Topological Charge of Elementary Excitations in Lieb-Liniger Model

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Outline

1. Lieb-Liniger model
2. Periodic boundary conditions
3. Thermodynamic limit at zero temperature
4. Excitations at zero temperature
5. Fermionization
6. Excitations at finite temperature

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Lieb-Liniger model

- Hamiltonian

$$H = \int dx \left[\partial_x \Psi^\dagger(x) \partial_x \Psi(x) + c \Psi^\dagger(x) \Psi^\dagger(x) \Psi(x) \Psi(x) \right]$$

- Canonical equal-time commutation relations:

$$[\Psi(x,t), \Psi^\dagger(y,t)] = \delta(x-y)$$
$$[\Psi(x,t), \Psi(y,t)] = [\Psi^\dagger(x,t), \Psi^\dagger(y,t)] = 0$$

- Heisenberg equation of motion

$$i\partial_t \Psi = -\partial_x^2 \Psi + 2c \Psi^\dagger \Psi \Psi$$

Eigenfunctions

- Eigenfunctions

$$|\psi_N(\lambda_1,\ldots,\lambda_N)\rangle = \frac{1}{\sqrt{N!}}\int d^N z\, \chi_N(z_1,\ldots,z_N|\lambda_1,\ldots,\lambda_N)\Psi^\dagger(z_1)\ldots\Psi^\dagger(z_N)\,|0\rangle$$

$$H\,|\psi_N\rangle = E_N\,|\psi_N\rangle$$

where χ_N is a symmetric function of all z_j.

- Quantum mechanical Hamiltonian

$$\mathcal{H}_N = \sum_{j=1}^{N}\left(-\frac{\partial^2}{\partial z_j^2}\right) + 2c\sum_{N\geq j>k\geq 1}\delta(z_j - z_k)\ ,\ c>0$$

$$\mathcal{H}_N\chi_N = E_N\chi_N$$

χ_N

- $$\chi_N = \left\{ N! \prod_{j>k} [(\lambda_j - \lambda_k)^2 + c^2] \right\}^{-1/2}$$
$$\sum_{\mathcal{P}} (-1)^{[\mathcal{P}]} \exp\left\{ i \sum_{n=1}^{N} z_n \lambda_{\mathcal{P}_n} \right\} \prod_{j>k} [\lambda_{\mathcal{P}_j} - \lambda_{\mathcal{P}_k} - ic\epsilon(z_j - z_k)]$$
$$= \frac{(-i)^{\frac{N(N-1)}{2}}}{\sqrt{N!}} \left\{ \prod_{N\geq j>k\geq 1} \epsilon(z_j - z_k) \right\} \sum_{\mathcal{P}} (-1)^{[\mathcal{P}]} \exp\left\{ i \sum_{k=1}^{N} z_k \lambda_{\mathcal{P}_k} \right\}$$
$$\times \exp\left\{ \frac{i}{2} \sum_{N\geq j>k\geq 1} \epsilon(z_j - z_k) \theta(\lambda_{\mathcal{P}_j} - \lambda_{\mathcal{P}_k}) \right\}$$
where $\theta(\lambda - \mu) = i \ln\left(\frac{ic+\lambda-\mu}{ic-\lambda+\mu} \right)$; $\epsilon(x) = \frac{x}{|x|}$
- χ_N is antisymmetric in momenta (Pauli principle in momentum space)
$$\chi_N(z_1, \ldots, z_N | \lambda_1, \ldots, \lambda_j, \ldots, \lambda_k, \ldots, \lambda_N)$$
$$= -\chi_N(z_1, \ldots, z_N | \lambda_1, \ldots, \lambda_k, \ldots, \lambda_j, \ldots, \lambda_N)$$

χ_N

- Normalization

$$\int_{-\infty}^{\infty} d^N z \, \chi_N^*(z_1, \ldots, z_N|\lambda_1, \ldots, \lambda_N)\chi_N(z_1, \ldots, z_N|\mu_1, \ldots, \mu_N) = (2\pi)^N \prod_{j=1}^{N} \delta(\lambda_j - \mu_j)$$

where the momenta $\{\lambda\}$ and $\{\mu\}$ are ordered:

$$\lambda_1 < \lambda_2 < \ldots < \lambda_N, \qquad \mu_1 < \mu_2 < \ldots < \mu_N$$

- Completeness

$$\int_{-\infty}^{\infty} d^N \lambda \, \chi_N^*(z_1, \ldots, z_N|\lambda_1, \ldots, \lambda_N)\chi_N(y_1, \ldots, y_N|\lambda_1, \ldots, \lambda_N) = (2\pi)^N \prod_{j=1}^{N} \delta(z_j - y_j)$$

where the coordinates $\{z\}$ and $\{y\}$ are ordered:

$$z_1 < z_2 < \ldots < z_N, \qquad y_1 < y_2 < \ldots < y_N$$

- Energy, $E_N = \sum_{j=1}^{N} \lambda_j^2$
- Momentum, $P_N = \sum_{j=1}^{N} \lambda_j$

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Periodic boundary conditions

- Traditionally, we have a periodic wave function:

$$\chi_N(z_1,\dots,z_j+L,\dots,z_N|\lambda_1,\dots,\lambda_N) = \chi_N(z_1,\dots,z_j,\dots,z_N|\lambda_1,\dots,\lambda_N)$$

- Bethe equations

$$\exp\{i\lambda_j L\} = -\prod_{k=1}^{N}\frac{\lambda_j-\lambda_k+ic}{\lambda_j-\lambda_k-ic}, \qquad j=1,\dots,N \tag{1}$$

- Log form of Bethe equations

$$\psi_j = 2\pi\tilde{n}_j, \quad j=1,\dots,N \tag{2}$$

where $\tilde{n}_j$ are integers, and

$$\psi_j = L\lambda_j + \sum_{\substack{k=1\\k\neq j}}^{N}\psi(\lambda_j-\lambda_k)$$

$$\psi = i\ln\left(\frac{\lambda+ic}{\lambda-ic}\right); \quad -2\pi<\psi(\lambda)<0, \quad \operatorname{Im}\lambda=0$$

Periodic boundary conditions

- Using the antisymmetric $\theta(\lambda)$ instead of $\psi(\lambda)$,

$$\theta(\lambda) = \psi(\lambda) + \pi; \quad \theta(\lambda) = -\theta(-\lambda)$$
$$\theta(\lambda) = i \ln \left(\frac{ic + \lambda}{ic - \lambda} \right)$$

- The Bethe equations become

$$L\lambda_j + \sum_{k=1}^{N} \theta(\lambda_j - \lambda_k) = 2\pi n_j \qquad (3)$$

where

$$n_j = \tilde{n}_j + \frac{N-1}{2}$$

and they can be integers (N odd) or half-integers (N even)

Existence of solutions to Bethe equations

- Theorem: Solutions to the Bethe equations (3) exist and can be uniquely parameterized by $\{n_j\}$
- Proof:
  - Yang-Yang action:

$$S = \frac{1}{2} L \sum_{j=1}^{N} \lambda_j^2 - 2\pi \sum_{j=1}^{N} n_j \lambda_j + \frac{1}{2} \sum_{j,k}^{N} \theta_1(\lambda_j - \lambda_k)$$

    where $\theta_1(\lambda) = \int_0^\lambda \theta(\mu) d\mu$
  - Extremum conditions (minima) for S give the Bethe equations (3) :

$$\frac{\partial S}{\partial \lambda_j} = 0$$

Existence of solutions to Bethe equations

- Consider

$$\frac{\partial^2 S}{\partial\lambda_j \partial\lambda_l} = \frac{\partial\psi_j}{\partial\lambda_l} = \psi'_{jl} = \delta_{jl}\left[L + \sum_{m=1}^{N} K(\lambda_j,\lambda_m)\right] - K(\lambda_j,\lambda_l)$$

- where

$$K(\lambda,\mu) = \psi'(\lambda-\mu) = \theta'(\lambda-\mu) = \frac{2c}{c^2+(\lambda-\mu)^2}$$

- So, the Yang-Yang action is convex:

$$\sum_{j,l} \frac{\partial^2 S}{\partial\lambda_j \partial\lambda_l} v_j v_l = \sum_{j=1}^{N} L v_j^2 + \sum_{j>l=1}^{N} K(\lambda_j,\lambda_l)(v_j - v_l)^2 \geq L \sum_{j=1}^{N} v_j^2 > 0,$$

- which means that solutions to Bethe equations exist and are unique.
- If the wavefunction is non-zero, then the Bethe equations are non-degenerate:

$$\int_0^L d^N z\, |\chi_N|^2 = \det\left(\frac{\partial^2 S}{\partial\lambda_j \partial\lambda_l}\right) = \det\left(\frac{\partial\psi_j}{\partial\lambda_l}\right)$$

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Preliminaries

- To pass to the thermodynamic limit, we need:
  - λ_j's are separated by some interval

$$\frac{2\pi(n_j - n_k)}{L} \geq |\lambda_j - \lambda_k| \geq \frac{2\pi(n_j - n_k)}{L(1 + \frac{2D}{c})} \geq \frac{2\pi}{L(1 + \frac{2D}{c})}; \quad j \neq k$$

    where density $D = N/L$
  - Energy $\sum_{j=1}^{N} \lambda_j^2$ is minimized under the condition that $\{\lambda_j\}$ satisfy the Bethe equations, given that

$$n_j = -\left(\frac{N-1}{2}\right) + j - 1, \quad j = 1, \ldots, N$$

Some remarks at $c = +\infty$

- At $c = +\infty$, the Bethe equations become

$$e^{iL\lambda_j} = (-1)^{N+1}$$

or in log form

$$L\lambda_j = 2\pi\tilde{\tilde{n}}_j$$

where $\tilde{\tilde{n}}_j$ can be integers (N odd) or half-integers (N even)

- We get a non-interacting, free fermion model (due to Pauli principle) for $c = +\infty$, with

$$E = \sum_{j=1}^{N} \lambda_j^2 = \frac{2\pi}{L} \sum_{j=1}^{N} \tilde{\tilde{n}}_j^2$$

$$P = \sum_{j=1}^{N} \lambda_j = \frac{2\pi}{L} \sum_{j=1}^{N} \tilde{\tilde{n}}_j$$

Thermodynamic limit at zero temperature

- Thermodynamic limit:

$$N \to \infty; \quad L \to \infty; \quad D = \frac{N}{L} = \text{const.}$$

- Bethe equation of ground state (lowest energy):

$$L\lambda_j + \sum_{k=1}^{N} \theta(\lambda_j - \lambda_k) = 2\pi \left[j - \left(\frac{N+1}{2} \right) \right], \quad j = 1, \dots, N$$

- We define the density of particles in momentum space:

$$\rho(\lambda_k) = \lim \frac{1}{L(\lambda_{k+1} - \lambda_k)} > 0$$

- Integral equation for $\rho(\lambda)$ (Lieb-Liniger equation)

$$\rho(\lambda) - \frac{1}{2\pi} \int_{-q}^{q} K(\lambda, \mu) \rho(\mu) d\mu = \frac{1}{2\pi}$$

- And

$$D = \frac{N}{L} = \int_{-q}^{q} \rho(\lambda) d\lambda$$

Energy

- Let the ground state at $T = 0$ be $|\Omega\rangle$
- Microcanonical ensemble:

$$\frac{\langle\Omega| H |\Omega\rangle}{\langle\Omega|\Omega\rangle} = E_L = L \int_{-q}^{q} \lambda^2 \rho(\lambda) d\lambda$$

- Grand canonical ensemble:
  - Let

$$H_h = H - hQ$$

  where $Q = \int \Psi^\dagger(x)\Psi(x)dx$
  - Energy

$$E_N^h = \sum_{j=1}^{N} (\lambda_j^2 - h)$$

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Excitations

- We start with periodic boundary conditions (3):

$$L\lambda_j + \sum_{k=1}^{N} \theta(\lambda_j - \lambda_k) = 2\pi n_j$$

- We define the "shift function" F:

$$F(\lambda_j|\lambda_p, \lambda_h) \equiv \frac{(\lambda_j - \tilde{\lambda}_j)}{(\lambda_{j+1} - \lambda_j)}.$$

satisfying the following integral equation:

$$F(\lambda_j|\lambda_p, \lambda_h) - \int_{-q}^{q} \frac{d\nu}{2\pi} K(\mu, \nu) F(\nu|\lambda_p, \lambda_h) = \frac{\theta(\mu - \lambda_p) - \theta(\mu - \lambda_h)}{2\pi}$$

One particle + one hole

- We consider excitations consisting of a particle and a hole
- Observable energy (particle + hole):

$$\begin{aligned}\Delta E(\lambda_p, \lambda_h) &= E_{\text{excited}}(\lambda_p, \lambda_h) - E_{\text{gs}} \\ &= \varepsilon_0(\lambda_p) - \varepsilon_0(\lambda_h) + \sum_j [\varepsilon_0(\tilde{\lambda}_j) - \varepsilon_0(\lambda_j)] \\ &= \varepsilon_0(\lambda_p) - \varepsilon_0(\lambda_h) - \int_{-q}^{q} \varepsilon_0'(\mu) F(\mu|\lambda_p, \lambda_h) d\mu\end{aligned}$$

- Observable momentum (particle + hole)

$$\begin{aligned}\Delta P(\lambda_p, \lambda_h) &= P_{\text{excited}}(\lambda_p, \lambda_h) - P_{\text{gs}} \\ &= \lambda_p - \lambda_h - \int_{-q}^{q} F(\mu|\lambda_p, \lambda_h) d\mu \\ &= \lambda_p - \lambda_h + \int_{-q}^{q} [\theta(\lambda_p - \mu) - \theta(\lambda_h - \mu)] \rho(\mu) d\mu\end{aligned}$$

Excitations for even number of particles and holes

- Let us define $\varepsilon(\lambda)$ and $k(\lambda)$ as:

$$\varepsilon(\lambda) - \frac{1}{2\pi}\int_{-q}^{q} K(\lambda,\mu)\varepsilon(\mu)d\mu = \lambda^2 - h \equiv \varepsilon_0(\lambda)\ , \quad \varepsilon(q) = \varepsilon(-q) = 0$$

$$k(\lambda) = \lambda + \int_{-q}^{q} \theta(\lambda-\mu)\rho(\mu)d\mu$$

  where

$$K(\lambda,\mu) = \frac{2c}{c^2 + (\lambda-\mu)^2}\ ;\ \theta(\lambda-\mu) = i\ln\left(\frac{ic+\lambda-\mu}{ic-\lambda+\mu}\right)$$

- Nontrivial integral identities leads to (for 1 particle + 1 hole)

$$\Delta E(\lambda_p,\lambda_h) = \varepsilon(\lambda_p) - \varepsilon(\lambda_h)$$
$$\Delta P(\lambda_p,\lambda_h) = k(\lambda_p) - k(\lambda_h)$$

- For excitations for even number of particles + holes, we get

$$\Delta E = \sum_{\text{particles}} \varepsilon(\lambda_p) - \sum_{\text{holes}} \varepsilon(\lambda_h)$$

$$\Delta P = \sum_{\text{particles}} k(\lambda_p) - \sum_{\text{holes}} k(\lambda_h)$$

Single particle/hole excitation

- We consider single particle excitation first.
- Put λ_p ouside the Fermi sphere, $|\lambda_p| > q$, $\mathrm{Im}\,\lambda_p = 0$
- Bethe equations (3) becomes

$$L\tilde{\lambda}_j + \sum_{k=1}^{N} \theta(\tilde{\lambda}_j - \tilde{\lambda}_k) + \theta(\tilde{\lambda}_j - \tilde{\lambda}_{N+1}) = 2\pi \left[j - \left(\frac{N+1}{2} \right) \right] + \pi, \quad j = 1, \dots, N$$

- However, this excitation is not low-lying, so we implement anti-periodic boundary conditions:

$$\chi_N(z_1, \dots, z_j + L, \dots, z_N | \lambda_1, \dots, \lambda_N) = -\chi_N(z_1, \dots, z_j, \dots, z_N | \lambda_1, \dots, \lambda_N)$$

- And Bethe equations become

$$L\tilde{\lambda}_j + \sum_{k=1}^{N} \theta(\tilde{\lambda}_j - \tilde{\lambda}_k) + \theta(\tilde{\lambda}_j - \tilde{\lambda}_{N+1}) = 2\pi \left[j - \left(\frac{N+1}{2} \right) \right], \quad j = 1, \dots, N$$

$$L\tilde{\lambda}_{N+1} + \sum_{k=1}^{N} \theta(\tilde{\lambda}_{N+1} - \tilde{\lambda}_k) = 2\pi n_{N+1}, \quad n_{N+1} > \frac{N+1}{2}$$

Single particle/hole excitation

- Thus, for a single particle excitation, we get:
  - Energy of particle

$$\varepsilon(\lambda) - \frac{1}{2\pi} \int_{-q}^{q} K(\lambda, \mu) \varepsilon(\mu) d\mu = \lambda^2 - h \equiv \varepsilon_0(\lambda)$$

  - Momentum of particle

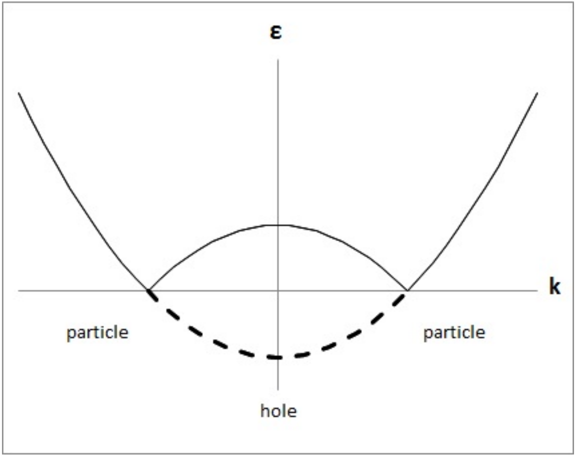
$$k(\lambda_p) = \lambda_p + \int_{-q}^{q} \theta(\lambda_p - \mu) \rho(\mu) d\mu$$

- The single hole excitation is similarly constructed, with
  - Energy of hole: $-\varepsilon(\lambda_h)$, where $-q \leq \lambda_h \leq q$
  - Momentum of hole:

$$\begin{aligned} k_h(\lambda_h) &= -\lambda_h - \int_{-q}^{q} \theta(\lambda_p - \mu) \rho(\mu) d\mu \\ &= -k(\lambda_h) \end{aligned}$$

Single particle/hole excitation

- We can put the particle/hole excitation together:



Excitations in general

- So, we constructed the elementary excitations.
- Arbitrary energy levels can be interpreted as scattering states of several elementary excitations
- In general, for all number of particles and holes,

$$\Delta E = \sum_{\text{particles}} \varepsilon(\lambda_p) - \sum_{\text{holes}} \varepsilon(\lambda_h)$$

$$\Delta P = \sum_{\text{particles}} k(\lambda_p) - \sum_{\text{holes}} k(\lambda_h)$$

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Fermionization

- Fermi-Bose correspondence:

$$\Psi_B^\dagger(x) = \Psi_F^\dagger(x) \exp\left\{ i\pi \int_{-\infty}^{x} \Psi_F^\dagger(z)\Psi_F(z)dz \right\}$$

$$\int d^N z\, \chi_N^B(z_1,\dots,z_N)\Psi_B^\dagger(z_1)\dots\Psi_B^\dagger(z_N)\,|0\rangle$$
$$\rightarrow \int d^N z\, \chi_N^F(z_1,\dots,z_N)\Psi_F^\dagger(z_1)\dots\Psi_F^\dagger(z_N)\,|0\rangle$$

where

$$\chi_N^F(z_1,\dots,z_N) = \prod_{1\le i<j\le N} \epsilon(z_j - z_i)\chi_N^B(z_1,\dots,z_N)\,, \quad \epsilon(x) = \frac{x}{|x|}$$

- Imposing periodic boundary conditions for the fermionic model imply the bosonic boundary conditions:

$$\chi_N^B(x_1 + L,\dots,x_N) = (-1)^{N-1}\chi_N^B(x_1,\dots,x_N)$$

Fermionization

- Elementary excitation is fermion:

$$\Psi_F^\dagger(x) = \Psi_B^\dagger(x) \exp\left\{ i\pi \int_{-\infty}^{x} \Psi_B^\dagger(z)\Psi_B(z)dz \right\}$$
$$\Psi_B^\dagger(x) = \Psi_F^\dagger(x) \exp\left\{ -i\pi \int_{-\infty}^{x} \Psi_F^\dagger(z)\Psi_F(z)dz \right\}$$

- Note that

$$\Psi_B^\dagger(x)\Psi_B(x) = \Psi_F^\dagger(x)\Psi_F(x)$$
$$\exp\left\{ 2\pi i \int_{-\infty}^{x} \Psi_F^\dagger(z)\Psi_F(z)dz \right\} = I = \exp\left\{ 2\pi i \int_{-\infty}^{x} \Psi_B^\dagger(z)\Psi_B(z)dz \right\}$$

Scattering matrix

- Phase of particle 2 going round the box, without particle 1:

$$\psi_2 = L\lambda_2 + \sum_{k=1}^{N} \theta(\lambda_2 - \tilde{\lambda}_k)$$

- Phase of particle 2 going round the box, with particle 1:

$$\psi_2 = L\lambda_2 + \sum_{k=1}^{N} \theta(\lambda_2 - \tilde{\tilde{\lambda}}_k) + \theta(\lambda_2 - \lambda_1)$$

- Scattering phase

$$\delta(\lambda_2, \lambda_1) = \psi_{21} - \psi_2$$

which satisfies the integral equation

$$\delta(\lambda, \mu) - \frac{1}{2\pi} \int_{-q}^{q} K(\lambda, \nu)\delta(\nu, \mu) d\nu = \theta(\theta - \mu)$$

- $\delta(\lambda, \mu) = 2\pi F(\lambda|\mu)$
- Scattering matrix

$$S = \exp\{i\delta(\lambda_2, \lambda_1)\}$$

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Space of thermal equilibrium

- Partition function

$$Z = \mathrm{tr}\left\{e^{-\frac{H}{T}}\right\} = \sum_E \exp\left\{S - \frac{E}{T}\right\}$$

- In thermodynamic limit ($L \to \infty$, $N \to \infty$, $N/L = D$ fixed), space of thermal equilibrium is obtained using steepest descent by solving:

$$\delta\left(S - \frac{E}{T}\right) = 0$$

- Dimension is $\exp\{S\}$
- Subspace shrinks to unique ground state as $T \to 0$
- Similar to typical subspace in quantum information
- To consider excitations, we choose a wavefunction within the space of thermal equilibrium and construct its energy, momentum and scattering matrix using the methods used at zero temperature.
- Expressions are found to be independent of the original choice of wavefunction within the subspace.

Excitations at finite temperature

- We start with periodic boundary conditions (3):

$$L\lambda(x) + \sum_{k=1}^{N} \theta(\lambda(x) - \lambda_k) = 2\pi L x$$

where λ_k are the values of particle momenta in the equilibrium state, and $\lambda(x)$ is λ_j extended to the real line

$$\lambda\left(\frac{n_j}{L}\right) = \lambda_j, \quad n_j \in \{n_k\}$$

- We consider excitations consisting of a particle and a hole
- Bethe equations become

$$L\tilde{\lambda}(x) + \sum_{k=1}^{N} \theta(\tilde{\lambda}(x) - \tilde{\lambda}_k) + \theta(\tilde{\lambda}(x) - \tilde{\lambda}_p) - \theta(\tilde{\lambda}(x) - \tilde{\lambda}_h) = 2\pi L x, \quad j = 1, \ldots, N$$

where $\tilde{\lambda}_k$ are the new momenta and λ_p and λ_h are the respective bare momenta of the particle and hole

Shift function

- We define the “shift function” F:

$$F\left(\lambda\left(\frac{j}{L}\right)\middle|\lambda_p,\lambda_h\right)\equiv\frac{\lambda\left(\frac{j}{L}\right)-\tilde{\lambda}\left(\frac{j}{L}\right)}{\lambda\left(\frac{j+1}{L}\right)-\tilde{\lambda}\left(\frac{j}{L}\right)}.$$

satisfying the following integral equation:

$$2\pi F(\lambda|\lambda_p,\lambda_h)-\int_{-\infty}^{\infty}K(\lambda,\mu)\vartheta(\mu)F(\mu|\lambda_p,\lambda_h)d\mu=\theta(\lambda-\lambda_p)-\theta(\lambda-\lambda_h)$$

which differ from that at zero temperature as:

$$\int_{-q}^{q}d\lambda\rightarrow\int_{-\infty}^{\infty}d\lambda\vartheta(\lambda)$$

where $\vartheta\lambda$ is the Fermi weight

$$\vartheta\lambda=\frac{\rho_p(\lambda)}{\rho_t(\lambda)}=\frac{1}{1+e^{\varepsilon(\lambda)/T}}$$

Energy and momenta

- Observable energy (particle + hole):

$$\Delta E(\lambda_p, \lambda_h) = \varepsilon_0(\lambda_p) - \varepsilon_0(\lambda_h) - \int_{-\infty}^{\infty} \varepsilon_0'(\mu) F(\mu|\lambda_p, \lambda_h) \vartheta(\mu) d\mu$$
$$= \varepsilon(\lambda_p) - \varepsilon(\lambda_h)$$

where $\varepsilon(\lambda)$ is the solution of the Yang-Yang equation:

$$\varepsilon(\lambda) = \lambda^2 - h - \frac{T}{2\pi} \int_{-\infty}^{\infty} K(\lambda, \mu) \ln\left(1 + e^{-\varepsilon(\mu)/T}\right) d\mu$$

- Observable momentum (particle + hole)

$$\Delta P(\lambda_p, \lambda_h) = \lambda_p - \lambda_h - \int_{-\infty}^{\infty} F(\mu|\lambda_p, \lambda_h) \vartheta(\mu) d\mu$$
$$= \lambda_p - \lambda_h + \int_{-\infty}^{\infty} [\theta(\lambda_p - \mu) - \theta(\lambda_h - \mu)] \rho(\mu) d\mu$$

Scattering matrix

- Scattering matrix

$$S = \exp\left\{ i\delta(\lambda_p, \lambda) \right\}, \quad \lambda_p > \lambda_h$$

where the scattering phase δ satisfies the integral equation

$$\delta(\lambda_p, \lambda_h) - \frac{1}{2\pi} \int_{-\infty}^{\infty} K(\lambda_p, \mu) \vartheta(\mu) \delta(\mu, \lambda_h) d\mu = \theta(\lambda_p - \lambda_h)$$

- Indeed, all the observables depend only on the macroscopic variables, and we have constructed stable excitations at finite temperatures.
- Stable excitations at $T > 0$ is possible due to infinitely many conservation laws that prevent particle from decay (lack of thermalization)

References

- For more information on space, time and temperature dependence of correlation functions at $c \to +\infty$, please refer to

Korepin, Vladimir E., N. M. Bogoliubov, and Anatolij G. Izergin. Quantum inverse scattering method and correlation functions. Cambridge university press, 1997.