Operator Algebras and Conformal Field Theory

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Operator algebraic approach to conformal field theory

$\rightarrow$ Interactions among von Neumann algebra theory, noncommutative geometry, vertex operator algebras and tensor categories through (super)conformal field theory (mainly with S. Carpi, R. Hillier, R. Longo and F. Xu)

Outline of the talk:

1. Chiral conformal field theory (CFT) and Wightman fields
2. Local conformal nets and vertex operator algebras
3. Examples, representation theory and classification
4. From chiral CFT to full CFT
5. From chiral CFT to boundary CFT
6. Supersymmetry and noncommutative geometry

Quantum Field Theory: (mathematical aspects/Wightman axioms)
Mathematical ingredients: Spacetime, its symmetry group, quantum fields on the spacetime.

Our basic spacetime is a Minkowski space since we deal with a relativistic structure. From a mathematical viewpoint, quantum fields (Wightman fields) are certain operator-valued distributions on the spacetime.

Mathematical axiomatization of such operator-valued distributions on a Hilbert space having the vacuum vector and an action of the symmetry group is given by the Wightman axioms. One family of Wightman fields gives one quantum field theory.

A (self-adjoint) smeared field $\langle T, f\rangle$ for a Wightman field T and a test function f supported in O gives an observable in O.

For a fixed O, let $A(O)$ be the von Neumann algebra generated by these observables. We have a family $\{A(O)\}$ of von Neumann algebras of bounded operators parameterized with regions O, and it is called a net. We work on a mathematical axiomatization for a net.

Recently, we have seen much progress in the theory on $(1+1)$-dimensional Minkowski space with a higher symmetry, rather than the classical approach to the 4-dimensional Minkowski space. This is the 2-dimensional conformal field theory.

Start with the $(1+1)$-dimensional Minkowski space. We now take one of the two light rays $\{x = \pm t\}$ and compactify it to S^1. This is now our spacetime. Then take the infinite dimensional Lie group $\mathrm{Diff}(S^1)$ of the orientation preserving diffeomorphism group of S^1 as our spacetime symmetry group.

The Lie group $\mathrm{Diff}(S^1)$ gives a Lie algebra generated by $L_n = -z^{n+1}\dfrac{\partial}{\partial z}$ with $|z| = 1$, $z \in \mathbb{C}$.

The Virasoro algebra is a central extension of its complexification. It is the infinite dimensional Lie algebra generated by $\{L_n \mid n \in \mathbb{Z}\}$ and a central element c with the following relations.

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}c.$$

We have a good understanding of its irreducible unitary highest weight representations, where the central charge c is mapped to a positive scalar. (This value is also called the central charge.)

Fix a nice representation π of the Virasoro algebra, called the vacuum representation, and simply write L_n for $\pi(L_n)$.

Consider $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$, the stress-energy tensor, for $z \in \mathbb{C}$ with $|z| = 1$. It is a Fourier expansion of an operator-valued distribution on S^1. This is a typical example of a quantum field.

Fix an interval (an open arc) $I \subset S^1$ and take a C^∞-function f supported in I. Let $A(I)$ be the von Neumann algebra of bounded linear operators generated by these smeared fields $\langle L, f\rangle$ with various test functions. The family $\{A(I)\}$ gives one realization of a chiral conformal field theory.

We now give operator algebraic axioms of a chiral conformal field theory. Our mathematical object is a family $\{A(I)\}$ parameterized by $I \subset S^1$ and called a local conformal net. Its representation theory will be important later.

Axioms:

1. $I_1 \subset I_2 \Rightarrow A(I_1) \subset A(I_2)$.
2. $I_1 \cap I_2 = \varnothing \Rightarrow [A(I_1), A(I_2)] = 0$. (locality)
3. $\mathrm{Diff}(S^1)$-covariance (conformal covariance)
4. Positive energy
5. Vacuum vector

It is difficult to construct even one example, and two basic sources are Kac-Moody Lie algebras (or the Virasoro algebra) and even lattices in Euclidean spaces.

We also have some methods to give more examples from known ones, notably the coset construction, the orbifold construction, the simple current extension and the extension by a Q-system.

Moonshine

Mysterious relations between the exceptional finite simple group Monster and elliptic modular functions, such as j-function.

A vertex operator algebra (VOA) has appeared as a natural infinite dimensional algebraic structure to understand them. This is an algebraic axiomatization of a family of Wightman fields on S^1. A general theory has been established to understand a single example, the Moonshine VOA constructed by Frenkel-Lepowsky-Meurman. The full Moonshine conjecture has been solved by Borcherds, but still many matters remain mysterious.

Since local conformal nets and vertex operator algebras both describe the same physical theory, we expect a bijective correspondence between the two classes, but no such theorems so far.

The Moonshine VOA

It has been recognized that the Moonshine VOA is an extension of the 48th tensor power of the Virasoro VOA $L(1/2, 0)$ with $c = 1/2$. (It has the corresponding Virasoro net.) Based on an interpretation from a lattice theory, a tensor power of $L(1/2, 0)$ is called a Virasoro frame in general, and its extension is called a framed VOA.

The operator algebraic counterpart of the Moonshine VOA has been constructed by K-Longo based on this theory.

Recently, Lam-Yamauchi constructed a large family of framed VOA's based on theory of binary codes including the Moonshine VOA. It looks difficult to translate their method directly to the operator algebraic setting, but K-Suthichitranont has recently found a quite different method to construct the corresponding local conformal nets.

Representation theory: Superselection sectors

We now consider a representation theory for a local conformal net $\{A(I)\}$ on another Hilbert space (without a vacuum vector).

Each representation is given with an endomorphism of a single von Neumann algebra $A(I_0)$ for an arbitrarily fixed interval I_0. Each $A(I_0)$ is a von Neumann algebra called a factor, so the image of the endomorphism is a subfactor of $A(I_0)$, and it has the Jones index. Its square root is defined to be the dimension of the representation π, whose value is in $[1, \infty]$.

We compose the two endomorphisms. This gives a notion of a tensor product. We have a braided tensor category.
(Doplicher-Haag-Roberts + Fredenhagen-Rehren-Schroer)

We sometimes have only finitely many irreducible representations. Such a situation is called rational. Theory of quantum groups have a similar notion.

K-Longo-Müger gave an operator algebraic characterization of such rationality for a local conformal net $\{A(I)\}$ without using any representation and called it complete rationality. This is characterized by the finiteness of the Jones index for a certain subfactor.

Recall we have a classical notion of induction of a representation. Now introduce a similar construction for local conformal nets.

Let $\{A(I) \subset B(I)\}$ be an inclusion of local conformal nets. We extend an endomorphism of $A(I)$ to a larger factor $B(I)$, using a braiding. ($\alpha^{\pm}$-induction: Longo-Rehren, Xu, Ocneanu, Böckenhauer-Evans-K)

Böckenhauer-Evans-K has shown natural appearance of a modular invariant as $Z_{\lambda,\mu} = \dim \mathrm{Hom}(\alpha^+_\lambda, \alpha^-_\mu)$ from the $\alpha^\pm$-induction, where λ, μ are irreducible representations of the smaller net. Based on this, K-Longo has obtained the following complete classification of local conformal nets with $c < 1$.

(1) Virasoro nets $\{\mathrm{Vir}_c(I)\}$ with $c < 1$.
(2) Their simple current extensions with index 2.
(3) Four exceptionals at $c = 21/22$, $25/26$, $144/145$, $154/155$.

Three exceptionals in the above (3) are identified with coset constructions, but the other one does not seem to be related to any other known constructions, and is given as an extension by a Q-system. Note that this appearance of modular invariants is different from its usual context.

A full conformal field theory

We also have a formulation of a full conformal field theory on a $(1+1)$-dimensional Minkowski space, using two chiral conformal field theories A_L and A_R on two light rays.

A basic example of a $(1+1)$-dimensional local conformal net is of the form $A_L(I) \otimes A_R(J)$ where I, J are intervals on S^1 with one point removed. A general one is an extension of such an example. Now the spacetime region is a rectangle $I \times J$, and it turns out that this type of regions are enough.

We have a satisfactory theory to study such extensions, again based on modular invariants. (This is closely related to the original appearance of modular invariant partition functions in conformal field theory.)

Maximal extensions of $A_L(I) \otimes A_R(J)$ directly produce modular invariants, and using classification of modular invariants, we can often classify maximal $(1+1)$-dimensional local conformal nets.

Note that this is more than just a classification of modular invariant partition functions, since the same partition function could correspond to different local conformal nets. This is known as a 2-cohomology problem, and basically corresponds to the fact that one algebra can have a different multiplicative structure.

Non-maximal extensions of $A_L(I) \otimes A_R(J)$ are intermediate ones. We have a Galois type theory to deal with this type of intermediate objects, and can often classify them. For example, K-Longo has completely classified all extensions of the tensor products of two Virasoro nets with $c < 1$.

Boundary conformal field theory

We now study a net of von Neumann algebras parameterized by certain rectangles $I \times J$ contained in the $(1+1)$-dimensional half Minkowski space $\{(x,t) \mid x > 0\}$. This is our framework of boundary conformal field theory due to Longo-Rehren. They have shown some holographic correspondence principle under the assumption of the so-called Haag duality.

It reduces some study on the half space to that on the boundary. Based on this correspondence, we have a complete classification result for the central charge $c < 1$ (K-Longo-Pennig-Rehren).

The relation between full and boundary conformal field theories have been also studied by Longo-Rehren by shifting the boundary to infinity.

They have basically studied a procedure to remove the boundary. We now consider the converse problem. That is, starting from a full conformal field theory, we would like to add the boundary without affecting the local structure of the theory.

That is, for rectangles in the half space away from the boundary, we would like to keep the original operator algebras, and create a consistent behavior near/at the boundary.

This has been done by Carpi-K-Longo recently with purely operator algebraic technique. A classification problem for boundary conformal field theory arising in this way seems to be feasible and we have some advances, since we have shown this is a finite problem and our previous classification machinery should be useful.

This classification problem is related to the one studied in the context of topological field theory by Fuchs, Kong, Runkel and Schweigert, in terms of tensor categories. We expect their algebraic approach and our operator algebraic approach are essentially the same and expect more relations.

Another related problem is a deformation procedure due to Longo-Witten. Though their method does not involve conformal symmetry and has only less symmetry, it provides a method to produce new examples. Such constructions have been studied by Bischoff and Tanimoto recently.

Another big open problem is to realize a local conformal net as a certain continuum limit of a quantum spin chain. This is a highly challenging one.

Local conformal nets and noncommutative geometry

We have a certain formal similarity between the Laplacian of a closed Riemannian manifold and the conformal Hamiltonian L_0, the generator of the rotation symmetry of S^1. The Dirac operator is a kind of a square root of the former, and its noncommutative generalization is the key notion in the Connes noncommutative geometry. (We consider a noncommutative analogue of the smooth function algebra $C^\infty(M)$ of a closed spin manifold M.)

The $N = 1$ and $N = 2$ super Virasoro algebras produce nice "square roots" of L_0, and this approach produces a nice new connection of our approach to noncommutative geometry through pairing of the K-theory and the entire cyclic cohomology (Carpi-Hillier-K-Longo-Xu).