Mathematical Statistical Physics, YITP

Kyoto, July 30th, 2013

RIGOROUS RESULTS ON SHORT-RANGE FINITE-DIMENSIONAL SPIN GLASSES

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Consider configurations of N Ising spins

$$\sigma = \{\sigma_i\}, \quad \tau = \{\tau_i\}, \quad \ldots \quad ,$$

introduce a centered Gaussian Hamiltonian $H_N(\sigma)$ defined by the covariance

$$\mathsf{Av}(H_N(\sigma)H_N(\tau)) = N c_N(\sigma,\tau) \; .$$

Examples of covariances:

Sherrington-Kirkpatrick and Edwards-Anderson model

$$\left(\frac{1}{N}\sum_i \sigma_i\tau_i\right)^2 \quad , \quad \frac{1}{N}\sum_{|i-j|=1} \sigma_i\sigma_j\tau_i\tau_j$$

We are interested in the large volume properties of the random probability measure

$$p_N(\sigma) = \frac{e^{-\beta H_N(\sigma)}}{\sum_\sigma e^{-\beta H_N(\sigma)}}$$

for all $\beta > 0$. Quantities of interest include:

the pressure

$$P_N(\beta) = \mathsf{Av} \log \sum_\sigma e^{-\beta H_N(\sigma)}$$

the covariance moments

$$\mathsf{Av} \left(\frac{\sum_{\sigma,\tau} c_N(\sigma,\tau) e^{-\beta[H_N(\sigma)+H_N(\tau)]}}{\sum_{\sigma,\tau} e^{-\beta[H_N(\sigma)+H_N(\tau)]}} \right) =$$

$$= < c >_N = \int c p_N(c) dc \ ,$$

$$< c_{12} c_{23} >_N = \int c_{12} c_{23} p_N^{(12),(23)}(c_{12}, c_{23}) \ ,$$

and especially the joint distribution (permutation invariant)

$$p_N^{(12),(23),...,(kl),...}(c_{12}, c_{23}, ..., c_{kl}, ...)$$

The joint distribution allows to compute internal energy, specify heat, etc.

What happens when $N \to \infty$?

The mean-field theory (Replica Symmetry Breaking) is characterised by two properties:

- $p(c)$ has a non-trivial support

- the joint distribution

$$p^{(12),(23),...,(kl),...}(c_{12}, c_{23}, ..., c_{kl}, ...)$$

fulfils a factorisation property and can be reconstructed starting from $p(c)$ through the *ultrametric* and *replica equivalence* rule.

Ultrametricity:

$$p^{(12),(23),(31)}(c_{12}, c_{23}, c_{31}) \;=$$

$$\delta(c_{12} - c_{23})\delta(c_{23} - c_{31})p(c_{12}) \int_0^{c_{12}} p(c)dc$$

$$+\, \theta(c_{12} - c_{23})\delta(c_{23} - c_{31})p(c_{12})p(c_{23})$$

$$+\, \theta(c_{23} - c_{31})\delta(c_{31} - c_{12})p(c_{23})p(c_{31})$$

$$+\, \theta(c_{31} - c_{12})\delta(c_{12} - c_{23})p(c_{31})p(c_{12})$$

no scalene triangles!

Replica Equivalence (Ghirlanda-Guerra)

$$p^{(12),(23)}(c_{12}, c_{23}) = \frac{1}{2} p(c_{12}) \delta(c_{12} - c_{23}) + \frac{1}{2} p(c_{12}) p(c_{23})$$

$$p^{(12),(34)}(c_{12}, c_{34}) = \frac{1}{3} p(c_{12}) \delta(c_{12} - c_{34}) + \frac{2}{3} p(c_{12}) p(c_{34})$$

Rigorous results on factorisation, mean-field models:

- 1997, M. Aizenman, P.C. (stochastic stability), 1998 F.Guerra, S.Ghirlanda (based on Pastur-Scherbina)

- 2005-2007, P.C. and C.Giardina (results for the first power moments, but hold also in finite-dimension short-range!), M. Talagrand (results in distribution, but only for the SK model)

* 2011 D.Panchenko proved that Ghirlanda-Guerra identities in distribution imply ultrametricity (proof by contradiction, geometrical methods)

Result:

Edwards-Anderson, and a wide class of finite-dimensional models, is ultrametric!

P.C., E.Mingione, S.Starr [JSP, 2013]

No claim on $p(c)$ is made

More precisely: consider, in a box $\Lambda \subset Z^d$, the model defined by the Hamiltonian

$$H_\Lambda(\sigma) = \sum_{X \subseteq \Lambda} J_{\Lambda,X}\sigma_X$$

if (thermodynamic stability)

$$\mathsf{Av}\,(H_\Lambda(\sigma)H_\Lambda(\sigma)) \leq cN$$

then for all power p, any number of system copies n, any bounded measurable function f of the $\{c_{l,m}\}_{l,m=1}^n$ in the thermodynamic limit it holds:

$$< f c^p_{n+1,n+1} >= \frac{1}{n+1}\sum_{k=1}^{n} < f c^p_{k,n+1} > + \frac{1}{n+1} < f >< c^p_{1,2} > ,$$

and (by Panchenko result) is ultrametric.

Main idea:

- *Stochastic Stability* and its extension (P.C., C.Giardina, C.Giberti, EPL 2011)

Consider the quenched equilibrium state

$$< F >_N = \text{Av}\left(\frac{\sum_\sigma F(\sigma) e^{-\beta H_N}}{\sum_\sigma e^{-\beta H_N}}\right)$$

and the smooth deformation g of the Hamiltonian density h:

$$< F >_N^{(\lambda)} = \frac{< F e^{-\lambda g(h)} >}{< e^{-\lambda g(h)} >}$$

The spin glass quenched equilibrium state is stochastically stable:

$$< F >_N^{(\lambda)} \to < F >$$

(check that the perturbation doesn't spoil thermodynamic stability). This implies bounds on thermal and disorder fluctuation

$$\mathsf{Av}(\omega(H^2)) - \mathsf{Av}(\omega(H)^2) \leq c_1 N$$

$$\mathsf{Av}(\omega(H)^2) - \mathsf{Av}(\omega(H))^2 \leq c_2 N$$

What's next?

- Overlap Equivalence:

$$V_N(Q|q) \to 0$$

numerically seen in PRL 2006 by P.C., Cristian Giardina, Claudio Giberti, Cecilia Vernia

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- Triviality? Study $P(Q)$