Free Parafermions

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Free fermions

- The fundamental system in theoretical physics
- Many properties can be computed exactly
- Keeps on keeping on
  e.g. topological classification, entanglement, quenches…
- Appear even in some non-obvious guises

For example, spin models sometimes can be mapped onto free-fermionic systems:

1d quantum transverse-field/2d classical Ising

Kauffman, Onsager; now known in its fermionic version as the ``Kitaev chain’’

1d quantum XY

Jordan-Wigner; Lieb-Schultz-Mattis

2d honeycomb model

Kitaev

Such models typically remain solvable even for spatially inhomogenous couplings.

Free fermions

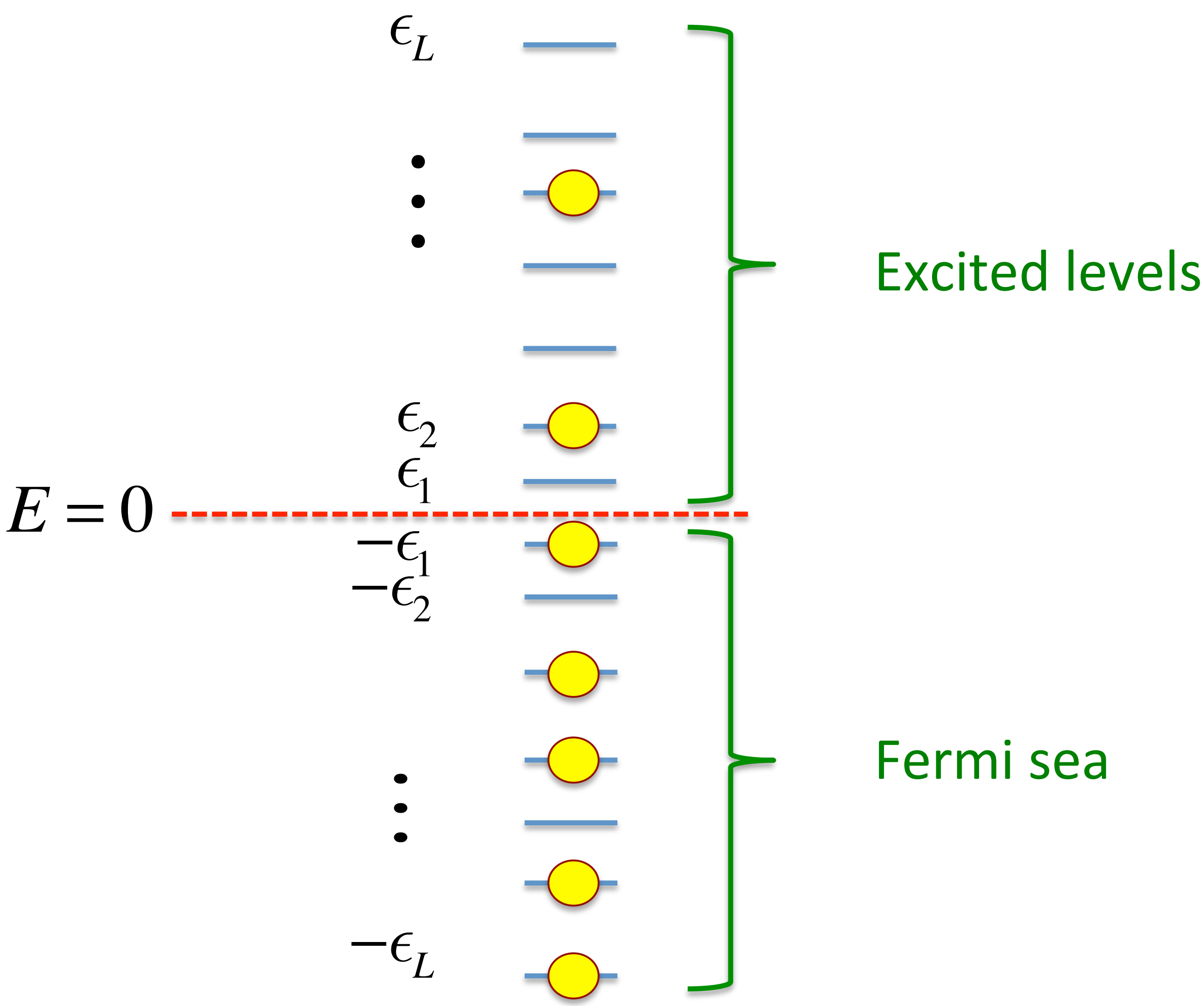
Forget statistics, forget operators, forget fields…the basic property of a free-fermion system is that the spectrum is

$$E = \pm\epsilon_1 \pm \epsilon_2 \pm \ \dots \ \pm \epsilon_L$$

Levels are either filled or empty.

The choice of a given $\pm$ is independent of the remaining choices, and does not effect the value of any ϵ_l.

$$E = \pm\epsilon_1 \pm \epsilon_2 \pm \ \ldots \ \pm \epsilon_L$$



Can this be generalized?

The free-fermion approach relies on a Clifford algebra.

Integrable models provide a generalization, but the algebraic structure (Yang-Baxter etc.) is much more complicated, and you work much harder for less.

Conformal field theory is also a generalization, but applies only to Lorentz-invariant critical models.

Typically a free-fermion model has a $\mathbb{Z}_2$ symmetry: $[(-1)^F, H] = 0$ where $(-1)^F$ counts the number of fermions mod 2. In Ising this is simply symmetry under flipping all spins.

So why isn't there a $\mathbb{Z}_n$ version?

- Fradkin and Kadanoff showed long ago that 1+1d clock models with $\mathbb{Z}_n$ symmetry can be written in terms of parafermions.

- Fateev and Zamolodchikov found integrable critical self-dual lattice spin models with $\mathbb{Z}_n$ symmetry. Later they found an elegant CFT description of the continuum limit.

- Read and Rezayi constructed fractional quantum Hall wavefunctions using the CFT parafermion correlators.

But these models are definitely not free. The lattice models are not even integrable unless critical and/or chiral.

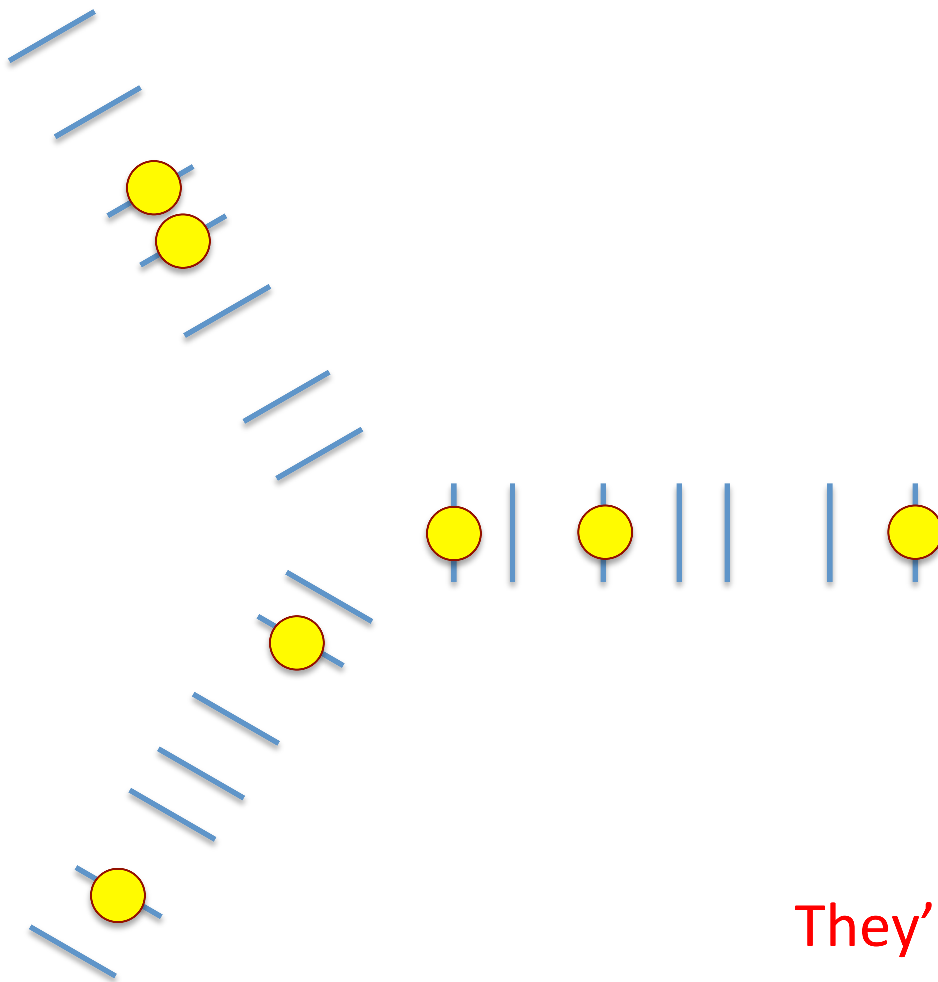
Nonetheless, Baxter found a non-Hermitian Hamiltonian with spectrum

$$E = \omega^{s_1}\epsilon_1 + \omega^{s_2}\epsilon_2 \pm \ \dots \ \pm \omega^{s_L}\epsilon_L \qquad \omega = e^{2\pi i/n}$$

$$s_j = 0,1,\dots n-1$$

A free parafermion sea?

For $\mathbb{Z}_3$:



They’re exclusons!

Baxter's proof is very indirect.

In particular, he asks:

For the Ising model this property follows from Kaufman's solution in terms of spinor operators [10], i.e. a Clifford algebra.[11, p.189] Whether there is some generalization of such spinor operators to handle the $\tau_2(t_q)$ model with open boundaries remains a fascinating speculation.[12]

The purpose of this talk is to display this structure, and so give a useful generalization of a Clifford algebra.

A key tool is the use of parafermions.

Useful for what?

- Parafermions play a nice role in the study of topological order, e.g. they give a way to prove existence of an edge zero mode.
- Baxter's Hamiltonian is related to the integrable chiral Potts model. Parafermions give an easy direct proof of the shift mode (the reason why people started studying the model).
- Non-hermitian Hamiltonians can arise as anisotropic limits of geometrical models (e.g. percolation, self-avoiding walks).
- The Clifford algebra plays a major role in mathematics (e.g. K-theory).
- Use as building blocks for Hermitian models.

Jordan-Wigner transformation to Majorana fermions:

The Hilbert space is a chain of two-state systems $(\mathbb{C}^2)^{\otimes L}$

The fermions are written in terms of strings of spin flips:

$$\psi_{2j-1} = \sigma_j^z \prod_{k=1}^{j-1} \sigma_k^x \qquad \psi_{2j} = i\sigma_j^x \psi_{2j-1}$$

String flips all spins behind site j

$$\{\psi_a, \psi_b\} = 2\delta_{ab}$$

The 1d Ising Hamiltonian is bilinear in fermions:

$$H = -\sum_{j=1}^{L} t_{2j-1}\,\sigma_j^x - \sum_{j=1}^{L-1} t_{2j}\,\sigma_j^z \sigma_{j+1}^z$$

$$= i \sum_{a=1}^{2L-1} t_a \psi_a \psi_{a+1}$$

These are open boundary conditions and arbitrary couplings t_a .

$\mathbb{Z}_2$ symmetry operator flips all spins:

$$(-1)^F = \prod_{j=1}^{L} \sigma_j^x = (-1)^L \prod_{a=1}^{2L} \psi_a$$

Solving the Ising chain in one slide

Let $\Psi = \sum_{a=1}^{2L} \mu_a \psi_a$ so that $[H, \Psi] = \Psi'$ with

$$\begin{pmatrix} \mu_1' \\ \mu_2' \\ \vdots \\ \mu_{2L}' \end{pmatrix} = 2i \begin{pmatrix} 0 & t_1 & 0 & \dots & & \\ -t_1 & 0 & t_2 & & & \\ 0 & -t_2 & 0 & & & \\ \vdots & & & & & t_{2L-1} \\ & & & & -t_{2L-1} & 0 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_{2L} \end{pmatrix}$$

because commuting bilinears in the fermions with linears gives linears.

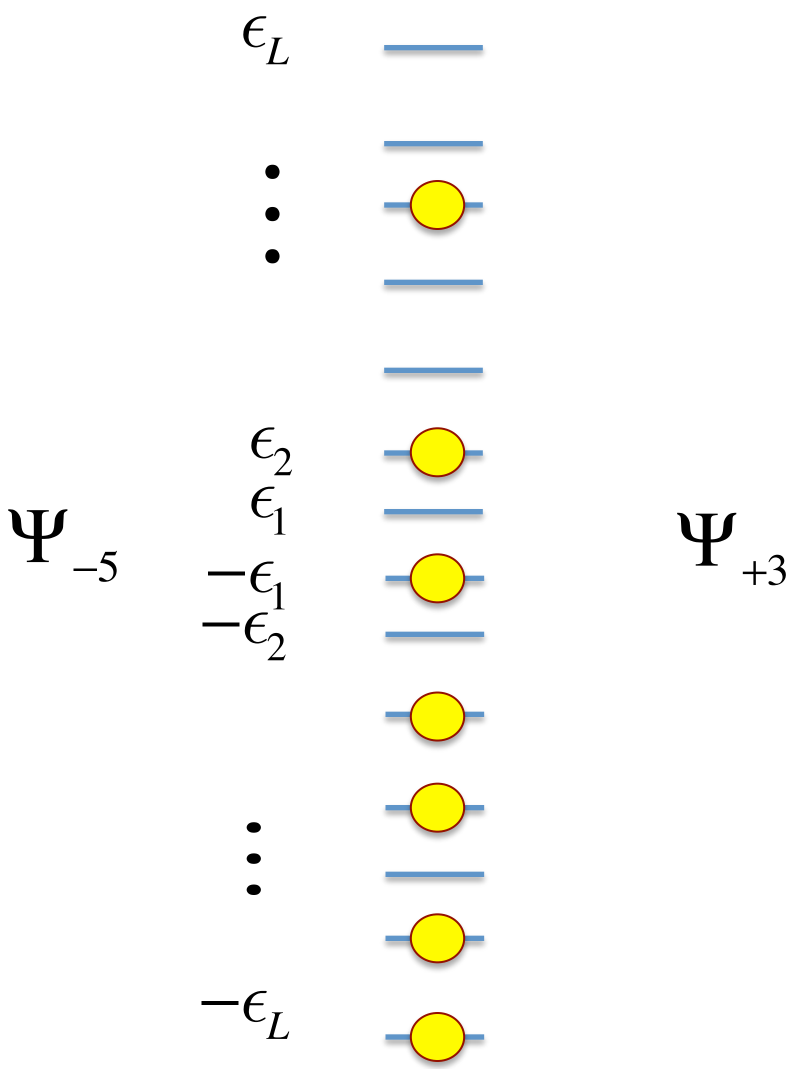
Diagonalizing this matrix gives $[H, \Psi_{\pm k}] = \pm 2\epsilon_k \Psi_k$

2L raising/lowering operators obey the Clifford algebra $\{\Psi_k, \Psi_l\} = 2\delta_{k,-l}$

Because $(\Psi_k + \Psi_{-k})^2 = 2$ no state is annihilated by both. Consistency requires

$$E = \pm\epsilon_1 \pm \epsilon_2 \pm \ \dots \ \pm \epsilon_L$$

$$[H, \Psi_{\pm k}] = \pm 2\epsilon_k \Psi_k$$



On to n-state models

For 3 states, i.e. a Hilbert space of $(\mathbb{C}^3)^{\otimes L}$:

flip is now "shift" ↓

Spin up or down is now "clock" ↓

$$\tau = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{-2\pi i/3} \end{pmatrix}$$

$$\tau^3 = \sigma^3 = 1, \quad \tau^2 = \tau^\dagger, \quad \sigma^2 = \sigma^\dagger$$

$$\boxed{\tau\sigma = e^{2\pi i/3}\sigma\tau}$$

Parafermions from the Fradkin-Kadanoff transformation:

In a 2d classical theory, they're the product of order and disorder operators. In the quantum chain with $\omega = e^{2\pi i/n}$

$$\psi_{2j-1} = \sigma_j \prod_{k=1}^{j-1} \tau_k \qquad \psi_{2j} = \omega^{(n-1)/2}\ \tau_j \psi_{2j-1}$$

$$\psi_a^n = 1, \qquad \psi_a^{n-1} = \psi_a^\dagger$$

Instead of anticommutators:

$$\psi_a \psi_b = \omega \psi_b \psi_a$$

for $a < b$

Baxter's Hamiltonian

shift — clock interaction

$$H = -\sum_{j=1}^{L} t_{2j-1}\,\tau_j - \sum_{j=1}^{L-1} t_{2j}\,\sigma_j^{\dagger}\sigma_{j+1}$$

I did not forget the h.c. – the Hamiltonian is not Hermitian.

This is the Hamiltonian limit of the $\tau_2(t_q)$ model, whose transfer matrix commutes with that of the integrable chiral Potts model

Bazhanov and Stroganov

Non-local commuting currents

$$[J^{(j)}, J^{(k)}] = 0$$

Let $h_a \equiv t_a \psi_a^2 \psi_{a+1}$

so $$J^{(1)} = H = \sum_{a=1}^{2L-1} t_a \psi_a^2 \psi_{a+1} = \sum_{a=1}^{2L-1} h_a$$

then $$J^{(2)} = \sum_{a=1}^{2L-3} \sum_{b=a+2}^{2L-1} h_a h_b$$

$$J^{(3)} = \sum_{a=1}^{2L-5} \sum_{b=a+2}^{2L-3} \sum_{c=b+2}^{2L-1} h_a h_b h_c$$ etc.

Note ``exclusion'' rule! Only one h_a for every 2 adjacent sites

``Higher'' commuting Hamiltonians

$$[H^{(j)}, H^{(k)}] = 0$$

Let $\quad T = 1 - J^{(1)}u + J^{(2)}u^2 - J^{(3)}u^3 + \ldots + J^{(L)}u^L$

The generating function for the local higher Hamiltonians is

$$-u\frac{d}{du}\ln T = -u\frac{T'}{T} = H\,u + H^{(2)}u^2 + H^{(3)}u^3 + \ldots$$

so e.g. $\quad H^{(2)} = \sum_{a=1}^{2L-1} h_a^2 + (1+\omega)\sum_{a=1}^{2L-2} h_{a+1}h_a$

These are indeed local:

$$H^{(2)} = \sum_{a=1}^{2L-1} h_a^2 \; + \; (1+\omega) \sum_{a=1}^{2L-2} h_{a+1} h_a$$

$$H^{(3)} = \sum_{a=1}^{2L-1} h_a^3 \; + \; (1+\omega+\omega^2)\Big(\sum_{a=1}^{2L-2} (h_{a+1}^2 h_a + h_{a+1} h_a^2) \; + \; \sum_{a=1}^{2L-3} h_{a+2} h_{a+1} h_a \Big)$$

Another exclusion rule: at most *n-1* h_a allowed on 2 adjacent sites.

For CFT aficianados: parafermion correlators obey analogous clustering properties.

Read-Rezayi

To find the energies and generalized Clifford algebra, we need the raising/lowering operators.

What worked so well for the fermions doesn't seem to work here:

Not linear in the parafermions!

For example: $$[H, \psi_1] \propto \psi_2$$

$$[H, \psi_2] \sim \psi_3 + \psi_1^{n-1} \psi_2^2$$

Ignoring constants

It starts to look nasty very quickly.

But staring at this long enough, a pattern emerges.

For 3 states, find the same exclusion rule: only terms are of the form

$$h_{b_1} h_{b_2} \ldots h_{b_l} \psi_a$$

with

$$b_l \leq a$$

$$|b_i - b_j| = 2, 3, \ldots$$

For *n* states, only *n-2* adjacent h_{b_i} .

So repeatedly commuting with H doesn't generate all n^{2L} operators.

In fact…

Let $$v_0 \equiv \psi_1$$

$$v_1 \equiv [H, v_0]$$

$$v_2 \equiv [H, v_1]$$

$$\vdots$$

$$v_{nj} \equiv [H, v_{nj-1}] - t^n_{2j-1} v_{nj-n} (\bar{\omega} - 1)^3$$

$$v_{nj+1} \equiv [H, v_{nj}] - t^n_{2j} v_{nj-2} (\bar{\omega} - 1)^3$$ j=1,2,3...

$$v_{nj+k} \equiv [H, v_{nj+k-1}]$$ k=2,3...,n-1

Then $$v_{nL} = 0$$

I've proved this by brute force for n=3, and it will be easy to generalize for low values of n by doing a bit of combinatorics.

To come up with a more elegant general proof, two key conjectures:

$$[H^{(m)}, v_0] = \frac{1-\omega^m}{(1-\omega)^m}\left[H,\left[H,\left[H,\ldots,\left[H,v_0\right]\ldots\right]\right]\right]$$

Because $h^n \propto 1$, a truncation : $H^{(n)} \propto 1$

Linear combinations of the v_a are analogous to linear linear combinations of the fermions: commuting with *H* gives a closed set of operators:

$$\Psi = \sum_{a=0}^{3L-1} \mu_a v_a \qquad [H, \Psi] = \Psi' \qquad \text{with}$$

$$\begin{pmatrix} \mu_1' \\ \mu_2' \\ \\ \vdots \\ \\ \mu_{3L}' \end{pmatrix} = \begin{pmatrix} 0 & t_1^3 & 0 & 0 & 0 & 0 & \dots & & & \\ 0 & 0 & t_2^3 & 0 & 0 & 0 & \dots & & & \\ 1 & 0 & 0 & 0 & 0 & 0 & & & & \\ 0 & 1 & 0 & 0 & t_3^3 & 0 & & & & \\ 0 & 0 & 1 & 0 & 0 & t_4^3 & & & & \\ \vdots & & & & & & & & & \\ & & & & & & & & 0 & t_{2L-1}^3 \\ & & & & & & & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \\ \vdots \\ \\ \mu_{3L} \end{pmatrix}$$

for 3 states

Diagonalize this to give ``rotating'' operators

$$\Psi_{1,k}, \Psi_{\omega,k}, \; \ldots \;, \Psi_{\bar{\omega},k}$$

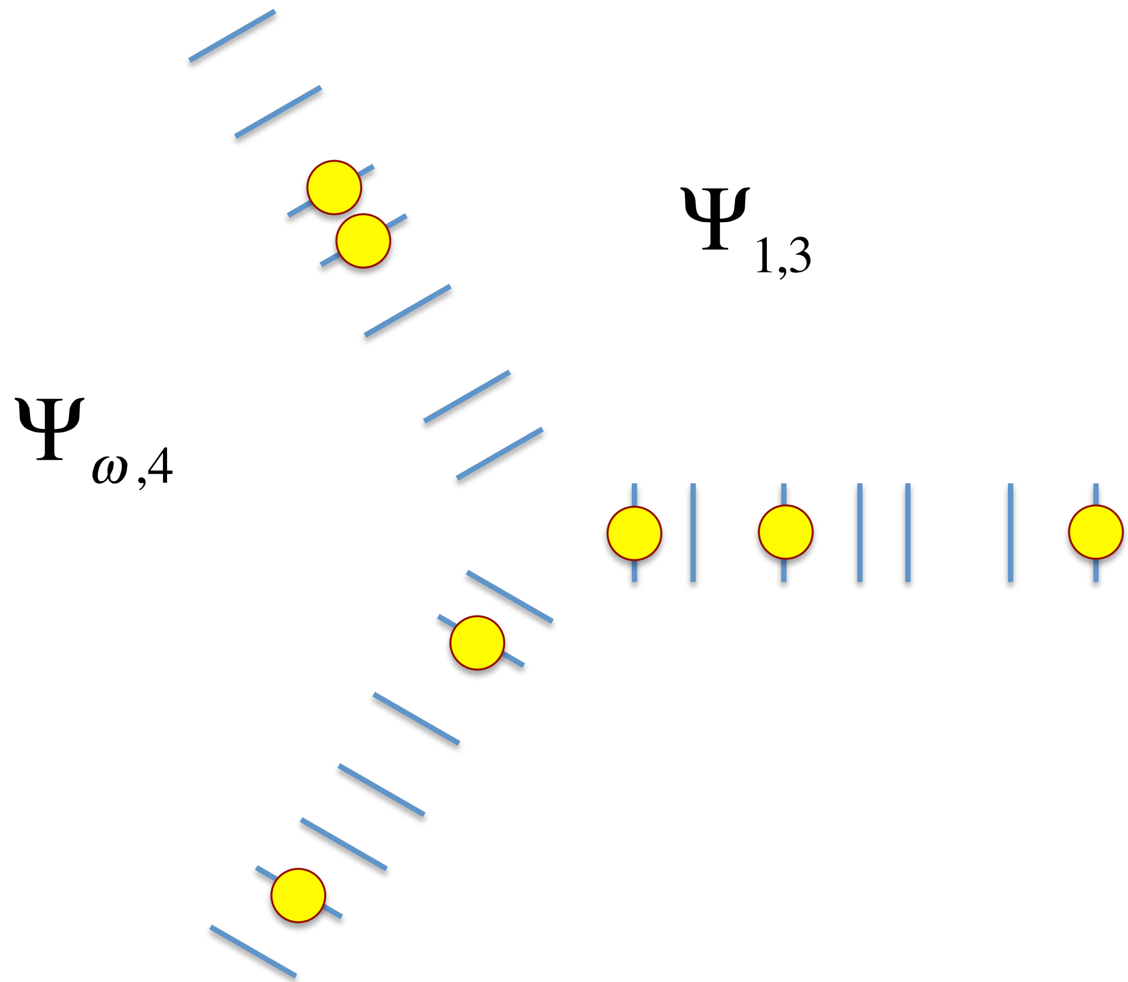
obeying $$[H, \Psi_{\omega^s,k}] = (\omega^{s+1} - \omega^s)\epsilon_k \Psi_{\omega^s,k}$$

The ϵ_k are positive and real: they are the positive eigenvalues of

$$\begin{pmatrix} 0 & t_1^{n/2} & 0 & \ldots & & \\ t_1^{n/2} & 0 & t_2^{n/2} & & & \\ 0 & t_2^{n/2} & 0 & & & \\ \vdots & & & & & t_{2L-1}^{n/2} \\ & & & & t_{2L-1}^{n/2} & 0 \end{pmatrix}$$

which is Baxter's result.

$$[H, \Psi_{\omega^s, k}] = (\omega^{s+1} - \omega^s)\epsilon_k \Psi_{\omega^s, k}$$



These rotating operators satisfy the generalized Clifford algebra

$$\Psi^2_{\omega^s,k} = 0 \qquad \Psi_{\omega^{s-1},k}\Psi_{\omega^s,k} = 0$$

$$\Psi_{\omega^s,k}\Psi_{\omega^{s'},k'} \propto \Psi_{\omega^{s'},k'}\Psi_{\omega^s,k} \qquad k \neq k'$$

$$(\Psi_{1,k} + \Psi_{\omega,k} + \Psi_{\bar{\omega},k})^3 \propto 1 \qquad \text{for 3 states}$$

Conjecture that first three can be subsumed in

$$(\epsilon_{k'}\omega^{s'} - \epsilon_k\omega^s)\Psi_{\omega^s,k}\Psi_{\omega^{s'},k'} = (\epsilon_k\omega^s - \epsilon_{k'}\omega^{s'+1})\Psi_{\omega^{s'},k'}\Psi_{\omega^s,k}$$

This algebra is independent of the Hamiltonian we used – these operators can be defined for any $\mathbb{Z}_n$-invariant spin system.

Future directions

- Take copies and fill pairs of levels to make a parafermion sea with real energy?
- Is this the chiral part of a CFT?
- Zero modes! Topological order!
  Mong, Clarke, Lindner, Alicea, Fendley, Nayak, Oreg, Stern, Berg, Shtengel, Fisher
- Solvable models of interacting parafermions?
- Redo for 2d classical models, any interesting geometric problems?
  Does the Pfaffian generalize to a Read-Rezayian?
- Use to build a (presumably gapless) 2d wavefunction?
- Closed boundary conditions? Full Bazhanov-Strogaonov model?
- The Clifford algebra plays a major role in mathematics and e.g. in the classification of topological systems. Is there a $\mathbb{Z}_n$ version?