Logarithmic correlations in geometrical critical phenomena

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Logarithms in critical phenomeana

- Scale invariance $\Rightarrow$ correlations are power-law or logarithmic

Introduction

Logarithms in critical phenomeana

- Scale invariance $\Rightarrow$ correlations are power-law or logarithmic

Two possibilities for logarithms

1. Marginally irrelevant operator:
   Gives logs upon approach to fixed point theory.
2. Dilatation operator not diagonalisable:
   Logs directly in the fixed point theory.

Non-diagonalisable dilatation operator

- Happens when dimensions of two operators collide
- Resonance phenomenon produces a log from two power laws

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Where do such logarithms appear?

- CFT with $c = 0$ [Gurarie, Gurarie-Ludwig, Cardy, ...]
  - Percolation, self-avoiding polymers ($c \to 0$ catastrophe)
  - Quenched random systems (replica limit catastrophe)

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- Logarithmic minimal models [Pearce-Rasmussen-Zuber, Read-Saleur]
- For any $d \leq$ upper critical dimension

Logarithms and non-unitarity [Cardy 1999]

Standard unitary CFT

- Expand local density $\Phi(r)$ on sum of scaling operators $\varphi(r)$

$$\langle \Phi(r)\Phi(0)\rangle \sim \sum_{ij} \frac{A_{ij}}{r^{\Delta_i+\Delta_j}}$$

- $A_{ij} \propto \delta_{ij}$ by conformal symmetry [Polyakov 1970]
- $A_{ii} \geq 0$ by reflection positivity
- Hence only power laws appear

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The non-unitary case

- Cancellations may occur
- Suppose $A_{ii} \sim -A_{jj} \to \infty$ with $A_{ii}(\Delta_i - \Delta_j)$ finite
- Then leading term is $r^{-2\Delta_i} \log r$

Application to geometrical models

Q-state Potts model

- Hamiltonian $H = J \sum_{\langle ij \rangle} \delta(\sigma_i, \sigma_j)$ with $\sigma_i = 1, 2, \ldots, Q$
- Reformulation in terms of Fortuin-Kasteleyn clusters

$$Z = \sum_{A \subseteq \langle ij \rangle} Q^{k(A)} (\mathrm{e}^K - 1)^{|A|}$$

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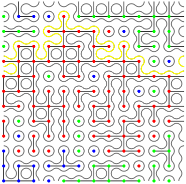

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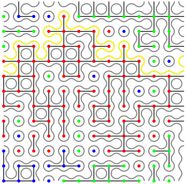
- Here shown for $Q = 3$
- The limit $Q \to 1$ is percolation
- Surrounding loops (**grey**) satisfy the Temperley-Lieb algebra



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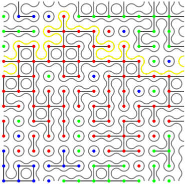

Continuum limit described by (L)CFT or SLE_κ

- Critical exponent in two dimensions exactly computable

Reminders

- 2 and 3-point functions fixed in any d by global conformal invariance alone [Polyakov 1970]
- Extra discrete symmetries must be taken into account as well
- Physical operators are irreducible under such symmetries [Cardy 1999]
  - $O(n)$ symmetry for polymers ($n \to 0$)
  - S_n replica symmetry for systems with quenched disorder ($n \to 0$)

Logarithmic correlations in percolation

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Correlators in bulk percolation in any dimension

- Two and three-point functions in bulk percolation
- Limit $Q \to 1$ of Potts model with S_Q symmetry
- Structure for any d; but universal prefactors only for $d = 2$

Potts model

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Operators acting on one spin

- Most general one-spin operator: $\mathcal{O}(r_i) \equiv \mathcal{O}(\sigma_i) = \sum_{a=1}^{Q} \mathcal{O}_a \delta_{a,\sigma_i}$

$$\underbrace{\delta_{a,\sigma_i}}_{\text{reducible}} = \underbrace{\frac{1}{Q}}_{\text{invariant}} + \underbrace{\left(\delta_{a,\sigma_i} - \frac{1}{Q}\right)}_{\varphi_a(\sigma_i)}$$

- Dimensions of representations: $(Q) = (1) \oplus (Q-1)$
  - Identity operator $1 = \sum_a \delta_{a,\sigma_i}$
  - Order parameter $\varphi_a(\sigma_i)$ satisfies the constraint $\sum_a \varphi_a(\sigma_i) = 0$

Operators acting on two spins

- $Q \times Q$ matrices $\mathcal{O}(r_i) \equiv \mathcal{O}(\sigma_i, \sigma_j) = \sum_{a=1}^{Q} \sum_{b=1}^{Q} \mathcal{O}_{ab} \delta_{a,\sigma_i} \delta_{b,\sigma_j}$
- The Q operators with $\sigma_i = \sigma_j$ decompose as before: $(1) \oplus (Q-1)$
- Other $\frac{Q(Q-1)}{2}$ operators with $\sigma_i \neq \sigma_j$: $(1) + (Q-1) + \left(\frac{Q(Q-3)}{2}\right)$

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Easy representation theory exercise

$$E = \delta_{\sigma_i \neq \sigma_j} = 1 - \delta_{\sigma_i, \sigma_j}$$

$$\phi_a = \delta_{\sigma_i \neq \sigma_j} \left(\varphi_a(\sigma_i) + \varphi_a(\sigma_j) \right)$$

$$\hat{\psi}_{ab} = \delta_{\sigma_i,a} \delta_{\sigma_j,b} + \delta_{\sigma_i,b} \delta_{\sigma_j,a} - \frac{1}{Q-2} (\phi_a + \phi_b) - \frac{2}{Q(Q-1)} E$$

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- Scalar E (energy), vector φ_a (order parameter) and tensor $\hat{\psi}_{ab}$
- Highest-rank tensor obtained from symmetrised combinations of δ's by subtracting suitable multiples of lower-rank tensors
- Constraint $\sum_{a=1}^{Q} \phi_a = 0$ and $\sum_{a(\neq b)} \hat{\psi}_{ab} = 0$

Switch to simpler notation

- $t^{(k,N)}$ is the rank-k tensor acting on N spins $\sigma_1, \ldots, \sigma_N$. By definition it vanishes if any two spins coincide.

$$t^{(1,1)} = (1\delta) - \frac{1}{Q}\left(1t^{(0,1)}\right)$$
$$t^{(1,2)} = (2\delta) - \frac{2}{Q}\left(1t^{(0,2)}\right) ,$$
$$t^{(2,2)} = (2\delta) - \frac{1}{Q-2}\left(2t^{(1,2)}\right) - \frac{2}{Q(Q-1)}\left(1t^{(0,2)}\right)$$

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Extension to rank-k tensors for all $k \leq N$

$$t^{(k,N)} = (\alpha_k \delta) - \sum_{i=0}^{k-1} \gamma_{k,i}\left(\beta_{k,i} t^{(i,N)}\right)$$

$$\alpha_k = \frac{N!}{(N-k)!}, \quad \beta_{k,i} = \frac{k!}{(k-i)!\,i!}, \quad \gamma_{k,i} = \frac{(N-i)!}{(N-k)!}\frac{(Q-i-k)!}{(Q-2i)!}.$$

One-spin results

$$\left\langle t^{(0,1)} t^{(0,1)} \right\rangle = 1 \,,$$

$$\left\langle t_a^{(1,1)} t_b^{(1,1)} \right\rangle = \frac{1}{Q} \left(\delta_{a,b} - \frac{1}{Q} \right) \mathbb{P} \left(\Big| \right) \,.$$

Geometrical interpretation of $t^{(k,N)}$

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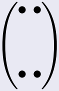
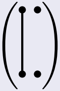
- In general we do not know the probability $\mathbb{P}\left(\Big| \right)$ that the two spins belong to the same Fortuin-Kasteleyn cluster.
- But its large-distance asymptotics is predicted from CFT.

Two-spin results

$$\left\langle t^{(0,2)} t^{(0,2)} \right\rangle = \left(\frac{Q-1}{Q}\right)^2 \left(\mathbb{P}\right.$$

$$+ \mathbb{P}$$

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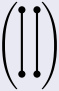
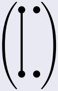

$$,$$

$$\left\langle t_a^{(1,2)} t_b^{(1,2)} \right\rangle = \frac{Q-2}{Q^2} \left(\delta_{a,b} - \frac{1}{Q} \right) \left(\frac{Q-2}{Q} \mathbb{P}\right.$$

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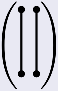
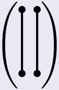

$$\left.\right),$$

$$\left\langle t_{ab}^{(2,2)} t_{cd}^{(2,2)} \right\rangle = \frac{2}{Q^2} \left(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} - \frac{1}{Q-2}(\delta_{ac} + \delta_{bd} + \delta_{ad} + \delta_{bc}) + \frac{2}{(Q-2)(Q-1)} \right) \mathbb{P}$$

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$$\left. + \frac{2}{(Q-2)(Q-1)} \right) \mathbb{P}\left(\begin{smallmatrix} \bullet & \bullet \\ | & | \\ \bullet & \bullet \end{smallmatrix}\right).$$

Physical interpretation

- For $k = N$, the operator $t^{(k,N)}$ makes k clusters propagate
- In 2D equivalent to $2k$-leg watermelon operator

Continuum limit

Energy operator $\varepsilon_i = E - \langle E \rangle$, with $E = \delta_{\sigma_i \neq \sigma_{i+1}}$ invariant

$$\langle \varepsilon(r) \varepsilon(0) \rangle = (Q-1)\tilde{A}(Q) r^{-2\Delta_\varepsilon(Q)},$$

- All correlators of ε_i vanish at $Q = 1$ (true already on the lattice)
- In 2D: exponent $\Delta_\varepsilon(Q) = d - \nu^{-1}$ known exactly

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Two-cluster operator $\hat{\psi}_{ab}(\sigma_i, \sigma_{i+1})$

$$\langle \hat{\psi}_{ab}(r)\hat{\psi}_{cd}(0) \rangle = \frac{2A(Q)}{Q^2} \Bigg(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} - \frac{1}{Q-2}(\delta_{ac} + \delta_{ad} + \delta_{bc} + \delta_{bd}) + \frac{2}{(Q-1)(Q-2)} \Bigg) \times \underbrace{r^{-2\Delta_2(Q)}}_{\text{CFT part}},$$

- In 2D: exponent $\Delta_2 = \frac{(4+g)(3g-4)}{8g}$ known from Coulomb gas

Percolation limit $Q \to 1$

Avoiding the $Q \to 1$ catastrophe

- The "scalar" part of $\langle \hat{\psi}_{ab}(r) \hat{\psi}_{cd}(0) \rangle$ diverges
- But $\Delta_2 = \Delta_\varepsilon = \frac{5}{4}$ at $Q = 1$ in 2D
  - And actually $\Leftrightarrow$ $d^F_{\rm red\ bonds} = \nu^{-1}$ for all $2 \leq d \leq d_{\rm u.c.}$ [Coniglio 1982]
- So we can cure the divergence by mixing the two operators:

$$\tilde{\psi}_{ab}(r) = \hat{\psi}_{ab}(r) + \frac{2}{Q(Q-1)} \varepsilon(r).$$

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Using $\langle \hat{\psi}_{ab} \varepsilon \rangle = 0$, we find a finite limit at $Q = 1$

$$\langle \tilde{\psi}_{ab}(r) \tilde{\psi}_{cd}(0) \rangle = 2A(1) r^{-5/2} \left(\delta_{ac} + \delta_{ad} + \delta_{bc} + \delta_{bd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} \right) + 4A(1) \frac{2\sqrt{3}}{\pi} r^{-5/2} \times \log r,$$

where we assumed that $A(1) = \tilde{A}(1)$.

Where does the log come from?

$$\frac{1}{Q-1}\left(r^{-2\Delta_\varepsilon(Q)} - r^{-2\Delta_2(Q)}\right) \sim 2\,\frac{\mathrm{d}(\Delta_2 - \Delta_\varepsilon)}{\mathrm{d}Q}\bigg|_{Q=1} r^{-5/2}\log r$$

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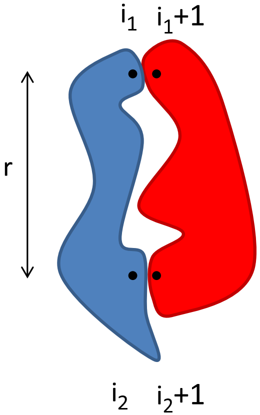
Geometrical interpretation of this logarithmic correlator?

- Idea: Translate the spin expressions into FK cluster formulation
- In addition to the above results, it follows from the representation theory that
  $\langle \varepsilon \hat{\psi}_{ab} \rangle = \langle \varepsilon \phi_a \rangle = \langle \hat{\psi}_{ab} \phi_c \rangle = 0$, and also $\langle \hat{\psi}_{ab} \rangle = \langle \phi_a \rangle = \langle \varepsilon \rangle = 0$.
- All correlators take a simple form in terms of FK clusters

For example we find:

$$\langle \hat{\psi}_{ab}(\sigma_{i_1}, \sigma_{i_1+1}) \hat{\psi}_{cd}(\sigma_{i_2}, \sigma_{i_2+1}) \rangle \propto \mathbb{P}_2(r = r_1 - r_2).$$



$$\mathbb{P}_2(r_1 - r_2) = \mathbb{P}\left[\begin{array}{l} (i_1, i_1+1) \notin \text{ same cluster} \\ (i_2, i_2+1) \notin \text{ same cluster} \\ \text{two clusters } 1 \to 2 \end{array}\right].$$

This probability should thus behave as $r^{-2\Delta_2}$

- Just like in the CFT limit, we introduce

$$\tilde{\psi}_{ab}(r_i) \equiv \tilde{\psi}_{ab}(\sigma_i, \sigma_{i+1}) = \hat{\psi}_{ab}(\sigma_i, \sigma_{i+1}) + \frac{2}{Q(Q-1)} \varepsilon(\sigma_i, \sigma_{i+1})$$

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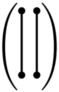
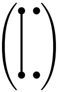
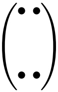
- Exact discrete expression for $\langle \tilde{\psi}_{ab}(r_1) \tilde{\psi}_{cd}(r_2) \rangle$ at $Q = 1$
- Expression in terms of simple percolation probabilities

$\mathbb{P}_2 = \mathbb{P}$ , $\mathbb{P}_1 = \mathbb{P}$ , $\mathbb{P}_0 = \mathbb{P}$ , and $\mathbb{P}_{\neq}$

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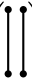
$\mathbb{P}_2 = \mathbb{P}\Bigg($  $\Bigg)$, $\mathbb{P}_1 = \mathbb{P}\Bigg($  $\Bigg)$, $\mathbb{P}_0 = \mathbb{P}\Bigg($  $\Bigg)$, and $\mathbb{P}_{\neq}$

Exact two-point function of $\tilde{\psi}_{ab}$ at $Q = 1$

$$\langle \tilde{\psi}_{ab}(r_1)\tilde{\psi}_{cd}(r_2)\rangle = 2\,(\delta_{ac} + \delta_{ad} + \delta_{bc} + \delta_{bd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) \times \mathbb{P}_2(r) + 4\left[\mathbb{P}_0(r) + \mathbb{P}_1(r) - 2\mathbb{P}_2(r) - \mathbb{P}_{\neq}^2\right].$$

Putting it all together

Exact Two-point function of $\tilde{\psi}_{ab}$ at $Q = 1$

$$\langle \tilde{\psi}_{ab}(r_1) \tilde{\psi}_{cd}(r_2) \rangle = 2 \left(\delta_{ac} + \delta_{ad} + \delta_{bc} + \delta_{bd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} \right) \times \mathbb{P}_2(r) + 4 \left[\mathbb{P}_0(r) + \mathbb{P}_1(r) - 2\mathbb{P}_2(r) - \mathbb{P}^2_{\neq} \right] .$$

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Reminder: CFT Expression

$$\langle \tilde{\psi}_{ab}(r)\tilde{\psi}_{cd}(0)\rangle = 2A(1)r^{-5/2}\left(\delta_{ac} + \delta_{ad} + \delta_{bc} + \delta_{bd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\right) + 4A(1)\frac{2\sqrt{3}}{\pi}r^{-5/2} \times \log r,$$

Numerical check

Comparison with the CFT expression yields geometrical interpretation

$$F(r) \equiv \frac{\mathbb{P}_0(r) + \mathbb{P}_1(r) - \mathbb{P}_{\neq}^2}{\mathbb{P}_2(r)} \sim \underbrace{\frac{2\sqrt{3}}{\pi}}_{\text{universal}} \log r \,,$$

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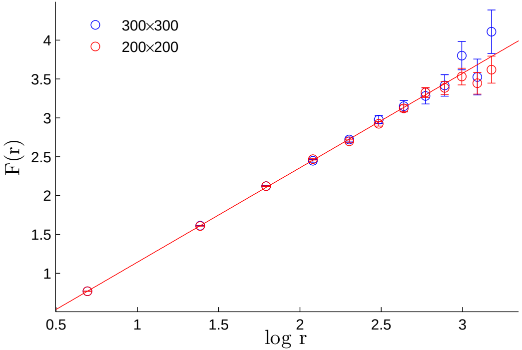

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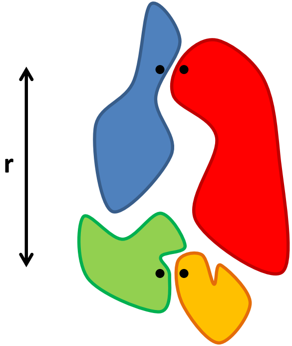

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- Also true for polymers and disordered systems [Cardy 1999]
- Should hold for $2 \leq d \leq d_{\rm u.c.}$, but prefactor depends on d
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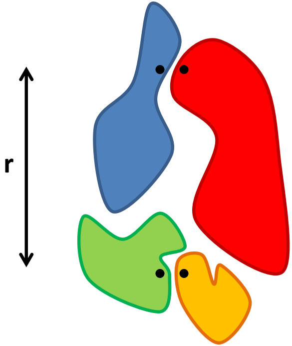

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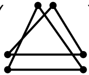
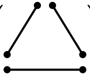
Other interesting logarithmic limits

- $Q \to 0$ (spanning trees, dense polymers, resistor networks . . .)
- $Q \to 2$ (Ising model)
- Logarithms for any integer Q.

Three-point functions on two spins (for $Q = 1$)

Just example, but we have complete results… $\left(\delta = \lim_{Q\to 1} \frac{\Delta_{\hat{\psi}} - \Delta_\varepsilon}{Q-1}\right)$

$\mathbb{P}\Bigg($  $\Bigg) \sim \frac{F_1(1)}{(r_{12}r_{23}r_{31})^{\Delta_{\hat{\psi}}(1)}}$

$\mathbb{P}\Bigg($  $\Bigg) \sim \frac{F_2(1)}{(r_{12}r_{23}r_{31})^{\Delta_{\hat{\psi}}(1)}}$

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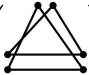
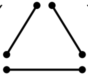
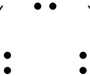
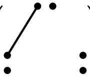
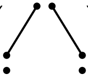
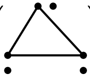
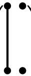
$\mathbb{P}\Big($  $\Big) \sim \frac{F_1(1)}{(r_{12}r_{23}r_{31})^{\Delta_{\hat{\psi}}(1)}}$ $\qquad$ $\mathbb{P}\Big($  $\Big) \sim \frac{F_2(1)}{(r_{12}r_{23}r_{31})^{\Delta_{\hat{\psi}}(1)}}$

$\mathbb{P}\Big($  $\Big) + \mathbb{P}\Big($  $\Big) + \mathbb{P}\Big($  $\Big) + \mathbb{P}\Big($  $\Big)$

$- \mathbb{P}_{\neq}\Big[\mathbb{P}\Big($  $\Big) + \mathbb{P}\Big($  $\Big)\Big] + 2\mathbb{P}_{\neq}^{3}$

$$\sim \frac{F_1(1) - F_2(1)}{(r_{12}r_{23}r_{31})^{\Delta_{\hat{\psi}}(1)}} \left[\mathrm{cst} - \delta^2 \log\left(\frac{r_{12}r_{23}r_{31}}{a^3}\right)^2\right]$$

Conclusion

- Logarithmic observables specific to percolation ($Q = 1$)
  $\Rightarrow$ LCFT as limits of ordinary CFT
- Completion of [Polyakov 1970]'s program, here only for percolation
- Logarithms tend to appear in *disconnected* observables
- Logarithmic dependence can be checked numerically
- Universal prefactor in front of the log closely related to β in LCFT
- In 2D: operator mixing between a primary and a descendent.
  In $d > 2$: S_Q repr. theory predicts mixing between two primaries.
- In 2D: Extremely fertile link to representation theory of non-semisimple algebras, both on the lattice (Temperley-Lieb algebra) and in the continuum limit (Virasoro algebra).

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