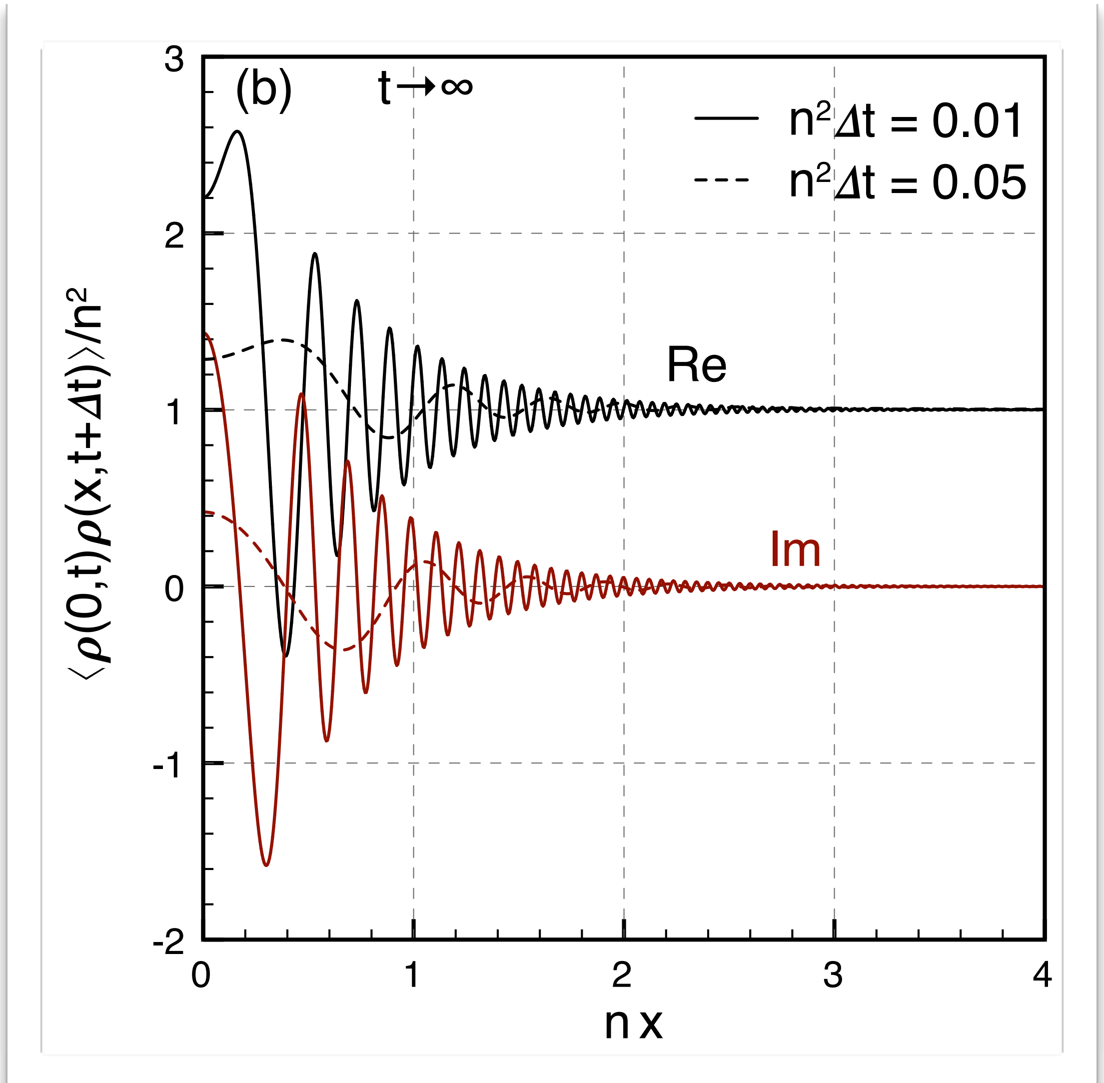
$$\langle\hat\rho(x_1,t)\hat\rho(x_2,t)\rangle = n^2 + n\mathrm{e}^{-2n|x_1-x_2|}\delta(x_2-x_1) - n^2\mathrm{e}^{-4n|x_1-x_2|} + \left|\frac{1}{2n}\int\frac{dk}{2\pi}\mathrm{e}^{ik(x_1-x_2)+ik^2 2t}kn(k)\right|^2$$



Truncated form factors data from Gritsev et al

Dynamical density-density correlation function





Dynamical structure factor in GGE:

$$S(q,\omega) = \frac{8n^2(q^2+\omega)^2|q|}{[(4nq)^2+(q^2-\omega)^2][(4nq)^2+(q^2+\omega)^2]}$$

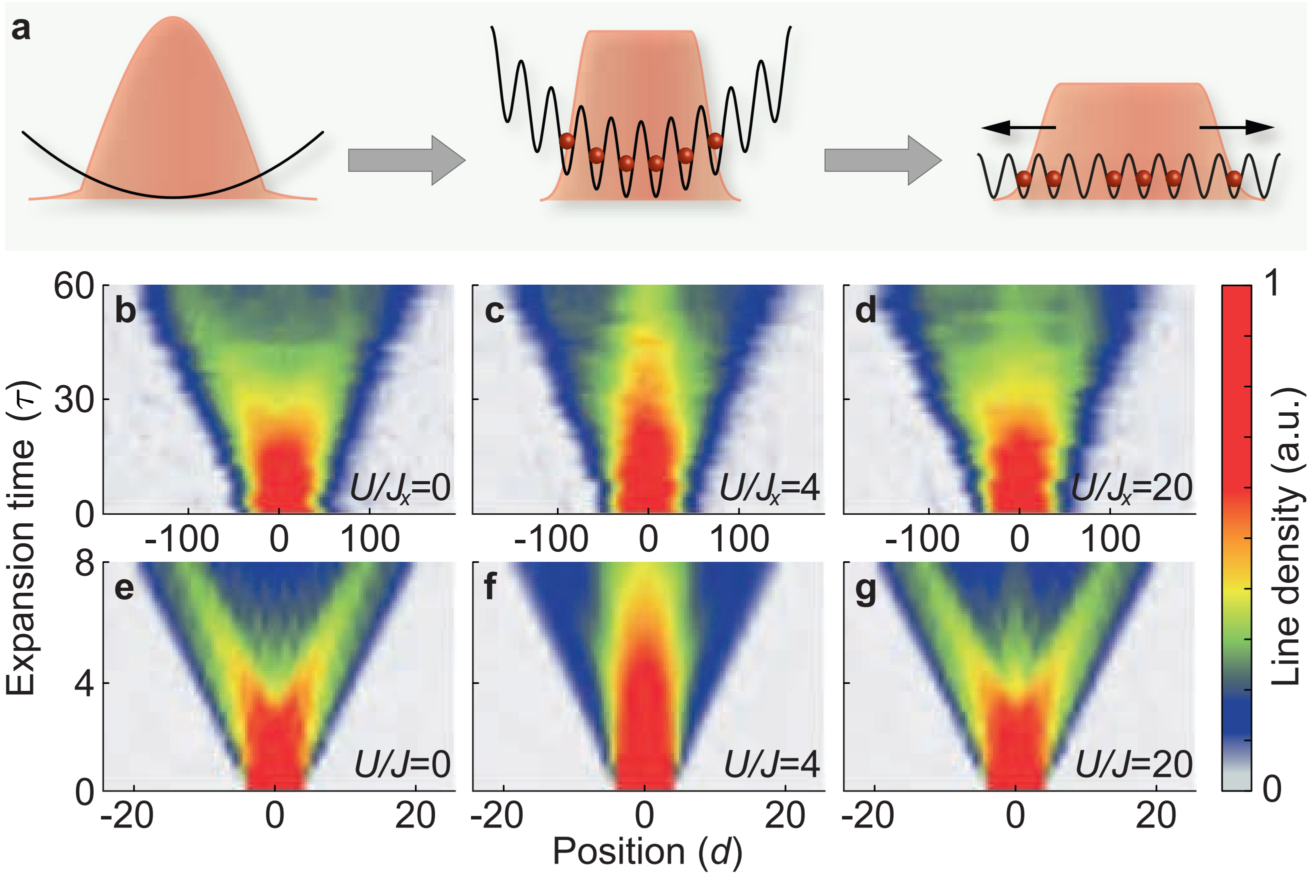
f sum-rule

$$\int d\omega S(q,\omega)\omega = 2\pi n q^2$$

A non homogeneous initial state:
Expansion of an interacting gas

Expansion of initially localized ultracold bosons in 1D and 2D optical lattices.

J.P.Ronzheimer *et al*, PRL 110, 205301 (2013)



1) Integrable system: Ballistic Expansion
2) Not-integrable: Diffusive Expansion

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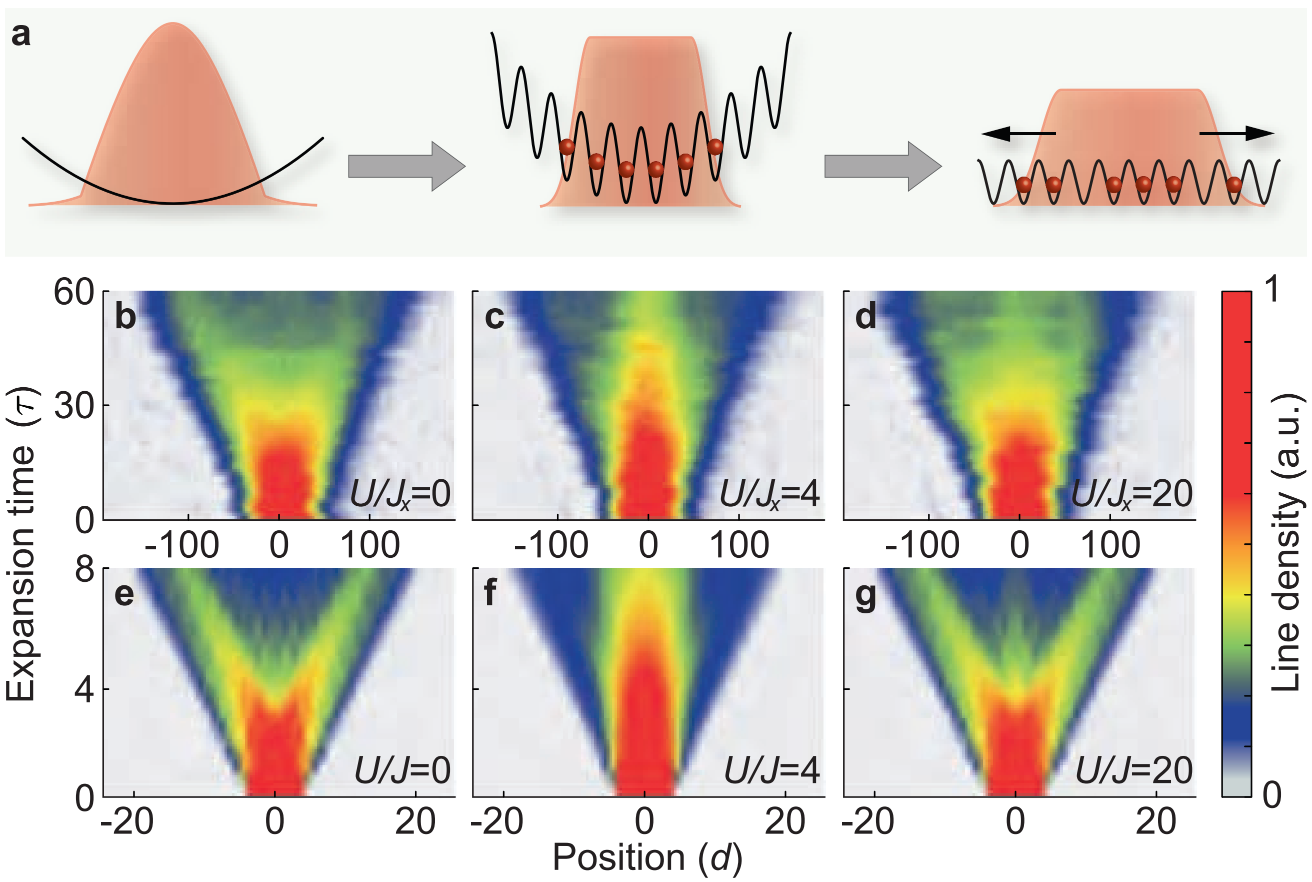
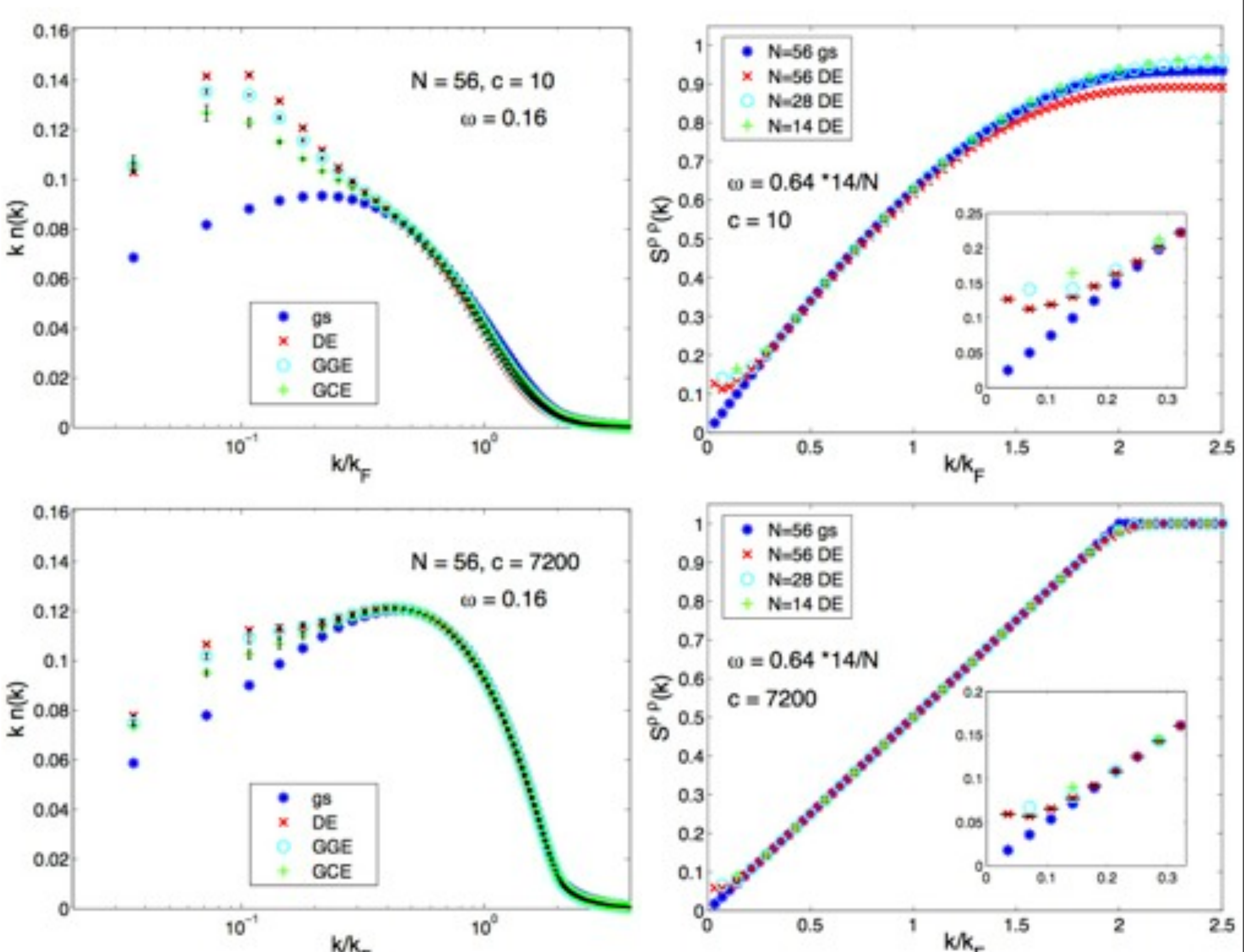
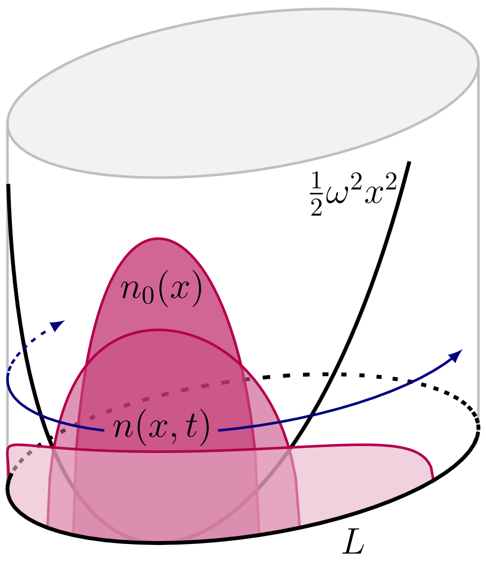


1) Integrable system: Ballistic Expansion
2) Not-integrable: Diffusive Expansion

JS **Caux** and R **Konik** exploited integrability to numerically study the non-equilibrium dynamics of the **Lieb-Liniger** model after the release of a parabolic trap into a circle [PRL 109, 175301 (2012)]



Set up



The initial state is the ground state of TG gas in harmonic trap

$$H = \int dx\, \hat{\Psi}^\dagger(x) \left[-\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}\omega^2 x^2 \right] \hat{\Psi}(x)$$

in JW fermions

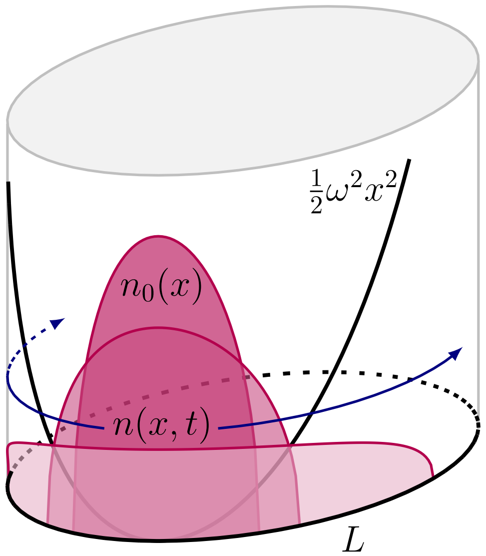
$$\hat{\Psi}(x) = \exp\left\{ i\pi \int_0^x dz\, \hat{\Psi}^\dagger(z)\hat{\Psi}(z) \right\} \hat{\Phi}(x)$$

In terms of the one-particle eigenfunctions $\chi_j(x)$ of the 1D harmonic oscillator

$$H = \sum_{j=0}^{\infty} \epsilon_j \hat{\xi}_j^\dagger \hat{\xi}_j, \quad \epsilon_j = \omega(j + 1/2) \qquad \hat{\Psi}(x) = \sum_{j=0}^{\infty} \chi_j(x)\hat{\xi}_j, \quad \hat{\xi}_j = \int_{-\infty}^{\infty} dx\, \chi_j^*(x)\hat{\Psi}(x)$$

Many body initial state: $|\Psi_0\rangle = \prod_{j=0}^{N-1} \hat{\xi}_j^\dagger |\emptyset\rangle$ Slater determinant in fermions

Set up



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Many body initial state: $|\Psi_0\rangle = \prod_{j=0}^{N-1} \hat{\xi}_j^\dagger |\emptyset\rangle$ Slater determinant in fermions

QUENCH PROTOCOL: At time *t=0* we release the harmonic trap.

The evolution is governed by the free-fermion Hamiltonian with PBC:

$$H_0 = \int dx\, \hat{\Psi}^\dagger(x) \left[-\frac{1}{2}\frac{\partial^2}{\partial x^2} \right] \hat{\Psi}(x) \xrightarrow{\text{diagonalization}} H_0 = \sum_{k=-\infty}^{\infty} \frac{k^2}{2} \hat{\eta}_k^\dagger \hat{\eta}_k, \; \hat{\eta}_k = \int_{-L/2}^{L/2} dx\, \varphi_k^*(x)\hat{\Psi}(x), \; \varphi_k(x) = \frac{e^{-ikx}}{\sqrt{L}}$$

TD and large time limits

The TD limit for a proper quench is defined as

$N, L \to \infty$ with $N/L = n$ but at the same time $\omega \to 0$ with ωN constant

Caux-Konik '12

TD and large time limits

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What about Periodic Boundary Conditions on the initial state??

The TD initial density profile $n_0(x) = \frac{\sqrt{2N\omega - \omega^2 x^2}}{\pi}\theta(\ell - |x|)$ $\ell = \sqrt{2N/\omega}$ → $\propto N$ Thomas-Fermi

vanishes for $x > \ell$

We require the additional (physical) condition

$L > 2\ell$ → $\sqrt{\omega N} > 2\sqrt{2}n$ → $n_0 > n$ → initial average density

TD and large time limits

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vanishes for $x>\ell$

We require the additional (physical) condition

$L>2\ell \to \sqrt{\omega N} > 2\sqrt{2}n \to n_0>n$ → initial average density

In which sense there is a long time limit??

In global quenches we consider always $\lim_{t\to\infty}\lim_{L\to\infty} O(t)$ to have a limit and avoid revivals

In finite systems this is t,L large with $vt<L$

but, in this case, we'd get infinite line expansion, i.e. zero density, i.e. no particles and no GGE

The revival time is $t\propto L^2$ [also Kaminishi, Sato, Deguchi 2013], thus we require

$vt \gg L$ with $t<L^2$

Interpretation: Stationarity comes from the interference of the particles going around L many times

Time evolution

The initial state is a Slater determinant

Free fermions Hamiltonian governing evolution

At any time the many-body state is a Slater Det and Wick theorem holds

TWO-POINT FERMIONIC CORRELATORS

$$C(x,y;t) \equiv \langle \hat{\Psi}^\dagger(x,t)\hat{\Psi}(y,t)\rangle = \sum_{j=0}^{N-1} \phi_j^*(x,t)\phi_j(y,t)$$

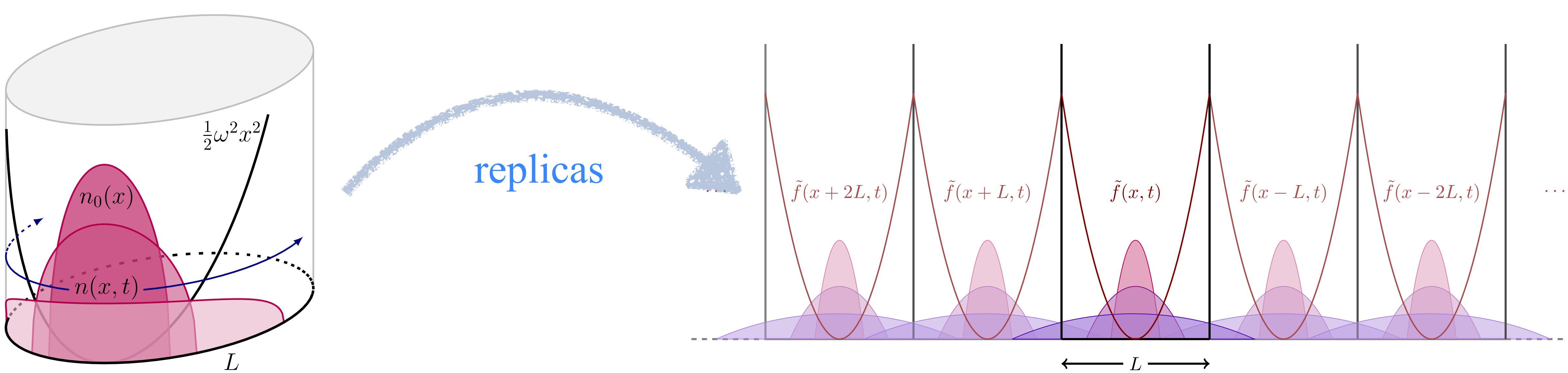
$\phi_j(x,t)$ is the solution of the one-particle problem

Write $\phi_j(x,t)$ with PBC in terms of the solution in infinite space $\phi_j^\infty(x,t)$

$$\phi_j(x,t) = \sum_{p=-\infty}^{\infty} \phi_j^\infty(x+pL,t) \qquad \phi_j^\infty(x,t) = \frac{1}{\sqrt{1+i\omega t}}\left(\frac{1-i\omega t}{1+i\omega t}\right)^{j/2} \mathrm{e}^{-i\frac{t\omega^2x^2}{2(1+\omega^2t^2)}}\chi_j\left(\frac{x}{\sqrt{1+\omega^2t^2}}\right)$$

Minguzzi-Gangardt 2005

Physical Interpretation:



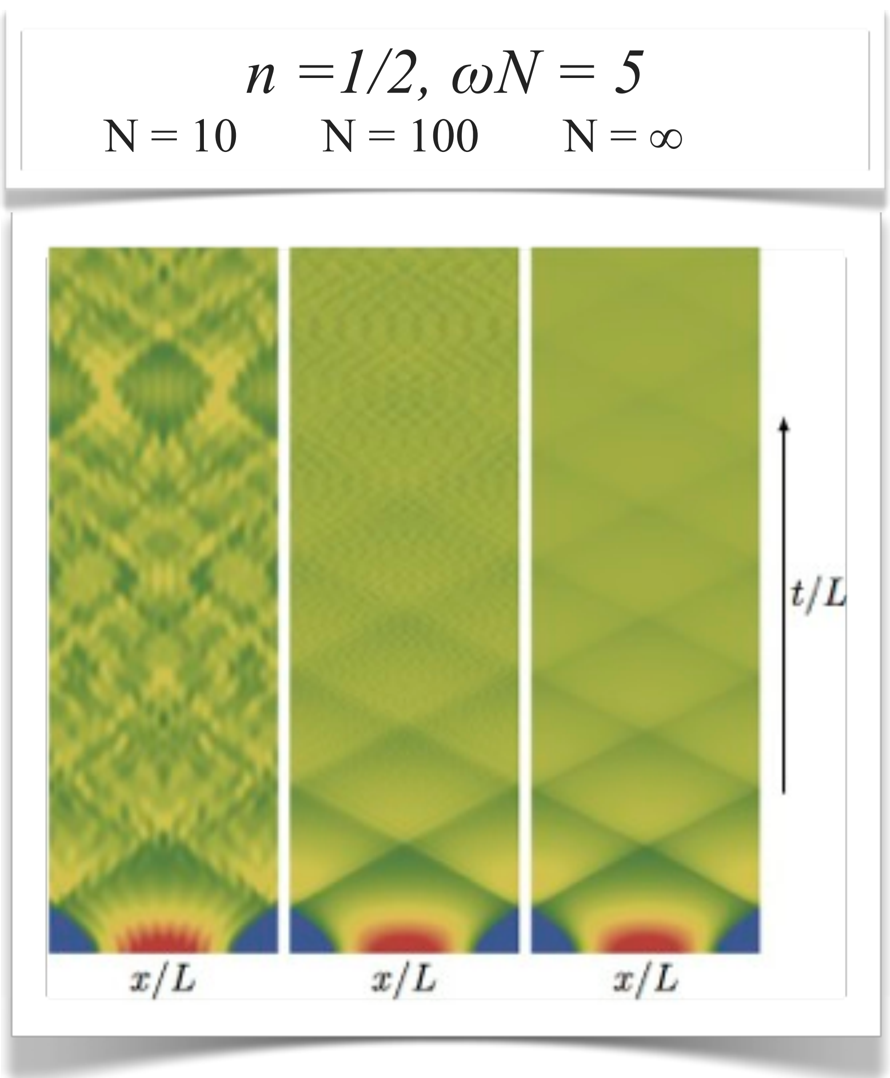
Density profile

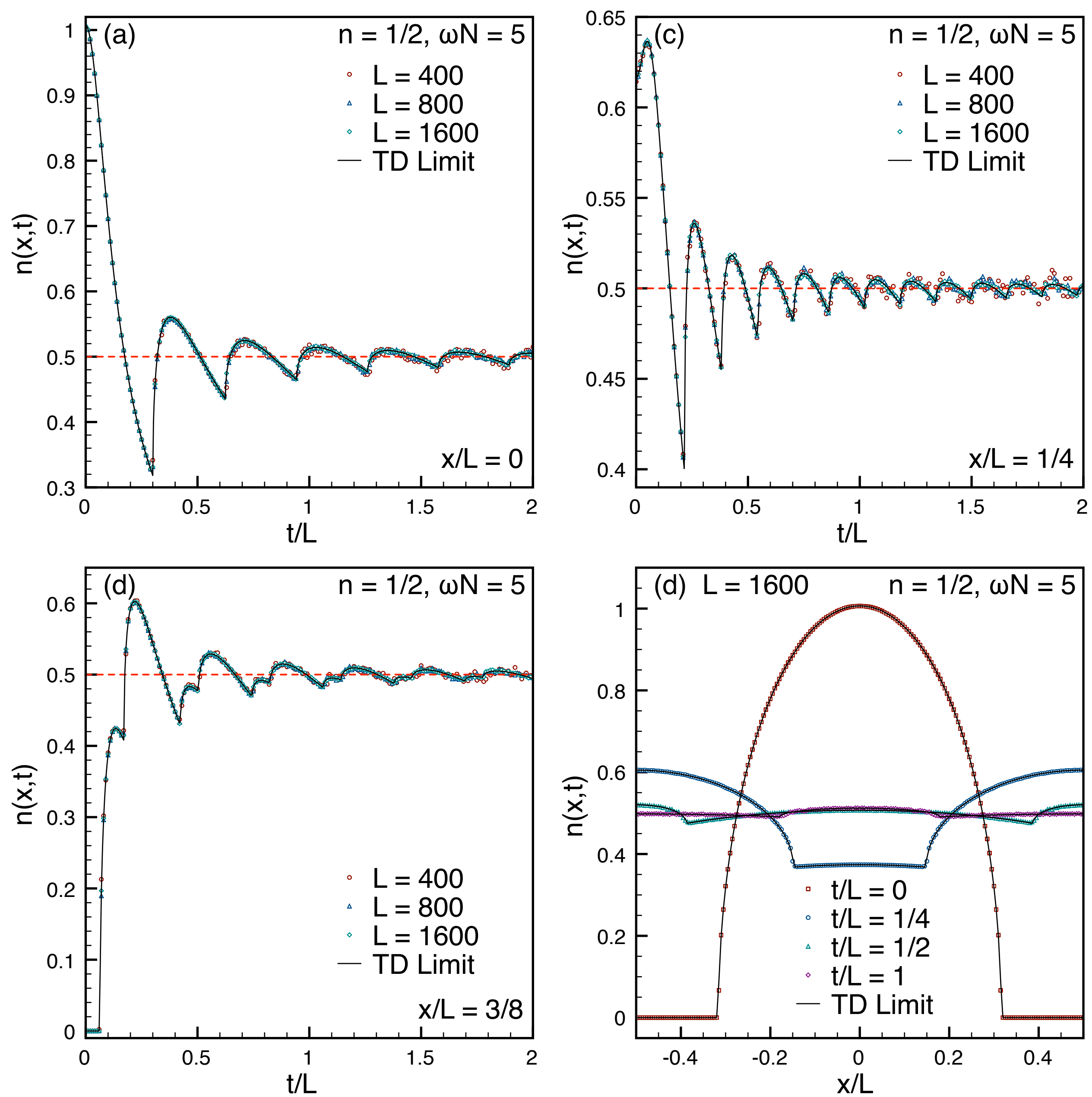
Density is simple! In the TD limit:

$$n(x,t) = \frac{1}{\sqrt{1+\omega^2 t^2}} \sum_{p=-\infty}^{\infty} n_0\left(\frac{x+pL}{\sqrt{1+\omega^2 t^2}}\right)$$

$n_0(x)$ is density at initial time

$$n_0(x) = \sqrt{2N\omega - \omega^2 x^2}/\pi$$



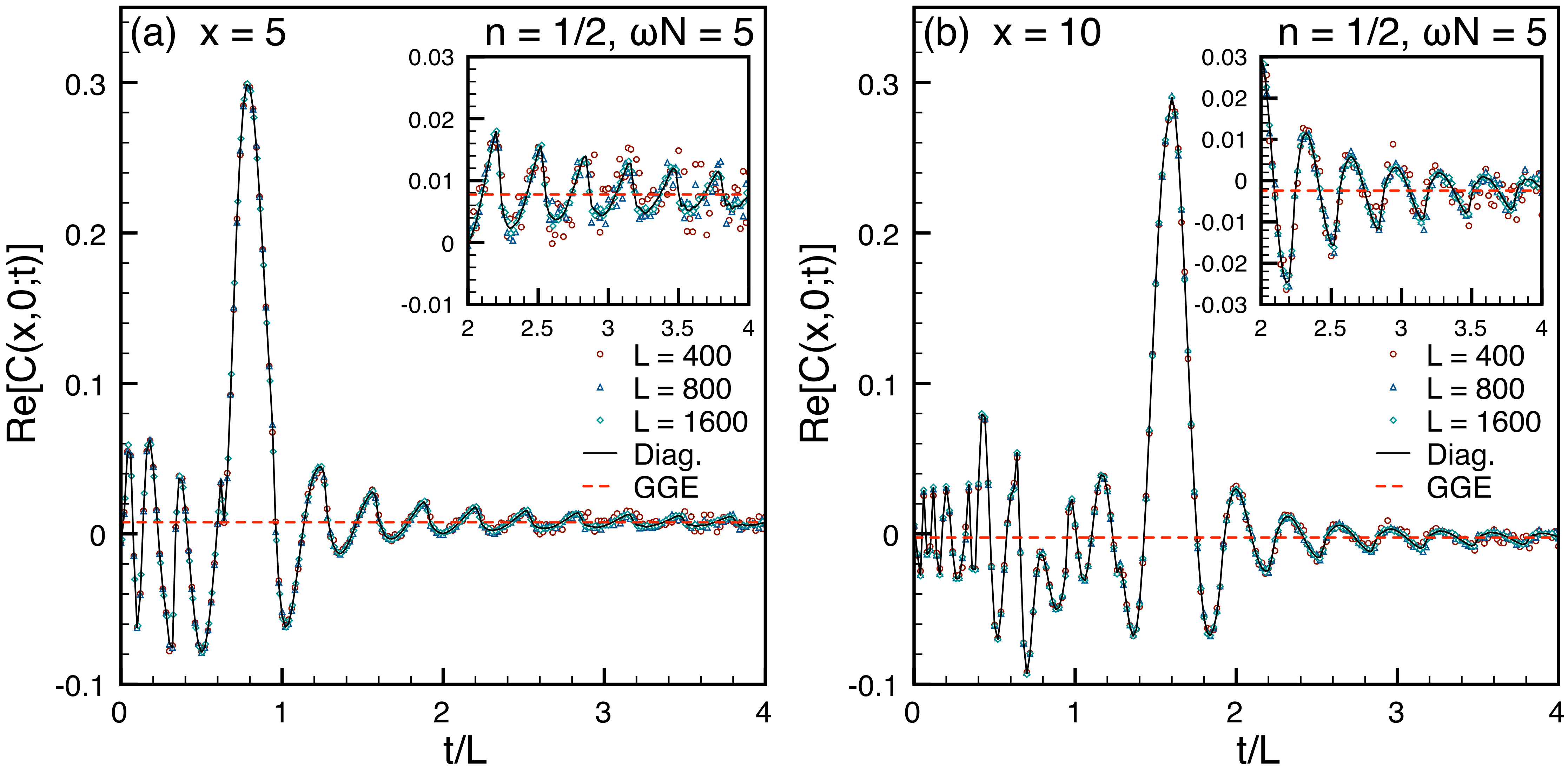


Numerical evidence it approaches to the TD Limit as N and L increase

Fermionic correlation

In the TD limit:

$$C(x,y;t) = \frac{\mathrm{e}^{i\frac{\omega^2 t(x^2-y^2)}{2(1+\omega^2t^2)}}}{\sqrt{1+\omega^2t^2}} \sum_{p=-\infty}^{\infty} e^{i\frac{\omega^2 t(x-y)pL}{1+\omega^2t^2}} \sum_{j=0}^{N-1} \chi_j\left(\frac{x+pL}{\sqrt{1+\omega^2t^2}}\right)\chi_j\left(\frac{y+pL}{\sqrt{1+\omega^2t^2}}\right)$$



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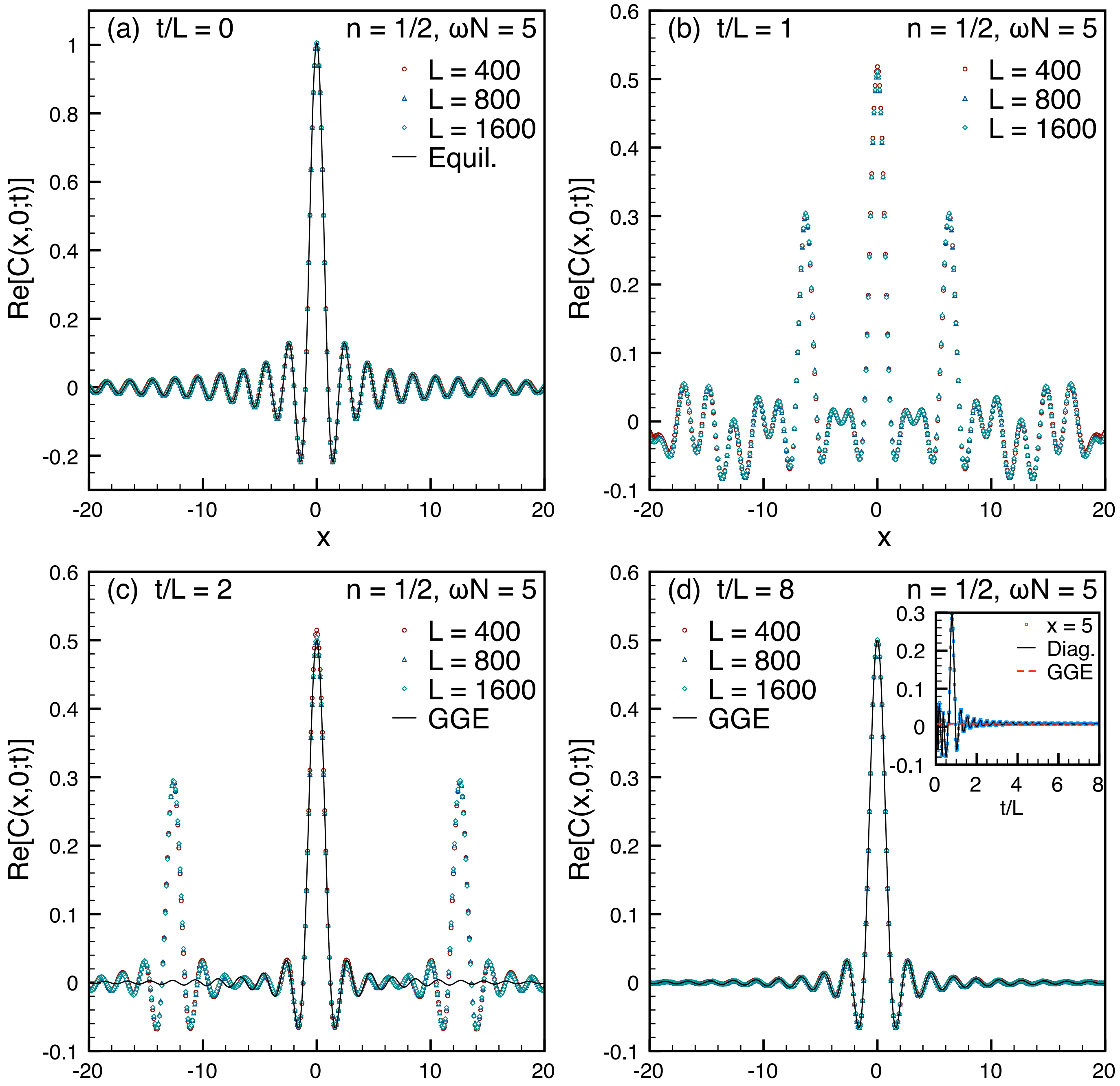

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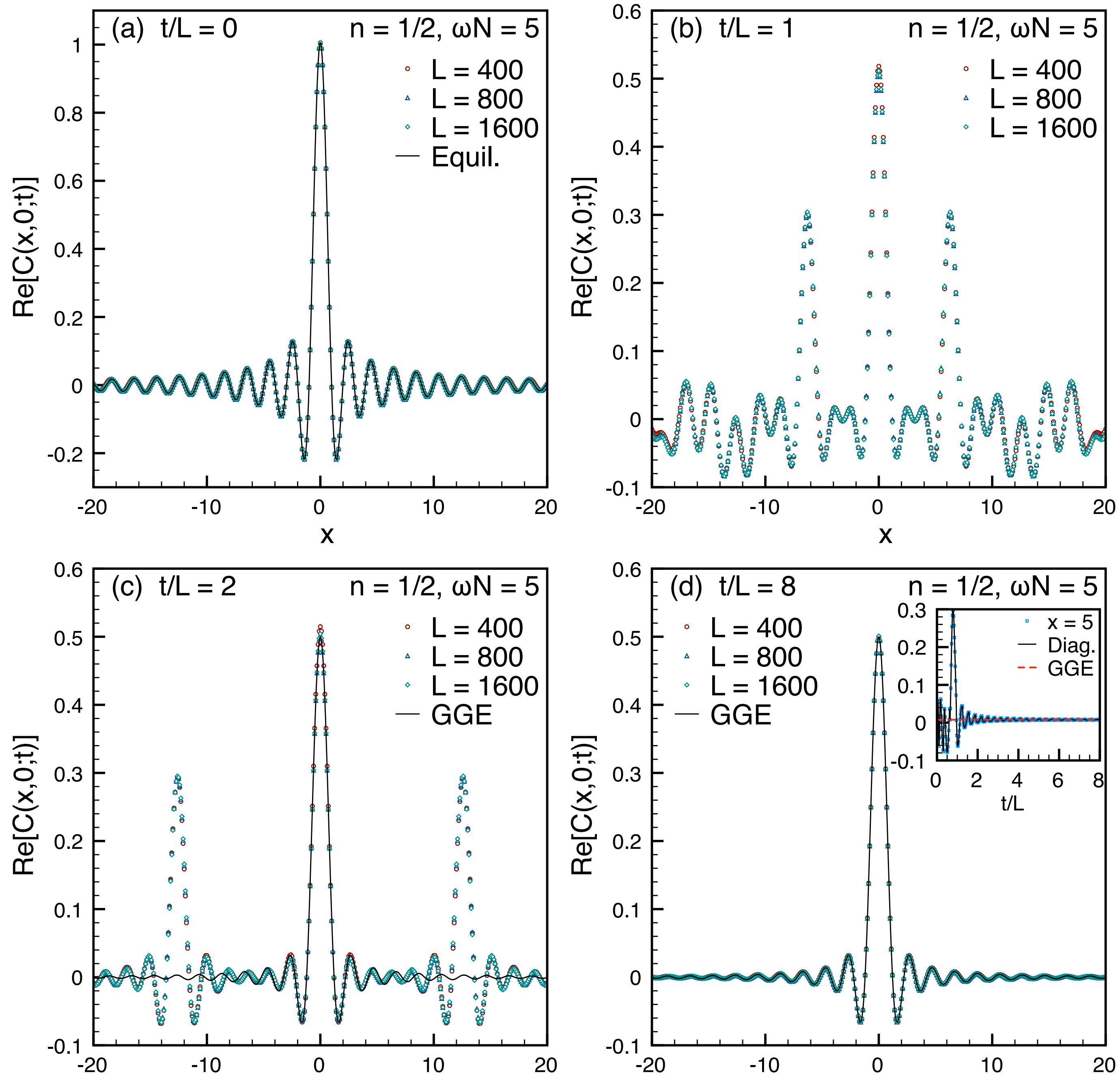
In the **large-time** limit translational invariance is recovered and

$$C(x,y;t\to\infty) = 2n\frac{J_1[\sqrt{2\omega N}(x-y)]}{\sqrt{2\omega N}(x-y)}$$



Fermionic correlation

In the TD limit:

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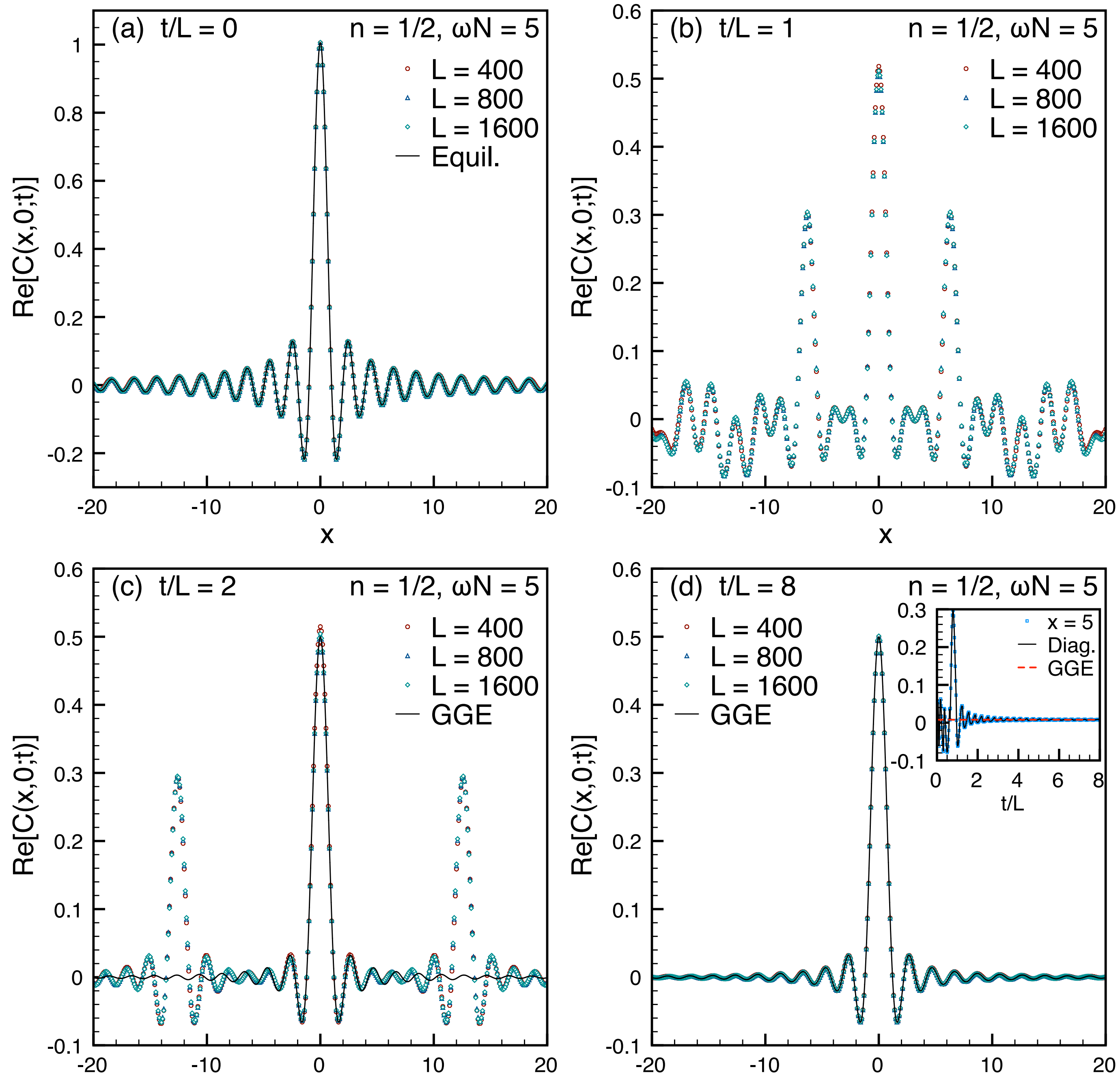
Fourier transforming, we have the momentum distribution, i.e. the conserved charges

$$n_{GGE}(k) \equiv \langle \hat{n}_k \rangle_0 = \frac{2}{L}\sqrt{\frac{2N}{\omega}}\sqrt{1-\frac{k^2}{2\omega N}}$$

$$\rho_{GGE} = Z^{-1}\mathrm{e}^{-\sum \lambda_k \hat{n}_k}$$

Fourier transform does not give the charges at finite time

➡ $C(x,y;t)$ evolves



Non-local vs local GGE

Wick theorem allows to rewrite any observable in terms of 2-pt function, in particular the FULL reduced density matrix, which turns out to be GGE with

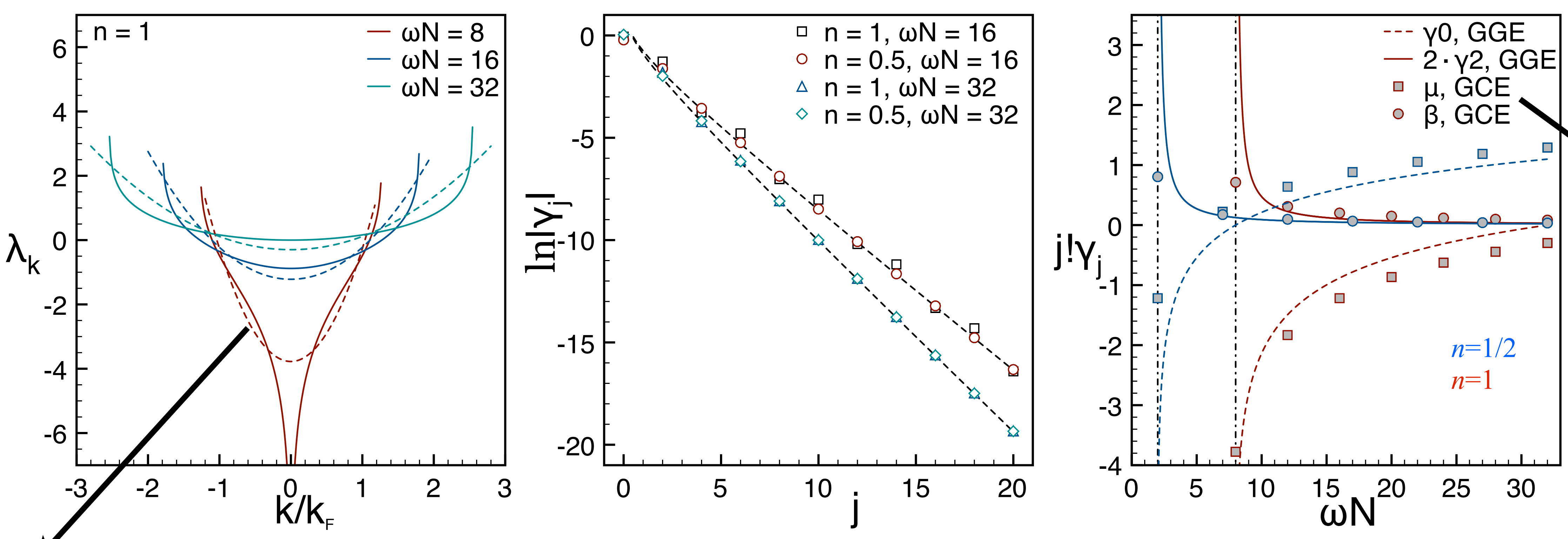
$$n_{\mathrm{GGE}}(k) = \frac{2}{L}\sqrt{\frac{2N}{\omega}}\sqrt{1-\frac{k^2}{2\omega N}}$$

$$n_{\mathrm{GGE}}(k) = \frac{1}{e^{\lambda_k}+1}$$

$$\Longrightarrow \quad \lambda_k = \ln\left[\frac{L\omega}{2}\frac{1}{\sqrt{2\omega N - k^2}} - 1\right]$$

$$\hat{I}_j = \int \frac{dk}{2\pi} k^j n_k$$

$$\int_{-\infty}^{\infty}\frac{dk}{2\pi}\lambda_k \hat{n}_k = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\sum_{j=0}^{\infty}\frac{1}{j!}\left[\frac{d^j}{dk^j}\lambda_k\right]_{k=0} k^j \hat{n}_k = \sum_{j=0}^{\infty}\frac{1}{j!}\left[\frac{d^j}{dk^j}\lambda_k\right]_{k=0}\hat{I}_j = \gamma_0\hat{N} + 2\gamma_2\hat{H} + \cdots$$



grancanonical (2 multipliers)

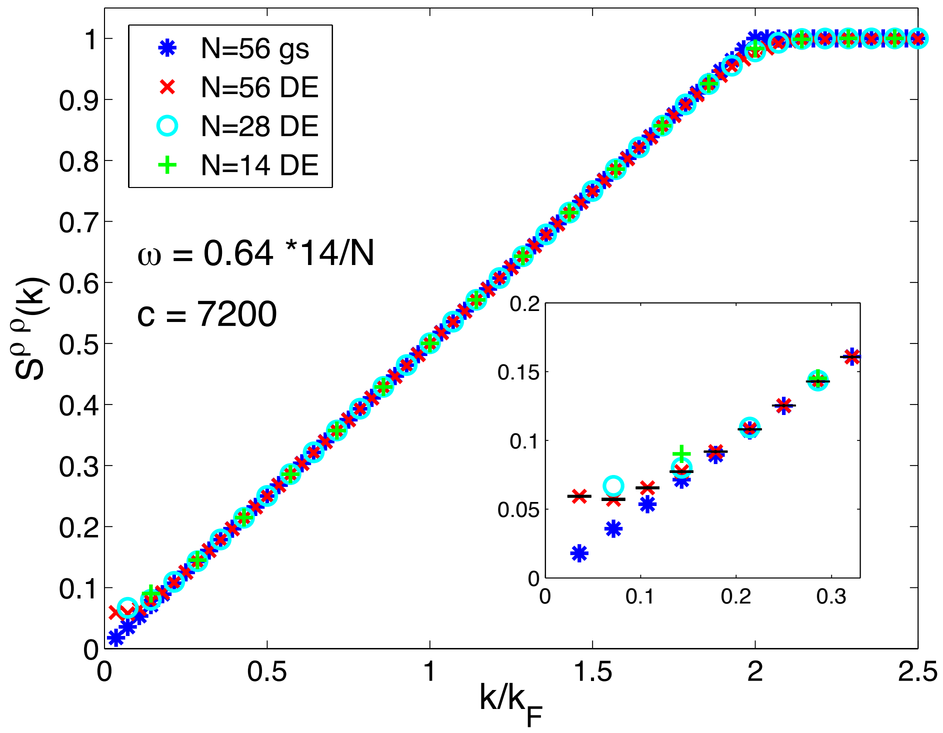
dashed = canonical (1 multiplier)

Static structure factor

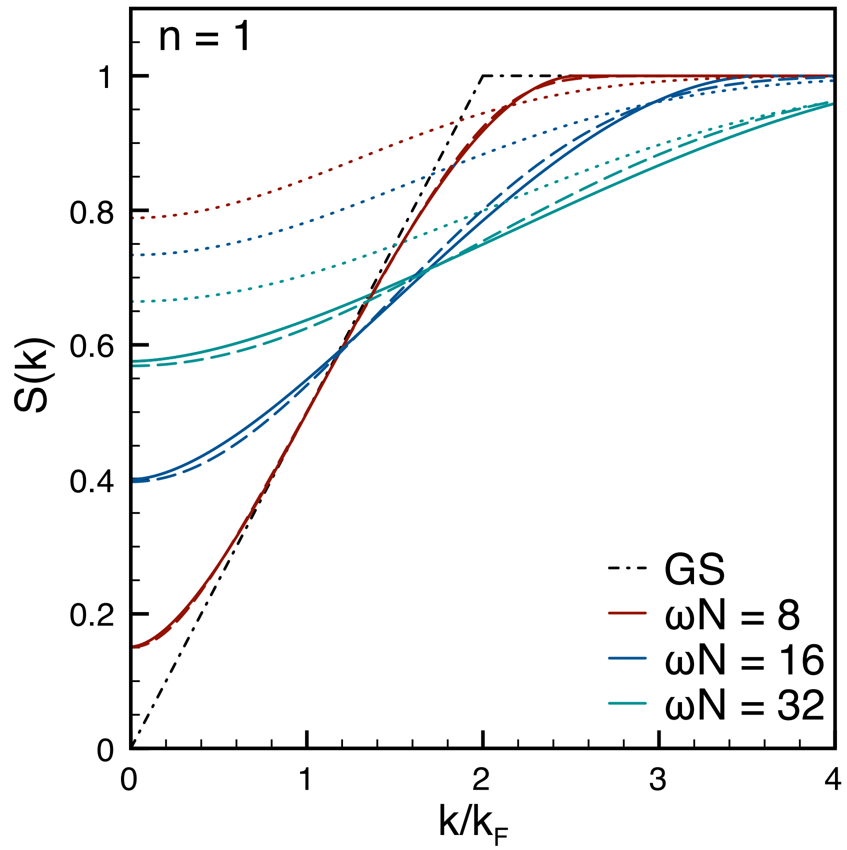
Steady state values of observables can be written in terms of $C(x,y;t\to\infty)$, e.g

$$S(k) = 1 - \frac{L}{N}\int \frac{dq}{2\pi}\, n_q n_{k+q} = 1 - \frac{4\sqrt{2}n}{\pi\sqrt{\omega N}} f\Big(\frac{k}{\sqrt{2\omega N}}\Big)$$

$$f(x) = \begin{cases} \left[(4+x^2)E\Big(1-\frac{4}{x^2}\Big) - 8K\Big(1-\frac{4}{x^2}\Big)\right]\frac{|x|}{6} & \text{if } |x|<2 \\ 0 & \text{if } |x|>2, \end{cases}$$

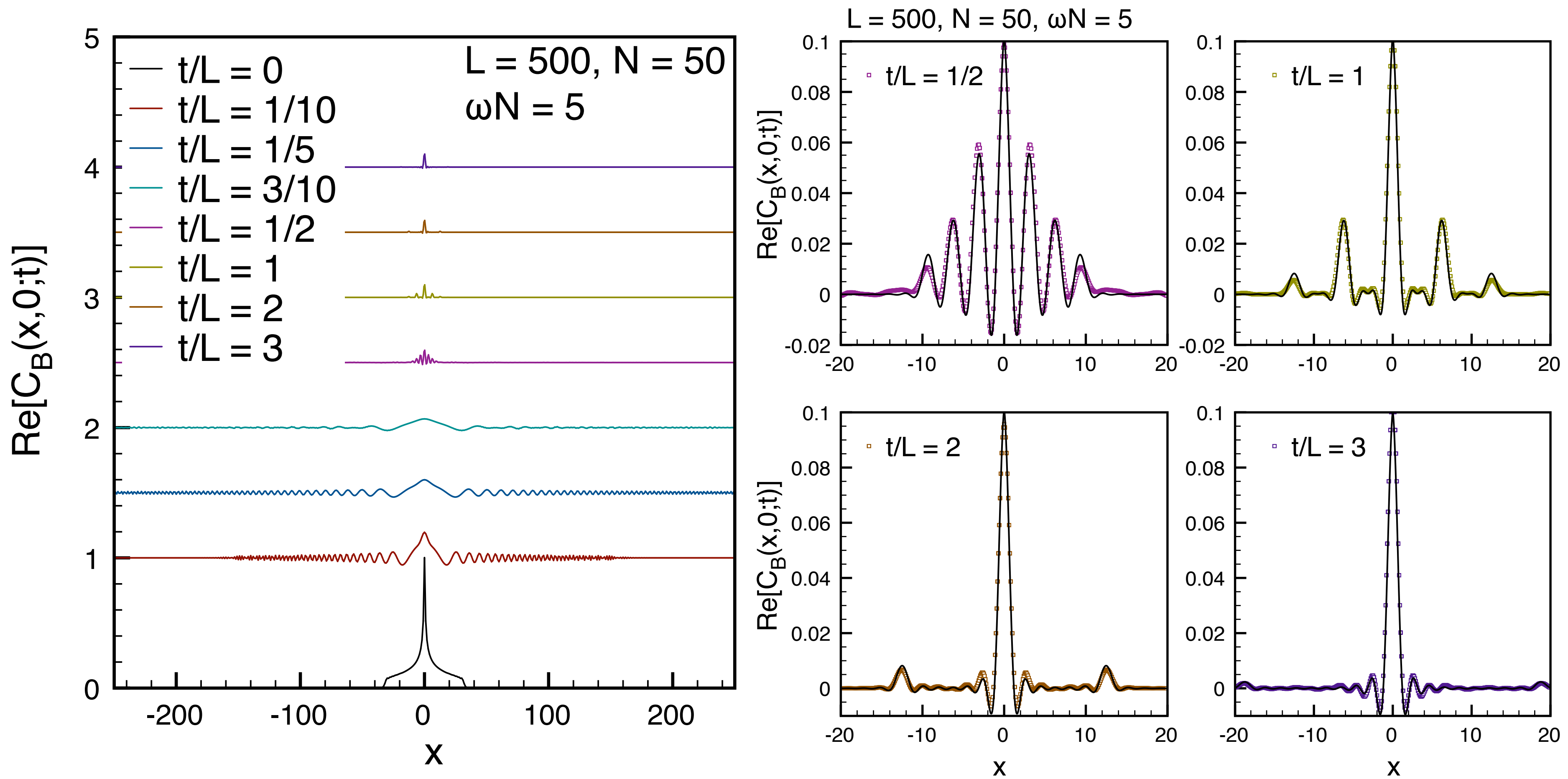


from J.S. Caux and R.M. Konik, Phys. Rev. Lett. **109**, 175301 (2012)



Bosonic Correlation

Bosonic correlation is a Fredholm minor involving $C(x,y;t\to\infty)$



For infinite time in the TD limit:

$$C_B(x,y;t\to\infty) = C_F(x,y;t\to\infty)\mathrm{e}^{-2n|x-y|} = 2n\frac{J_1\left[\sqrt{2\omega N}(x-y)\right]}{\sqrt{2\omega N}(x-y)}\mathrm{e}^{-2n|x-y|}$$

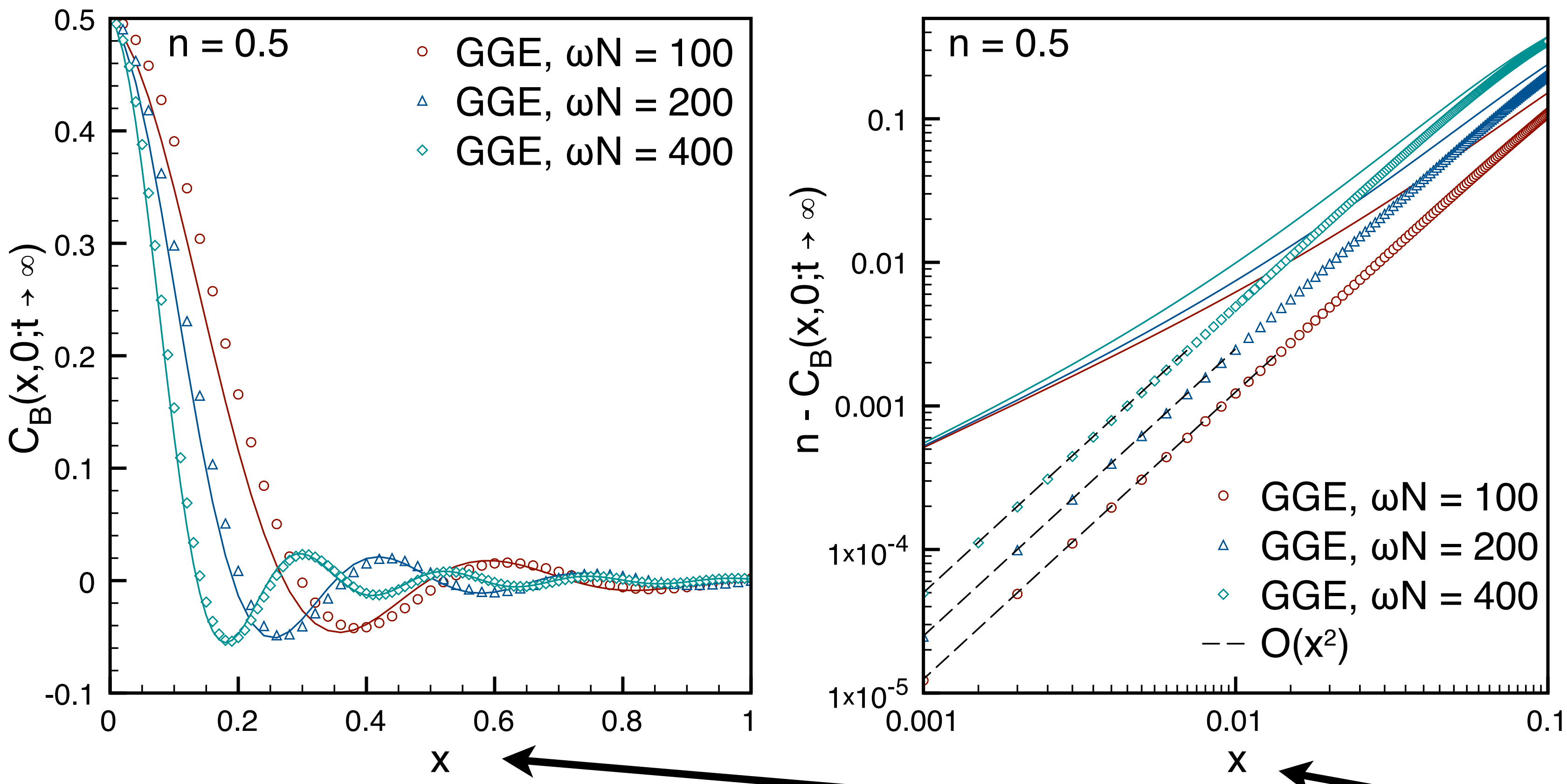
Fourier transform bosonic MDF

$$n_B(k) = \int_{-\sqrt{2\omega N}}^{\sqrt{2\omega N}} \frac{dq}{2\pi}\, n_{GGE}(q) \frac{1/n}{1+(k-q)^2/4n^2} \xrightarrow{\text{large } k} \frac{4n^2}{k^2}$$

No k^{-4} “Tan-tail”?

Bosonic Correlation II

Numerical evaluation of the Fredholm minor:



For small x, at any finite N, there is a crossover to

this is x, not x/L

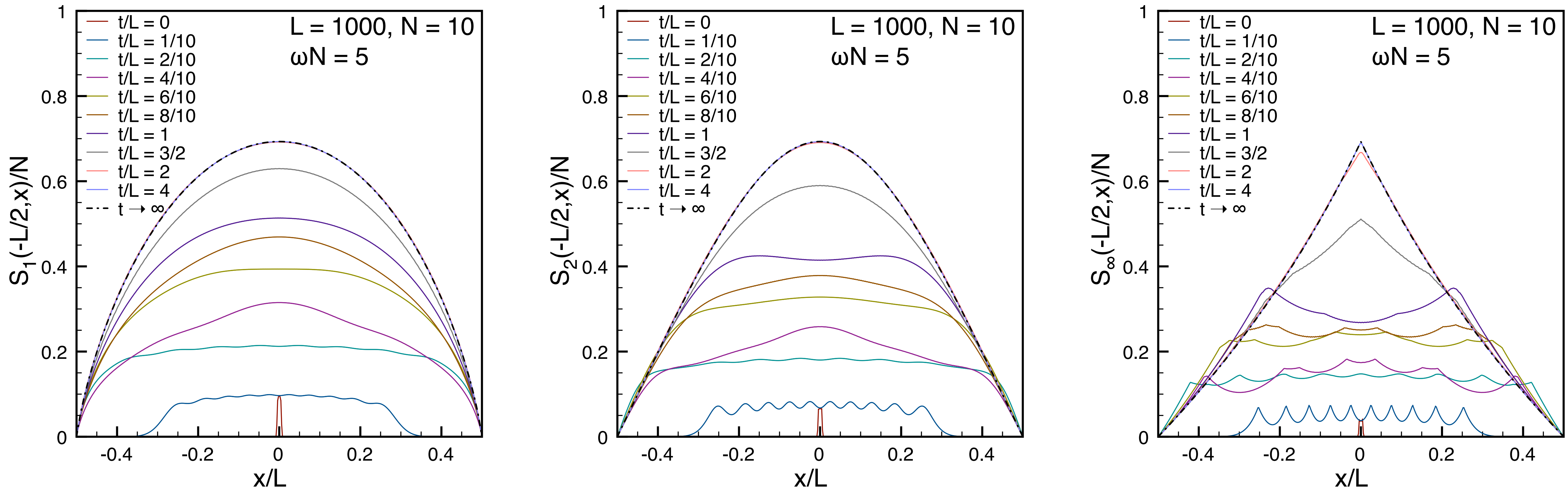
$$C_B(x,y;t\to\infty) \sim n - \frac{n\omega N}{4}(x-y)^2 - \frac{n^2\omega N}{6}|x-y|^3 + O((x-y)^4)$$

resulting in a standard k^{-4} “Tan-tail” in MDF

Entanglement entropy

In the TD and long time limit, very simple result (for ℓ/L~O(1)):

$$S_\alpha(\ell; t \to \infty) = \frac{\ln \mathrm{Tr}\rho^\alpha_{[\ell;t\to\infty]}}{1-\alpha} = \frac{N}{1-\alpha} \ln\left[\left(\frac{\ell}{L}\right)^\alpha + \left(1-\frac{\ell}{L}\right)^\alpha\right]$$



Two different regimes $(v = \sqrt{2\omega N}\,)$

❶ $t/L < 1/v$ ➡ expansion in full space: $S_\alpha(\ell; t) = S_\alpha(\ell/\gamma(t); 0), \quad \gamma(t) = \sqrt{1+\omega^2 t^2}$ [Vicari 2012]

❷ $1/v < t/L \ll L/2\pi$, geometry (PBC) leads to equilibration

Partial table of results

	Trap $\to$ Ring (GGE)	BEC $\to$ TG (GGE)	Ground-state
Fermion correlator	$2n\frac{J_1[\sqrt{2\omega N}(x-y)]}{\sqrt{2\omega N}(x-y)}$	$ne^{-2n\|x-y\|}$	$\frac{\sin(\pi n(x-y))}{\pi(x-y)}$
Boson correlator	$e^{-2n\|x-y\|}C(x,y)$	$ne^{-2n\|x-y\|}$	$\frac{A}{\|x-y\|^{\frac{1}{2}}}$

Conclusions

- Simple results on quenches provide important insights for general integrable models
- GGE states are candidates for novel phases of matter with unusual correlations
- Many open problems:
  - ★ *Is GGE valid for interacting integrable systems?*
    Cardy, Caux, Eisert, Essler, Mussardo, Konik, Rigol, Silva, Sotiriadis...
  - ★ *Will a generic system have a thermal steady state?*
    Caux, Cirac, Kollath, Konik, Mussardo, Rigol, Silva...
  - ★ *Connection with "typicality"?*

Thank you for your attention