

Quantum Quenches in Integrable Models

Fabian Essler (Oxford)

Collaborators: J.S. Caux (Amsterdam), M. Fagotti (Oxford)

earlier work: P. Calabrese (Pisa), S. Evangelisti (Oxford), D. Schuricht (Aachen)

M. Fagotti and F.H.L. Essler, JSTAT P07012 (2013)

J.S. Caux and F.H.L. Essler, PRL 110, 257203 (2013)

M. Fagotti and F.H.L. Essler, PRB 87, 245107 (2013)

F.H.L. Essler, S. Evangelisti & M. Fagotti, PRL 109, 247206 (2012)

Outline

1. Introduction to Quantum Quenches and some general notions.
2. A new formalism for analyzing quenches in integrable models.
3. Generalized Gibbs Ensembles and the late-time behaviour after quenches in integrable models
 - GGE and dynamical correlations.
 - Time evolution of reduced density matrices (TFIM).
 - GGE expectation values of local observables for quenches in the spin-1/2 XXZ chain.

A. Introduction and some general notions

Quantum Quenches in isolated many-particle systems

- A. Consider an **isolated** quantum system in the **thermodynamic limit**; Hamiltonian $H(h)$ (short-ranged), h e.g. bulk magnetic field
- B. Prepare the system in the ground state $|\psi\rangle$ of $H(h_0)$
- C. At time $t=0$ change the Hamiltonian to $H(h)$
- D. (Unitary) time evolution $|\psi(t)\rangle = \exp(-iH(h)t) |\psi\rangle$
- E. Goal: study time evolution of local (in space) observables
 $\langle \psi(t) | O(x) | \psi(t) \rangle$, $\langle \psi(t) | O_1(x) O_2(y) | \psi(t) \rangle$, $\langle \psi(t) | O_1(x, t_1) O_2(y, t_2) | \psi(t) \rangle$

Local Relaxation

Given that we are considering an **isolated** system, does the system relax in some way ?

- It can never relax as a whole.

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Initial state $|\psi\rangle$ after the quench is a **pure state**

$$|\psi(t)\rangle = \exp(-iH(\hbar)t) |\psi\rangle = \sum_n \exp(-iE_n t) \langle n | \psi \rangle |n\rangle.$$

Can **always** choose "observables" O that never relax, e.g.

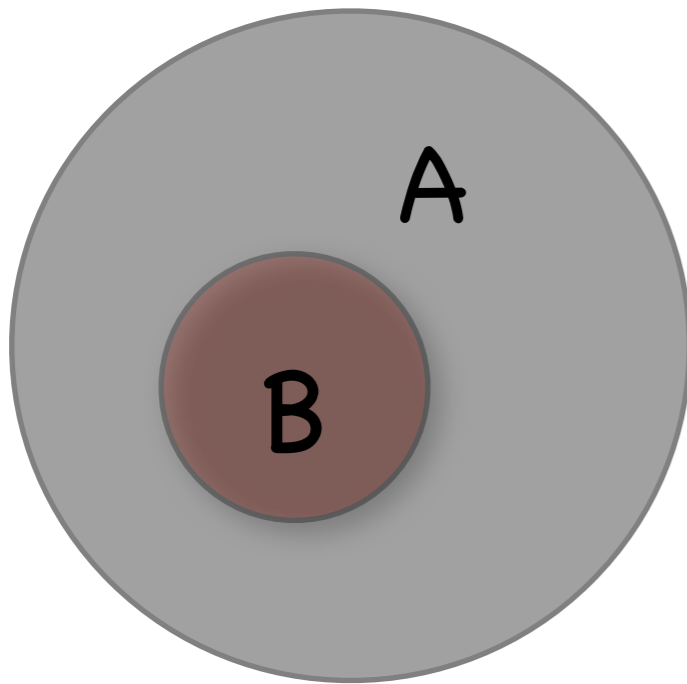
$$O=O^\dagger = |1\rangle\langle 2| + |2\rangle\langle 1|$$

$$\langle \psi(t) | O | \psi(t) \rangle = A \cos([E_1 - E_2]t + \phi)$$

Local Relaxation

Given that we are considering an **isolated** system, does the system relax in some way ?

- It can never relax as a whole.
- It can relax **locally** (in space).



- Entire System: $A \cup B$
- Take A infinite, B finite
- Ask questions only about B:

Expectation values
of **local** ops:

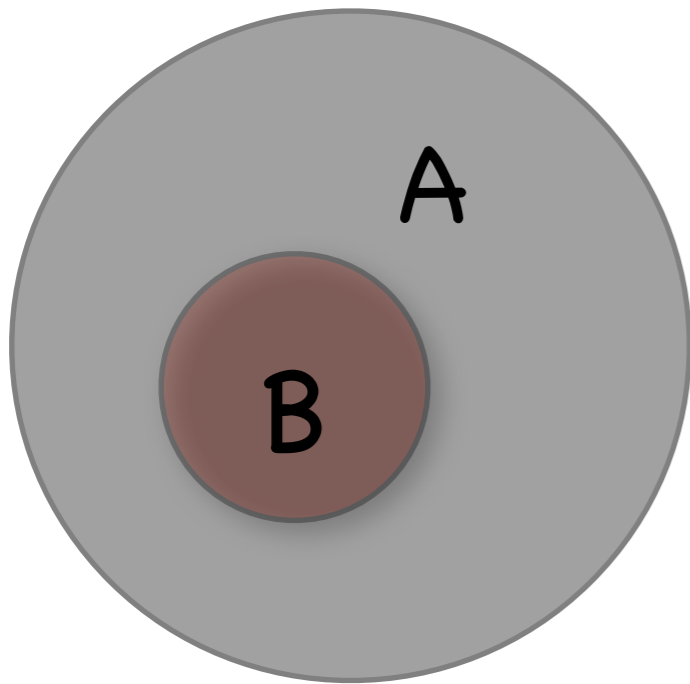
$$\langle \Psi(t) | O_B(x) | \Psi(t) \rangle$$

Physical Picture: A acts like a bath for B.

Local Relaxation

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No time-averaging involved !!!

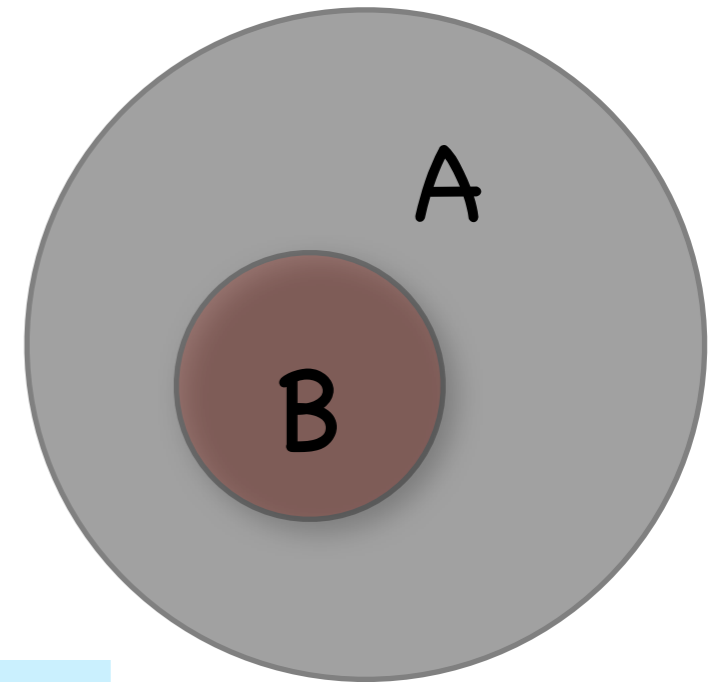
Physical Picture: A acts like a bath for B.

Subsystems and Reduced Density Matrices

$|\psi\rangle$ = initial (pure) state of the entire system $A \cup B$ (A infinite)

Density matrix: $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$

Reduced density matrix: $\rho_B(t) = \text{tr}_A \rho(t)$



ρ_B contains **all** local correlation functions in B:

$$\rho_B(t) = \frac{1}{2^\ell} \sum_{\alpha_1, \dots, \alpha_\ell} \langle \Psi(t) | \sigma_1^{\alpha_1} \dots \sigma_\ell^{\alpha_\ell} | \Psi(t) \rangle \sigma_1^{\alpha_1} \dots \sigma_\ell^{\alpha_\ell}$$

$\alpha_j = 0, x, y, z$

for $B = [1, \dots, \ell]$ in a spin-1/2 quantum spin chain

Definition of the Stationary State

If $\lim_{t \rightarrow \infty} \rho_B(t) = \rho_B(\infty)$ exists for any finite subsystem B:

→ system approaches a stationary state; $\langle \psi(t) | O_B(x) | \psi(t) \rangle$ become time-independent for all local operators.

How about quenches in quantum integrable models?

How about quenches in quantum integrable models?

Have **local** integrals of motion

$$[I_m, I_n] = [I_m, H(h)] = 0.$$

Example: transverse-field Ising chain

Grady '82
Prosen '98

define operators $S_{j,j+l}^{\alpha\beta} = \sigma_j^\alpha [\sigma_{j+1}^z \cdots \sigma_{j+l-1}^z] \sigma_{j+l}^\beta$

$$I_0^+ = H(h) = -J \sum_j S_{j,j+1}^{xx} + h\sigma_j^z$$

$$I_1^+ = -J \sum_j S_{j,j+2}^{xx} + 2\sigma_j^z - h[S_{j,j+1}^{xx} + S_{j,j+1}^{yy}]$$

$$I_{n \geq 2}^+ = -J \sum_j S_{j,j+n+1}^{xx} + S_{j,j+n-1}^{yy} - h[S_{j,j+n}^{xx} + S_{j,j+n}^{yy}]$$

$$I_n^- = -J \sum_j S_{j,j+n}^{xy} - S_{j,j+n}^{yx}$$

I_n involve spins
on $n+2$
neighbouring
sites

How about quenches in quantum integrable models?

Have **local** integrals of motion $[I_m, I_n]=[I_m, H(h)]=0$.

→ $\langle \psi(t) | I_m | \psi(t) \rangle$ independent of time

Expectation: Time evolution of local operators/stationary state should be “special”.

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Stationary state:

M. Rigol, V. Dunjko, V. Yurosvki
& M. Olshanii, PRL **98**, 050405 (2007)

Generalized Gibbs Ensemble

RDM formulation of the Generalized Gibbs Ensemble

Let I_m be **local** integrals of motion $[I_m, I_n] = [I_m, H(h)] = 0$

Define GGE density matrix by:

$$\rho_{gG} = \exp(-\sum \lambda_m I_m) / Z_{gG}$$

λ_m fixed by

$$\text{tr}[\rho_{gG} I_m] = \langle \psi(0) | I_m | \psi(0) \rangle$$

Reduced density matrix of B:

$$\rho_{gG,B} = \text{tr}_A \rho_{gG}$$

The system is described by a GGE if for any finite subsystem B

$$\rho_B(\infty) = \rho_{gG,B}$$

cf Barthelemy &
Schollwöck '09

B. A new formalism for quenches in integrable models

B. A new formalism for quenches in integrable models

In integrable models we can construct a basis of eigenstates

$$H(h)|\Phi_n\rangle = E_n|\Phi_n\rangle$$

We then want to calculate

$$\langle\Psi(t)|\mathcal{O}|\Psi(t)\rangle = \sum_n \sum_m \langle\Psi(0)|\Phi_n\rangle \langle\Phi_n|\mathcal{O}|\Phi_m\rangle \langle\Phi_m|\Psi(0)\rangle e^{i(E_m - E_n)t}$$

This is difficult. To study thermodynamic limit we must sum infinitely many terms & in general must deal with late time (“infrared”) divergencies.

cf Calabrese, Essler & Fagotti '11, '12

Idea: follow Yang&Yang approach to thermodynamics

1. $|\Phi\rangle \rightarrow |\rho\rangle, \quad \sum_{\Phi}(\dots) \rightarrow \int \mathcal{D}[\rho]e^{S_{\rho}}(\dots).$

2. evaluate path integral by saddle point approximation

Caux&Essler '13

$$\langle \Psi(t) | \mathcal{O} | \Psi(t) \rangle = \frac{\langle \Psi(t) | \mathcal{O} | \Phi(t) \rangle}{2 \langle \Psi(t) | \Phi(t) \rangle} + \frac{\langle \Phi(t) | \mathcal{O} | \Psi(t) \rangle}{2 \langle \Phi(t) | \Psi(t) \rangle}$$

$|\Phi\rangle$ = simultaneous eigenstate of all local conservation laws

$$I_m |\Phi\rangle = \alpha_m |\Phi\rangle$$

such that

$$\langle \psi(0) | I_m | \psi(0) \rangle = \alpha_m$$

$|\Phi\rangle$ can be constructed e.g. by a **generalized TBA** once the expansion of $|\psi(0)\rangle$ in eigenstates of $H(h)$ is known.

Mossel &
Caux '12

Caux&Essler '13

$$\langle \Psi(t) | \mathcal{O} | \Psi(t) \rangle = \frac{\langle \Psi(t) | \mathcal{O} | \Phi(t) \rangle}{2\langle \Psi(t) | \Phi(t) \rangle} + \frac{\langle \Phi(t) | \mathcal{O} | \Psi(t) \rangle}{2\langle \Phi(t) | \Psi(t) \rangle}$$

Much more efficient way of calculating dynamics!

e.g.
$$\langle \Psi(t) | \mathcal{O} | \Phi(t) \rangle = \sum_m \langle \Psi(0) | \Phi_m \rangle \langle \Phi_m | \mathcal{O} | \Phi \rangle e^{-i(E_\Phi - E_m)t}$$

Late-time dynamics dominated by small “excitations” over $|\Phi\rangle$

$$E_m \approx E_\Phi$$

Have reproduced some known results for Ising using this formalism (rather non-trivial checks). Applications to sine-Gordon and Lieb-Liniger models under way.

talk by J.-S.
Caux

Bertini, Essler,
Schuricht

Description of the stationary state

GGE:

$$\lim_{t \rightarrow \infty} \text{tr}_A |\psi(t)\rangle \langle \psi(t)| = \text{tr}_A [\rho_{\text{GGE}}]$$

Our description:

$$\lim_{t \rightarrow \infty} \text{tr}_A |\psi(t)\rangle \langle \psi(t)| = \text{tr}_A |\Phi\rangle \langle \Phi|$$

So expectation values of local operators in the stationary state are given by using a **single simultaneous eigenstate of all conservation laws**.

- analogous result for Gibbs ensemble:

Goldstein, Lebowitz,
Tumulka, Zanghi '06

- Similar to “generalized microcanonical ensemble”

Cassidy, Clark
& Rigol '11

So **locally** (in space) $|\Phi\rangle$ looks the same as the GGE (and the “diagonal ensemble”). Globally they all differ.

C. Some new results on GGEs & integrable models

1. The GGE applies to dynamical correlation functions.

Essler, Evangelisti & Fagotti '12

More generally: if $\lim_{t \rightarrow \infty} \langle \Psi(t) | \mathcal{O}_1 \dots \mathcal{O}_n | \Psi(t) \rangle = \text{Tr} [\rho_{\text{stat}} \mathcal{O}_1 \dots \mathcal{O}_n]$

then $\lim_{t \rightarrow \infty} \langle \Psi(t) | \mathcal{O}_1(t_1) \dots \mathcal{O}_n(t_n) | \Psi(t) \rangle = \text{Tr} [\rho_{\text{stat}} \mathcal{O}_1(t_1) \dots \mathcal{O}_n(t_n)]$

for fixed t_1, t_2, \dots, t_n

ultimately follows from

Lieb&Robinson '72

Bravyi, Hastings& Verstraete '06

Explicit results for $\langle \Psi(t) | \sigma_j^\alpha(t_1) \sigma_n^\alpha(t_2) | \Psi(t) \rangle$ in Ising.

2. Some conservation laws are more important than others.



Transverse Field Ising Chain

Fagotti & Essler'13

Transverse Field Ising Chain

Hamiltonian:

$$H(h_0) = -J \sum_{j=1}^L \sigma_j^z \sigma_{j+1}^z + h_0 \sigma_j^x$$

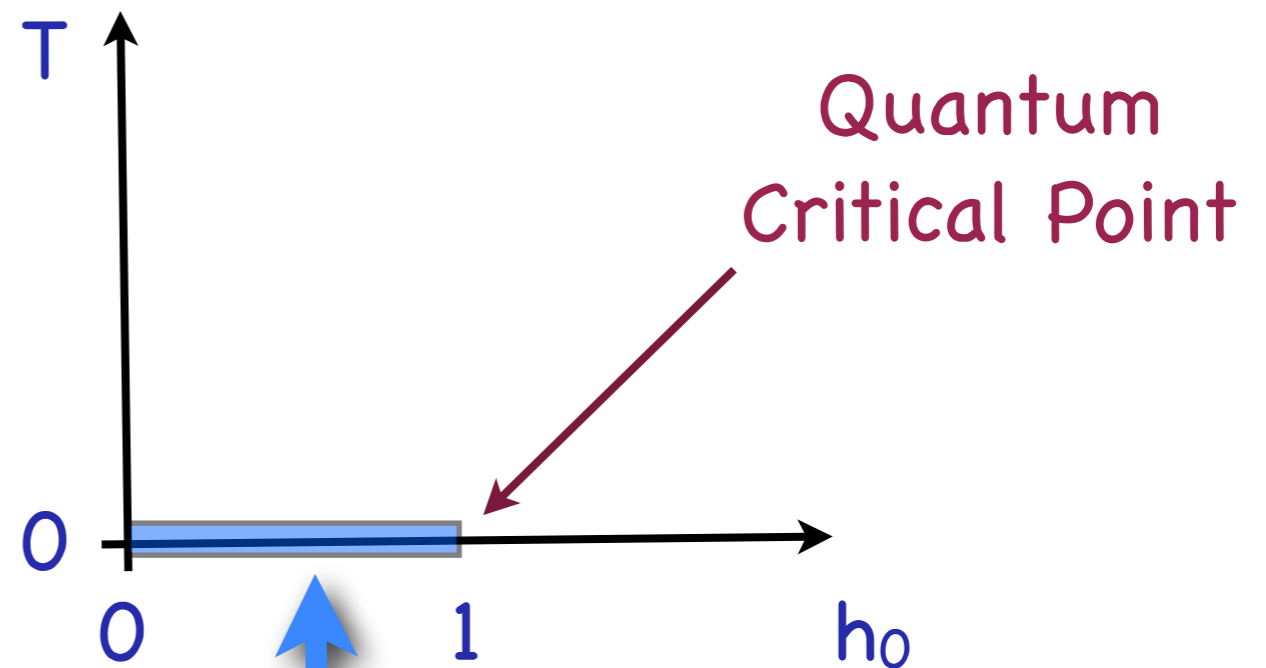
\mathbb{Z}_2 symmetry: rotation by π around x-axis.

$$\sigma_j^\alpha \rightarrow -\sigma_j^\alpha, \quad \alpha = y, z.$$

Phase Diagram:

order parameter: $\langle \sigma_j^z \rangle$

($\langle \sigma_j^x \rangle \neq 0$ always)



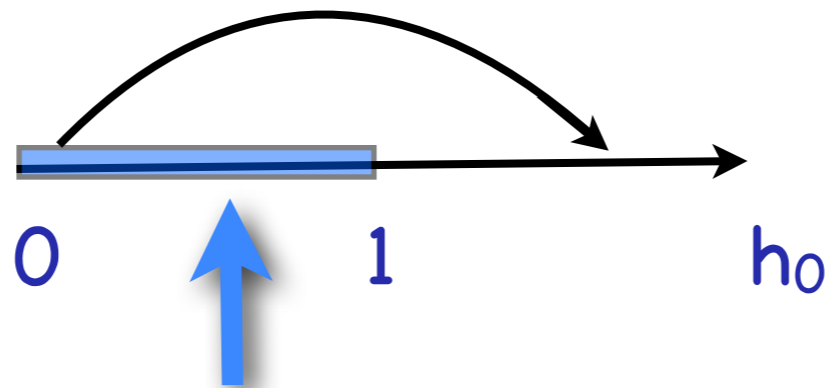
$T > 0$: order melts

Transverse Field Ising Chain

Hamiltonian:

$$H(h_0) = -J \sum_{j=1}^L \sigma_j^z \sigma_{j+1}^z + h_0 \sigma_j^x$$

quenches: $h_0 \rightarrow h$



$$\langle \sigma_j^z \rangle \neq 0$$

Barouch, McCoy & Dresden '70
Igloi & Rieger '00, '11
Rossini et al '09, '10
Calabrese, Essler & Fagotti '11, '12
Schuricht & Essler '12
Essler, Evangelisti & Fagotti '12
Foini, Cugliandolo & Gambassi '12
Heyl, Polkovnikov & Kehrein '12
Viehmann et al '13

How fast is the approach to the $t \rightarrow \infty$ limit?

- How close is $\rho_B(t)$ to $\rho_{gG,B}$?

Define a distance:

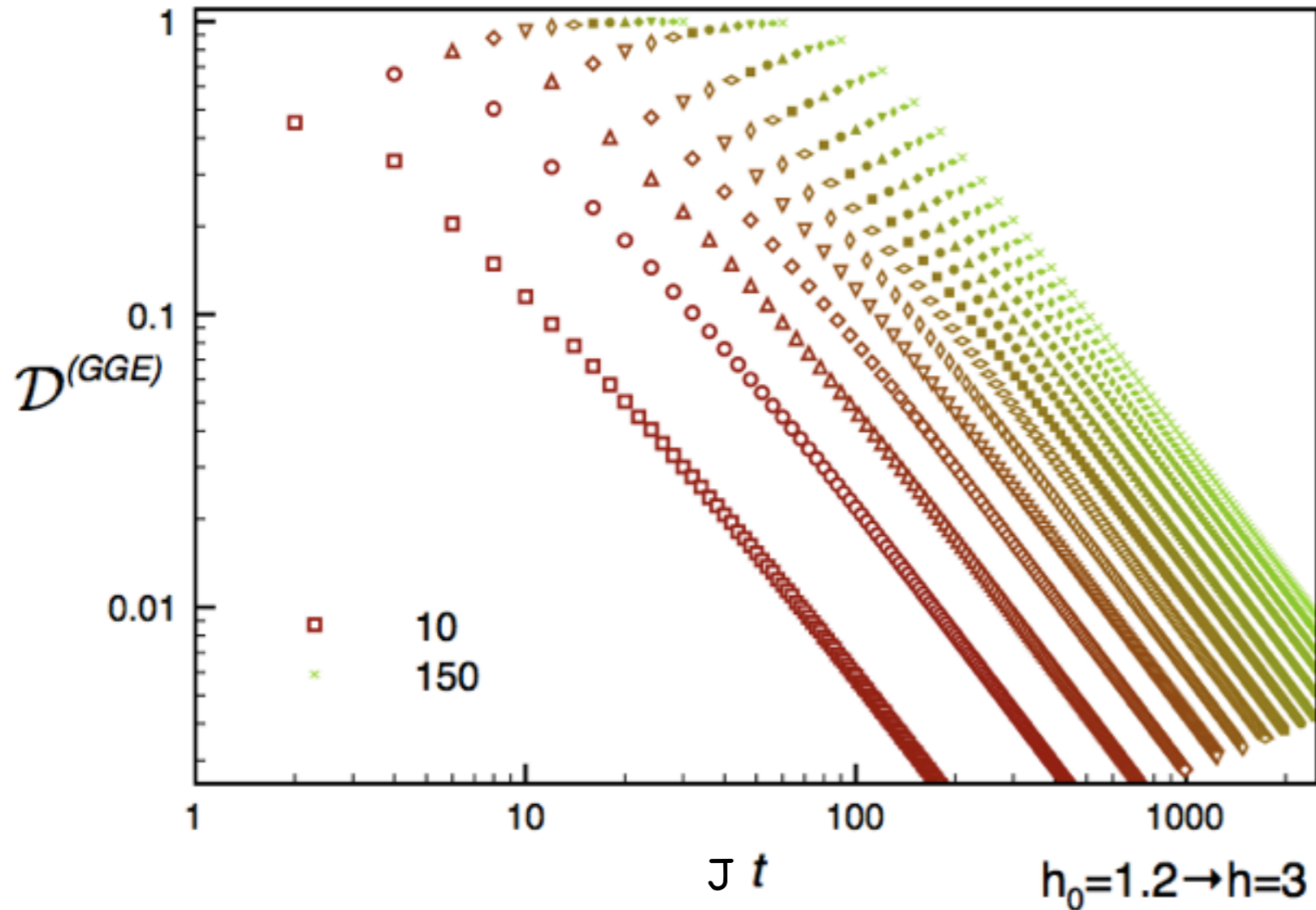
$$\mathcal{D}(\rho, \rho') \equiv \frac{\|\rho - \rho'\|_F}{\sqrt{\|\rho\|_F^2 + \|\rho'\|_F^2}}.$$

$$\|A\|_F \equiv \sqrt{\text{Tr}[A^\dagger A]}$$

- $h_0 > 1$ (Z_2 unbroken): Can reduce this to expression in terms of $2N \times 2N$ matrix (ρ_B is $2^N \times 2^N$ matrix).
- $h_0 < 1$ (Z_2 broken): ρ_B is **not** Gaussian \rightarrow use cluster decomposition + causality ("Calabrese-Cardy horizon")

Calabrese &
Cardy '05

Distance $\mathcal{D}^{(\text{GGE})} = \mathcal{D}(\rho_\ell(t), \rho_{\text{gG},\ell})$ between quench and generalized Gibbs reduced density matrices for sub-system sizes $\ell = 10, \dots, 150$

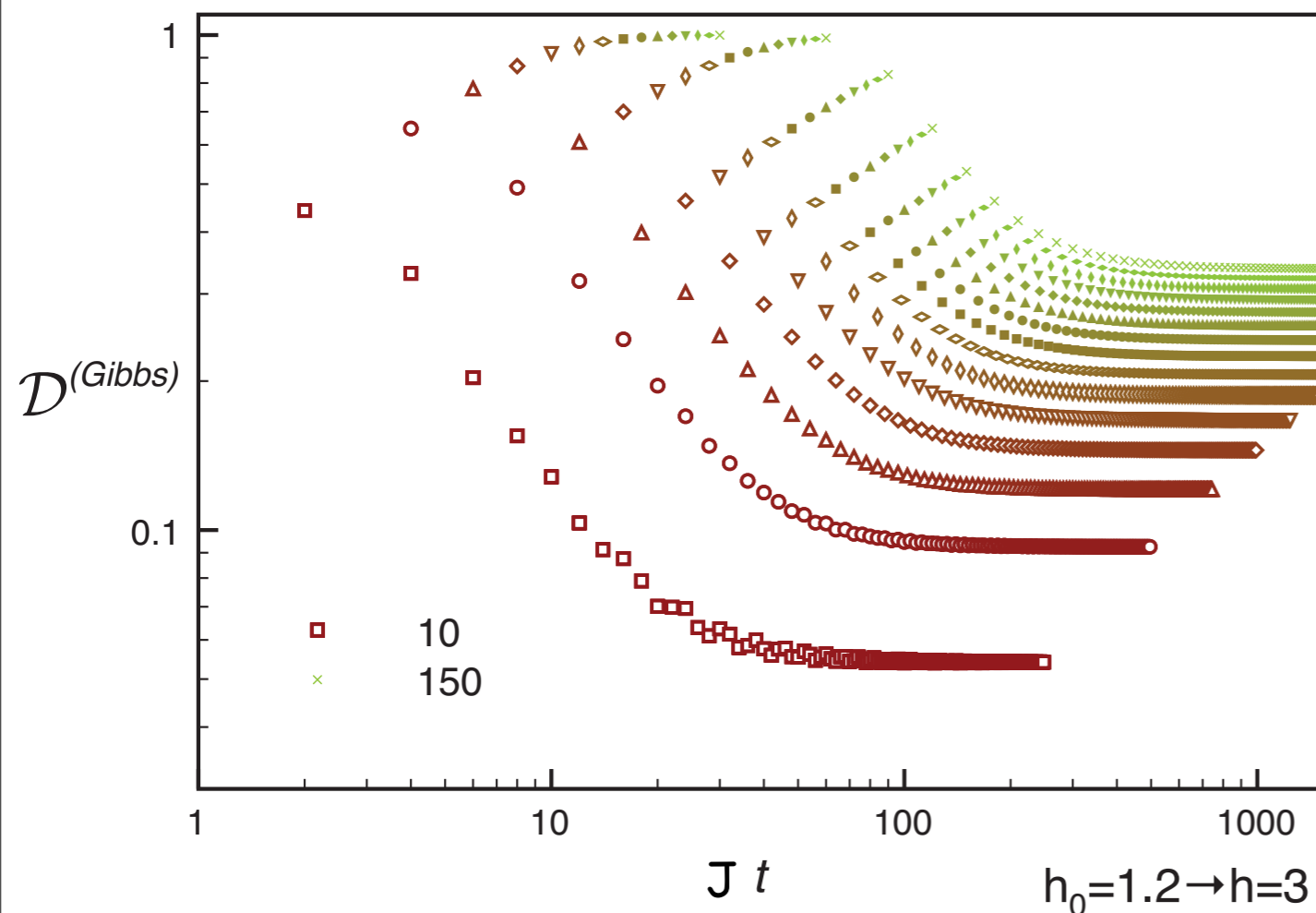


$\mathcal{D}^{(\text{GGE})} \propto t^{-3/2}$ at late times.

Holds for any quench $h_0 \rightarrow h$

Difference between GGE and thermalization (Gibbs)?

Distance to a Gibbs ensemble at the appropriate temperature



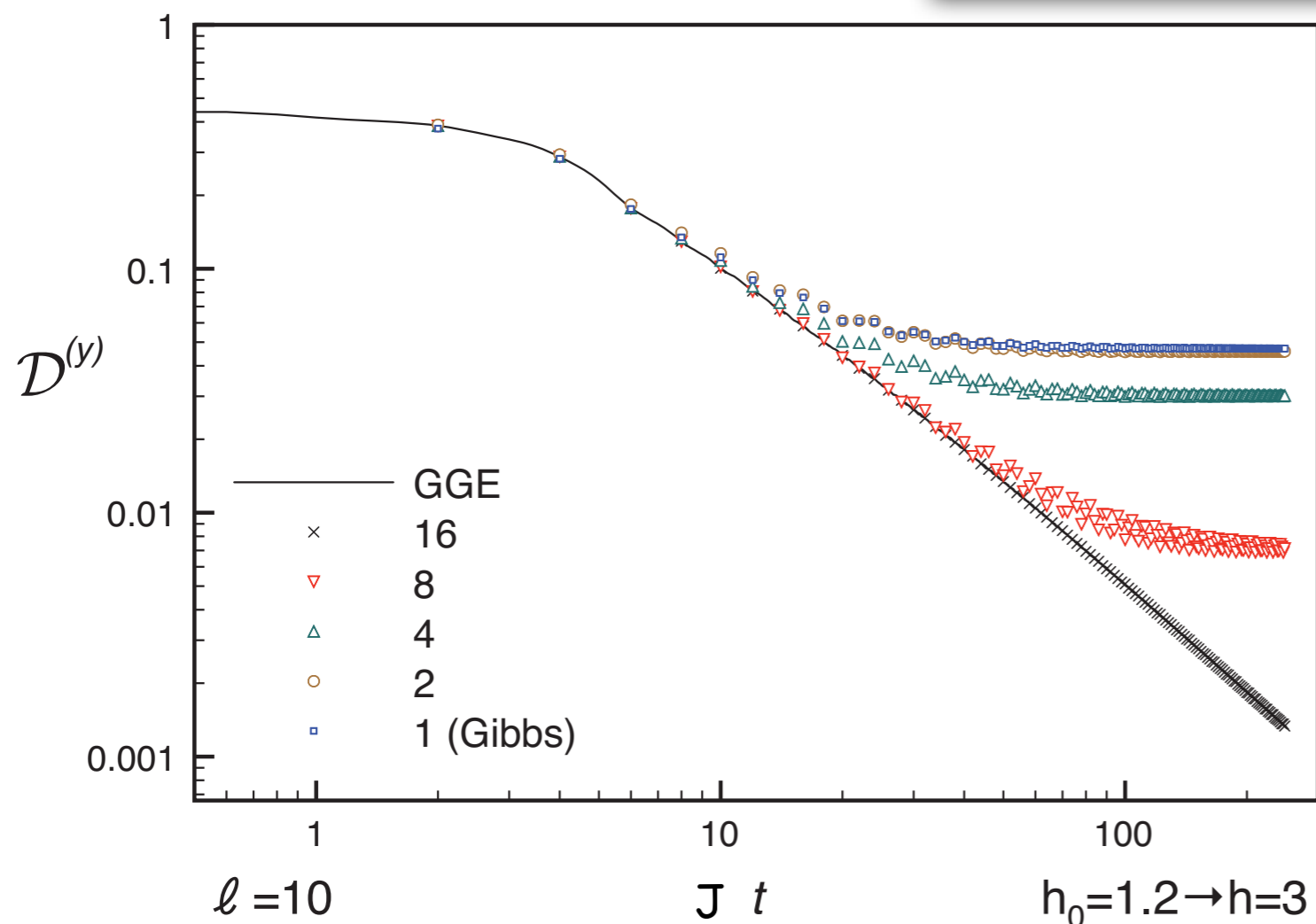
Distance between quench and Gibbs reduced density matrices for sub-system sizes 10–150.

$\mathcal{D}^{(Gibbs)} \propto \text{const}$ at late times.

Do we really need all conservation laws?

Define a truncated GGE by keeping only the y most local conservation laws:

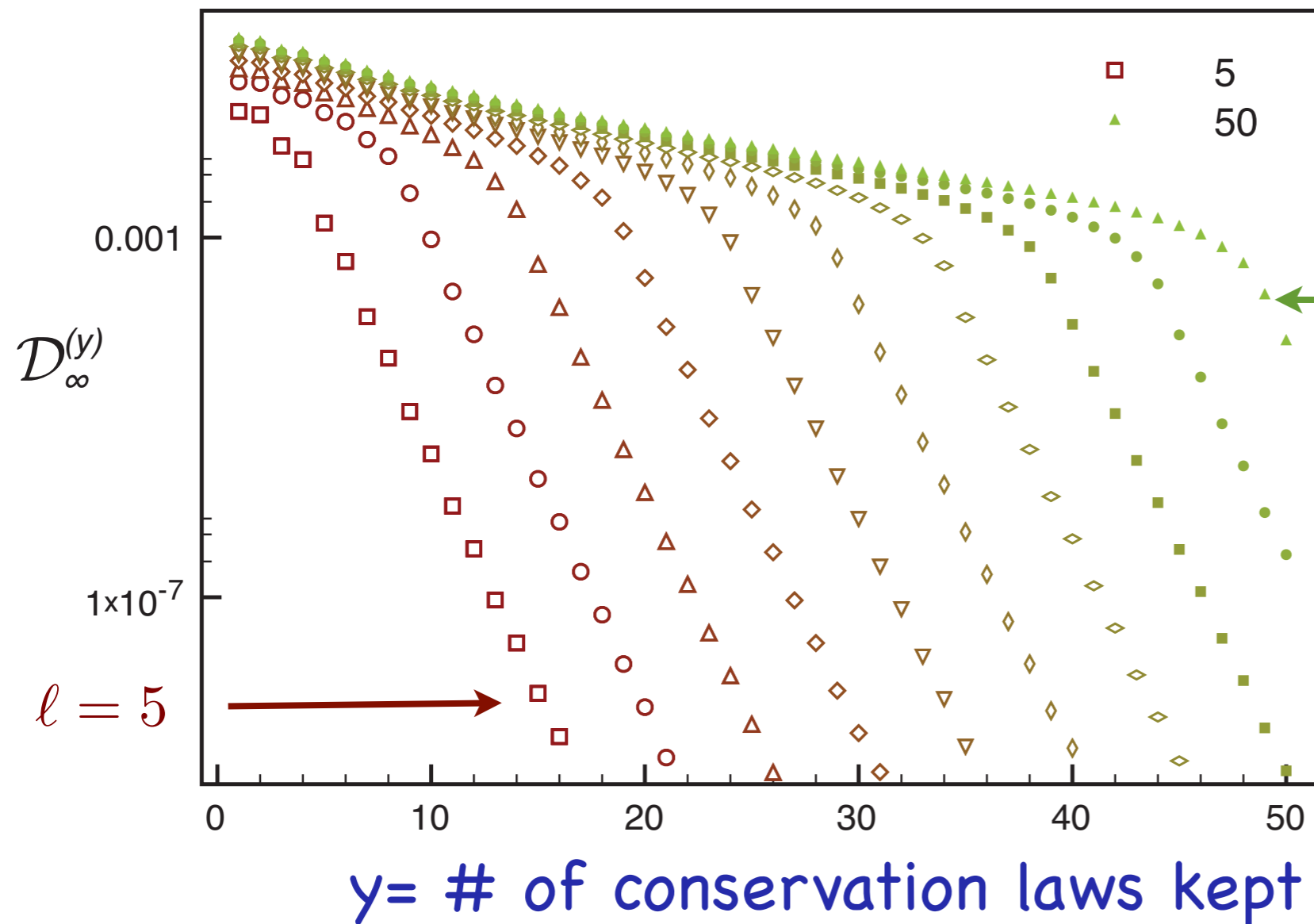
$$\rho_{\text{tGGE}}^{(y)} = \frac{1}{Z_y} \exp\left(\sum_{n=0}^{y-1} \sum_{\sigma=\pm} [\lambda_{n,y}^{\sigma} I_n^{\sigma}]\right).$$



Distance $\mathcal{D}^{(y)} = \mathcal{D}(\rho_{\ell}(t), \rho_{\text{tGGE},\ell}^{(y)})$ between quench and truncated GGE reduced density matrices for subsystem size $\ell=10$.

- Keeping more conservation laws gives a better description

Do we really need all conservation laws?



$$\mathcal{D}_{\infty}^{(y)} = \lim_{t \rightarrow \infty} \mathcal{D}(\rho_{\ell}(t), \rho_{t\text{GGE},\ell}^{(y)})$$

$\ell = 50$

$\ell = 5$

$$h_0 = 1.2 \rightarrow h = 3$$

$$5 \leq \ell \leq 50$$

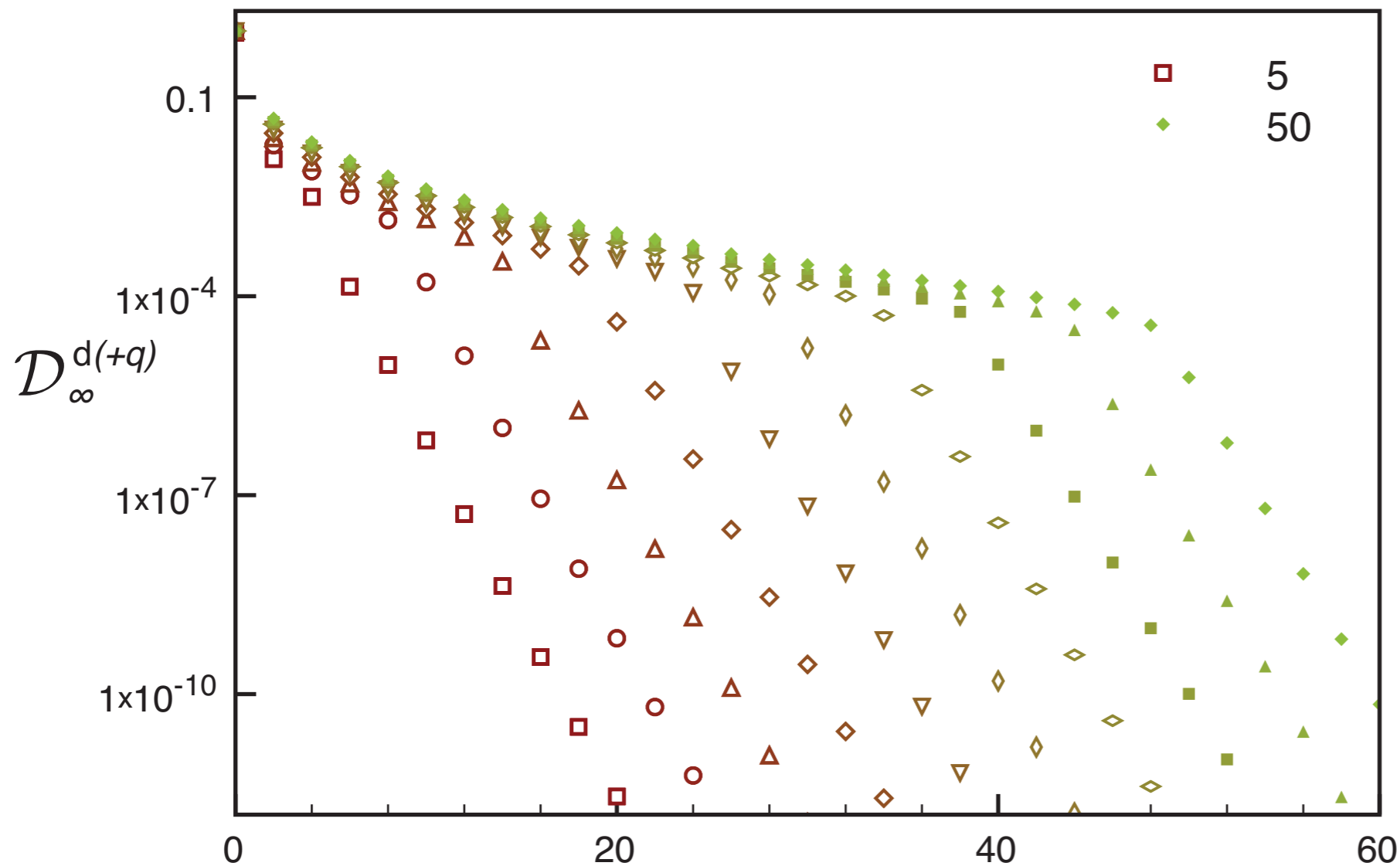
- Good description as soon as $y \geq \ell$!

Which conservation laws are most important?

Leave out the q 'th conservation law:

$$\rho_{\text{dGGE}}^{(+q)} = \frac{1}{\mathcal{Z}_{(+q)}} \exp\left(\sum_{n=0}^{\infty} \sum_{\sigma=\pm} [\lambda_{n,(+q)}^{\sigma} I_n^{\sigma}]\right).$$

$$\lim_{t \rightarrow \infty} \mathcal{D}(\rho_{\ell}(t), \rho_{\text{dGGE},\ell}^{(+q)}) \equiv \mathcal{D}_{\infty}^{\text{d}(+q)}.$$



$q =$ index of removed conservation law

$$5 \leq \ell \leq 50$$

$$h_0 = 1.2 \rightarrow h = 3$$

The more local the conservation law, the more important it is!!!

Which conservation laws are most important?

Conservation laws

$$I_m = \sum_j f_m^{\alpha_1 \dots \alpha_{m+1}} \sigma_j^{\alpha_1} \sigma_{j+1}^{\alpha_2} \dots \sigma_{j+m}^{\alpha_{m+1}}$$

Subsystem size ℓ : must keep all I_m with $m < \ell + n_0(h, h_0)$

3. GGE and quenches in the spin-1/2 Heisenberg chain.

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$$H^{(1)} = J \sum_j S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z - \frac{1}{4}, \quad \Delta > 1$$

Higher conservation laws are known; generated by transfer matrix

$$H^{(k)} = i \left(\frac{\sin \gamma}{\gamma} \frac{\partial}{\partial \lambda} \right)^k \log \tau(i + \lambda) \Big|_{\lambda=0}. \quad [H^{(k)}, H^{(n)}] = 0.$$

$$\tau(i + \lambda) = \text{Tr} [\mathcal{L}_L(\lambda) \mathcal{L}_{L-1}(\lambda) \dots \mathcal{L}_1(\lambda)],$$

$$\mathcal{L}_j(\lambda) = \frac{1 + \tau^z \sigma_j^z}{2} + \frac{\sinh(\frac{\gamma\lambda}{2})}{\sinh(i\gamma + \frac{\gamma\lambda}{2})} \frac{1 - \tau^z \sigma_j^z}{2} + \frac{\sinh(i\gamma)}{\sinh(i\gamma + \frac{\gamma\lambda}{2})} (\tau^+ \sigma_j^- + \tau^- \sigma_j^+),$$

GGE density matrix

$$\rho_{\text{GGE}} = \frac{1}{Z_{\text{GGE}}} \exp \left(- \sum_{l=1} \lambda_l H^{(l)} \right).$$

Can be viewed as **thermal** density matrix of integrable Hamiltonian

$$\mathcal{H} = \sum_{l=1} \frac{\lambda_l}{\lambda_1} H^{(l)}$$

Can use (Quantum Transfer Matrix) formalism developed for finite temperature correlators!

Boos, Göhmann,
Klümper et al '04-'10

Boos, Miwa, Jimbo,
Smirnov, Takeyama '06-'09

Two tasks remain:

1. Construct Quantum Transfer Matrix formalism for generalized Hamiltonian.

→ straightforward generalization of

$$\mathcal{H} = \sum_{l=1}^2 \frac{\lambda_l}{\lambda_1} H^{(l)}$$

Klümper and Sakai '02

2. Determine the λ_m

must solve

$$\lim_{L \rightarrow \infty} \frac{\langle \Psi_0 | H^{(l)} | \Psi_0 \rangle}{L} = \lim_{L \rightarrow \infty} \frac{\text{Tr} [\rho_{\text{GGE}} H^{(l)}]}{L}.$$

This is hard: cf Poszgay '13

Idea: evaluate generating function

$$\Omega_{\Psi_0}(\lambda) = \frac{1}{L} \langle \Psi_0 | \tau'(i + \lambda) \tau^{-1}(i + \lambda) | \Psi_0 \rangle = -i \sum_{k=1} \left(\frac{\gamma}{\sin \gamma} \right)^k \frac{\lambda^{k-1}}{(k-1)!} \frac{\langle \Psi_0 | H^{(k)} | \Psi_0 \rangle}{L},$$

using QISM



$$\Omega_{\Psi_0}(\lambda) \sim \frac{1}{L} \frac{\partial}{\partial x} \Big|_{x=\lambda} \text{Sp} \langle \Psi_0 | V_L(x, \lambda) \dots V_1(x, \lambda) | \Psi_0 \rangle, \quad [V_n(x, \lambda)]_{cd}^{ab} = [\mathcal{L}_n(x)]^{ab} [M_n(x)]^{cd}$$

$$M_j(\lambda) = \frac{1 + \tau^z \sigma_j^z}{2} + \frac{\sinh(\frac{\gamma^* \lambda}{2})}{\sinh(-i\gamma^* + \frac{\gamma^* \lambda}{2})} \frac{1 - \tau^z \sigma_j^z}{2} + \frac{\sinh(-i\gamma^*)}{\sinh(-i\gamma^* + \frac{\gamma^* \lambda}{2})} (\tau^+ \sigma_j^+ + \tau^- \sigma_j^-).$$

The point:

$$\Omega_{\Psi_0}(\lambda) \sim \frac{1}{L} \frac{\partial}{\partial x} \Big|_{x=\lambda} \text{Sp} \langle \Psi_0 | V_L(x, \lambda) \dots V_1(x, \lambda) | \Psi_0 \rangle ,$$

can be evaluated explicitly for matrix product states!!!

e.g. for product states we only need to diagonalize a 4x4 matrix.

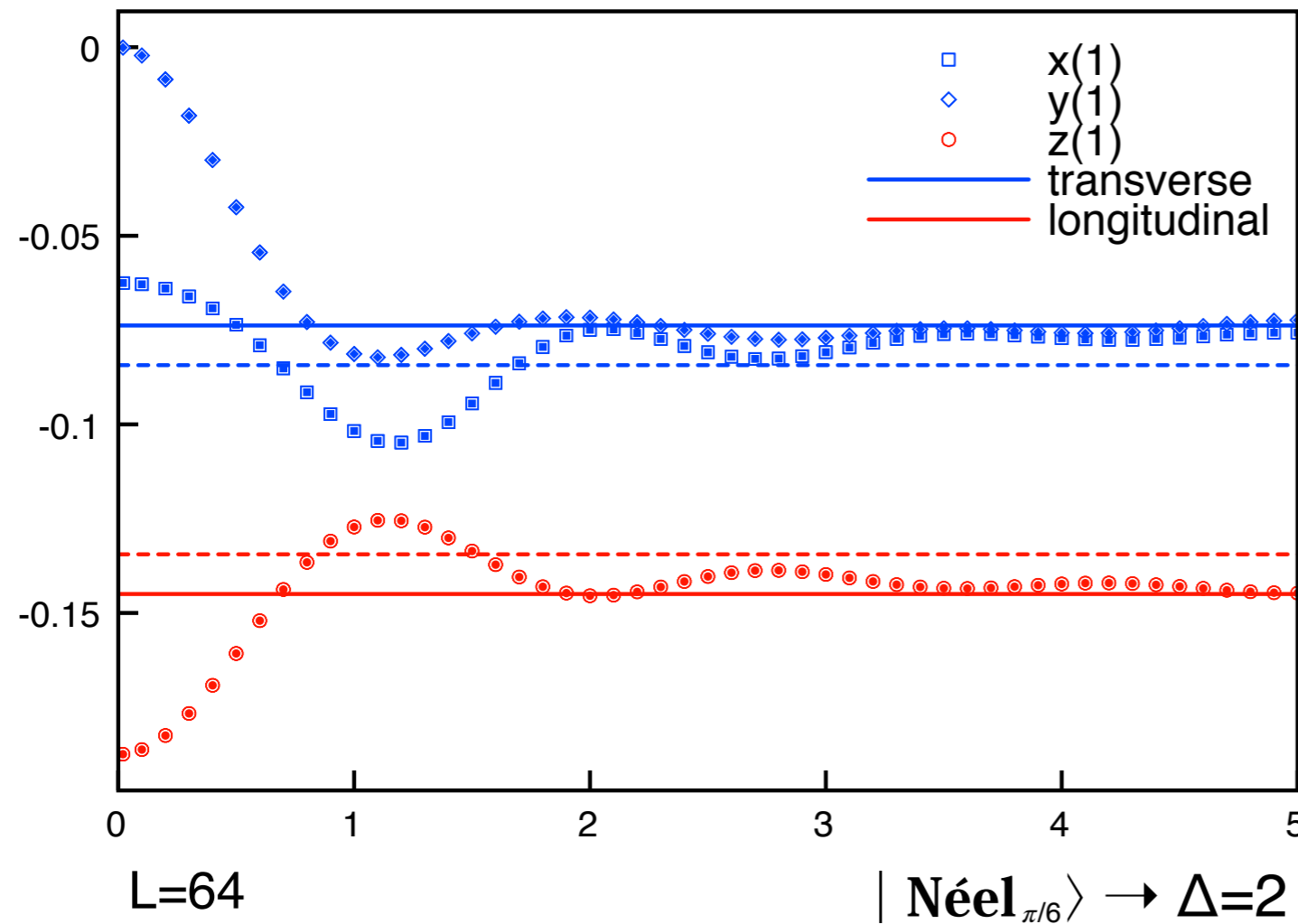
Results:

(short-distance) correlation function of spins in the GGE for quenches from a variety of initial states.

Interaction quenches from large Δ included!

Comparison to numerics (TDMRG):

Fagotti, Calabrese, Collura, Essler



$$\langle \Psi(t) | \sigma_j^\alpha \sigma_{j+1}^\alpha | \Psi(t) \rangle$$

GGE

thermal

initial state $|\text{Néel}_\alpha\rangle = e^{-i\alpha \sum_j \sigma_j^x} |\uparrow\downarrow\uparrow\downarrow \dots\rangle$

Conclusions

1. In the thermodynamic limit can describe **local physics** through a **single eigenstate** of all conservation laws.
2. Late-time dynamics given by small excitations around this state.
3. The GGE gives both **static** and **dynamic** correlators at stationarity.
4. The **most local** conservation laws are most important for describing the stationary state.
5. Have determined local correlators in stationary state of XXZ after quenching from certain initial states.

Physical interpretation of the distance

It measures the average mean relative difference of all expectation values of local operators:

$$D(\rho_1, \rho_2) = \left(\overline{(R(\mathcal{O}))^2} \right)^{1/2}$$

$$R(\mathcal{O}) \equiv \frac{|\langle \mathcal{O} \rangle_1 - \langle \mathcal{O} \rangle_2|}{\sqrt{\langle \mathcal{O} \rangle_1^2 + \langle \mathcal{O} \rangle_2^2}}$$

Average defined by:

$$\overline{f(\mathcal{O})} \equiv \sum_{\mathcal{O}} P(\mathcal{O}) f(\mathcal{O}) ,$$

$$P(\mathcal{O}) = \frac{\langle \mathcal{O} \rangle_1^2 + \langle \mathcal{O} \rangle_2^2}{\sum_{\mathcal{Q}} \langle \mathcal{Q} \rangle_1^2 + \langle \mathcal{Q} \rangle_2^2}$$