Mutual Information in Conformal Field Theories in Higher Dimensions

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Outline

- Quantum entanglement in general and its quantification
- Path integral approach
- Area law in higher dimensions
- Mutual information for a general CFT
- Results for a gaussian free field
- Universal logarithmic corrections

Quantum Entanglement (Bipartite, Pure State)

- quantum system in a pure state $|\Psi\rangle$, density matrix $\rho = |\Psi\rangle\langle\Psi|$
- $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$
- Alice can make unitary transformations and measurements only in A, Bob only in the complement B
- in general Alice's measurements are entangled with those of Bob
- example: two spin-$\frac{1}{2}$ degrees of freedom

$$|\psi\rangle = \cos\theta \, |\uparrow\rangle_A |\downarrow\rangle_B + \sin\theta \, |\downarrow\rangle_A |\uparrow\rangle_B$$

Measuring bipartite entanglement in pure states

- Schmidt decomposition:

$$|\Psi\rangle = \sum_j c_j \, |\psi_j\rangle_A \otimes |\psi_j\rangle_B$$

with $c_j \geq 0$, $\sum_j c_j^2 = 1$.

- one quantifier of the amount of entanglement is the entropy

$$S_A \equiv -\sum_j |c_j|^2 \log |c_j|^2 = S_B$$

- if $c_1 = 1$, rest zero, $S = 0$ and $|\Psi\rangle$ is unentangled
- if all c_j equal, $S \sim \log \min(\dim \mathcal{H}_A, \dim \mathcal{H}_B)$ – maximal entanglement

- equivalently, in terms of Alice's reduced density matrix:

$$\rho_A \equiv \mathrm{Tr}_B \, |\Psi\rangle\langle\Psi|$$

$$S_A = -\mathrm{Tr}_A \, \rho_A \log \rho_A = S_B$$

- the von Neumann entropy: similar information is contained in the Rényi entropies

$$S_A{}^{(n)} = (1-n)^{-1} \log \mathrm{Tr}_A \, \rho_A{}^n$$

- $S_A = \lim_{n \to 1} S_A{}^{(n)}$

- other measures of entanglement exist, but entropy has several nice properties: additivity, convexity, ...
- it increases under Local Operations and Classical Communication (LOCC)
- it gives the amount of classical information required to specify ρ_A (important for numerical computations)
- it gives a basis-independent way of identifying and characterising quantum phase transitions
- in a relativistic theory the entanglement in the vacuum encodes all the data of the theory (spectrum, anomalous dimensions, ...)

Entanglement entropy in a (lattice) QFT

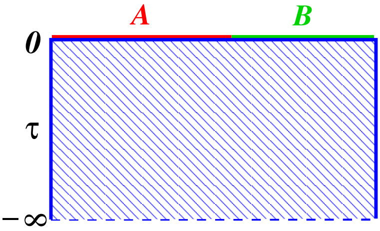
In this talk we consider the case when:

- the degrees of freedom are those of a local relativistic QFT in large region $\mathcal{R}$ in $\mathbb{R}^d$
- the whole system is in the vacuum state $|0\rangle$
- A is the set of degrees of freedom in some large (compact) subset of $\mathcal{R}$, so we can decompose the Hilbert space as

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

- in fact this makes sense only in a cut-off QFT (e.g. a lattice), and some of the results will in fact be cut-off dependent
- How does S_A depend on the size and geometry of A and the universal data of the QFT?

Rényi entropies from the path integral ($d = 1$)



- wave functional $\Psi(\{a\}, \{b\})$ is proportional to the conditioned path integral in imaginary time from $\tau = -\infty$ to $\tau = 0$:

$$\Psi(\{a\}, \{b\}) = Z_1^{-1/2} \int_{a(0)=a, b(0)=b} [da(\tau)][db(\tau)]\, e^{-(1/\hbar) S[\{a(\tau)\},\{b(\tau)\}]}$$

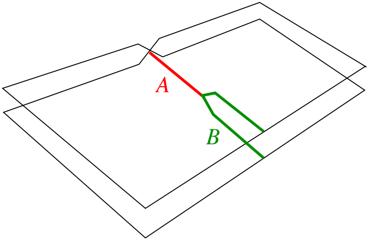
where $S = \int_{-\infty}^{0} L\big(a(\tau), b(\tau)\big)\, d\tau$

- similarly $\Psi^*(\{a\}, \{b\})$ is given by the path integral from $\tau = 0$ to $+\infty$

Example: $n = 2$

$$\rho_A(a_1, a_2) = \int db\, \Psi(a_1, b)\Psi^*(a_2, b)$$

$$\mathrm{Tr}_A\, \rho_A^2 = \int da_1 da_2 db_1 db_2\, \Psi(a_1, b_1)\Psi^*(a_2, b_1)\Psi(a_2, b_2)\Psi^*(a_1, b_2)$$



$$\mathrm{Tr}_A\, {\rho_A}^2 = Z(\mathcal{C}^{(2)})/Z_1^2$$

where $Z(\mathcal{C}^{(2)})$ is the euclidean path integral (partition function) on an 2-sheeted conifold $\mathcal{C}^{(2)}$

- in general

$$\mathrm{Tr}_A\, {\rho_A}^n = Z(\mathcal{C}^{(n)})/Z_1^n$$

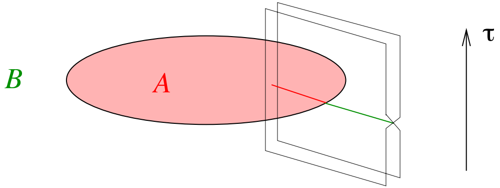
where the half-spaces are connected as

to form $\mathcal{C}^{(n)}$.

- conical singularity of opening angle $2\pi n$ at the boundary of A and B on $\tau = 0$
- in 1+1 dimensions many results are known, e.g for a single interval of length ℓ in a CFT (Holzhey et al., Calabrese-JC)

$$S_A^{(n)} \sim (c/6)(1+n^{-1})\log(\ell/\epsilon)$$

Higher dimensions $d > 1$

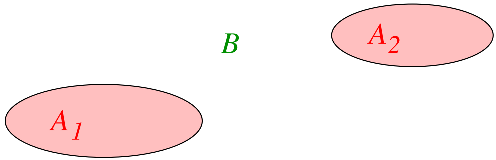


- the conifold $\mathcal{C}_A^{(n)}$ is now locally $\{2d \text{ conifold}\} \times \mathbb{R}^{d-1}$, formed by sewing together n copies of $\{\tau > 0\} \times \mathbb{R}^{d-1}$ to n copies of $\{\tau < 0\} \times \mathbb{R}^{d-1}$ along $\tau = 0$, so that copy j is sewn to $j+1$ for $r \in A$, and j to j for $r \in B$

$$S_A^{(n)} \propto \log(Z(\mathcal{C}_A^{(n)})/Z^n) \sim \text{Vol}(\partial A) \cdot \epsilon^{-(d-1)}$$

- this is the 'area law' in 3+1 dimensions [Srednicki 1992]
- coefficient is non-universal

Mutual Information of multiple regions



- the non-universal 'area' terms cancel in

$$I^{(n)}(A_1, A_2) = S^{(n)}_{A_1} + S^{(n)}_{A_2} - S^{(n)}_{A_1 \cup A_2}$$

- this mutual Rényi information is expected to be universal depending only on the geometry and the data of the CFT
- however this dependence is very difficult to compute, even in 1+1 dimensions (Calabrese-JC-Tonni)

Operator Expansion Method

For any region X

$$S_X^{(n)} = (1-n)^{-1} \log \left(\frac{Z(\mathcal{C}_X^{(n)})}{Z^n} \right)$$

So

$$I^{(n)}(A_1, A_2) \equiv S_{A_1}^{(n)} + S_{A_2}^{(n)} - S_{A_1 \cup A_2}^{(n)} = (n-1)^{-1} \log \left(\frac{Z(\mathcal{C}_{A_1 \cup A_2}^{(n)}) Z^n}{Z(\mathcal{C}_{A_1}^{(n)}) Z(\mathcal{C}_{A_2}^{(n)})} \right)$$

Write

$$\frac{Z(\mathcal{C}_{A_1 \cup A_2}^{(n)})}{Z^n} = \langle \Sigma_{A_1}^{(n)} \Sigma_{A_2}^{(n)} \rangle_{(\mathbb{R}^{d+1})^n}$$

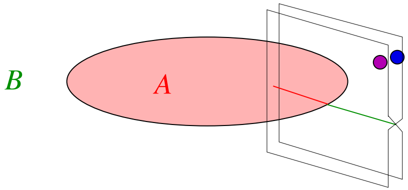
where

$$\Sigma_A^{(n)} = \frac{Z(\mathcal{C}_A^{(n)})}{Z^n} \sum_{\{k_j\}} C_{\{k_j\}}^A \prod_{j=0}^{n-1} \Phi_{k_j}(r_A^{(j)})$$

$$
\begin{aligned}
\frac{Z(\mathcal{C}^{(n)}_{A_1\cup A_2})}{Z^n} &= \frac{Z(\mathcal{C}^{(n)}_{A_1})}{Z^n}\frac{Z(\mathcal{C}^{(n)}_{A_2})}{Z^n}\sum_{\{k_j\},\{k'_j\}} C^{A_1}_{\{k_j\}} C^{A_2}_{\{k'_j\}} \prod_{j=0}^{n-1}\langle \Phi_{k_j}(r^{(j)}_{A_1})\Phi_{k'_j}(r^{(j)}_{A_2})\rangle \\
&= \frac{Z(\mathcal{C}^{(n)}_{A_1})}{Z^n}\frac{Z(\mathcal{C}^{(n)}_{A_2})}{Z^n}\sum_{\{k_j\}} C^{A_1}_{\{k_j\}} C^{A_2}_{\{k_j\}}\, r^{-2\sum_j x_{k_j}}
\end{aligned}
$$

- last equation flows from orthonormality of 2-point functions, valid in any CFT
- this gives an expansion of $I^{(n)}(A_1, A_2)$ in increasing powers of $1/r$, valid for large r
- first term comes from the identity operator with $x_{k_j} = 0\ \forall j$, but this cancels in $I^{(n)}(A_1, A_2)$
- leading terms come from taking either 1 or 2 of the $x_{k_j} \neq 0$

The coefficients $C^{A}{}_{\{k_j\}}$



These may be computed by inserting a complete set of operators on a single conifold $\mathcal{C}_A^{(n)}$:

$$\langle \prod_{j'} \Phi_{k'_{j'}}(r^{(j')}) \rangle_{\mathcal{C}_A^{(n)}} = \left\langle \left(\prod_{j'} \Phi_{k'_{j'}}(r^{(j')}) \right) \left(\sum_{\{k_j\}} C^{A}_{\{k_j\}} \prod_{j=0}^{n-1} \Phi_{k_j}(r^{(j)}) \right) \right\rangle_{(\mathbb{R}^{d+1})^n}$$

Using orthonormality

$$C^{A}{}_{\{k_j\}} = \lim_{\{r^{(j)}\} \to \infty_j} |r^{(j)}|^{\sum_j x_{k_j}} \langle \prod_j \Phi_{k_j}(r^{(j)}) \rangle_{\mathcal{C}_A^{(n)}}$$

- note that ${C^{A}}_{\{k_j\}} \propto {R_A}^{\sum_j x_{k_j}}$ by dimensional analysis
- the 1- and 2-point functions on $\mathcal{C}_A^{(n)}$ are still very hard to compute, and we have succeeded only for a free field theory

Free scalar field theory (gaussian free field)

Action is proportional to $\int(\partial\phi)^2 d^{d+1}x$, and we normalise so 2-point function in $\mathbb{R}^{d+1}$ is

$$\langle\phi(x)\phi(x')\rangle \equiv G_0(x-x') = |x-x'|^{-(d-1)}\,.$$

We need to compute

$$\lim_{x,x'\to\infty}(xx')^{d-1}\langle\phi_j(x)\phi_{j'}(x')\rangle_{\mathcal{C}_A^{(n)}} \qquad (j\neq j')$$

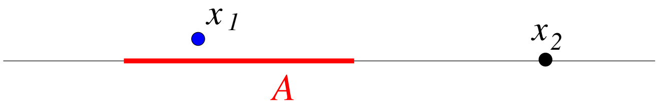
$$\lim_{x\to\infty}x^{2(d-1)}\langle:\phi_j^2(x):\rangle_{\mathcal{C}_A^{(n)}}$$

where $\phi_j(x,0-)=\phi_{j+1}(x,0+)$ for $x\in A$, and $\phi_j(x,0-)=\phi_j(x,0+)$ for $x\notin A$.
These can be though of as the potential at x' on copy j' due to a unit charge at x on copy j, and the self-energy of a unit charge at x.

The case $n = 2$

Define $\phi_\pm = 2^{-1/2}(\phi_0 \pm \phi_1)$

- ϕ_+ is continuous everywhere and so $\langle\phi_+(x)\phi_+(x')\rangle = G_0(x - x')$
- ϕ_- changes sign across $A \cap \{\tau = 0\}$; on the other hand, if the source x lies on $\tau = 0$ then $\langle\phi_-(x)\phi_-(x')\rangle$ must be symmetric under $\tau' \to -\tau'$, so it vanishes on $A \cap \{\tau = 0\}$



- $\langle\phi_-(x)\phi_-(x')\rangle$ is the potential at x' due to a unit charge at x in the presence of a conductor held at zero potential at $A \cap \{\tau = 0\}$

As $x, x' \to \infty$

$$\langle \phi_-(x)\phi_-(x') \rangle - G_0(x - x') \sim -\mathbf{C}_A |x|^{-(d-1)} |x'|^{-(d-1)}$$

where $\mathbf{C}_A$ is the electrostatic capacitance of $A \cap \{\tau = 0\}$.
This gives

$$I^{(2)}(A_1, A_2) \sim \frac{\mathbf{C}_{A_1}\mathbf{C}_{A_2}}{2r^{2(d-1)}}$$

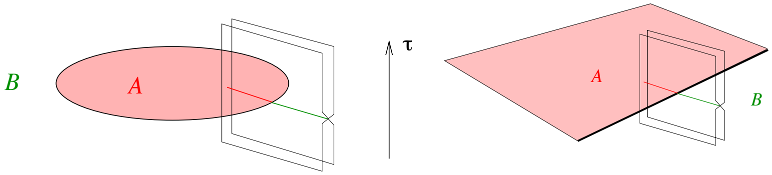
If A is a sphere of radius R_A, the generalisation of a classic result of W. Thomson gives

$$\mathbf{C}_A = \frac{\Gamma(d/2)\Gamma(1/2)}{\pi\Gamma((d+1)/2)}\, {R_A}^{d-1}$$

but in general the result depends on the shape of A.

Case when A_1 and A_2 are both spheres, general n

If A is the interior of a sphere S^d, we can make a conformal mapping in $\mathbb{R}^{d+1}$ so that the boundary of A becomes $\mathbb{R}^2$



- the conifold is now a 2d conical singularity $\times\mathbb{R}^{d-1}$ so we have cylindrical symmetry. We want the potential $G^{(n)}(\rho,\theta,z)$ due to the unit charge at $((2R_A)^{-1},0,0)$
- for the moment suppose that $n=1/m$, where m is a positive integer, so the cone has opening angle $2\pi/m$

Method of images gives

$$G^{(1/m)}(\rho, \theta, z) = \sum_{k=0}^{m-1} G_0(\rho, \theta + 2\pi k/m, z)$$

Specialising to $\rho = 1$, $z = 0$,

$$G^{(1/m)}(1, \theta, 0) = \sum_{k=0}^{m-1} \frac{1}{\big(2 - 2\cos(\theta + 2\pi k/m)\big)^{(d-1)/2}}$$

This is straightforward for $d + 1$ even, a little harder for $d + 1$ odd. E.g. for $d = 3$

$$G^{(1/m)}(1, \theta, 0) = \frac{m^2}{2 - 2\cos m\theta}$$

This can now be continued back to $n = 1/m > 1$.

Self-energy

$$\langle :\phi_0^2(1): \rangle_{\mathcal{C'}_A^{(n)}} = \lim_{\theta \to 0} \left(\frac{1/n^2}{2 - 2\cos(\theta/n)} - \frac{1}{2 - 2\cos\theta} \right) = \frac{1 - n^2}{12n^2}$$

The leading term in the mutual entropy involves (this piece)^2 and

$$\sum_{j=1}^{n-1} G^{(n)}(1, 2\pi j/n, 0)^2 = \frac{1}{n^4} \sum_{j=1}^{n-1} \frac{1}{\big(2 - 2\cos(2\pi j/n)\big)^2}$$

Once again this can be done analytically.

Final result in $d = 3$

$$I^{(n)}(A_1, A_2) \sim \frac{n^4 - 1}{15n^3(n-1)} \left(\frac{R_1 R_2}{r^2}\right)^2$$

Taking the limit $n \to 1$ gives the mutual information

$$I(A_1, A_2) \sim \frac{4}{15} \left(\frac{R_1 R_2}{r^2}\right)^2$$

- this can computed another way: for a gaussian state, the correlation functions determine the density matrix

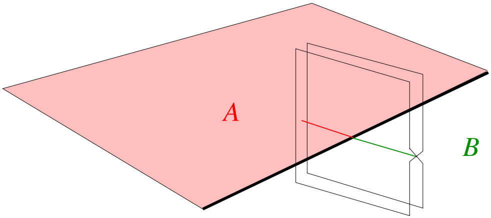
  (Bombelli et al., Casini-Huerta)

- but the matrix computations must still be carried out numerically for finite r and extrapolated
- this was carried out by N. Shiba who found ≈ 0.26 compared with $\frac{4}{15} = 0.26\dot{6}$

For $d = 2$ we find

$$I(A_1, A_2) \sim \frac{1}{3} \left(\frac{R_1 R_2}{r^2}\right)$$

Logarithmic corrections to area law

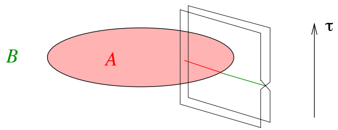


- we can use the same methods to compute the stress tensor in the cylindrically symmetric geometry. e.g. in $d = 3$

$$\langle T_{\rho\rho} \rangle \propto \frac{(1 - 1/n^4)a}{\rho^4} \qquad \text{`}a\text{-anomaly'}$$

$$\epsilon(\partial/\partial\epsilon)\log Z(\mathcal{C}^{(n)}) = n\int \langle T_{\rho\rho}\rangle \rho d\rho d\theta d^2 z \sim \epsilon^{-2} \times \mathrm{Area}(\partial A)$$

- but when we map back to the sphere



$$\langle T_{\rho\rho}\rangle \propto a\,(1-1/n^4)\left(\frac{1}{\rho^4}+\frac{1}{R_A^2\rho^2}+\cdots\right)$$

$$\epsilon(\partial/\partial\epsilon)\log Z(\mathcal{C}^{(n)}) \sim \epsilon^{-2}\times(4\pi R_A^2) + \text{universal } O(1) \text{ term}$$

$$S_A^{(n)} \sim \epsilon^{-2}\mathrm{Area}(\partial A) + \#a\,(n-1/n^3)\log(R_A/\epsilon)$$

[Casini/Huerta, Fursaev/Soludukhin,. . .]

- similar result whenever $d+1$ is even
- relation to a-theorem?

Summary

- mutual information in the ground state of relativistic field theory encodes data (scaling dimensions, OPE coefficients...) of general CFTs (= critical systems) in higher dimensions
- we have treated example of free field theory, difficult to go further quantitatively
- universal log corrections to area law in even $d + 1$ encode the a-anomaly