Gibbs-non-Gibbs dynamical transitions. A large-deviation paradigm

R. Fernández — Utrecht
F. den Hollander — Leiden
J. Martínez — Leiden

Kyoto, July 2013

The Gibbs – non-Gibbs saga

Intuitively, μ Gibbs if

$$\mu \propto e^{-\beta H}$$

This is, however, valid only on finite regions

To pass to the thermodynamic limit must introduce:

- *Interactions*
- *Specifications*

Gibbs measures designed to describe *equilibrium*
No reason to expect them out of equilibrium, e.g. under evolutions

The Gibbs – non-Gibbs saga

Intuitively, μ Gibbs if

$$\mu \propto e^{-\beta H}$$

This is, however, valid only on finite regions

To pass to the thermodynamic limit must introduce:

- *Interactions*
- *Specifications*

Gibbs measures designed to describe *equilibrium*

No reason to expect them out of equilibrium, e.g. under evolutions

The Gibbs – non-Gibbs saga

Intuitively, μ Gibbs if

$$\mu \propto e^{-\beta H}$$

This is, however, valid only on finite regions

To pass to the thermodynamic limit must introduce:

- *Interactions*
- *Specifications*

Gibbs measures designed to describe *equilibrium*
No reason to expect them out of equilibrium, e.g. under evolutions

Examples of non-Gibsianness

Renormalization transformations

- Block-renormalization: blocks of spins $\to$ effective spins
- Renormalized measure: coarser, blurred
- In many instances: renormalized measure non-Gibbsian
- Reason: hidden variables bringing info from infinite

Stochastic evolutions

Gibbs measures subjected to Glauber dynamics

- Can loose Gibsianness at some finite time
- Gibbsianness recovered in some cases

Examples of non-Gibsianness

Renormalization transformations

- Block-renormalization: blocks of spins $\to$ effective spins
- Renormalized measure: coarser, blurred
- In many instances: renormalized measure non-Gibbsian
- Reason: hidden variables bringing info from infinite

Stochastic evolutions

Gibbs measures subjected to Glauber dynamics

- Can loose Gibsianness at some finite time
- Gibbsianness recovered in some cases

Examples of non-Gibsianness

Renormalization transformations

- Block-renormalization: blocks of spins $\to$ effective spins
- Renormalized measure: coarser, blurred
- In many instances: renormalized measure non-Gibbsian
- Reason: hidden variables bringing info from infinite

Stochastic evolutions

Gibbs measures subjected to Glauber dynamics

- Can loose Gibsianness at some finite time
- Gibbsianness recovered in some cases

Dynamic non-Gibbsianness

Original explanation

- Two-slice system: past acts as hidden variables for present
- Two-slice system $\sim$ equilibrium duplicated variables

Alternative paradigm

- Intuitively: most probable history of an improbable state
- Formally: large deviations in trajectory space
- Non-Gibbs = multiple optimal trajectories $\rightarrow$ discontinuity

Dynamic non-Gibbsianness

Original explanation

- Two-slice system: past acts as hidden variables for present
- Two-slice system $\sim$ equilibrium duplicated variables

Alternative paradigm

- Intuitively: most probable history of an improbable state
- Formally: large deviations in trajectory space
- Non-Gibbs = multiple optimal trajectories $\rightarrow$ discontinuity

Dynamic non-Gibbsianness

Original explanation

- Two-slice system: past acts as hidden variables for present
- Two-slice system $\sim$ equilibrium duplicated variables

Alternative paradigm

- Intuitively: most probable history of an improbable state
- Formally: large deviations in trajectory space
- Non-Gibbs = multiple optimal trajectories $\rightarrow$ discontinuity

Plan and credits

In this talk:

- Review of Gibbsianness
- Review of original proof of dynamical non-Gibbsianness
- New paradigm for dynamical non-Gibbsianness
- Rigorous results for
  - Mean-field spin models
  - Kac models

Acknowledgemens:

Aernout van Enter (Groningen)
Franck Redig (Delft)
Christoff Külske (Bochum)
Victor Ermolaev (Nedap)

Plan and credits

In this talk:

- Review of Gibbsianness
- Review of original proof of dynamical non-Gibbsianness
- New paradigm for dynamical non-Gibbsianness
- Rigorous results for
  - Mean-field spin models
  - Kac models

Acknowledgemens:

Aernout van Enter (Groningen)
Franck Redig (Delft)
Christoff Külske (Bochum)
Victor Ermolaev (Nedap)

Plan and credits

In this talk:

- Review of Gibbsianness
- Review of original proof of dynamical non-Gibbsianness
- New paradigm for dynamical non-Gibbsianness
- Rigorous results for
  - Mean-field spin models
  - Kac models

Acknowledgemens:

Aernout van Enter (Groningen)
Franck Redig (Delft)
Christoff Külske (Bochum)
Victor Ermolaev (Nedap)

Lattice systems

Basic ingredients:

- *Lattice* $\mathbb{L}$: e.g. $\mathbb{Z}^d$
- *Single-spin space* S: e.g. $\{-1, 1\}$
- *Configuration space* $\Omega = S^{\mathbb{L}}$

  Topology and σ-algebra $\mathcal{F}$ generated by cylinders:

$$C_{\omega_\Lambda} = \left\{\omega \in \Omega : \omega_\Lambda = \sigma_\Lambda\right\} , \ \Lambda \subset\subset \mathbb{L} \quad \left[\omega_\Lambda = (\omega_x)_{x\in\Lambda}\right]$$

Interaction: Family of local functions (=local contributions)

$$\Phi = \left\{\phi_B : \Omega \to \mathbb{R} \ , \ \mathcal{F}_B\text{−measurable}\right\} \quad \left[\phi_B(\omega) = \phi_B(\sigma) \text{ if } \omega_B = \sigma_B\right]$$

Hamiltonian on region Λ given σ outside:

$$H_\Lambda(\omega \mid \sigma) = \sum_{B : B\cap\Lambda\neq\emptyset} \phi_B(\omega_\Lambda\sigma)$$

Lattice systems

Basic ingredients:

- *Lattice* $\mathbb{L}$: e.g. $\mathbb{Z}^d$
- *Single-spin space* S: e.g. $\{-1,1\}$
- *Configuration space* $\Omega = S^{\mathbb{L}}$
  Topology and σ-algebra $\mathcal{F}$ generated by cylinders:

$$C_{\omega_\Lambda} = \left\{\omega \in \Omega : \omega_\Lambda = \sigma_\Lambda\right\}, \ \Lambda \subset\subset \mathbb{L} \quad \left[\omega_\Lambda = (\omega_x)_{x\in\Lambda}\right]$$

Interaction: Family of local functions (=local contributions)

$$\Phi = \left\{\phi_B : \Omega \to \mathbb{R}\ ,\ \mathcal{F}_B\text{−measurable}\right\} \quad \left[\phi_B(\omega) = \phi_B(\sigma) \text{ if } \omega_B = \sigma_B\right]$$

Hamiltonian on region Λ given σ outside:

$$H_\Lambda(\omega \mid \sigma) = \sum_{B: B\cap\Lambda\neq\emptyset} \phi_B(\omega_\Lambda\sigma)$$

Lattice systems

Basic ingredients:

- *Lattice* $\mathbb{L}$: e.g. $\mathbb{Z}^d$
- *Single-spin space* S: e.g. $\{-1,1\}$
- *Configuration space* $\Omega = S^{\mathbb{L}}$
  Topology and σ-algebra $\mathcal{F}$ generated by cylinders:

$$C_{\omega_\Lambda} = \left\{\omega \in \Omega : \omega_\Lambda = \sigma_\Lambda\right\} \,,\; \Lambda \subset\subset \mathbb{L} \quad \left[\omega_\Lambda = (\omega_x)_{x\in\Lambda}\right]$$

Interaction: Family of local functions (=local contributions)

$$\Phi = \left\{\phi_B : \Omega \to \mathbb{R} \,,\; \mathcal{F}_B\text{−measurable}\right\} \quad \left[\phi_B(\omega) = \phi_B(\sigma) \text{ if } \omega_B = \sigma_B\right]$$

Hamiltonian on region Λ given σ outside:

$$H_\Lambda(\omega \mid \sigma) = \sum_{B:B\cap\Lambda\neq\emptyset} \phi_B(\omega_\Lambda\sigma)$$

Lattice systems

Basic ingredients:

- *Lattice* $\mathbb{L}$: e.g. $\mathbb{Z}^d$
- *Single-spin space* S: e.g. $\{-1,1\}$
- *Configuration space* $\Omega = S^{\mathbb{L}}$
  Topology and σ-algebra $\mathcal{F}$ generated by cylinders:

$$C_{\omega_\Lambda} = \left\{\omega \in \Omega : \omega_\Lambda = \sigma_\Lambda\right\} , \ \Lambda \subset\subset \mathbb{L} \quad \left[\omega_\Lambda = (\omega_x)_{x\in\Lambda}\right]$$

Interaction: Family of local functions (=local contributions)

$$\Phi = \left\{\phi_B : \Omega \to \mathbb{R} \ , \ \mathcal{F}_B\text{−measurable}\right\} \quad \left[\phi_B(\omega) = \phi_B(\sigma) \text{ if } \omega_B = \sigma_B\right]$$

Hamiltonian on region Λ given σ outside:

$$H_\Lambda(\omega \mid \sigma) = \sum_{B:B\cap\Lambda\neq\emptyset} \phi_B(\omega_\Lambda\sigma)$$

(Lattice) Gibbs measures: formal definition

Gibbsian specification: Family $\Pi^\Phi = \{\pi^\Phi_\Lambda : \Lambda \subset\subset \mathbb{L}\}$ with

$$\pi^\Phi_\Lambda(C_{\omega_\Lambda}) = \frac{\mathrm{e}^{-\beta H_\Lambda(\omega|\sigma)}}{\text{Norm.}}$$

[$\pi^\Phi_\Lambda(\cdot \mid \sigma)$ = equilibrium in Λ given σ]

Gibbs measures: μ is Gibbs for Φ if, equivalently,

- μ is left invariant by Π^Φ:

$$\int \pi^\Phi_\Lambda(C_{\omega_\Lambda})\, \mu(d\omega) = \mu(C_{\omega_\Lambda})$$

  [μ = equilibrium in $\mathbb{L}$ = every Λ in equilibrium]

- $\mu = w - \lim_{\Lambda\to\mathbb{L}} \pi^\Phi_\Lambda(\cdot \mid \sigma)$ + convex combinations [thermodynamic limit]

(Lattice) Gibbs measures: formal definition

Gibbsian specification: Family $\Pi^\Phi = \{\pi_\Lambda^\Phi : \Lambda \subset\subset \mathbb{L}\}$ with

$$\pi_\Lambda^\Phi(C_{\omega_\Lambda}) = \frac{e^{-\beta H_\Lambda(\omega|\sigma)}}{\text{Norm.}}$$

$[\pi_\Lambda^\Phi(\cdot \mid \sigma) = \text{equilibrium in } \Lambda \text{ given } \sigma]$

Gibbs measures: μ is Gibbs for Φ if, equivalently,

- μ is left invariant by Π^Φ:

$$\int \pi_\Lambda^\Phi(C_{\omega_\Lambda})\, \mu(d\omega) = \mu(C_{\omega_\Lambda})$$

  $[\mu = \text{equilibrium in } \mathbb{L} = \text{every } \Lambda \text{ in equilibrium}]$

- $\mu = w - \lim_{\Lambda \to \mathbb{L}} \pi_\Lambda^\Phi(\cdot \mid \sigma)$ + convex combinations [thermodynamic limit]

(Lattice) Gibbs measures: formal definition

Gibbsian specification: Family $\Pi^\Phi = \{\pi_\Lambda^\Phi : \Lambda \subset\subset \mathbb{L}\}$ with

$$\pi_\Lambda^\Phi(C_{\omega_\Lambda}) = \frac{e^{-\beta H_\Lambda(\omega|\sigma)}}{\text{Norm.}}$$

[$\pi_\Lambda^\Phi(\cdot \mid \sigma)$ = equilibrium in Λ given σ]

Gibbs measures: μ is Gibbs for Φ if, equivalently,

- μ is left invariant by Π^Φ:

$$\int \pi_\Lambda^\Phi(C_{\omega_\Lambda})\, \mu(d\omega) = \mu(C_{\omega_\Lambda})$$

  [μ = equilibrium in $\mathbb{L}$ = every Λ in equilibrium]

- $\mu = w - \lim_{\Lambda \to \mathbb{L}} \pi_\Lambda^\Phi(\cdot \mid \sigma)$ + convex combinations
  [thermodynamic limit]

How to recognize Gibbsianness

Kozlov – Sullivan: μ is Gibbs iff it is

- *Non-null:* $\mu(C_{\omega_\Lambda}) > 0$ for every cylinder C_{ω_Λ}
- *Quasilocal:* If $\Lambda \subset \Gamma \Subset \mathbb{L}$,

$$\sup_{\sigma,\omega,\xi^\pm} \left| \mu\big(C_{\omega_\Lambda} \mid \sigma_\Gamma\, \xi^+\big) - \mu\big(C_{\omega_\Lambda} \mid \sigma_\Gamma\, \xi^-\big) \right| \xrightarrow[\gamma\to\mathbb{L}]{} 0$$

Physics in Λ does not depend on state of the Andromeda galaxy

How to recognize Gibbsianness

Kozlov – Sullivan: μ is Gibbs iff it is

- *Non-null:* $\mu(C_{\omega_\Lambda}) > 0$ for every cylinder C_{ω_Λ}
- *Quasilocal:* If $\Lambda \subset \Gamma \subset\subset \mathbb{L}$,

$$\sup_{\sigma,\omega,\xi^\pm} \Big| \mu\big(C_{\omega_\Lambda} \mid \sigma_\Gamma\, \xi^+\big) - \mu\big(C_{\omega_\Lambda} \mid \sigma_\Gamma\, \xi^-\big) \Big| \xrightarrow[\gamma \to \mathbb{L}]{} 0$$

Physics in Λ does not depend on state of the Andromeda galaxy

How to recognize Gibbsianness

Kozlov – Sullivan: μ is Gibbs iff it is

- *Non-null:* $\mu(C_{\omega_\Lambda}) > 0$ for every cylinder C_{ω_Λ}
- *Quasilocal:* If $\Lambda \subset \Gamma \subset\subset \mathbb{L}$,

$$\sup_{\sigma,\omega,\xi^\pm} \Big| \mu\big(C_{\omega_\Lambda} \mid \sigma_\Gamma\, \xi^+\big) - \mu\big(C_{\omega_\Lambda} \mid \sigma_\Gamma\, \xi^-\big) \Big| \xrightarrow[\gamma\to\mathbb{L}]{} 0$$

Physics in Λ does not depend on state of the Andromeda galaxy

Essential non-quasilocality

μ is Gibbs if $\exists\ \Lambda$ and w^{sp} s.t: $\exists\ \xi^{\pm}$ for which

$$\sup_{\sigma,\omega,\xi^{\pm}} \Big| \mu\big(C_{\omega_\Lambda} \mid \sigma_\Gamma\, \xi^{+}\big) - \mu\big(C_{\omega_\Lambda} \mid \sigma_\Gamma\, \xi^{-}\big) \Big| \underset{\gamma \to \mathbb{L}}{\not\longrightarrow} 0$$

for *every* realisation of $\mu\big(C_{\omega_\Lambda} \mid \cdot\big)$

- Quasilocality = continuity w.r.t. external conditions
- Non-quasilocality = essential discontinuity w.r.t. external conditions

Interpretation: Info from ∞ despite frozen fluctuations
Possible explanation: hidden variables

Essential non-quasilocality

μ is Gibbs if $\exists\ \Lambda$ and w^{sp} s.t: $\exists\ \xi^{\pm}$ for which

$$\sup_{\sigma,\omega,\xi^{\pm}} \left| \mu\big(C_{\omega_\Lambda} \mid \sigma_\Gamma\, \xi^{+}\big) - \mu\big(C_{\omega_\Lambda} \mid \sigma_\Gamma\, \xi^{-}\big) \right| \underset{\gamma \to \mathbb{L}}{\not\longrightarrow} 0$$

for *every* realisation of $\mu\big(C_{\omega_\Lambda} \mid \cdot\big)$

- Quasilocality = continuity w.r.t. external conditions
- Non-quasilocality = essential discontinuity w.r.t. external conditions

Interpretation: Info from ∞ despite frozen fluctuations
Possible explanation: hidden variables

Essential non-quasilocality

μ is Gibbs if $\exists\ \Lambda$ and w^{sp} s.t: $\exists\ \xi^{\pm}$ for which

$$\sup_{\sigma,\omega,\xi^{\pm}} \Big| \mu\big(C_{\omega_\Lambda} \mid \sigma_\Gamma\, \xi^{+}\big) - \mu\big(C_{\omega_\Lambda} \mid \sigma_\Gamma\, \xi^{-}\big) \Big| \underset{\gamma \to \mathbb{L}}{\not\longrightarrow} 0$$

for *every* realisation of $\mu\big(C_{\omega_\Lambda} \mid \cdot\big)$

- Quasilocality = continuity w.r.t. external conditions
- Non-quasilocality = essential discontinuity w.r.t. external conditions

Interpretation: Info from ∞ despite frozen fluctuations

Possible explanation: hidden variables

Essential non-quasilocality

μ is Gibbs if $\exists\ \Lambda$ and w^{sp} s.t: $\exists\ \xi^{\pm}$ for which

$$\sup_{\sigma,\omega,\xi^{\pm}} \Big| \mu\big(C_{\omega_\Lambda} \mid \sigma_\Gamma\, \xi^{+}\big) - \mu\big(C_{\omega_\Lambda} \mid \sigma_\Gamma\, \xi^{-}\big) \Big| \underset{\gamma \to \mathbb{L}}{\not\longrightarrow} 0$$

for *every* realisation of $\mu\big(C_{\omega_\Lambda} \mid \cdot\big)$

- Quasilocality = continuity w.r.t. external conditions
- Non-quasilocality = essential discontinuity w.r.t. external conditions

Interpretation: Info from ∞ despite frozen fluctuations
Possible explanation: hidden variables

Renormalization transformations

General definition: A (stochastic) RT is a map

$$\begin{array}{ccc} \mathrm{Prob}(\Omega) & \longrightarrow & \mathrm{Prob}(\Omega') \\ \mu & \longmapsto & \mu'(\,\cdot\,) = \int K(\,\cdot \mid \omega)\, \mu(d\omega) \end{array}$$

where K is a probability kernel

The transformation is *deterministic* if $\exists\ f : \Omega \to \Omega'$ s.t.

$$K(\,\cdot \mid \omega) = \delta_{f(\omega)}(\,\cdot\,)$$

A *block* RT is of the form

$$K(d\omega' \mid \omega) = \prod_{x'} K'_x(d\omega'_{x'} \mid \omega_{B_{x'}})$$

each $B_{x'} \subset\subset \mathbb{L}$ is the block associated to x'

Renormalization transformations

General definition: A (stochastic) RT is a map

$$\begin{aligned} \mathrm{Prob}(\Omega) &\longrightarrow \mathrm{Prob}(\Omega') \\ \mu &\longmapsto \mu'(\,\cdot\,) = \int K(\,\cdot \mid \omega)\, \mu(d\omega) \end{aligned}$$

where K is a probability kernel

The transformation is *deterministic* if $\exists\; f : \Omega \to \Omega'$ s.t.

$$K(\,\cdot \mid \omega) = \delta_{f(\omega)}(\,\cdot\,)$$

A *block* RT is of the form

$$K(d\omega' \mid \omega) = \prod_{x'} K'_x(d\omega'_{x'} \mid \omega_{B_{x'}})$$

each $B_{x'} \subset\subset \mathbb{L}$ is the block associated to x'

Renormalization transformations

General definition: A (stochastic) RT is a map

$$\begin{array}{rcl} \mathrm{Prob}(\Omega) & \longrightarrow & \mathrm{Prob}(\Omega') \\ \mu & \longmapsto & \mu'(\,\cdot\,) = \int K(\,\cdot \mid \omega)\, \mu(d\omega) \end{array}$$

where K is a probability kernel

The transformation is *deterministic* if $\exists\ f : \Omega \to \Omega'$ s.t.

$$K(\,\cdot \mid \omega) = \delta_{f(\omega)}(\,\cdot\,)$$

A *block* RT is of the form

$$K(d\omega' \mid \omega) = \prod_{x'} K'_x(d\omega'_{x'} \mid \omega_{B_{x'}})$$

each $B_{x'} \subset\subset \mathbb{L}$ is the block associated to x'

Examples of block transformations

Deterministic transformations:

- Decimation
- Majority (odd block)

Stochastic transformations:

- Majority (even block)
- Kadanoff:

$$K'_x(d\omega'_{x'} \mid \omega_{B_{x'}}) = \frac{\exp\left\{p\,\omega'_{x'} \sum_{x\in B_{x'}} \omega_x\right\}}{\text{Norm}}\, d\omega'_{x'}$$

[weighted majority; $\to$ majority as $p \to \infty$]

Hidden variables and non-quasilocality

Hidden-variables mechanism:

- Each fixed ω'_{Λ^c} determines a constrained Ω system
- ω'^{sp} is s.t. the constrained system has a phase transition
- ξ' far away decides the phase $\rightarrow$ info form ∞

Two-slice point of view:

- Ω = original slice = hidden variables
- Ω' = present slice = observed variables

Non-quasilocality: Hidden variables constrained by observed variables exhibit a phase transition

Hidden variables and non-quasilocality

Hidden-variables mechanism:

- Each fixed ω'_{Λ^c} determines a constrained Ω system
- ω'^{sp} is s.t. the constrained system has a phase transition
- ξ' far away decides the phase $\to$ info form ∞

Two-slice point of view:

- Ω = original slice = hidden variables
- Ω' = present slice = observed variables

Non-quasilocality: Hidden variables constrained by observed variables exhibit a phase transition

Hidden variables and non-quasilocality

Hidden-variables mechanism:

- Each fixed ω'_{Λ^c} determines a constrained Ω system
- ω'^{sp} is s.t. the constrained system has a phase transition
- ξ' far away decides the phase $\to$ info form ∞

Two-slice point of view:

- Ω = original slice = hidden variables
- Ω' = present slice = observed variables

Non-quasilocality: Hidden variables constrained by observed variables exhibit a phase transition

Hidden variables and non-quasilocality

Hidden-variables mechanism:

- Each fixed ω'_{Λ^c} determines a constrained Ω system
- ω'^{sp} is s.t. the constrained system has a phase transition
- ξ' far away decides the phase $\to$ info form ∞

Two-slice point of view:

- Ω = original slice = hidden variables
- Ω' = present slice = observed variables

Non-quasilocality: Hidden variables constrained by observed variables exhibit a phase transition

Single-site Kadanoff transformations ($B_{x'} = \{x'\}$)

On finite volumes, the two-slice measures are of the form

$$K_\Lambda(d\omega' \mid \omega)\, \mu_\Lambda(d\omega) \;\propto\; \exp\Big\{\beta\big[H_\Lambda^{\rm Kad}(\omega,\omega') + H_\Lambda(\omega)\big]\Big\}\, d\omega'_\Lambda\, d\omega_\Lambda$$

where

$$H_\Lambda^{\rm Kad}(\omega,\omega') = \sum_{x'} \Big\{\frac{p}{\beta}\,\omega'_x\omega_x - \frac{1}{\beta}\log\big[2\cosh(p\,\omega_x)\big]\Big\}$$

acts on the original spins as an extra magnetic field

Constrained internal spins have phase transition if

$$\frac{p}{\beta}\omega'_x \text{ compensates } h \text{ in the average} \qquad (*)$$

and β is large enough

Single-site Kadanoff transformations ($B_{x'} = \{x'\}$)

On finite volumes, the two-slice measures are of the form

$$K_\Lambda(d\omega' \mid \omega)\, \mu_\Lambda(d\omega) \;\propto\; \exp\Big\{\beta\big[H_\Lambda^{\mathrm{Kad}}(\omega,\omega') + H_\Lambda(\omega)\big]\Big\}\, d\omega'_\Lambda\, d\omega_\Lambda$$

where

$$H_\Lambda^{\mathrm{Kad}}(\omega,\omega') = \sum_{x'} \Big\{ \frac{p}{\beta}\, \omega'_x \omega_x - \frac{1}{\beta} \log\big[2\cosh(p\,\omega_x)\big] \Big\}$$

acts on the original spins as an extra magnetic field

Constrained internal spins have phase transition if

$$\frac{p}{\beta}\omega'_x \text{ compensates } h \text{ in the average} \qquad (*)$$

and β is large enough

Single-site Kadanoff transformations ($B_{x'} = \{x'\}$)

On finite volumes, the two-slice measures are of the form

$$K_\Lambda(d\omega' \mid \omega)\, \mu_\Lambda(d\omega) \;\propto\; \exp\Big\{\beta\big[H_\Lambda^{\rm Kad}(\omega,\omega') + H_\Lambda(\omega)\big]\Big\}\, d\omega'_\Lambda\, d\omega_\Lambda$$

where

$$H_\Lambda^{\rm Kad}(\omega,\omega') = \sum_{x'} \Big\{\frac{p}{\beta}\,\omega'_x\omega_x - \frac{1}{\beta}\log\big[2\cosh(p\,\omega_x)\big]\Big\}$$

acts on the original spins as an extra magnetic field

Constrained internal spins have phase transition if

$$\frac{p}{\beta}\omega'_x \text{ compensates } h \text{ in the average} \qquad (*)$$

and β is large enough

Single-site Kadanoff transformations (cont.)

Let h be the original Ising field and fix β large enough s.t.

- Original model with $h = 0$ has phase transition
- Pirogov-Sinai theory holds

If $h = 0$, alternated $\omega' \Longrightarrow (*)$ for p/β small enough
Hence, $\exists\ p_1 > p_2$ s.t.

- μ' is Gibbs for $p > p_1$
- μ' is not Gibbs for $_2 > p$

If $h \neq 0$, $\exists\ \omega'$ s.t. $(*)$ *only* for a range of p/β
Hence, $\exists\ p_1 \geq p_2 > p_3 \geq p_4$ s.t.

- μ' is Gibbs for $p > p_1$
- μ' is not Gibbs for $p_2 > p > p_3$
- μ' is not Gibbs for $p_4 > p$

Single-site Kadanoff transformations (cont.)

Let h be the original Ising field and fix β large enough s.t.

- Original model with $h = 0$ has phase transition
- Pirogov-Sinai theory holds

If $h = 0$, alternated $\omega' \Longrightarrow (*)$ for p/β small enough
Hence, $\exists\ p_1 > p_2$ s.t.

- μ' is Gibbs for $p > p_1$
- μ' is not Gibbs for $_2 > p$

If $h \neq 0$, $\exists\ \omega'$ s.t. $(*)$ *only* for a range of p/β
Hence, $\exists\ p_1 \geq p_2 > p_3 \geq p_4$ s.t.

- μ' is Gibbs for $p > p_1$
- μ' is not Gibbs for $p_2 > p > p_3$
- μ' is not Gibbs for $p_4 > p$

Single-site Kadanoff transformations (cont.)

Let h be the original Ising field and fix β large enough s.t.

- Original model with $h = 0$ has phase transition
- Pirogov-Sinai theory holds

If $h = 0$, alternated $\omega' \Longrightarrow (*)$ for p/β small enough
Hence, $\exists\; p_1 > p_2$ s.t.

- μ' is Gibbs for $p > p_1$
- μ' is not Gibbs for $_2 > p$

If $h \neq 0$, $\exists\; \omega'$ s.t. $(*)$ *only* for a range of p/β
Hence, $\exists\; p_1 \geq p_2 > p_3 \geq p_4$ s.t.

- μ' is Gibbs for $p > p_1$
- μ' is not Gibbs for $p_2 > p > p_3$
- μ' is not Gibbs for $p_4 > p$

Single-site Kadanoff transformations (cont.)

Let h be the original Ising field and fix β large enough s.t.

- Original model with $h = 0$ has phase transition
- Pirogov-Sinai theory holds

If $h = 0$, alternated $\omega' \implies (*)$ for p/β small enough
Hence, $\exists\ p_1 > p_2$ s.t.

- μ' is Gibbs for $p > p_1$
- μ' is not Gibbs for $_2 > p$

If $h \neq 0$, $\exists\ \omega'$ s.t. $(*)$ *only* for a range of p/β
Hence, $\exists\ p_1 \geq p_2 > p_3 \geq p_4$ s.t.

- μ' is Gibbs for $p > p_1$
- μ' is not Gibbs for $p_2 > p > p_3$
- μ' is not Gibbs for $p_4 > p$

Single-site Kadanoff transformations (cont.)

Let h be the original Ising field and fix β large enough s.t.

- Original model with $h = 0$ has phase transition
- Pirogov-Sinai theory holds

If $h = 0$, alternated $\omega' \Longrightarrow (*)$ for p/β small enough
Hence, $\exists\ p_1 > p_2$ s.t.

- μ' is Gibbs for $p > p_1$
- μ' is not Gibbs for $_2 > p$

If $h \neq 0$, $\exists\ \omega'$ s.t. $(*)$ *only* for a range of p/β
Hence, $\exists\ p_1 \geq p_2 > p_3 \geq p_4$ s.t.

- μ' is Gibbs for $p > p_1$
- μ' is not Gibbs for $p_2 > p > p_3$
- μ' is not Gibbs for $p_4 > p$

Dynamic Gibbs – non-Gibbs transitions

Simulations, particle systems, PCA:

$$\mu_t = S_t \mu$$

with S_t= semigroup of operators.

Dynamic G–non-G: μ Gibbs but μ_t non-Gibbs at some t

Example:

- μ=low-T Ising model
- $S_t = S^n$ infinite-T discrete-time Glauber

$$S = \prod_x S_{\{x\}} \quad \text{with} \quad \begin{array}{l} S_x(\omega_x \mid \omega_x) = 1 - \epsilon \\ S_x(-\omega_x \mid \omega_x) = \epsilon \end{array}$$

[invariant measure = product measure = infinite-T Gibbs]

Unquenching: heating up a low-T Ising model

Dynamic Gibbs – non-Gibbs transitions

Simulations, particle systems, PCA:

$$\mu_t = S_t \mu$$

with S_t= semigroup of operators.
Dynamic G–non-G: μ Gibbs but μ_t non-Gibbs at some t

Example:

- μ=low-T Ising model
- $S_t = S^n$ infinite-T discrete-time Glauber

$$S = \prod_x S_{\{x\}} \quad \text{with} \quad \begin{array}{l} S_x(\omega_x \mid \omega_x) = 1 - \epsilon \\ S_x(-\omega_x \mid \omega_x) = \epsilon \end{array}$$

[invariant measure = product measure = infinite-T Gibbs]

Unquenching: heating up a low-T Ising model

Dynamic Gibbs – non-Gibbs transitions

Simulations, particle systems, PCA:

$$\mu_t = S_t \mu$$

with S_t= semigroup of operators.

Dynamic G–non-G: μ Gibbs but μ_t non-Gibbs at some t

Example:

- μ=low-T Ising model
- $S_t = S^n$ infinite-T discrete-time Glauber

$$S = \prod_x S_{\{x\}} \quad \text{with} \quad \begin{array}{l} S_x(\omega_x \mid \omega_x) = 1 - \epsilon \\ S_x(-\omega_x \mid \omega_x) = \epsilon \end{array}$$

[invariant measure = product measure = infinite-T Gibbs]

Unquenching: heating up a low-T Ising model

Dynamic Gibbs – non-Gibbs transitions

Simulations, particle systems, PCA:

$$\mu_t = S_t \mu$$

with S_t= semigroup of operators.
Dynamic G–non-G: μ Gibbs but μ_t non-Gibbs at some t

Example:

- μ=low-T Ising model
- $S_t = S^n$ infinite-T discrete-time Glauber

$$S = \prod_x S_{\{x\}} \quad \text{with} \quad \begin{array}{l} S_x(\omega_x \mid \omega_x) = 1 - \epsilon \\ S_x(-\omega_x \mid \omega_x) = \epsilon \end{array}$$

[invariant measure = product measure = infinite-T Gibbs]

Unquenching: heating up a low-T Ising model

Un-quenching G–non-G transitions

In matrix form $(S_x)_{\omega\,\omega'} \equiv S_x(\omega'_x \mid \omega_x)$

$$S_x = \begin{pmatrix} 1-\epsilon & \epsilon \\ \epsilon & 1-\epsilon \end{pmatrix} \quad , \quad S_x^n = \frac{1}{2}\begin{pmatrix} 1+a_n & 1-a_n \\ 1-a_n & 1+a_n \end{pmatrix}$$

with $a_n = (1-2\epsilon)^n$. Hence

$$S_x^n(\omega'_x \mid \omega_x) = A_n\, e^{p_n \omega'_x \omega_x} \quad , \quad p_n = \log\left(\frac{1+a_n}{1-a_n}\right)$$

Kadanoff with $p_n \xrightarrow[n\to 0]{} \infty$ and $p_n \xrightarrow[n\to\infty]{} 0$

Un-quenching G–non-G transitions

In matrix form $\left(S_x\right)_{\omega\,\omega'} \equiv S_x(\omega'_x \mid \omega_x)$

$$S_x = \begin{pmatrix} 1-\epsilon & \epsilon \\ \epsilon & 1-\epsilon \end{pmatrix} \quad , \quad S_x^n = \frac{1}{2}\begin{pmatrix} 1+a_n & 1-a_n \\ 1-a_n & 1+a_n \end{pmatrix}$$

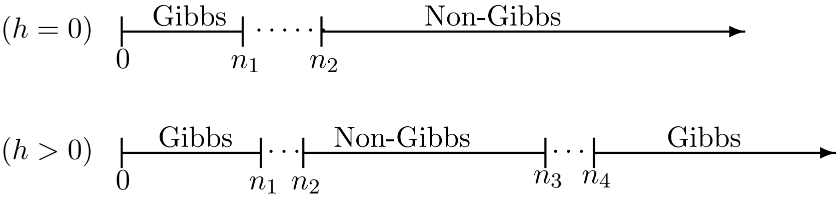
with $a_n = (1-2\epsilon)^n$. Hence

$$S_x^n(\omega'_x \mid \omega_x) = A_n\,\mathrm{e}^{p_n \omega'_x \omega_x} \quad , \quad p_n = \log\left(\frac{1+a_n}{1-a_n}\right)$$

Kadanoff with $p_n \xrightarrow[n\to 0]{} \infty$ and $p_n \xrightarrow[n\to\infty]{} 0$

Un-quenching G–non-G transitions (cont.)

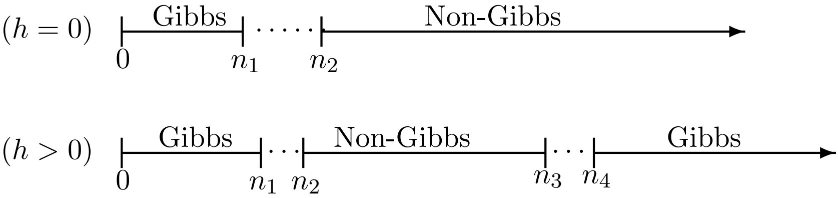
Using previous results on Kadanoff-renormalized measures:



Mathematical mechanism: hidden variables (two-slice view)
Physical mechanism?

Un-quenching G–non-G transitions (cont.)

Using previous results on Kadanoff-renormalized measures:



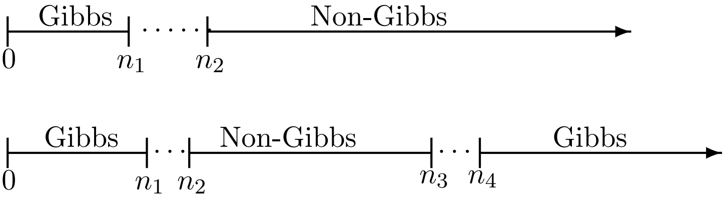
Mathematical mechanism: hidden variables (two-slice view)

Physical mechanism?

Un-quenching G–non-G transitions (cont.)

Using previous results on Kadanoff-renormalized measures:

$(h = 0)$
$(h > 0)$



Mathematical mechanism: hidden variables (two-slice view)
Physical mechanism?

Alternative paradigm: Heuristic version

A. van Enter: *most probable history of an improbable state*

Given a large improbable droplet. How did it get there?

- **Nurture:** Created by the dynamics (cost exp-volume)
- **Nature:** Present at $t = 0$ and survived
  To compete: typical of the other phase (cost exp-perimeter)

Heuristic version:

- Short t: Only nature, no time to change much
- Mid t:
  - ω^{sp} nurtured, but $\xi^{\pm}$ nature
  - Hence $\xi^{\pm}$ determines original phase $\rightarrow$ discontinuity
- Long t: If $h \neq 0$ only one phase $\rightarrow$ no tilting mechanism

Alternative paradigm: Heuristic version

A. van Enter: *most probable history of an improbable state*

Given a large improbable droplet. How did it get there?

- **Nurture:** Created by the dynamics (cost exp-volume)
- **Nature:** Present at $t = 0$ and survived
  To compete: typical of the other phase (cost exp-perimeter)

Heuristic version:

- Short t: Only nature, no time to change much
- Mid t:
  - ω^{sp} nurtured, but $\xi^{\pm}$ nature
  - Hence $\xi^{\pm}$ determines original phase $\to$ discontinuity
- Long t: If $h \neq 0$ only one phase $\to$ no tilting mechanism

Alternative paradigm: Heuristic version

A. van Enter: *most probable history of an improbable state*

Given a large improbable droplet. How did it get there?

- **Nurture:** Created by the dynamics (cost exp-volume)
- **Nature:** Present at $t = 0$ and survived
  To compete: typical of the other phase (cost exp-perimeter)

Heuristic version:

- Short t: Only nature, no time to change much
- Mid t:
  - ω^{sp} nurtured, but $\xi^{\pm}$ nature
  - Hence $\xi^{\pm}$ determines original phase $\rightarrow$ discontinuity
- Long t: If $h \neq 0$ only one phase $\rightarrow$ no tilting mechanism

Alternative paradigm: Heuristic version

A. van Enter: *most probable history of an improbable state*

Given a large improbable droplet. How did it get there?

- **Nurture:** Created by the dynamics (cost exp-volume)
- **Nature:** Present at $t = 0$ and survived
  To compete: typical of the other phase (cost exp-perimeter)

Heuristic version:

- Short t: Only nature, no time to change much
- Mid t:
  - ω^{sp} nurtured, but $\xi^{\pm}$ nature
  - Hence $\xi^{\pm}$ determines original phase $\rightarrow$ discontinuity
- Long t: If $h \neq 0$ only one phase $\rightarrow$ no tilting mechanism

Alternative paradigm: Heuristic version

A. van Enter: *most probable history of an improbable state*

Given a large improbable droplet. How did it get there?

- **Nurture:** Created by the dynamics (cost exp-volume)
- **Nature:** Present at $t = 0$ and survived
  To compete: typical of the other phase (cost exp-perimeter)

Heuristic version:

- Short t: Only nature, no time to change much
- Mid t:
  - ω^{sp} nurtured, but $\xi^{\pm}$ nature
  - Hence $\xi^{\pm}$ determines original phase $\rightarrow$ discontinuity
- Long t: If $h \neq 0$ only one phase $\rightarrow$ no tilting mechanism

Alternative paradigm: Heuristic version

A. van Enter: *most probable history of an improbable state*

Given a large improbable droplet. How did it get there?

- **Nurture:** Created by the dynamics (cost exp-volume)
- **Nature:** Present at $t = 0$ and survived
  To compete: typical of the other phase (cost exp-perimeter)

Heuristic version:

- Short t: Only nature, no time to change much
- Mid t:
  - ω^{sp} nurtured, but $\xi^{\pm}$ nature
  - Hence $\xi^{\pm}$ determines original phase $\to$ discontinuity
- Long t: If $h \neq 0$ only one phase $\to$ no tilting mechanism

Alternative paradigm: rigorous version

most probable history = most probable trajectory

most probable = minimizer of the large-deviation rate

Paradigm: Establish a large-deviation principle for trajectories of *measures conditioned* to a given final *empirical measure*

- Single minimizer = Gibbsianness
- Multiple minimizers = non-Gibbsianness

  Perturbation of conditioning

  $\rightarrow$ discontinuous choice of trajectory

Alternative paradigm: rigorous version

most probable history = most probable trajectory

most probable = minimizer of the large-deviation rate

Paradigm: Establish a large-deviation principle for trajectories of *measures conditioned* to a given final *empirical measure*

- Single minimizer = Gibbsianness
- Multiple minimizers = non-Gibbsianness

Perturbation of conditioning
$\to$ discontinuous choice of trajectory

Alternative paradigm: rigorous version

most probable history = most probable trajectory

most probable = minimizer of the large-deviation rate

Paradigm: Establish a large-deviation principle for trajectories of *measures conditioned* to a given final *empirical measure*

- Single minimizer = Gibbsianness
- Multiple minimizers = non-Gibbsianness

Perturbation of conditioning
$\rightarrow$ discontinuous choice of trajectory

Alternative paradigm: rigorous version

most probable history = most probable trajectory

most probable = minimizer of the large-deviation rate

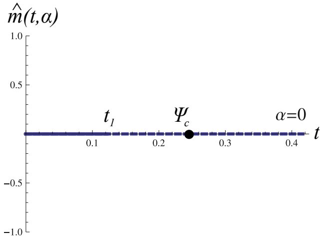
Paradigm: Establish a large-deviation principle for trajectories of *measures conditioned* to a given final *empirical measure*

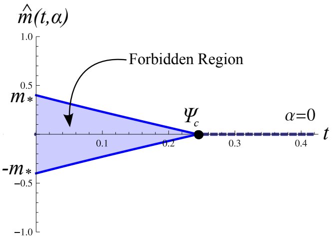
- Single minimizer = Gibbsianness
- Multiple minimizers = non-Gibbsianness

  Perturbation of conditioning
  $\rightarrow$ discontinuous choice of trajectory

Alternative paradigm: graphical summary

$[h = 0]$





One trajectory = Gibbs Many trajectories = non-Gibbs

The program

Prove rigorously the previous paradigm.

Steps:

(i) Mean-field models
(ii) Kac models
(iii) Finite-range models

At present: (i) and (ii) for Ising under independent dynamics

(i) Mean-field:

- No geometry – no notion of neighbourhood
- Everything in terms of empirical magnetization

The program

Prove rigorously the previous paradigm.

Steps:

(i) Mean-field models
(ii) Kac models
(iii) Finite-range models

At present: (i) and (ii) for Ising under independent dynamics

(i) Mean-field:

- No geometry – no notion of neighbourhood
- Everything in terms of empirical magnetization

Mean-field Ising model

N Ising spins ($\omega_i \in \{-1, 1\}$)

$$\begin{aligned} H^N(\sigma) &= -\frac{J}{2N}\sum_{i,j=1}^{N}\sigma_i\sigma_j - h\sum_{i=1}^{N}\sigma_i \\ &= N\overline{H}(m_N(\sigma)) \end{aligned}$$

where m_N is the *empirical magnetization*

$$m_N(\sigma) = \frac{1}{N}\sum_{i=1} N\sigma_i$$

and, if $m \in \mathcal{M}_N := \{-1, -1 + 2N^{-1}, \ldots, +1 - 2N^{-1}, +1\}$,

$$\overline{H}(m) := -\tfrac{1}{2}Jm^2 - hm$$

Mean-field Ising model

N Ising spins ($\omega_i \in \{-1, 1\}$)

$$\begin{aligned} H^N(\sigma) &= -\tfrac{J}{2N} \sum_{i,j=1}^{N} \sigma_i \sigma_j - h \sum_{i=1}^{N} \sigma_i \\ &= N\overline{H}(m_N(\sigma)) \end{aligned}$$

where m_N is the *empirical magnetization*

$$m_N(\sigma) = \frac{1}{N} \sum_{i=1} N\sigma_i$$

and, if $m \in \mathcal{M}_N := \{-1, -1 + 2N^{-1}, \ldots, +1 - 2N^{-1}, +1\}$,

$$\overline{H}(m) := -\tfrac{1}{2} J m^2 - hm$$

Mean-field measures and evolution

The H^N-Gibbs measure [β absorbed]

$$\mu^N(d\sigma) = \frac{\mathrm{e}^{-H^N(\sigma)}}{Z^N}\, d\sigma$$

induces a measure on $\mathcal{M}_N$

$$\overline{\mu}^N(dm) := \binom{N}{\frac{1+m}{2}N} \frac{e^{-N\overline{H}(m)}}{\overline{Z}^N}\, dm$$

[$\mu^N \longleftrightarrow \overline{\mu}^N$ + permutation invariance]

Mean-field measures and evolution

The H^N-Gibbs measure [β absorbed]

$$\mu^N(d\sigma) = \frac{e^{-H^N(\sigma)}}{Z^N}\, d\sigma$$

induces a measure on $\mathcal{M}_N$

$$\overline{\mu}^N(dm) := \binom{N}{\frac{1+m}{2}N} \frac{e^{-N\overline{H}(m)}}{\overline{Z}^N}\, dm$$

[$\mu^N \longleftrightarrow \overline{\mu}^N$ + permutation invariance]

Mean-field evolution

Independent ($T = \infty$) dynamics on Ω_N induces on $\mathcal{M}_N$ a continuous-time Markov chain $(m_t^N)_{t\geq 0}$ with generator

$$\begin{aligned}(\overline{L}_N f)(m) \quad = \quad & \frac{1+m}{2} N\big[f(m-2N^{-1}) - f(m)\big] \\ & + \frac{1-m}{2} N\big[f(m+2N^{-1}) - f(m)\big]\end{aligned}$$

This induces a dynamics on measures on $\mathcal{M}_N$

$$\overline{\mu}_t^N(f) = \overline{\mu}^N\big(e^{t\overline{L}_N} f\big)$$

Mean-field evolution

Independent ($T = \infty$) dynamics on Ω_N induces on $\mathcal{M}_N$ a continuous-time Markov chain $(m_t^N)_{t\geq 0}$ with generator

$$\begin{aligned}(\overline{L}_N f)(m) \quad &= \quad \frac{1+m}{2} N\big[f(m-2N^{-1}) - f(m)\big] \\ &\qquad + \frac{1-m}{2} N\big[f(m+2N^{-1}) - f(m)\big]\end{aligned}$$

This induces a dynamics on measures on $\mathcal{M}_N$

$$\overline{\mu}_t^N(f) = \overline{\mu}^N\big(\mathrm{e}^{t\overline{L}_N} f\big)$$

Mean-field single-site specification
(Külske and Le Ny)

Consider the single-spin conditional probabilities

$$\gamma_t^N(\sigma_1 \mid \alpha_{N-1}) := \mu_t^N(\sigma_1 \mid \sigma_{N-1}) \ ,$$

with

- $\sigma_1 \in \{-1, +1\}$,
- $\alpha_{N-1} \in \mathcal{M}_{N-1}$,
- $\sigma_{N-1} \in \Omega_{N-1}$ any configuration s.t. $m_{N-1}(\sigma_{N-1}) = \alpha_{N-1}$

[By permutation invariance RHS independ of choice of σ_{N-1}]

Gibbs and Non-Gibbs mean-field models

(Külske and Le Ny)

For fixed $t \geq 0$:

(a) A magnetization $\alpha \in [-1,1]$ is **good** for μ_t if

$$\gamma_t(\cdot \mid \widetilde{\alpha}) := \lim_{\substack{N\to\infty \\ \alpha_N \to \widetilde{\alpha}}} \gamma_t^N(\cdot \mid \alpha_{N-1}),$$

- exists and is independent of the sequence $\alpha_N \to \widetilde{\alpha}$
- it is continuous in $\widetilde{\alpha}$

for $\widetilde{\alpha}$ in a neighbourhood of α

(b) A magnetization $\alpha \in [-1,+1]$ is bad if it is not good

(c) μ_t is **Gibbs** if it has no bad magnetizations

Gibbs and Non-Gibbs mean-field models

(Külske and Le Ny)

For fixed $t \geq 0$:

(a) A magnetization $\alpha \in [-1, 1]$ is **good** for μ_t if

$$\gamma_t(\cdot \mid \widetilde{\alpha}) := \lim_{\substack{N\to\infty \\ \alpha_N \to \widetilde{\alpha}}} \gamma_t^N(\cdot \mid \alpha_{N-1}),$$

- exists and is independent of the sequence $\alpha_N \to \widetilde{\alpha}$
- it is continuous in $\widetilde{\alpha}$

for $\widetilde{\alpha}$ in a neighbourhood of α

(b) A magnetization $\alpha \in [-1, +1]$ is bad if it is not good

(c) μ_t is **Gibbs** if it has no bad magnetizations

Large-deviations: General definition

Informally:

A family of measures (ν^N) satisfies a *large-deviation principle* if

$$\nu^N(A) \sim \mathrm{e}^{-N\, I(A)}$$

- N is the LDP *speed*, I the *rate function*
- As a consequence, $\mathrm{supp}(\nu^N) \to \mathrm{argmin}(I)$

Formally:

(ν^N) on a Borel space satisf. LDP with rate fcn I and speed N if

$$\liminf_{N\to\infty} \frac{1}{N} \log \nu^N(A) \geq -\inf_{x\in A} I(x) \qquad \text{for } A \text{ open}$$

$$\limsup_{N\to\infty} \frac{1}{N} \log \nu^N(A) \leq -\sup_{x\in A} I(x) \qquad \text{for } A \text{ closed}$$

Large-deviations: General definition

Informally:

A family of measures (ν^N) satisfies a *large-deviation principle* if

$$\nu^N(A) \sim \mathrm{e}^{-N\, I(A)}$$

- N is the LDP *speed*, I the *rate function*
- As a consequence, $\mathrm{supp}(\nu^N) \to \mathrm{argmin}(I)$

Formally:

(ν^N) on a Borel space satisf. LDP with rate fcn I and speed N if

$$\begin{aligned}
\liminf_{N\to\infty} \frac{1}{N} \log \nu^N(A) \quad &\geq \quad -\inf_{x\in A} I(x) \qquad \text{for } A \text{ open} \\
\limsup_{N\to\infty} \frac{1}{N} \log \nu^N(A) \quad &\leq \quad -\sup_{x\in A} I(x) \qquad \text{for } A \text{ closed}
\end{aligned}$$

Large-deviations: General definition

Informally:

A family of measures (ν^N) satisfies a *large-deviation principle* if

$$\nu^N(A) \sim \mathrm{e}^{-N\, I(A)}$$

- N is the LDP *speed*, I the *rate function*
- As a consequence, $\mathrm{supp}(\nu^N) \to \mathrm{argmin}(I)$

Formally:

(ν^N) on a Borel space satisf. LDP with rate fcn I and speed N if

$$\begin{aligned}
\liminf_{N\to\infty} \frac{1}{N} \log \nu^N(A) \quad &\geq \quad -\inf_{x\in A} I(x) \qquad \text{for } A \text{ open} \\
\limsup_{N\to\infty} \frac{1}{N} \log \nu^N(A) \quad &\leq \quad -\sup_{x\in A} I(x) \qquad \text{for } A \text{ closed}
\end{aligned}$$

LDP for mean-field Ising:

“Static” part:

The family $(\overline{\mu}^N)$ satisfies a LDP with speed N and rate $I_S - \inf(I_S)$ with

$$I_S(m) := \overline{H}(m) + \frac{1+m}{2}\log(1+m) + \frac{1-m}{2}\log(1-m).$$

Independent evolutions:

Let P^N = law of $(m_t^N)_{t\geq 0}$

Defined on the space of càdlàg trajectories; Skorohod topology

LDP for mean-field Ising:

"Static" part:

The family $(\overline{\mu}^N)$ satisfies a LDP with speed N and rate $I_S - \inf(I_S)$ with

$$I_S(m) := \overline{H}(m) + \frac{1+m}{2}\log(1+m) + \frac{1-m}{2}\log(1-m).$$

Independent evolutions:

Let P^N = law of $(m_t^N)_{t\geq 0}$

Defined on the space of càdlàg trajectories; Skorohod topology

LDP for mean-field evolutions

(Ermolaev and Külske)

(P^N) restricted to $[0,T]$ satisfies LDP with speed N and rate $I^T - \inf(I^T)$ given by

$$I^T(\phi) := I_S(\phi(0)) + I_D^T(\phi),$$

where

$$I_D^T(\phi) := \begin{cases} \int_0^T L(\phi(s), \dot\phi(s))\, ds & \text{if } \dot\phi \text{ exists} \\ \infty & \text{otherwise} \end{cases}$$

is the action integral with Lagrangian

$$L(m,\dot m) = -\frac{1}{2}\sqrt{4\,(1-m^2)+\dot m^2} + \frac{1}{2}\dot m \log\left(\frac{\sqrt{4\,(1-m^2)+\dot m^2}+\dot m}{2(1-m)}\right) + 1$$

LDP for mean-field evolutions

(Ermolaev and Külske)

(P^N) restricted to $[0,T]$ satisfies LDP with speed N and rate $I^T - \inf(I^T)$ given by

$$I^T(\phi) := I_S(\phi(0)) + I_D^T(\phi),$$

where

$$I_D^T(\phi) := \begin{cases} \int_0^T L(\phi(s), \dot\phi(s))\, ds & \text{if } \dot\phi \text{ exists} \\ \infty & \text{otherwise} \end{cases}$$

is the action integral with Lagrangian

$$\begin{aligned} L(m,\dot m) &= -\frac{1}{2}\sqrt{4\,(1-m^2)+\dot m^2} \\ &\quad + \frac{1}{2}\dot m \log\left(\frac{\sqrt{4\,(1-m^2)+\dot m^2}+\dot m}{2(1-m)}\right) + 1 \end{aligned}$$

LDP for conditioned mean-field evolutions

The family of measures on trajectory space

$$Q^N_{t,\alpha}(\,\cdot\,) := P^N\Big((m_N(s))_{0\le s\le t} = \,\cdot\, \;\Big|\; m_N(t) = \alpha\Big)$$

satisfies LDP with speed N and rate $I^{t,\alpha} - \inf(I^{t,\alpha})$, with

$$I^{t,\alpha}(m\phi) = \begin{cases} I^t(\phi) & \text{if } \phi_t = \alpha \\ \infty & \text{otherwise} \end{cases}$$

Hence, conditioned optimal trajectories correspond to

$$\underset{\phi:\ \phi(t)=\alpha}{\operatorname{argmin}}\ I^t(\phi)$$

LDP for conditioned mean-field evolutions

The family of measures on trajectory space

$$Q^N_{t,\alpha}(\,\cdot\,) := P^N\Big((m_N(s))_{0\le s\le t} = \,\cdot\, \Big|\, m_N(t) = \alpha\Big)$$

satisfies LDP with speed N and rate $I^{t,\alpha} - \inf(I^{t,\alpha})$, with

$$I^{t,\alpha}(m\phi) = \begin{cases} I^t(\phi) & \text{if } \phi_t = \alpha \\ \infty & \text{otherwise} \end{cases}$$

Hence, conditioned optimal trajectories correspond to

$$\underset{\phi:\ \phi(t)=\alpha}{\operatorname{argmin}}\ I^t(\phi)$$

The mean-field computational advantage

Simplifying feature:

- There is an explicit expression for

$$C_{t,\alpha}(m) := \inf_{\substack{\phi:\ \phi(0)=m,\\ \phi(t)=\alpha}} I^t(\phi)$$

- We have the identity

$$\inf_{m\in[-1,+1]} C_{t,\alpha}(m) = \inf_{\phi:\ \phi(t)=\alpha} I^t(\phi)$$

Hence,

multiple conditioned trajectories

$\iff$ multiple global minima of $C_{t,\alpha}(m)$

The mean-field computational advantage

Simplifying feature:

- There is an explicit expression for

$$C_{t,\alpha}(m) := \inf_{\substack{\phi:\ \phi(0)=m,\\ \phi(t)=\alpha}} I^t(\phi)$$

- We have the identity

$$\inf_{m\in[-1,+1]} C_{t,\alpha}(m) = \inf_{\phi:\ \phi(t)=\alpha} I^t(\phi)$$

Hence,

multiple conditioned trajectories

$\iff$ multiple global minima of $C_{t,\alpha}(m)$

I. Single optimal trajectory = Gibbsianness

$\alpha \mapsto \gamma_t(\sigma \mid \alpha)$ is continuous at α_0 if and only if

- $I^t(\phi)$ has a unique minimizing path $\widehat{\phi}$
- or, equivalently, $C_{t,\alpha_0}(m)$ has a unique minimizing m.

Furthermore, in this case, the specification kernel equals

$$\gamma_t(z \mid \alpha) = \frac{\sum_{x\in\{-1,+1\}} e^{x[J\widehat{\phi}(0)+h]}\, p_t(x,z)}{\sum_{x,y\in\{-1,+1\}} e^{x[J\widehat{\phi}(0)+h]}\, p_t(x,y)}$$

$p_t(\cdot,\cdot)$ = kernel of Markov jump process on $\{-1,+1\}$ with

- jumping rate 1
- jump probabilities $p_t(i,\pm i) = e^{-t}\begin{cases}\cosh(t)\\ \sinh(t)\end{cases}$

"If" part and for of γ_t proven by Ermolaev and Külske

I. Single optimal trajectory = Gibbsianness

$\alpha \mapsto \gamma_t(\sigma \mid \alpha)$ is continuous at α_0 if and only if

- $I^t(\phi)$ has a unique minimizing path $\widehat{\phi}$
- or, equivalently, $C_{t,\alpha_0}(m)$ has a unique minimizing m.

Furthermore, in this case, the specification kernel equals

$$\gamma_t(z \mid \alpha) = \frac{\sum_{x\in\{-1,+1\}} e^{x[J\widehat{\phi}(0)+h]}\, p_t(x,z)}{\sum_{x,y\in\{-1,+1\}} e^{x[J\widehat{\phi}(0)+h]}\, p_t(x,y)}$$

$p_t(\cdot,\cdot)$ = kernel of Markov jump process on $\{-1,+1\}$ with

- jumping rate 1
- jump probabilities $p_t(i,\pm i) = e^{-t}\begin{cases}\cosh(t)\\ \sinh(t)\end{cases}$

"If" part and for of γ_t proven by Ermolaev and Külske

I. Single optimal trajectory = Gibbsianness

$\alpha \mapsto \gamma_t(\sigma \mid \alpha)$ is continuous at α_0 if and only if

- $I^t(\phi)$ has a unique minimizing path $\widehat{\phi}$
- or, equivalently, $C_{t,\alpha_0}(m)$ has a unique minimizing m.

Furthermore, in this case, the specification kernel equals

$$\gamma_t(z \mid \alpha) = \frac{\sum_{x\in\{-1,+1\}} e^{x[J\widehat{\phi}(0)+h]}\, p_t(x,z)}{\sum_{x,y\in\{-1,+1\}} e^{x[J\widehat{\phi}(0)+h]}\, p_t(x,y)}$$

$p_t(\cdot,\cdot) =$ kernel of Markov jump process on $\{-1,+1\}$ with

- jumping rate 1
- jump probabilities $p_t(i,\pm i) = e^{-t}\begin{cases}\cosh(t)\\ \sinh(t)\end{cases}$

"If" part and for of γ_t proven by Ermolaev and Külske

II. Short-term Gibbsianness

Theorem

If $J \leq 1$ the evolved measures μ_t are Gibbs for all $t \geq 0$

- Proven by Külske and Le Ny and Külske and Ermolaev
- Note that $1 = \beta_{\text{cr}}^{\text{MF}}$

II. Short-term Gibbsianness

Theorem
If $J \leq 1$ the evolved measures μ_t are Gibbs for all $t \geq 0$

- Proven by Külske and Le Ny and Külske and Ermolaev
- Note that $1 = \beta_{\mathrm{cr}}^{\mathrm{MF}}$

III. Case $J > 1$, $h = 0$

Consider the critical time

$$\Psi_c(J) := \begin{cases} \frac{1}{2}\,\mathrm{acoth}(2J-1) & \text{if } 1 < J \leq \frac{3}{2}, \\ t_*(J) \text{ implicitly calculable} & \text{if } J > \frac{3}{2}, \end{cases}$$

Then:

- $t < \Psi_c$: Evolved measure μ_t is Gibbs
- $t > \Psi_c$: Discontinuity at $\alpha = 0$; two optimal trajectories $\pm\widetilde{\phi}$
- If $\Lambda_{t,0}(J)$ = cone between the trajectories $\pm\hat{\phi}$
  - No trajectory can penetrate $\Lambda_{t,0}(J)$
  - For $J \leq 3/2$ the map $t \mapsto \Lambda_{t,0}(J)$ is continuous
  - For $J > 3/2$ the map $t \mapsto \Lambda_{t,0}(J)$ is continuous except at $t = \Psi_c$ where it exhibits a right-continuous jump

III. Case $J > 1$, $h = 0$

Consider the critical time

$$\Psi_c(J) := \begin{cases} \frac{1}{2}\,\mathrm{acoth}(2J-1) & \text{if } 1 < J \leq \frac{3}{2}, \\ t_*(J) \text{ implicitly calculable} & \text{if } J > \frac{3}{2}, \end{cases}$$

Then:

- $t < \Psi_c$: Evolved measure μ_t is Gibbs
- $t > \Psi_c$: Discontinuity at $\alpha = 0$; two optimal trajectories $\pm\widetilde{\phi}$
- If $\Lambda_{t,0}(J)$ = cone between the trajectories $\pm\hat{\phi}$
  - No trajectory can penetrate $\Lambda_{t,0}(J)$
  - For $J \leq 3/2$ the map $t \mapsto \Lambda_{t,0}(J)$ is continuous
  - For $J > 3/2$ the map $t \mapsto \Lambda_{t,0}(J)$ is continuous except at $t = \Psi_c$ where it exhibits a right-continuous jump

III. Case $J > 1$, $h = 0$

Consider the critical time

$$\Psi_c(J) := \begin{cases} \frac{1}{2}\,\mathrm{acoth}(2J-1) & \text{if } 1 < J \leq \frac{3}{2}, \\ t_*(J) \text{ implicitly calculable} & \text{if } J > \frac{3}{2}, \end{cases}$$

Then:

- $t < \Psi_c$: Evolved measure μ_t is Gibbs
- $t > \Psi_c$: Discontinuity at $\alpha = 0$; two optimal trajectories $\pm\widetilde{\phi}$
- If $\Lambda_{t,0}(J)$ = cone between the trajectories $\pm\hat{\phi}$
  - No trajectory can penetrate $\Lambda_{t,0}(J)$
  - For $J \leq 3/2$ the map $t \mapsto \Lambda_{t,0}(J)$ is continuous
  - For $J > 3/2$ the map $t \mapsto \Lambda_{t,0}(J)$ is continuous except at $t = \Psi_c$ where it exhibits a right-continuous jump

III. Case $J > 1$, $h = 0$

Consider the critical time

$$\Psi_c(J) := \begin{cases} \frac{1}{2}\operatorname{acoth}(2J-1) & \text{if } 1 < J \leq \frac{3}{2}, \\ t_*(J) \text{ implicitly calculable} & \text{if } J > \frac{3}{2}, \end{cases}$$

Then:

- $t < \Psi_c$: Evolved measure μ_t is Gibbs
- $t > \Psi_c$: Discontinuity at $\alpha = 0$; two optimal trajectories $\pm\widetilde{\phi}$
- If $\Lambda_{t,0}(J)$ = cone between the trajectories $\pm\hat{\phi}$
  - No trajectory can penetrate $\Lambda_{t,0}(J)$
  - For $J \leq 3/2$ the map $t \mapsto \Lambda_{t,0}(J)$ is continuous
  - For $J > 3/2$ the map $t \mapsto \Lambda_{t,0}(J)$ is continuous except at $t = \Psi_c$ where it exhibits a right-continuous jump

III. Case $J > 1$, $h = 0$

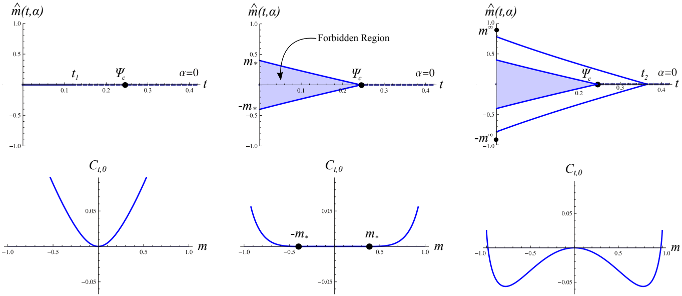
Consider the critical time

$$\Psi_c(J) := \begin{cases} \frac{1}{2}\,\mathrm{acoth}(2J-1) & \text{if } 1 < J \leq \frac{3}{2}, \\ t_*(J) \text{ implicitly calculable} & \text{if } J > \frac{3}{2}, \end{cases}$$

Then:

- $t < \Psi_c$: Evolved measure μ_t is Gibbs
- $t > \Psi_c$: Discontinuity at $\alpha = 0$; two optimal trajectories $\pm\widetilde{\phi}$
- If $\Lambda_{t,0}(J)$ = cone between the trajectories $\pm\hat{\phi}$
  - No trajectory can penetrate $\Lambda_{t,0}(J)$
  - For $J \leq 3/2$ the map $t \mapsto \Lambda_{t,0}(J)$ is continuous
  - For $J > 3/2$ the map $t \mapsto \Lambda_{t,0}(J)$ is continuous except at $t = \Psi_c$ where it exhibits a right-continuous jump

Graphic summary: $h = 0$, $\alpha = 0$

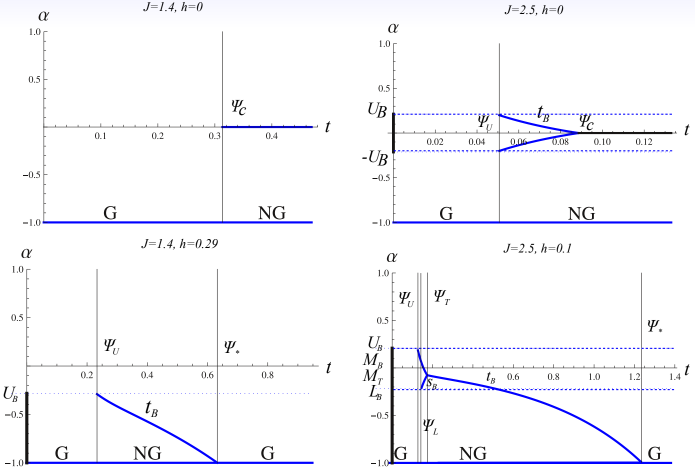


$t < \Psi_c$ $\quad$ $t = \Psi_c$ $\quad$ $t > \Psi_c$

First row: Minimizing trajectories for $(J, h) = (1.6, 0)$

Second row: Corresponding plots of $m \mapsto C_{t,0}(m)$

IV. Bad magnetizations as function of time



$1 < J \leq \frac{3}{2}$ $\qquad$ $J > \frac{3}{2}$

S_B = trifurcation point, rest of the line = bifurcation

Kac models: Basic definitions

- $\Delta_n^d := \mathbb{Z}^d / n\mathbb{Z}^d$ = the discrete torus of size n
- $\Omega_n := \{-1, +1\}^{\Delta_n^d}$ = Ising-spin configurations on Δ_n^d
- *Kac-type Hamiltonian:*

$$H^n(\sigma) := -\frac{1}{2n^d} \sum_{x,y \in \Delta_n^d} J\left(\frac{x-y}{n}\right) \sigma(x)\sigma(y) - \sum_{x \in \Delta_n^d} h\left(\frac{x}{n}\right) \sigma(x)$$

[$J \geq 0$ symmetric]

- *Gibbs measure* associated with H^n:

$$\mu^n(d\sigma) := \frac{e^{-\beta H^n(\sigma)}}{Z^n} \, d\sigma$$

Kac models: Basic definitions

- $\Delta_n^d := \mathbb{Z}^d / n\mathbb{Z}^d$ = the discrete torus of size n
- $\Omega_n := \{-1,+1\}^{\Delta_n^d}$ = Ising-spin configurations on Δ_n^d
- *Kac-type Hamiltonian:*

$$H^n(\sigma) := -\frac{1}{2n^d} \sum_{x,y \in \Delta_n^d} J\left(\frac{x-y}{n}\right) \sigma(x)\sigma(y) - \sum_{x \in \Delta_n^d} h\left(\frac{x}{n}\right) \sigma(x)$$

[$J \geq 0$ symmetric]

- *Gibbs measure* associated with H^n:

$$\mu^n(d\sigma) := \frac{e^{-\beta H^n(\sigma)}}{Z^n} \, d\sigma$$

Continuum limit

- $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$, the d-dimensional unit torus
- $\mathbb{T}^d_n := \frac{1}{n}\Delta^d_n = (1/n)$-discretization of $\mathbb{T}^d$
- $\mathcal{M}(\mathbb{T}^d_n)$ $[\mathcal{M}(\mathbb{T}^d)]$ = signed measures on $\mathbb{T}^d_n$ $[\mathbb{T}^d]$ (TV ≤ 1)

The *empirical density* of $\sigma \in \Omega_n$ inside $\Lambda \subseteq \Delta^d_n$ is

$$\pi^n_\Lambda \colon \Omega_n \to \mathcal{M}(\mathbb{T}^d_n) \subseteq \mathcal{M}(\mathbb{T}^d)$$

$$\pi^n_\Lambda(\sigma) := \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\sigma(x)\delta_{x/n}$$

Continuum limit

- $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$, the d-dimensional unit torus
- $\mathbb{T}^d_n := \frac{1}{n}\Delta^d_n = (1/n)$-discretization of $\mathbb{T}^d$
- $\mathcal{M}(\mathbb{T}^d_n)$ $[\mathcal{M}(\mathbb{T}^d)]$ = signed measures on $\mathbb{T}^d_n$ $[\mathbb{T}^d]$ (TV ≤ 1)

The *empirical density* of $\sigma \in \Omega_n$ inside $\Lambda \subseteq \Delta^d_n$ is

$$\pi^n_\Lambda \colon\ \Omega_n \to \mathcal{M}(\mathbb{T}^d_n) \subseteq \mathcal{M}(\mathbb{T}^d)$$

$$\pi^n_\Lambda(\sigma) := \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\sigma(x)\delta_{x/n}$$

Induced objects. Profiles

Via π^n we define induced Gibbs measures on $\mathcal{M}(\mathbb{T}^d_n)$:

$$\breve{\mu}^n = \mu^n \circ (\pi^n)^{-1}$$

and rewrite

$$H^n(\sigma) = n^d H(\pi^n(\sigma))$$

with

$$H(\nu) = -\left\langle \tfrac{1}{2} J * \nu + h, \nu \right\rangle$$

A measure on $\mathcal{M}(\mathbb{T}^d)$ of the form $\alpha\,\lambda$ with

- λ Lebesgue measure
- $\alpha \in B$ density function, with B=unit ball in $L^\infty(\mathbb{T}^d)$

will be referred as a **profile** $\alpha \in B$

Induced objects. Profiles

Via π^n we define induced Gibbs measures on $\mathcal{M}(\mathbb{T}_n^d)$:

$$\breve{\mu}^n = \mu^n \circ (\pi^n)^{-1}$$

and rewrite

$$H^n(\sigma) = n^d H(\pi^n(\sigma))$$

with

$$H(\nu) = -\left\langle \tfrac{1}{2} J * \nu + h, \nu \right\rangle$$

A measure on $\mathcal{M}(\mathbb{T}^d)$ of the form $\alpha\,\lambda$ with

- λ Lebesgue measure
- $\alpha \in B$ density function, with B=unit ball in $L^\infty(\mathbb{T}^d)$

will be referred as a **profile** $\alpha \in B$

Single-site Kac specifications

Given

- A (continuum) site $u \in \mathbb{T}^d$
- A probability measure ρ^n on Ω_n
- A measure $\alpha^u_{n-1} \in \mathcal{M}(\mathbb{T}^d_n \setminus \lfloor nu \rfloor)$

The *single-site conditional probability* at site $\lfloor nu \rfloor \in \mathbb{T}^d_n$ is

$$\gamma^{u,n}\left(\cdot \mid \alpha^u_{n-1}\right) := \rho^n\Big(\sigma(\lfloor nu \rfloor) = \cdot \;\Big|\; \pi^{u,n}(\sigma) = \alpha^u_{n-1}\Big)$$

Single-site Kac specifications

Given

- A (continuum) site $u \in \mathbb{T}^d$
- A probability measure ρ^n on Ω_n
- A measure $\alpha^u_{n-1} \in \mathcal{M}(\mathbb{T}^d_n \setminus \lfloor nu \rfloor)$

The *single-site conditional probability* at site $\lfloor nu \rfloor \in \mathbb{T}^d_n$ is

$$\gamma^{u,n}\big(\cdot \mid \alpha^u_{n-1} \big) := \rho^n \Big(\sigma(\lfloor nu \rfloor) = \cdot \; \Big| \; \pi^{u,n}(\sigma) = \alpha^u_{n-1} \Big)$$

Gibbs and no-Gibbs Kac measures

(a) A profile $\alpha \in B$ is **good** for (ρ^n)

$$\gamma^u(\,\cdot \mid \widetilde{\alpha}) := \lim_{n\to\infty} \gamma^{u,n}(\,\cdot \mid \alpha^u_{n-1})$$

- exists and is independent of the sequence $\alpha^u_{n-1} \to \widetilde{\alpha}\lambda$
- it is continuous in $\widetilde{\alpha}$

for $\widetilde{\alpha}$ in a neighbourhood of α

(b) A profile $\alpha \in B$ is bad if it is not good

(c) (ρ^n) is **Gibbs** if it has no bad profiles

Gibbs and no-Gibbs Kac measures

(a) A profile $\alpha \in B$ is **good** for (ρ^n)

$$\gamma^u\big(\cdot \,\big|\, \widetilde{\alpha}\big) := \lim_{n\to\infty} \gamma^{u,n}\big(\cdot \,\big|\, \alpha^u_{n-1}\big)$$

- exists and is independent of the sequence $\alpha^u_{n-1} \to \widetilde{\alpha}\lambda$
- it is continuous in $\widetilde{\alpha}$

for $\widetilde{\alpha}$ in a neighbourhood of α

(b) A profile $\alpha \in B$ is bad if it is not good

(c) (ρ^n) is **Gibbs** if it has no bad profiles

LDP for ("static") Kac measures

(Comets)

$(\check{\mu}^n)$ satisfies an LDP with

- speed n^d
- rate function $I_S - \inf_{\mathcal{M}(\mathbb{T}^d)} I_S$ with

$$I_S(\nu) := \begin{cases} -\beta\left\langle \frac{1}{2} J * \alpha + h, \alpha\lambda \right\rangle + \langle \Phi \circ \alpha, \lambda \rangle & \text{if } \nu = \alpha\lambda,\ \alpha \in B \\ \infty & \text{otherwise} \end{cases}$$

where Φ is the relative entropy

$$\Phi(m) := \tfrac{1+m}{2} \log(1+m) + \tfrac{1-m}{2} \log(1-m), \qquad m \in [-1, +1].$$

LDP for conditioned Kac evolutions

Let

$$P^n := \text{ law of } (\pi_s^n)_{s\geq 0} \text{ conditional on } \pi_0^n \sim \check{\mu}^n,$$

and

$$Q_{t,\alpha}^n(\cdot) := P^n\Big((\pi_s^n)_{s\in[0,t]} \in \cdot \mid \pi_t^n = \alpha_n\Big),$$

with $\alpha_n \in \mathcal{M}^n$ the element closest to $\alpha\lambda$. Then

For $t \geq 0$ and $\alpha \in B$, $(Q_{t,\alpha'}^n)_{n\in\mathbb{N}}$ satisfies an LDP with

- speed n^d
- rate function $I^{t,\alpha} - \inf_{D_{[0,t]}(\mathcal{M}(\mathbb{T}^d))} I^{t,\alpha}$ with

$$I^{t,\alpha}(\phi) := \begin{cases} I_S(\phi_0) + I_D^t(\phi) & \text{if } \phi_t \equiv \alpha \\ \infty, & \text{otherwise.} \end{cases}$$

with $I_D^t(\phi)$ given by the integral of an explicit Lagrangian

LDP for conditioned Kac evolutions

Let

$$P^n := \text{ law of } (\pi^n_s)_{s\geq 0} \text{ conditional on } \pi^n_0 \sim \check{\mu}^n,$$

and

$$Q^n_{t,\alpha}(\cdot) := P^n\Big((\pi^n_s)_{s\in[0,t]} \in \cdot \mid \pi^n_t = \alpha_n\Big),$$

with $\alpha_n \in \mathcal{M}^n$ the element closest to $\alpha\lambda$. Then

For $t \geq 0$ and $\alpha \in B$, $(Q^n_{t,\alpha'})_{n\in\mathbb{N}}$ satisfies an LDP with

- speed n^d
- rate function $I^{t,\alpha} - \inf_{D_{[0,t]}(\mathcal{M}(\mathbb{T}^d))} I^{t,\alpha}$ with

$$I^{t,\alpha}(\phi) := \begin{cases} I_S(\phi_0) + I^t_D(\phi) & \text{if } \phi_t \equiv \alpha \\ \infty, & \text{otherwise.} \end{cases}$$

with $I^t_D(\phi)$ given by the integral of an explicit Lagrangian

Results for Kac models

(A) For general Glauber dynamics:

$$\text{Gibbsianness} \iff \text{unique minimizing path}$$

(B) For independent spin flips:

Let $J := \int_{\mathbb{T}^d} J(u)du$, then

(i) *Short-time Gibbs:* $\exists\, t_0 = t_0(J, h)$ s.t. no bifurcation in $[0, t_0]$

(ii) *Mean-Field behaviour:* If

$$h \equiv c \in [0, \infty) \quad \text{and} \quad \alpha' \equiv c' \in [-1, +1]$$

then

$$\text{bifurcation} \sim \text{MF with } (J^{\text{MF}}, h^{\text{MF}}) = (\beta J, \beta c);\ \alpha = c'$$

Results for Kac models

(A) For general Glauber dynamics:

$$\text{Gibbsianness} \iff \text{unique minimizing path}$$

(B) For independent spin flips:

Let $J := \int_{\mathbb{T}^d} J(u)du$, then

(i) *Short-time Gibbs:* $\exists\, t_0 = t_0(J,h)$ s.t. no bifurcation in $[0, t_0]$

(ii) *Mean-Field behaviour:* If

$$h \equiv c \in [0,\infty) \quad \text{and} \quad \alpha' \equiv c' \in [-1,+1]$$

then

$$\text{bifurcation} \sim \text{MF with } (J^{\text{MF}}, h^{\text{MF}}) = (\beta J, \beta c);\ \alpha = c'$$

Results for Kac models

(A) For general Glauber dynamics:

$$\text{Gibbsianness} \iff \text{unique minimizing path}$$

(B) For independent spin flips:

Let $J := \int_{\mathbb{T}^d} J(u)du$, then

(i) *Short-time Gibbs:* $\exists\, t_0 = t_0(J, h)$ s.t. no bifurcation in $[0, t_0]$

(ii) *Mean-Field behaviour:* If

$$h \equiv c \in [0, \infty) \quad \text{and} \quad \alpha' \equiv c' \in [-1, +1]$$

then

$$\text{bifurcation} \sim \text{MF with } (J^{\text{MF}}, h^{\text{MF}}) = (\beta J, \beta c);\ \alpha = c'$$

Conclusions

- New paradigm seems to work
- However: needs LDP in spaces of trajectories of measure
- Practical consequences (numerics, other phenomena)?