# Stochastic integrable models and Grothendieck polynomials

**Mathematical Statistical Physics**

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joint work with
**Kazumitsu Sakai (University of Tokyo)**

based on

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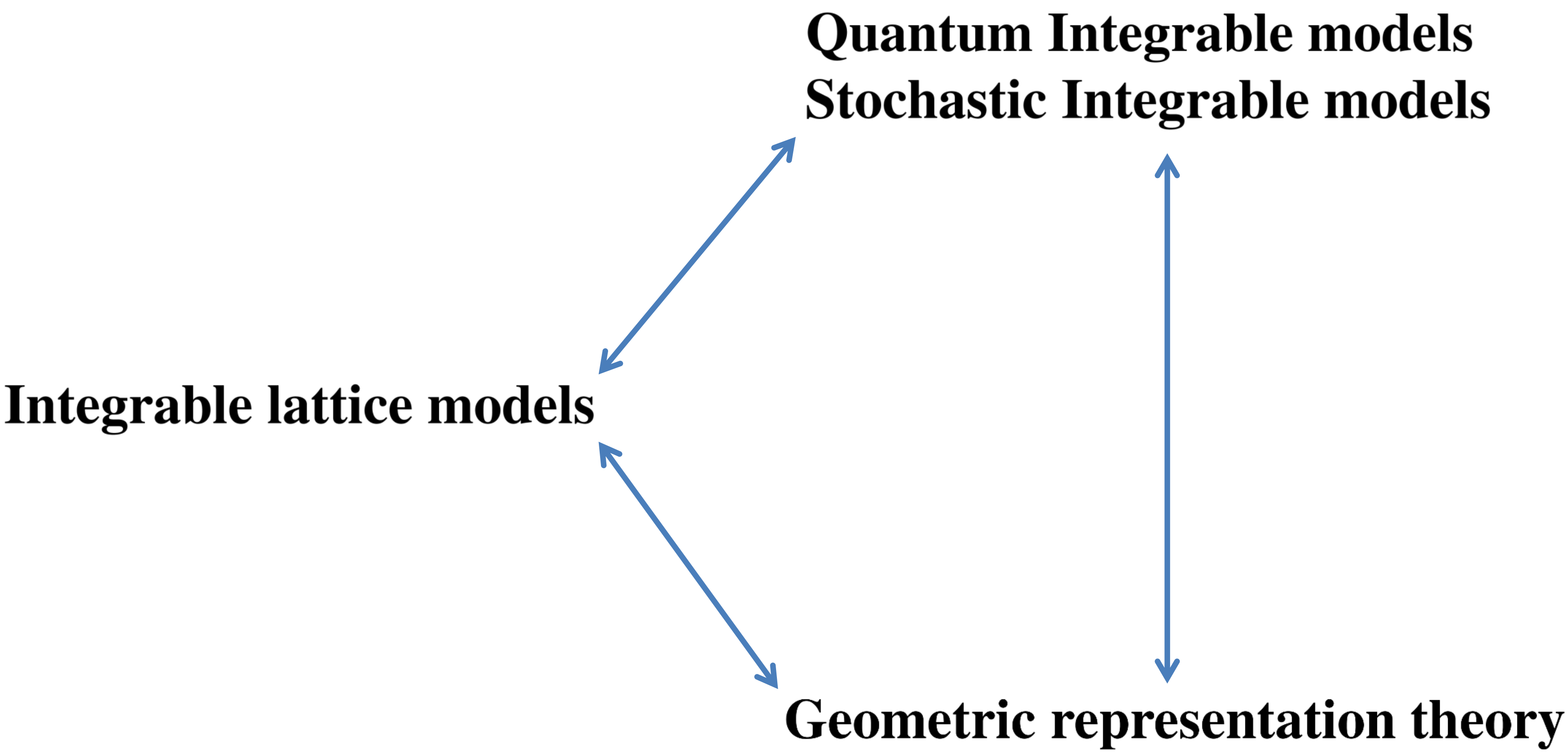
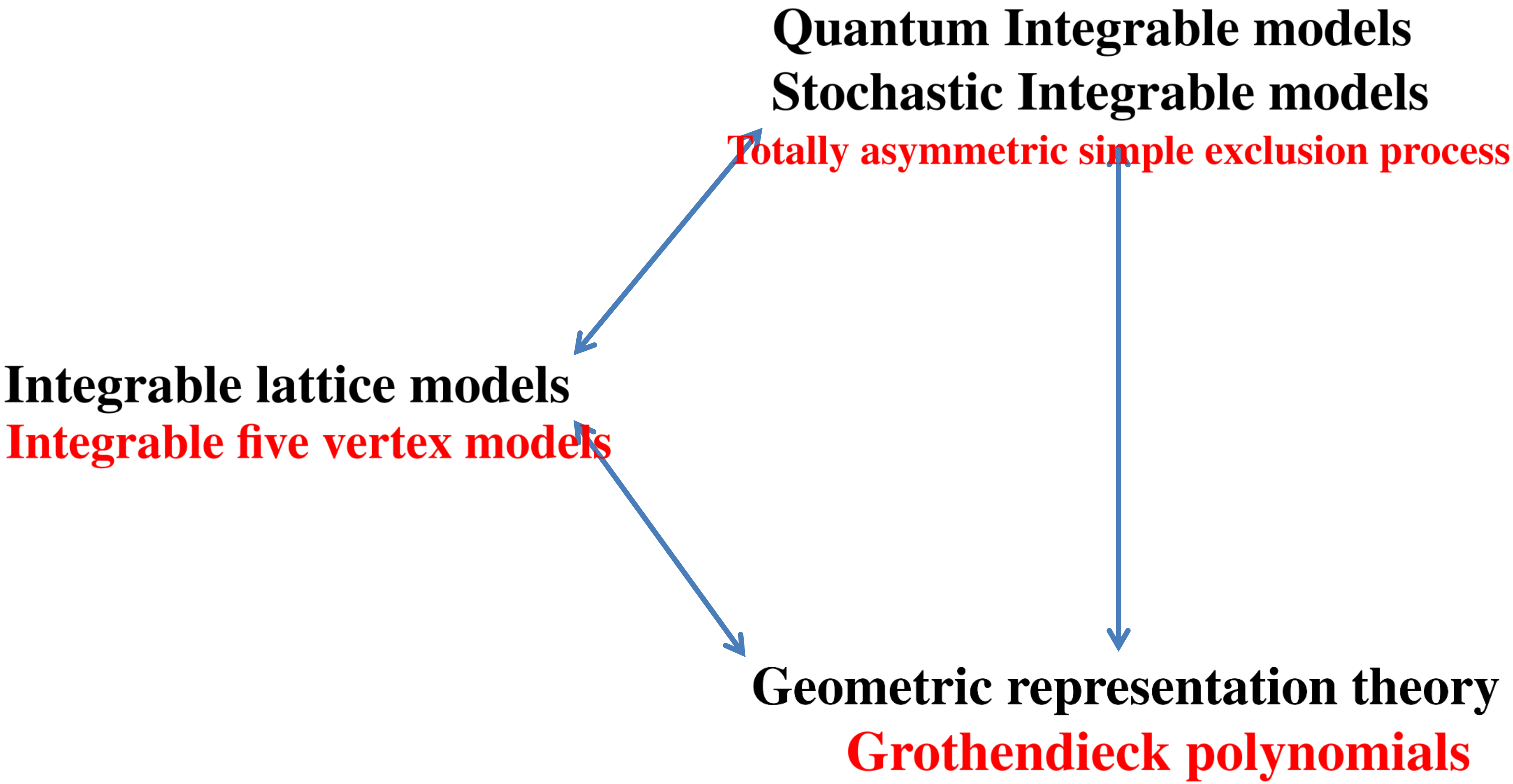

Quantum Integrable models
Stochastic Integrable models
Integrable lattice models
Geometric representation theory


Quantum Integrable models
Stochastic Integrable models
Totally asymmetric simple exclusion process
Integrable lattice models
Integrable five vertex models
Geometric representation theory
Grothendieck polynomials

**Quantum Integrable models**

**Stochastic Integrable models**

**Totally asymmetric simple exclusion process**

**Long time asymptotics**

(cf. M-Sakai-Sato, J. Phys. A 45 (2012) 465004)

**Integrable lattice models**

**Integrable five vertex models**

**Geometric representation theory**

**Grothendieck polynomials**

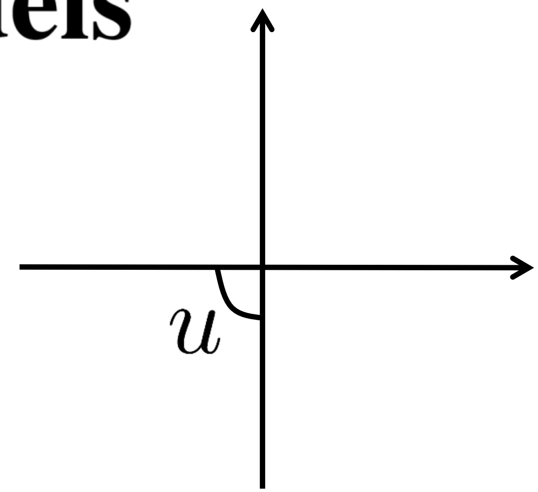
**Cauchy identity**

**Orthogonality**

# Integrable five vertex models

## $L$ operator

$$L_{aj}(u) \in \mathrm{End}(W_a \otimes V_j)$$




## $RLL$ relation

$$R_{ab}(u,v)L_{aj}(u)L_{bj}(v) = L_{bj}(v)L_{aj}(u)R_{ab}(u,v)$$
$$\in \mathrm{End}(W_a \otimes W_b \otimes V_j)$$

auxiliary space ($W_a$, $W_b$) quantum space ($V_j$)

$$W = V = \mathbb{C}^2$$

$$|0\rangle = \binom{1}{0} \quad |1\rangle = \binom{0}{1}$$

# Integrable five vertex models

## *RLL* relation

$$R_{ab}(u,v)L_{aj}(u)L_{bj}(v) = L_{bj}(v)L_{aj}(u)R_{ab}(u,v)$$

## *R* matrix

$$R_{ab}(u,v) \in \mathrm{End}(W_a \otimes W_b)$$

## Yang-Baxter relation

$$R_{ab}(u,v)R_{ac}(u,w)R_{bc}(v,w) = R_{bc}(v,w)R_{ac}(u,w)R_{ab}(u,v)$$

$$R(u,v) = \begin{pmatrix} f(v,u) & 0 & 0 & 0 \\ 0 & 0 & g(v,u) & 0 \\ 0 & g(v,u) & 1 & 0 \\ 0 & 0 & 0 & f(v,u) \end{pmatrix} \quad f(v,u) = \frac{u^2}{u^2-v^2}, \quad g(v,u) = \frac{uv}{u^2-v^2}$$

# Integrable five vertex models

## $L$ operator

$$L_{aj}(u) \in \mathrm{End}(W_a \otimes V_j)$$

## $RLL$ relation

$$R_{ab}(u,v)L_{aj}(u)L_{bj}(v) = L_{bj}(v)L_{aj}(u)R_{ab}(u,v)$$

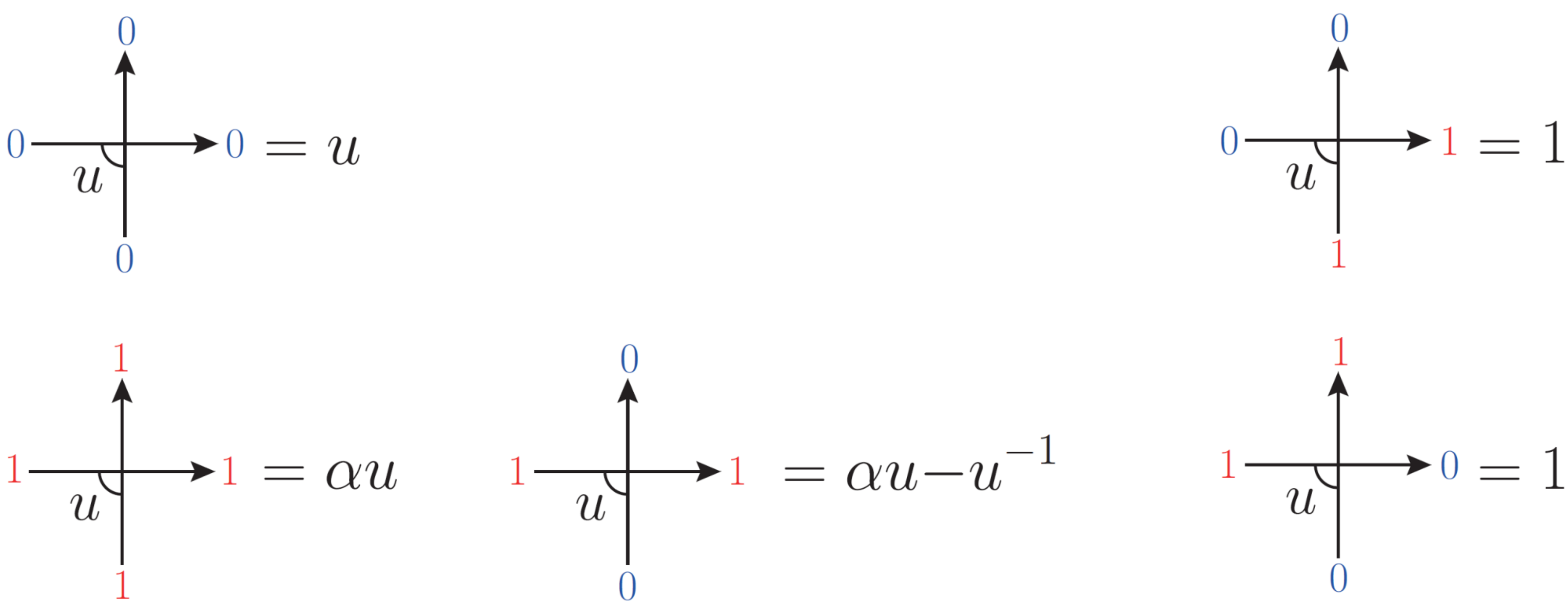
$$L_{aj}(u) = u s_a s_j + \sigma_a^- \sigma_j^+ + \sigma_a^+ \sigma_j^- + (\alpha u - u^{-1})\, n_a s_j + \alpha u\, n_a n_j$$

$$s = (1+\sigma^z)/2 \qquad\qquad n = (1-\sigma^z)/2$$

# Integrable five vertex models

## $L$ operator

$$L_{aj}(u) \in \mathrm{End}(W_a \otimes V_j)$$



# Integrable five vertex models

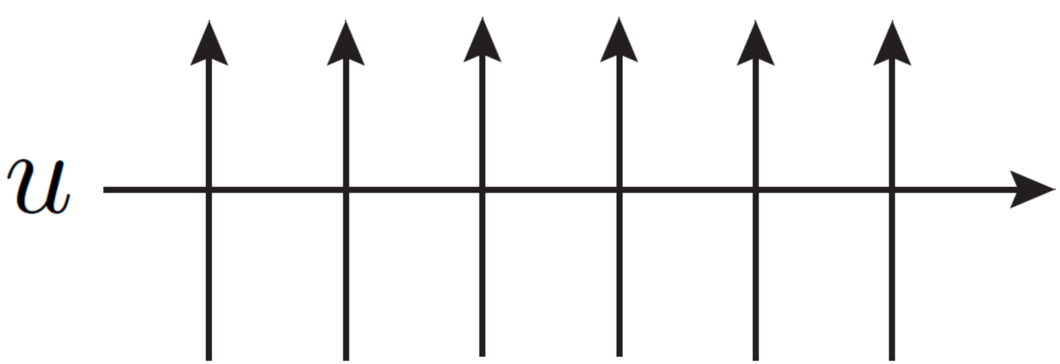
## $L$ operator

$$L_{aj}(u) \in \mathrm{End}(W_a \otimes V_j)$$

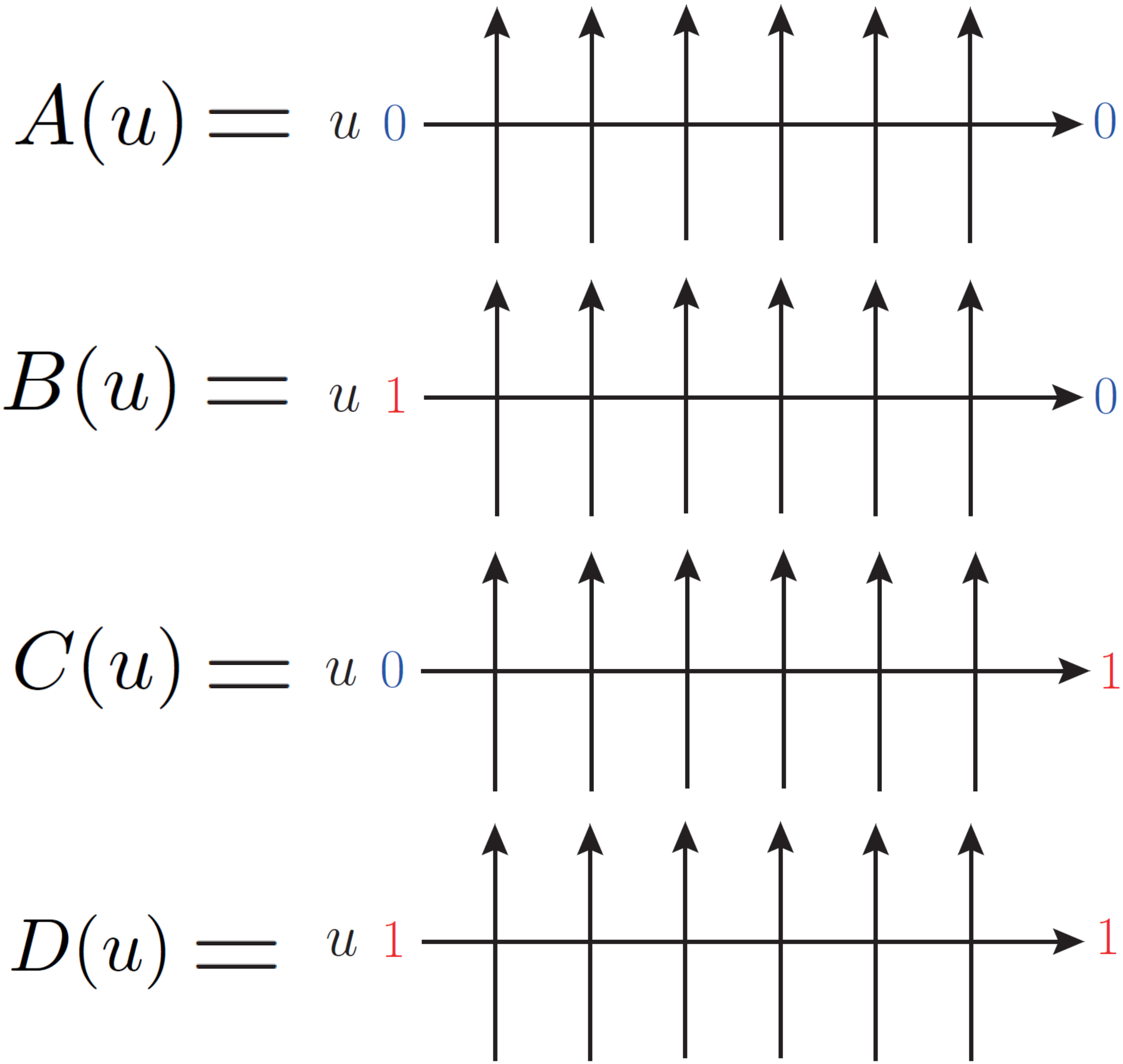
↓

## monodromy matrix

$$T_a(u) = \prod_{i=1}^{M} L_{aj}(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}_a$$



# Integrable five vertex models



# Integrable five vertex models

monodromy matrix

$$T_a(u)$$

$\downarrow$

transfer matrix

$$\tau(u) = \mathrm{Tr}_{W_a} T_a(u)$$

*RLL* relation

$$R_{ab}(u,v)L_{aj}(u)L_{bj}(v) = L_{bj}(v)L_{aj}(u)R_{ab}(u,v)$$

$$\longrightarrow \quad [\tau(u), \tau(v)] = 0$$

# Integrable five vertex models

transfer matrix $\tau(u) = \mathrm{Tr}_{W_a} T_a(u)$

$$\frac{1}{2\sqrt{\alpha}}\frac{\partial}{\partial u}\log\left\{\left(\sqrt{\alpha}u\right)^{-M}\tau(u)\right\}\Bigg|_{u=\frac{1}{\sqrt{\alpha}}}$$

$$=\sum_{j=1}^{M}\left\{\alpha\sigma_j^{+}\sigma_{j+1}^{-}+\frac{1}{4}(\sigma_j^{z}\sigma_{j+1}^{z}-1)\right\}=\mathcal{H}$$

# Integrable five vertex models

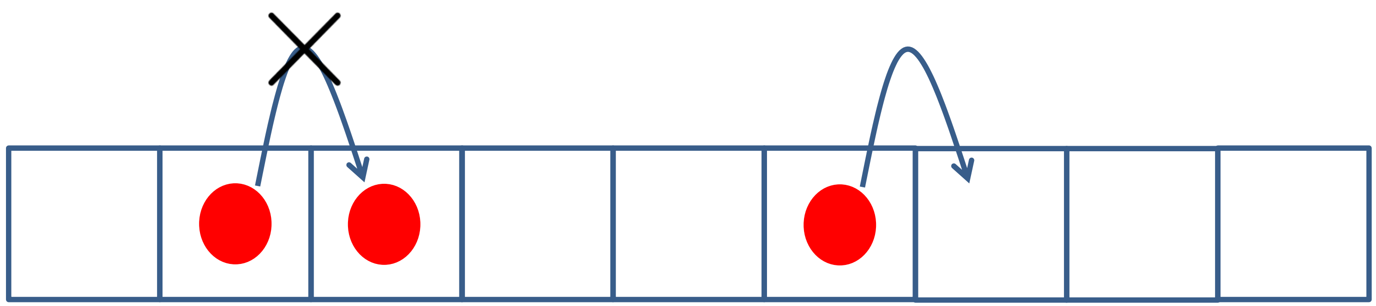
transfer matrix $\tau(u) = \mathrm{Tr}_{W_a} T_a(u)$

$$\frac{1}{2\sqrt{\alpha}} \frac{\partial}{\partial u} \log \left\{ \left( \sqrt{\alpha} u \right)^{-M} \tau(u) \right\} \Bigg|_{u=\frac{1}{\sqrt{\alpha}}}$$

$$= \sum_{j=1}^{M} \left\{ \alpha \sigma_j^+ \sigma_{j+1}^- + \frac{1}{4} (\sigma_j^z \sigma_{j+1}^z - 1) \right\} = \mathcal{H}$$

$\alpha = 1$

Markov matrix of the totally asymmetric simple exclusion process



$\alpha$ generic

(large deviation: Derrida-Lebowitz, Mallick-Prolhac, Simon-Popkov-Schuetz)

# Integrable five vertex models

## $N$ particle state

$$|\psi(\{u\}_N)\rangle = \prod_{j=1}^{N} B(u_j)|\Omega\rangle$$

$$\langle\psi(\{u\}_N)| = \langle\Omega| \prod_{j=1}^{N} C(u_j)$$

$$|\Omega\rangle = |0\rangle_1 \otimes \cdots \otimes |0\rangle_M$$

$$\langle\Omega| = {}_1\langle 0| \otimes \cdots \otimes {}_M\langle 0|$$

# Integrable five vertex models

## Algebraic Bethe ansatz

$$A(u)B(v) = f(u,v)B(v)A(u) + g(v,u)B(u)A(v)$$

$$D(u)B(v) = f(v,u)B(v)D(u) + g(u,v)B(u)D(v)$$

$A(u)|\Omega\rangle = a(u)|\Omega\rangle$ $D(u)|\Omega\rangle = d(u)|\Omega\rangle$ $a(u) = u^M$ $d(u) = (\alpha u - u^{-1})^M$

## Bethe ansatz equation

$$(\alpha - u_k^{-2})^{-M} u_k^{-2N} = (-1)^{N-1} \prod_{j=1}^{N} u_j^{-2}$$

**Cassini oval**

(Golinelli-Mallick)

$\longrightarrow$

$$\mathcal{H}\,|\psi(\{u\}_N)\rangle = \sum_{j=1}^{N} \frac{1}{\alpha u_j^2 - 1}\,|\psi(\{u\}_N)\rangle$$

$$\langle\psi(\{u\}_N)|\,\mathcal{H} = \sum_{j=1}^{N} \frac{1}{\alpha u_j^2 - 1}\,\langle\psi(\{u\}_N)|$$

# Integrable five vertex models

## wavefunctions

$$\langle x_1 \cdots x_N | \psi(\{v\}_N) \rangle$$

$$\langle \psi(\{u\}_N) | x_1 \cdots x_N \rangle$$

$x_1 < \cdots < x_N$ configuration of particles

# Integrable five vertex models

## wavefunctions

$$\langle x_1 \cdots x_N | \psi(\{v\}_N) \rangle = \frac{\prod_{j=1}^N v_j^{M-1} (\alpha v_j^2 - 1)^{-1}}{\prod_{1 \le j < k \le N} (v_k^2 - v_j^2)} \det{}_N \left( v_j^{2k} (\alpha - v_j^{-2})^{x_k} \right)$$

$$\langle \psi(\{u\}_N) | x_1 \cdots x_N \rangle = \frac{\prod_{j=1}^N (\alpha u_j - u_j^{-1})^M u_j^{2N-1}}{\prod_{1 \le j < k \le N} (u_j^2 - u_k^2)} \det{}_N \left( u_j^{-2k} (\alpha - u_j^{-2})^{-x_k} \right)$$

# Integrable five vertex models

$$\langle\psi(\{u\}_N)|x_1\cdots x_N\rangle=\frac{\prod_{j=1}^{N}(\alpha u_j-u_j^{-1})^M u_j^{2N-1}}{\prod_{1\le j<k\le N}(u_j^2-u_k^2)}\mathrm{det}_N(u_j^{-2k}(\alpha-u_j^{-2})^{-x_k})$$

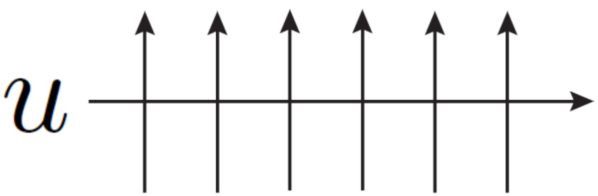
# Integrable five vertex models

$$\langle \psi(\{u\}_N)|x_1\cdots x_N\rangle = \frac{\prod_{j=1}^{N}(\alpha u_j - u_j^{-1})^M u_j^{2N-1}}{\prod_{1\le j<k\le N}(u_j^2-u_k^2)}\det_N(u_j^{-2k}(\alpha-u_j^{-2})^{-x_k})$$

## 1 change the monodromy matrix

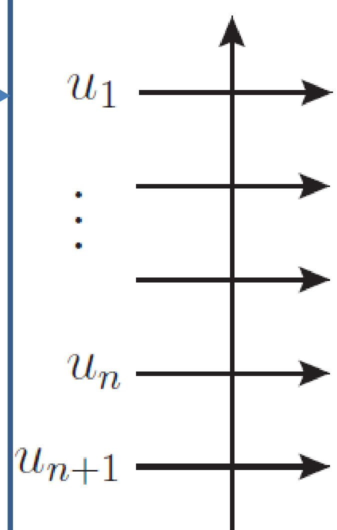
(Golinelli-Mallick cf. quantum transfer matrix)

$$T_a(u) = \prod_{i=1}^{M} L_{aj}(u)$$




$$\mathcal{T}_j(\{u\}_N) = \prod_{a=1}^{N} L_{aj}(u_a)$$

$$= \begin{pmatrix} \mathcal{A}_N(\{u\}_N) & \mathcal{B}_N(\{u\}_N) \\ \mathcal{C}_N(\{u\}_N) & \mathcal{D}_N(\{u\}_N) \end{pmatrix}_j$$

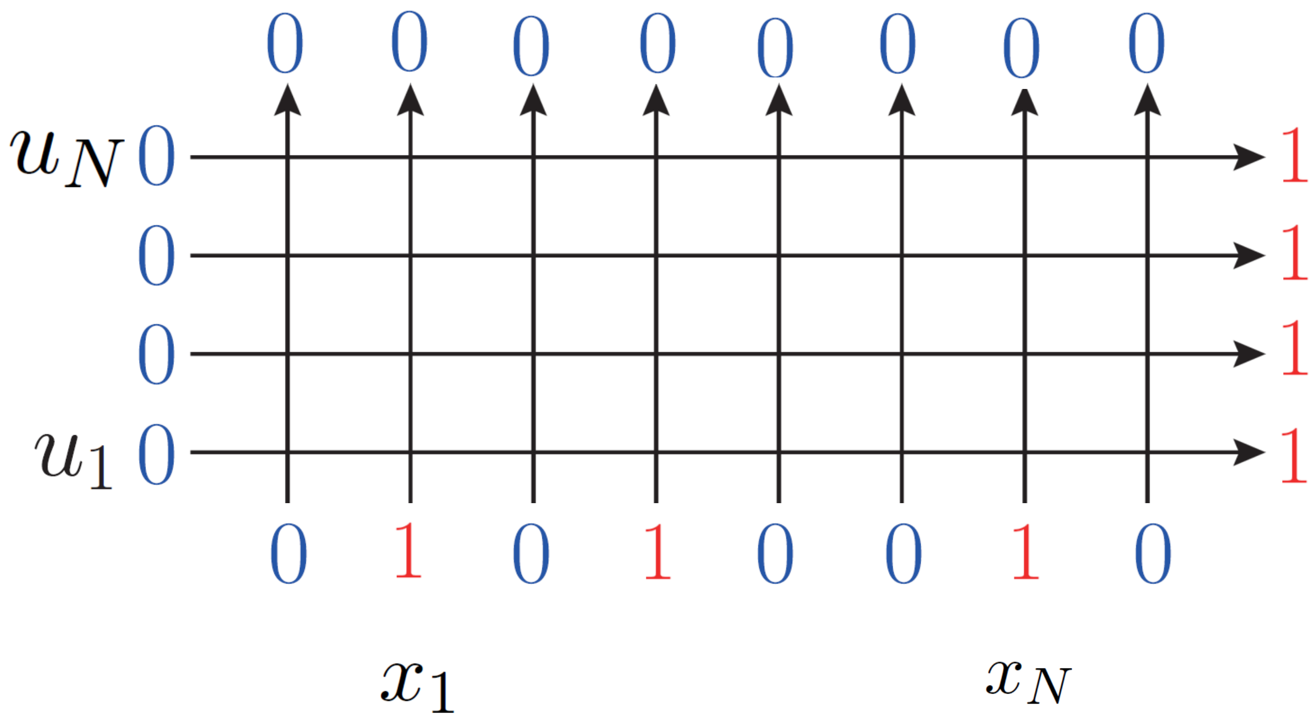


# Integrable five vertex models

$$\langle\psi(\{u\}_N)|x_1\cdots x_N\rangle = \frac{\prod_{j=1}^N(\alpha u_j - u_j^{-1})^M u_j^{2N-1}}{\prod_{1\le j<k\le N}(u_j^2-u_k^2)}\det_N(u_j^{-2k}(\alpha - u_j^{-2})^{-x_k})$$

## 1 change the monodromy matrix

$$\langle\psi(\{u\}_N)|x_1\ldots x_N\rangle =$$

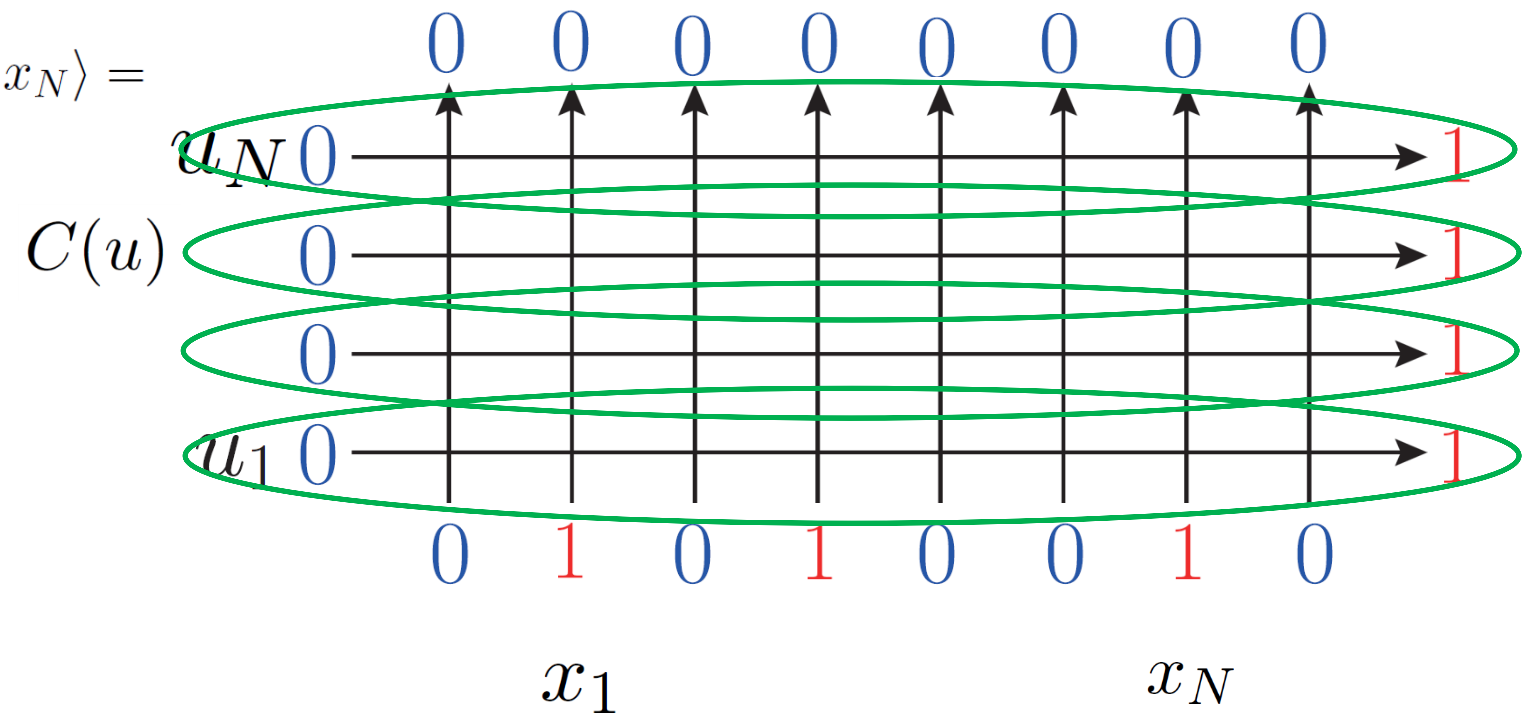


# Integrable five vertex models

$$\langle \psi(\{u\}_N)|x_1 \cdots x_N \rangle = \frac{\prod_{j=1}^{N} (\alpha u_j - u_j^{-1})^M u_j^{2N-1}}{\prod_{1 \le j < k \le N} (u_j^2 - u_k^2)} \det_N (u_j^{-2k} (\alpha - u_j^{-2})^{-x_k})$$

## 1 change the monodromy matrix

$$\langle \psi(\{u\}_N)|x_1 \ldots x_N \rangle =$$



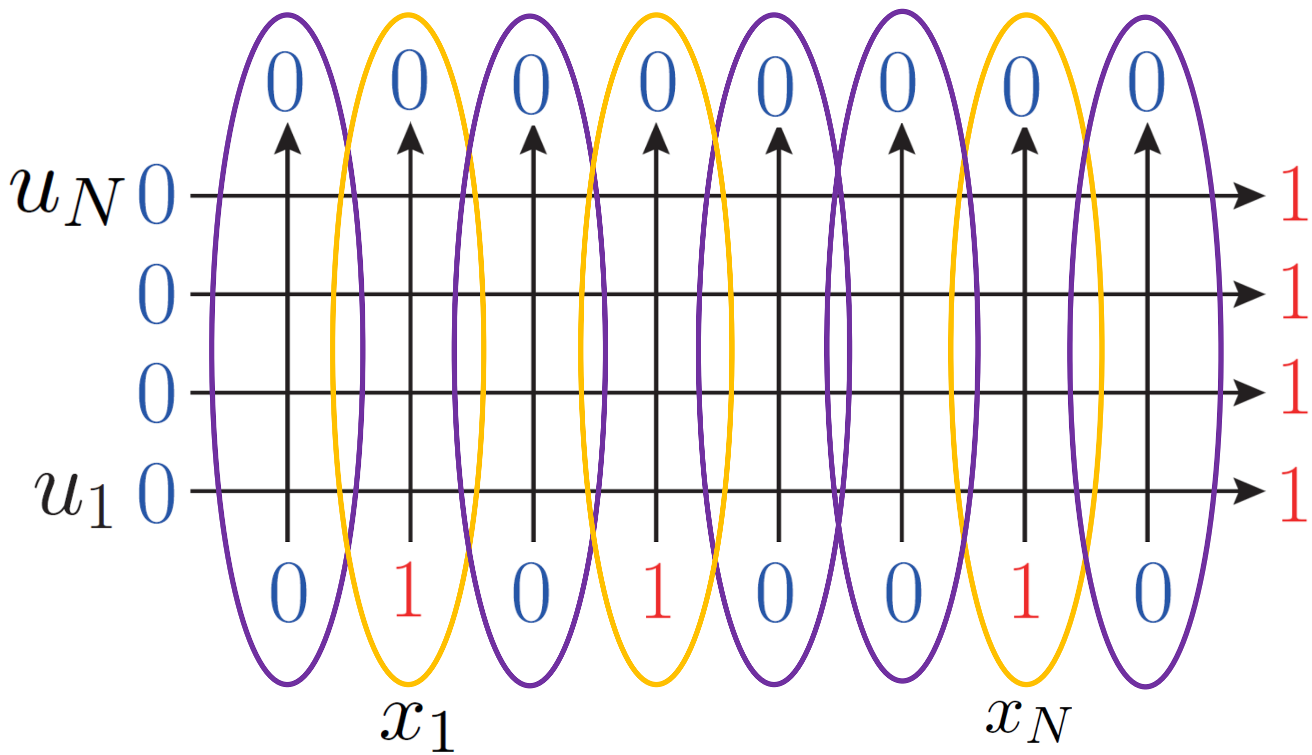
# Integrable five vertex models

$$\langle \psi(\{u\}_N)|x_1 \cdots x_N\rangle = \frac{\prod_{j=1}^N (\alpha u_j - u_j^{-1})^M u_j^{2N-1}}{\prod_{1\le j<k\le N}(u_j^2 - u_k^2)} \det_N (u_j^{-2k}(\alpha - u_j^{-2})^{-x_k})$$

## 1 change the monodromy matrix

$\mathcal{A}_N(\{u\}_N)$ $\mathcal{B}_N(\{u\}_N)$

$\langle \psi(\{u\}_N)|x_1 \ldots x_N\rangle =$



# Integrable five vertex models

$$\langle \psi(\{u\}_N)|x_1 \cdots x_N\rangle = \frac{\prod_{j=1}^{N}(\alpha u_j - u_j^{-1})^M u_j^{2N-1}}{\prod_{1\le j<k\le N}(u_j^2 - u_k^2)} \det{}_N (u_j^{-2k}(\alpha - u_j^{-2})^{-x_k})$$

# 1 change the monodromy matrix

$$\begin{aligned}\langle \psi(\{u\}_N)|x_1 \ldots x_N\rangle &= \mathrm{Tr}_{W^{\otimes N}} \left[ \langle \Omega| \prod_{j=1}^{M} \mathcal{T}_j(\{u\}_N)|x_1 \cdots x_N\rangle P \right] \\ &= \mathrm{Tr}_{W^{\otimes N}} \left[ \mathcal{A}_N^{M-x_N} \mathcal{B}_N \mathcal{A}_N^{x_N - x_{N-1}-1} \ldots \mathcal{B}_N \mathcal{A}_N^{x_2-x_1-1} \mathcal{B}_N \mathcal{A}_N^{x_1-1} P \right]\end{aligned}$$

$$P = |0^N\rangle\langle 1^N|$$

# Integrable five vertex models

$$\langle\psi(\{u\}_N)|x_1\cdots x_N\rangle=\frac{\prod_{j=1}^N(\alpha u_j-u_j^{-1})^M u_j^{2N-1}}{\prod_{1\le j<k\le N}(u_j^2-u_k^2)}\det_N(u_j^{-2k}(\alpha-u_j^{-2})^{-x_k})$$

## 2 turn into determinant forms

$$\langle\psi(\{u\}_N)|x_1\dots x_N\rangle=\mathrm{Tr}_{W^{\otimes N}}\left[\mathcal{A}_N^{M-x_N}\mathcal{B}_N\mathcal{A}_N^{x_N-x_{N-1}-1}\dots\mathcal{B}_N\mathcal{A}_N^{x_2-x_1-1}\mathcal{B}_N\mathcal{A}_N^{x_1-1}P\right]$$

$$\mathcal{B}_n=\sum_{j=1}^n\mathcal{B}_n^{(j)}\qquad\qquad \mathcal{B}_n^{(j)}\mathcal{A}_n=\frac{u_j}{\alpha u_j-u_j^{-1}}\mathcal{A}_n\mathcal{B}_n^{(j)}$$

$$(\mathcal{B}_n^{(j)})^2=0\qquad(\alpha u_j^2-1)\mathcal{B}_n^{(j)}\mathcal{B}_n^{(k)}=-(\alpha u_k^2-1)\mathcal{B}_n^{(k)}\mathcal{B}_n^{(j)},\quad(j\neq k)$$

$$\langle\psi(\{u\}_N)|x_1\dots x_N\rangle=K\det_N\left[u_j^{-2k}\left(\alpha-u_j^{-2}\right)^{-x_k}\right]$$

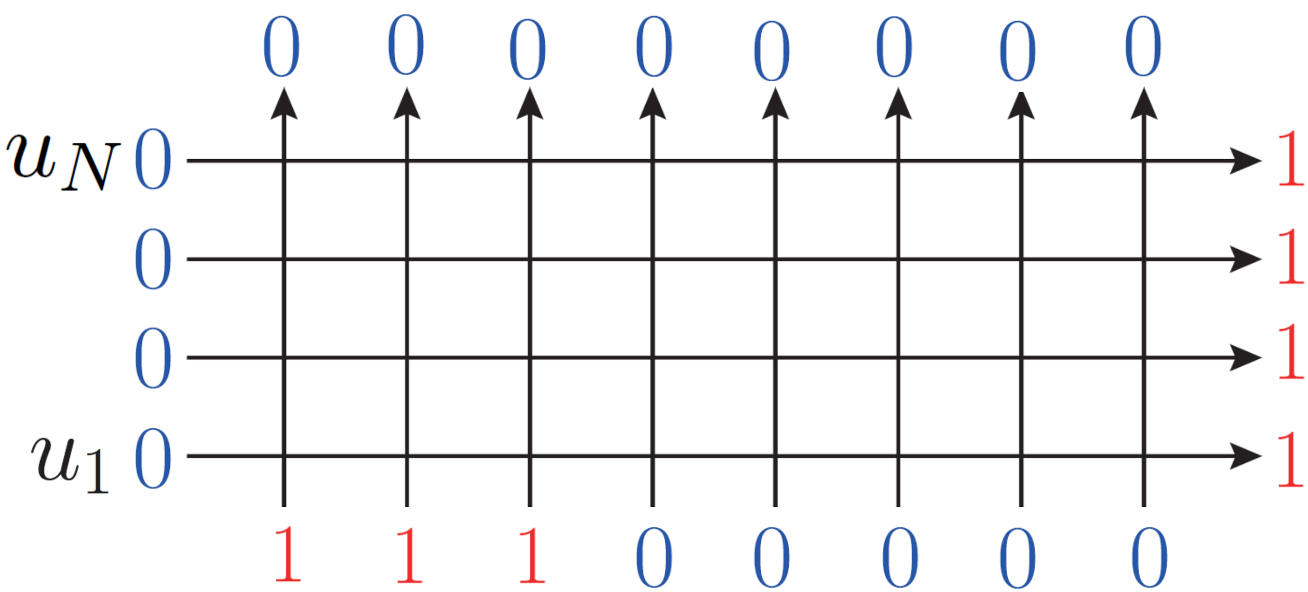
independent of the configurations

# Integrable five vertex models

$$\langle\psi(\{u\}_N)|x_1\cdots x_N\rangle=\frac{\prod_{j=1}^N(\alpha u_j-u_j^{-1})^M u_j^{2N-1}}{\prod_{1\le j<k\le N}(u_j^2-u_k^2)}\det_N(u_j^{-2k}(\alpha-u_j^{-2})^{-x_k})$$

3 fix the overall factor *K* by $x_j=j\ (1\le j\le n)$

$$\langle\psi(\{u\}_N)|12\dots N\rangle=\alpha^{N(N-1)/2}\prod_{j=1}^N u_j^{N-1}(\alpha u_j-u_j^{-1})^{M-N}$$

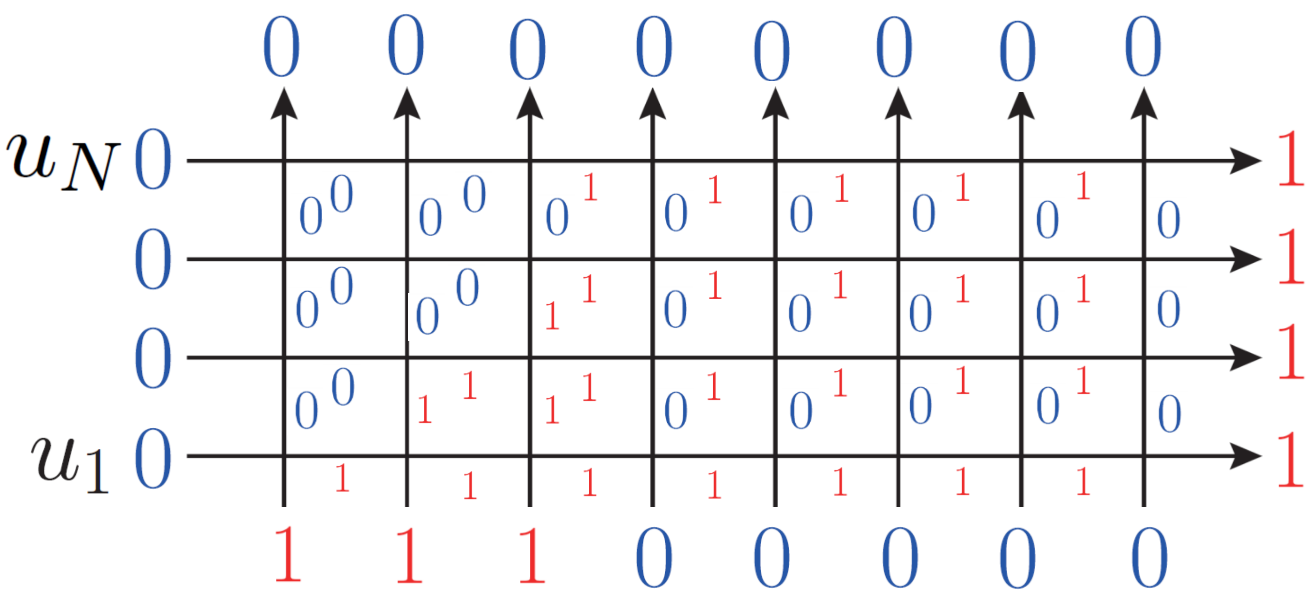


# Integrable five vertex models

$$\langle \psi(\{u\}_N) | x_1 \cdots x_N \rangle = \frac{\prod_{j=1}^{N} (\alpha u_j - u_j^{-1})^M u_j^{2N-1}}{\prod_{1 \le j < k \le N} (u_j^2 - u_k^2)} \det_N (u_j^{-2k} (\alpha - u_j^{-2})^{-x_k})$$

3 fix the overall factor $K$ by $x_j = j \ (1 \le j \le n)$

$$\langle \psi(\{u\}_N) | 12 \ldots N \rangle = \alpha^{N(N-1)/2} \prod_{j=1}^{N} u_j^{N-1} (\alpha u_j - u_j^{-1})^{M-N}$$



# Integrable five vertex models

$$\langle\psi(\{u\}_N)|x_1\cdots x_N\rangle = \frac{\prod_{j=1}^N(\alpha u_j - u_j^{-1})^M u_j^{2N-1}}{\prod_{1\le j<k\le N}(u_j^2-u_k^2)}\det_N(u_j^{-2k}(\alpha - u_j^{-2})^{-x_k})$$

3 fix the overall factor $K$ by $x_j = j\ (1\le j\le n)$

$$\langle\psi(\{u\}_N)|12\ldots N\rangle = \alpha^{N(N-1)/2}\prod_{j=1}^N u_j^{N-1}(\alpha u_j - u_j^{-1})^{M-N}$$

$$\Updownarrow$$

$$\langle\psi(\{u\}_N)|x_1\ldots x_N\rangle = K\det_N\left[u_j^{-2k}\left(\alpha - u_j^{-2}\right)^{-x_k}\right]$$

$$\longrightarrow\quad K = \frac{\prod_{j=1}^N(\alpha u_j - u_j^{-1})^M u_j^{2N-1}}{\prod_{1\le j<k\le 1}(u_j^2-u_k^2)}$$

# Integrable five vertex models

## wavefunctions

$$\langle x_1 \cdots x_N | \psi(\{v\}_N) \rangle = \frac{\prod_{j=1}^N v_j^{M-1} (\alpha v_j^2 - 1)^{-1}}{\prod_{1 \le j < k \le N} (v_k^2 - v_j^2)} \det_N \left( v_j^{2k} (\alpha - v_j^{-2})^{x_k} \right)$$

$$\langle \psi(\{u\}_N) | x_1 \cdots x_N \rangle = \frac{\prod_{j=1}^N (\alpha u_j - u_j^{-1})^M u_j^{2N-1}}{\prod_{1 \le j < k \le N} (u_j^2 - u_k^2)} \det_N \left( u_j^{-2k} (\alpha - u_j^{-2})^{-x_k} \right)$$

# Integrable five vertex models

## wavefunctions

$$\langle x_1 \cdots x_N | \psi(\{v\}_N) \rangle = \frac{\prod_{j=1}^{N} v_j^{M-1} (\alpha v_j^2 - 1)^{-1}}{\prod_{1 \le j < k \le N} (v_k^2 - v_j^2)} \det_N (v_j^{2k} (\alpha - v_j^{-2})^{x_k})$$

$$\langle \psi(\{u\}_N) | x_1 \cdots x_N \rangle = \frac{\prod_{j=1}^{N} (\alpha u_j - u_j^{-1})^M u_j^{2N-1}}{\prod_{1 \le j < k \le N} (u_j^2 - u_k^2)} \det_N (u_j^{-2k} (\alpha - u_j^{-2})^{-x_k})$$

↕

## Grothendieck polynomials

# Grothendieck polynomials

[Lascoux-Schuetzenberger, Fomin-Kirillov, Ikeda-Naruse]

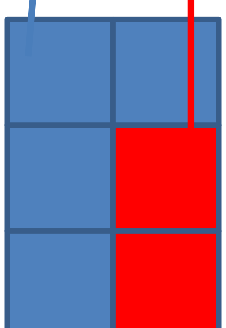
$$G_\lambda(\boldsymbol{z};\beta)=\frac{\det_N\left(z_j^{\lambda_k+N-k}(1+\beta z_j)^{k-1}\right)}{\prod_{1\le j<k\le N}(z_j-z_k)}$$

$$\overline{G}_\lambda(\boldsymbol{z};\beta)=\frac{\det_N\left(z_j^{\lambda_k+N-k}(1+\beta z_j^{-1})^{1-k}\right)}{\prod_{1\le j<k\le N}(z_j-z_k)}$$

structure sheaf of the Schubert variety

in the $K$-theory of the Grassmannian $\mathrm{Gr}(N,\mathbb{C}^M)$

$\lambda_1\le M-N$



$$G_\lambda(\boldsymbol{z};0)=\overline{G}_\lambda(\boldsymbol{z};0)=s_\lambda(\boldsymbol{z})$$

Schur polynomials

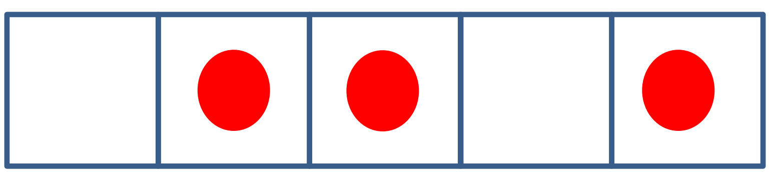
# Integrable five vertex models

wavefunctions

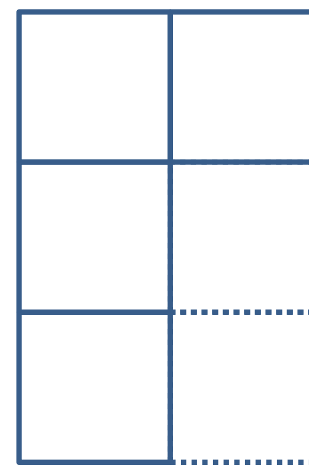
↓

Grothendieck polynomials

$\{x_1,\dots,x_N\}\ (1\le x_1<\cdots<x_N\le M)$ $\qquad$ $\lambda=(\lambda_1,\dots,\lambda_N)\subseteq(M-N)^N$



$$\lambda_j = x_{N-j+1} - N + j - 1$$



$$z_j = \alpha - v_j^{-2}, \quad y_j^{-1} = \alpha - u_j^{-2}, \quad \beta = -1/\alpha$$

# Integrable five vertex models

wavefunctions ⟷ Grothendieck polynomials

$$\langle x_1 \dots x_N | \psi(\{v\}_N) \rangle = \alpha^{N(N-1)/2} \prod_{j=1}^{N} v_j^{M-1} G_\lambda(\boldsymbol{z};\beta)$$

$$\langle \psi(\{u\}_N) | x_1 \dots x_N \rangle = \alpha^{N(N-1)/2} \prod_{j=1}^{N} u_j^{M-1} y_j^{-M+N} (1+\beta y_j^{-1})^{N-1} \overline{G}_\lambda(\boldsymbol{y};\beta)$$

# Integrable five vertex models

wavefunctions ⟷ Grothendieck polynomials

something about integrable five vertex models ⟷ something about Grothendieck polynomials

# Integrable five vertex models

wavefunctions ⟷ Grothendieck polynomials

integrable five vertex models

scalar products

Orthogonality

⟺

Grothendieck polynomials

Cauchy identity

Orthogonality

# Integrable five vertex models

## scalar products

$$\langle \psi(\{u\}_N) | \psi(\{v\}_N) \rangle$$

# Integrable five vertex models

## scalar products

$$\langle\psi(\{u\}_N)|\psi(\{v\}_N)\rangle = \prod_{1\leq j<k\leq N} \frac{1}{(u_j^2-u_k^2)(v_k^2-v_j^2)} \det{}_N Q(\{u\}_N|\{v\}_N)$$

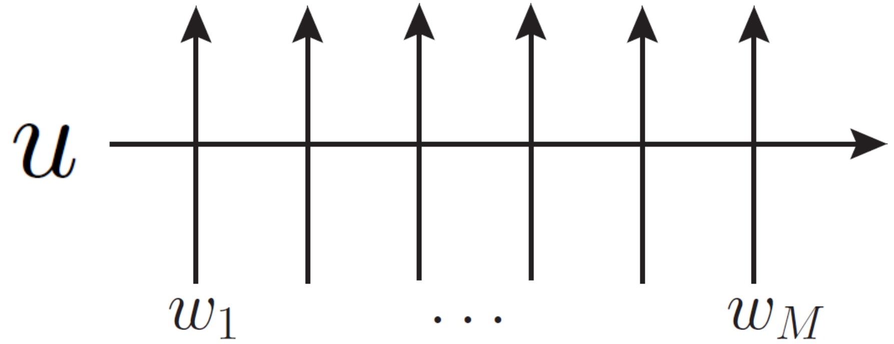
$$Q(\{u\}_N|\{v\}_N)_{jk} = \frac{a(u_j)d(v_k)v_k^{2(N-1)} - a(v_k)d(u_j)u_j^{2(N-1)}}{v_k/u_j - u_j/v_k}$$

# Integrable five vertex models

# scalar products

1 introduce inhomogeneous parameters and prove the generalization from the recursive relation (Izergin-Korepin, Wheeler)

$T_a(u,\{w\}) = \prod_{j=1}^{M} L_{aj}(u/w_j)$




$$\langle \psi(\{u\}_N,\{w\}) | \psi(\{v\}_N,\{w\}) \rangle = \prod_{1\le j<k\le n} \frac{1}{(u_j^2-u_k^2)(v_k^2-v_j^2)} \det{}_N Q(\{u\}_N|\{v\}_N|\{w\})$$

$$Q(\{u\}_N|\{v\}_N|\{w\})_{jk} = \frac{a(u_j,\{w\})d(v_k,\{w\})v_k^{2(N-1)} - a(v_k,\{w\})d(u_j,\{w\})u_j^{2(N-1)}}{v_k/u_j - u_j/v_k}$$

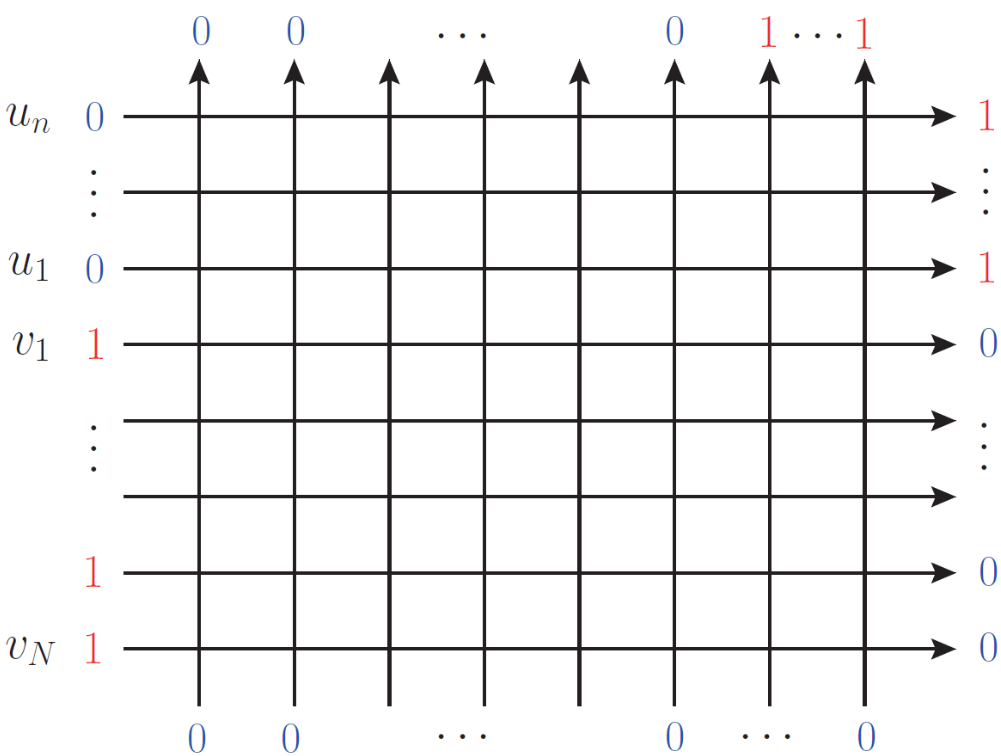
$$a(u,\{w\}) = \prod_{j=1}^{M} \frac{u}{w_j} \qquad d(u,\{w\}) = \prod_{j=1}^{M} \left( \frac{\alpha u}{w_j} - \frac{w_j}{u} \right)$$

# Integrable five vertex models

## scalar products

## 2 introduce the intermediate scalar products

$$S(\{u\}_n|\{v\}_N|\{w\}) = \langle 0^{M-N+n}1^{N-n}|\prod_{j=1}^{n} C(u_j,\{w\})\prod_{k=1}^{N} B(v_k,\{w\})|\Omega\rangle$$




$n = N$ scalar products

$n = 0$

domain wall boundary partition function

# Integrable five vertex models

# scalar products

3 the intermediate scalar products are uniquely determined by the following properties

1. $S(\{u\}_n|\{v\}_N|\{w\})$ *is symmetric with respect to the variables* $\{w_1, \dots, w_{M-N+n}\}$.
2. $\prod_{j=1}^n u_j^{M+2n-2N-1} S(\{u\}_n|\{v\}_N|\{w\})$ *is a polynomial of degree* $M-N+n-1$ *in* $u_n^2$.
3. *The following recursive relations between the intermediate scalar products hold*

$$S(\{u\}_n|\{v\}_N|\{w\})|_{u_n=\pm\alpha^{-1/2}w_{M-N+n}} = \alpha^{N-n-(M-1)/2}(\pm 1)^{M-1}\frac{w_{M+n-N}^M}{\prod_{j=1}^M w_j} S(\{u\}_{n-1}|\{v\}_N|\{w\}).$$

4. *The case* $n=0$ *of the intermediate scalar products has the following form:*

$$S(\{u\}_0|\{v\}_N|\{w\}) = \alpha^{N(N-1)/2}\prod_{j=1}^{N}\prod_{k=1}^{M-N}\left(\frac{\alpha v_j}{w_k}-\frac{w_k}{v_j}\right)\frac{\prod_{j=1}^N v_j^{N-1}}{\prod_{j=M-N+1}^M w_j^{N-1}}.$$

# Integrable five vertex models

# scalar products

3 the intermediate scalar products are uniquely determined by the following properties

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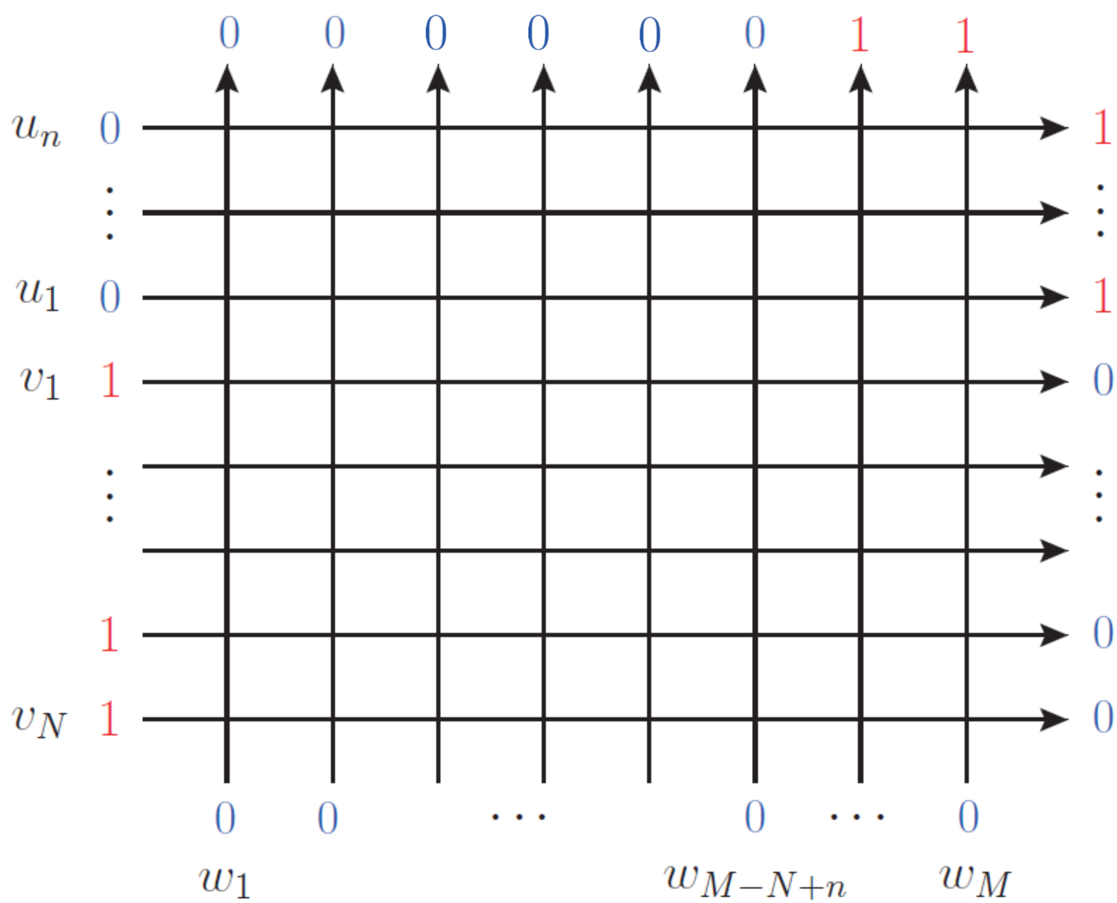
# Integrable five vertex models

# scalar products

$$S(\{u\}_n|\{v\}_N|\{w\}) =$$

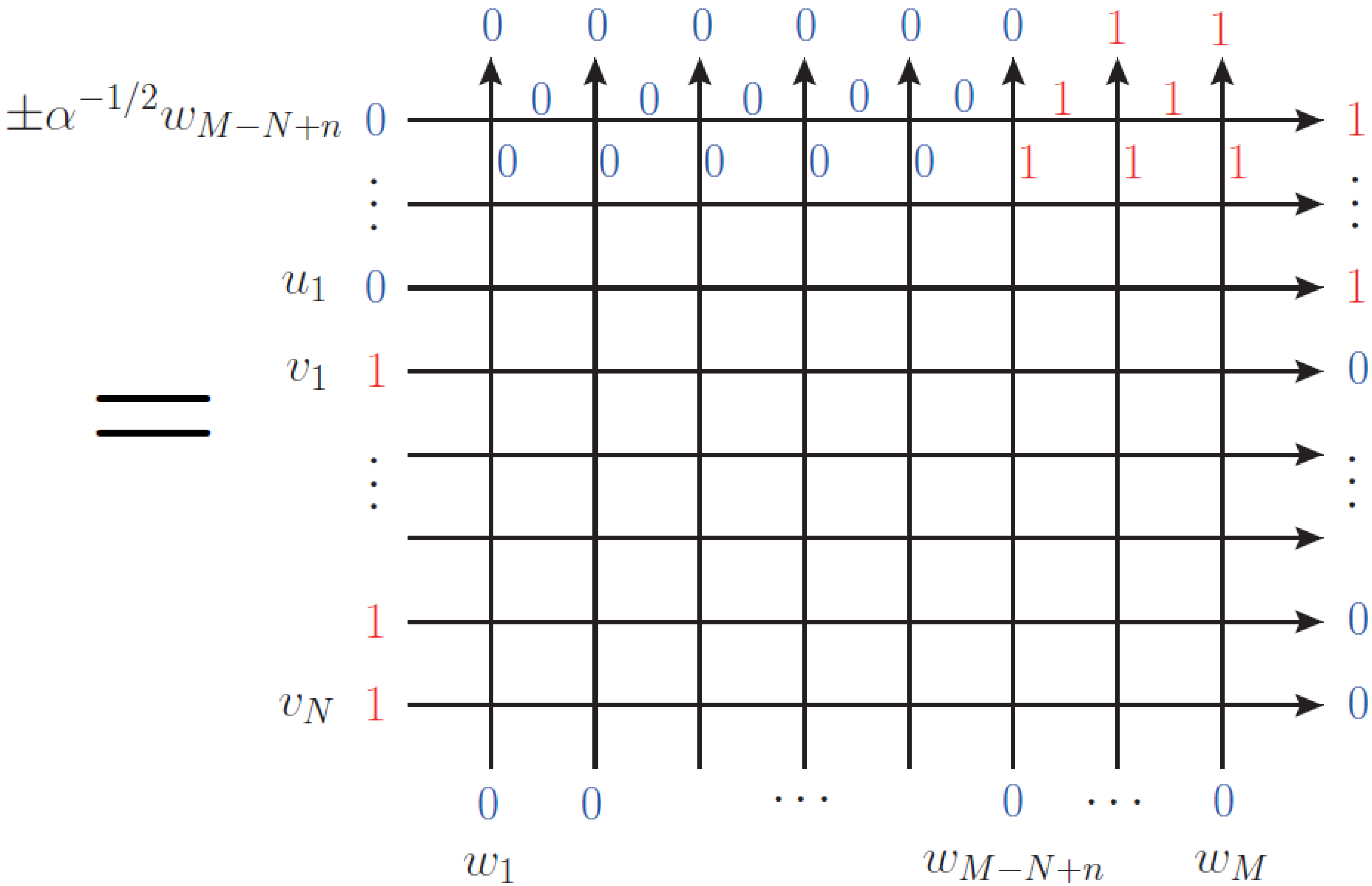


# Integrable five vertex models

## scalar products

$$S(\{u\}_n|\{v\}_N|\{w\})|_{u_n=\pm\alpha^{-1/2}w_{M-N+n}}$$

$$=$$



# Integrable five vertex models

## scalar products

$$S(\{u\}_n|\{v\}_N|\{w\})|_{u_n=\pm\alpha^{-1/2}w_{M-N+n}}$$

$$= \alpha^{N-n-(M-1)/2}(\pm 1)^{M-1}\frac{w_{M+n-N}^M}{\prod_{j=1}^M w_j}S(\{u\}_{n-1}|\{v\}_N|\{w\})$$

# Integrable five vertex models

# scalar products

4 show the determinant representations of the intermediate scalar products

$$S(\{u\}_n|\{v\}_N|\{w\}) = \prod_{M-N+n+1\le j<k\le M} \frac{1}{w_j^2 - w_k^2} \prod_{1\le j<k\le n} \frac{1}{u_j^2 - u_k^2} \prod_{1\le j<k\le N} \frac{1}{v_k^2 - v_j^2} \times \det{}_N Q(\{u\}_n|\{v\}_N|\{w\})$$

$$Q(\{u\}_n|\{v\}_N|\{w\})_{jk} = \begin{cases} \dfrac{a(u_j,\{w\})d(v_k,\{w\})v_k^{2(N-1)} - a(v_k,\{w\})d(u_j,\{w\})u_j^{2(N-1)}}{(v_k/u_j - u_j/v_k)\displaystyle\prod_{l=M-N+n+1}^{M}(u_j^2 - \alpha^{-1}w_l^2)}, & (1\le j\le n) \\ v_k^{2(N-1)} \displaystyle\prod_{\substack{l=1 \\ l\ne M-N+j}}^{M} \left(\frac{\alpha v_k}{w_l} - \frac{w_l}{v_k}\right), & (n+1\le j\le N) \end{cases}$$

# Integrable five vertex models

# scalar products

5 the scalar products is obtained as a special case of the intermediate scalar products

$n = N$ $\qquad S(\{u\}_n|\{v\}_N|\{w\})$

$$\langle \psi(\{u\}_N, \{w\}) | \psi(\{v\}_N, \{w\}) \rangle = \prod_{1 \le j < k \le n} \frac{1}{(u_j^2 - u_k^2)(v_k^2 - v_j^2)} \mathrm{det}_N Q(\{u\}_N|\{v\}_N|\{w\})$$

$$Q(\{u\}_N|\{v\}_N|\{w\})_{jk} = \frac{a(u_j, \{w\}) d(v_k, \{w\}) v_k^{2(N-1)} - a(v_k, \{w\}) d(u_j, \{w\}) u_j^{2(N-1)}}{v_k/u_j - u_j/v_k}$$

$$a(u, \{w\}) = \prod_{j=1}^{M} \frac{u}{w_j} \qquad d(u, \{w\}) = \prod_{j=1}^{M} \left( \frac{\alpha u}{w_j} - \frac{w_j}{u} \right)$$

the original scalar products is recovered by taking $w_j \to 1 \ (1 \le j \le M)$

# Integrable five vertex models

## scalar products

$$\langle \psi(\{u\}_N) | \psi(\{v\}_N) \rangle = \prod_{1 \le j < k \le N} \frac{1}{(u_j^2 - u_k^2)(v_k^2 - v_j^2)} \det_N Q(\{u\}_N | \{v\}_N)$$

$$Q(\{u\}_N | \{v\}_N)_{jk} = \frac{a(u_j) d(v_k) v_k^{2(N-1)} - a(v_k) d(u_j) u_j^{2(N-1)}}{v_k/u_j - u_j/v_k}$$

# Integrable five vertex models

## scalar products

$$\langle \psi(\{u\}_N) | \psi(\{v\}_N) \rangle = \prod_{1 \le j < k \le N} \frac{1}{(u_j^2 - u_k^2)(v_k^2 - v_j^2)} \det{}_N Q(\{u\}_N | \{v\}_N)$$

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$$\langle \psi(\{u\}_N) | \psi(\{v\}_N) \rangle = \sum_{1 \le x_1 < \cdots < x_N \le M} \langle \psi(\{u\}_N) | x_1 \ldots x_N \rangle \langle x_1 \ldots x_N | \psi(\{v\}_N) \rangle$$

# Integrable five vertex models

# scalar products

$$\langle\psi(\{u\}_N)|\psi(\{v\}_N)\rangle = \prod_{1\le j<k\le N} \frac{1}{(u_j^2-u_k^2)(v_k^2-v_j^2)} \det{}_N Q(\{u\}_N|\{v\}_N)$$

$$Q(\{u\}_N|\{v\}_N)_{jk} = \frac{a(u_j)d(v_k)v_k^{2(N-1)} - a(v_k)d(u_j)u_j^{2(N-1)}}{v_k/u_j - u_j/v_k}$$

$$\langle\psi(\{u\}_N)|\psi(\{v\}_N)\rangle = \sum_{1\le x_1<\cdots<x_N\le M} \langle\psi(\{u\}_N)|x_1\ldots x_N\rangle\langle x_1\ldots x_N|\psi(\{v\}_N)\rangle$$

$$\alpha^{N(N-1)/2}\prod_{j=1}^{N} u_j^{M-1} y_j^{-M+N}(1+\beta y_j^{-1})^{N-1}\overline{G}_\lambda(\boldsymbol{y};\beta)$$

$$\alpha^{N(N-1)/2}\prod_{j=1}^{N} v_j^{M-1} G_\lambda(\boldsymbol{z};\beta)$$

# Integrable five vertex models

# scalar products

$$\langle \psi(\{u\}_N) | \psi(\{v\}_N) \rangle = \prod_{1 \leq j < k \leq N} \frac{1}{(u_j^2 - u_k^2)(v_k^2 - v_j^2)} \det{}_N Q(\{u\}_N | \{v\}_N)$$

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$$\alpha^{N(N-1)/2} \prod_{j=1}^{N} u_j^{M-1} y_j^{-M+N} (1 + \beta y_j^{-1})^{N-1} \overline{G}_\lambda(\boldsymbol{y}; \beta)$$

$$\alpha^{N(N-1)/2} \prod_{j=1}^{N} v_j^{M-1} G_\lambda(\boldsymbol{z}; \beta)$$

# Integrable five vertex models

## scalar products

↕

## Cauchy identity for the Grothendieck polynomials

$$\sum_{\lambda \subseteq (M-N)^N} G_\lambda(\boldsymbol{z};\beta)\overline{G}_\lambda(\boldsymbol{y};\beta)$$

$$= \prod_{1 \le j < k \le N} \frac{1}{(z_j - z_k)(y_j - y_k)} \det_N \left[ \frac{(z_j y_k)^M - \{(1+\beta z_j)/(1+\beta y_k^{-1})\}^{N-1}}{z_j y_k - 1} \right]$$

# Integrable five vertex models

Cauchy identity

↓

summation formula for the Grothendieck polynomials

$$\sum_{\lambda \subseteq (M-N)^N} (-\beta)^{\sum_{j=1}^N \lambda_j} G_\lambda(\boldsymbol{z};\beta) = \prod_{1\le j<k\le N} \frac{1}{z_k - z_j} \mathrm{det}_N V^{(M)}$$

$$V_{jk}^{(M)} = \sum_{m=0}^{j-1} (-1)^m (-\beta)^{j-N} \binom{M}{m} (1+\beta z_k)^{m-j+N-1}$$

$$V_{Nk}^{(M)} = -\sum_{m=\max(N-1,1)}^{M} (-1)^m \binom{M}{m} (1+\beta z_k)^{m-1}$$

# Integrable five vertex models

## Orthogonality

# Integrable five vertex models

## Orthogonality

$$\langle x_1 \cdots x_N | x_1' \cdots x_N' \rangle = \prod_{j=1}^{N} \delta_{x_j x_j'}$$

# Integrable five vertex models

## Orthogonality

$$\langle x_1 \cdots x_N | x'_1 \cdots x'_N \rangle = \prod_{j=1}^{N} \delta_{x_j x'_j}$$

$$I = \sum_{\{u\}_N} \frac{|\psi(\{u\}_N)\rangle\langle\psi(\{u\}_N)|}{\langle\psi(\{u\}_N)|\psi(\{u\}_N)\rangle}$$

# Integrable five vertex models

## Orthogonality

$$\frac{\langle x_1 \cdots x_N | \psi(\{u\}_N)\rangle \langle \psi(\{u\}_N) | x_1' \cdots x_N' \rangle}{\langle \psi(\{u\}_N) | \psi(\{u\}_N)\rangle} = \prod_{j=1}^{N} \delta_{x_j x_j'}$$

# Integrable five vertex models

## Orthogonality

$$\frac{\langle x_1 \cdots x_N | \psi(\{u\}_N) \rangle \langle \psi(\{u\}_N) | x'_1 \cdots x'_N \rangle}{\langle \psi(\{u\}_N) | \psi(\{u\}_N) \rangle} = \prod_{j=1}^{N} \delta_{x_j x'_j}$$

$\Updownarrow$

$$\sum_{\{z\}_N} w(\{z\}_N) \overline{G}_\lambda(\boldsymbol{z}^{-1};\beta) G_\mu(\boldsymbol{z};\beta) = \delta_{\lambda\mu}$$

$$w(\{z\}_N) = \left(1 + \sum_{j=1}^{N} \frac{\beta z_j}{M + (M-N)\beta z_j}\right)^{-1} \prod_{\substack{j,k=1 \\ j \neq k}}^{N} (z_j - z_k) \prod_{j=1}^{N} \frac{z_j^{1-N}(1+\beta z_j)}{M + (M-N)\beta z_j}$$

# Integrable five vertex models

## Orthogonality

$$\sum_{\{z\}_N} w(\{z\}_N)\overline{G}_\lambda(\boldsymbol{z}^{-1};\beta)G_\mu(\boldsymbol{z};\beta)=\delta_{\lambda\mu}$$

$$w(\{z\}_N)=\left(1+\sum_{j=1}^{N}\frac{\beta z_j}{M+(M-N)\beta z_j}\right)^{-1}\prod_{\substack{j,k=1\\ j\neq k}}^{N}(z_j-z_k)\prod_{j=1}^{N}\frac{z_j^{1-N}(1+\beta z_j)}{M+(M-N)\beta z_j}$$

## Grothendieck polynomials

||

## discrete orthogonal polynomials on the Cassini oval

$$(1+\beta z_k)^N+(-1)^N z_k^M\prod_{j=1}^{N}(1+\beta z_j)=0$$

# Integrable five vertex models

## Orthogonality

$$\sum_{\{z\}_N} w(\{z\}_N)\overline{G}_\lambda(\boldsymbol{z}^{-1};\beta)G_\mu(\boldsymbol{z};\beta)=\delta_{\lambda\mu}$$

$$w(\{z\}_N)=\left(1+\sum_{j=1}^N \frac{\beta z_j}{M+(M-N)\beta z_j}\right)^{-1}\prod_{\substack{j,k=1\\ j\neq k}}^N (z_j-z_k)\prod_{j=1}^N \frac{z_j^{1-N}(1+\beta z_j)}{M+(M-N)\beta z_j}$$

$$\beta = 0$$

$$\sum_{\{z\}_N} s_\lambda(\boldsymbol{z}^{-1})s_\mu(\boldsymbol{z})\prod_{j=1}^N \frac{z_j}{M}\prod_{\substack{j,k=1\\ j\neq k}}^N (z_j-z_k)\prod_{j=1}^N z_j^{-N}=\delta_{\lambda\mu}$$

# Integrable five vertex models

## Orthogonality

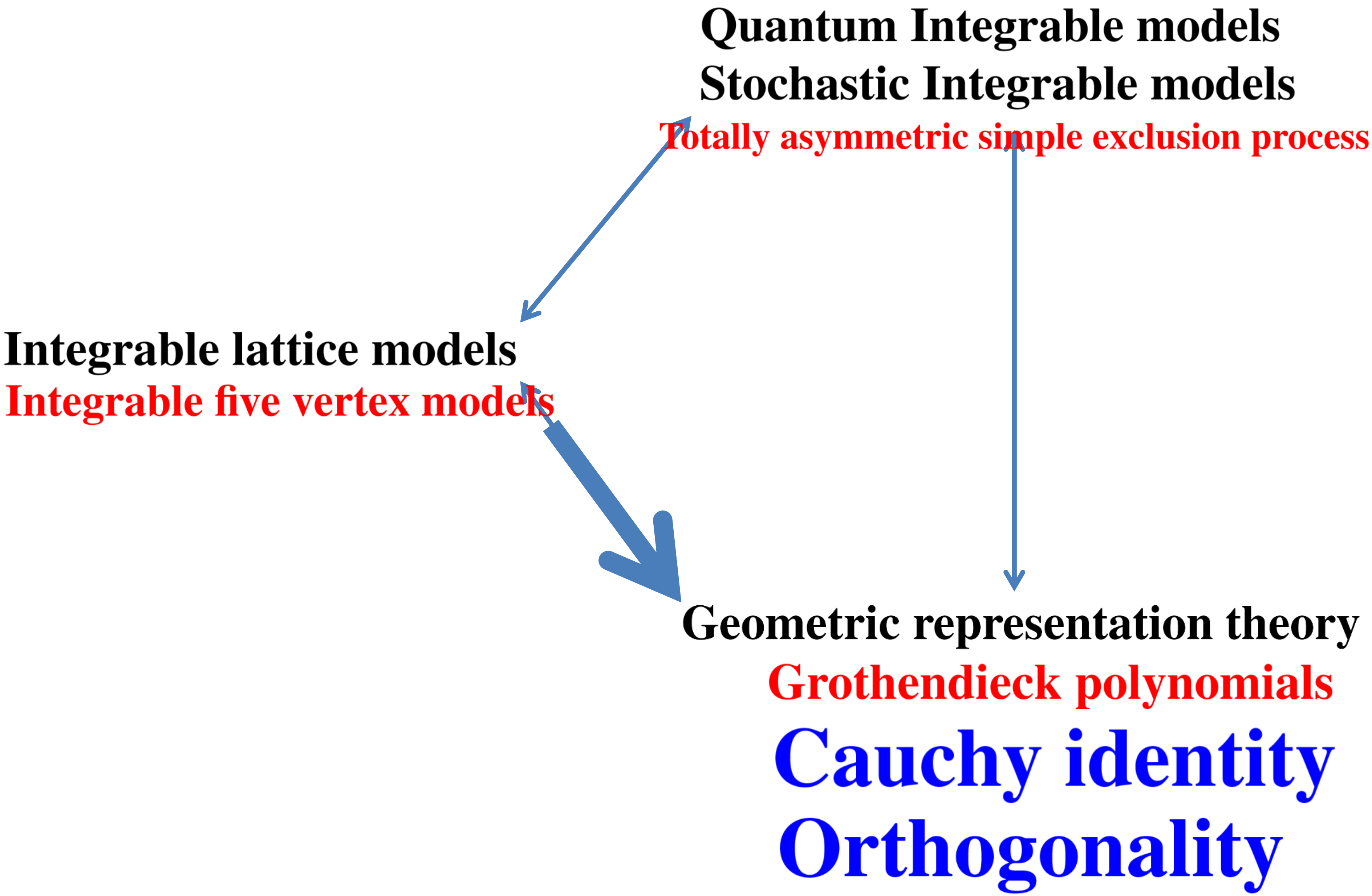
$$\sum_{\{z\}_N} w(\{z\}_N)\overline{G}_\lambda(\boldsymbol{z}^{-1};\beta)G_\mu(\boldsymbol{z};\beta) = \delta_{\lambda\mu}$$

$$w(\{z\}_N) = \left(1+\sum_{j=1}^N \frac{\beta z_j}{M+(M-N)\beta z_j}\right)^{-1} \prod_{\substack{j,k=1\\ j\neq k}}^N (z_j - z_k) \prod_{j=1}^N \frac{z_j^{1-N}(1+\beta z_j)}{M+(M-N)\beta z_j}$$

$$\beta = 0 \quad M \to \infty$$

$$\frac{1}{(2\pi \mathrm{i})^N N!}\oint_{|z_1|=1}\cdots\oint_{|z_N|=1}\prod_{j=1}^N \mathrm{d}z_j s_\lambda(\boldsymbol{z}^{-1}) s_\mu(\boldsymbol{z}) \prod_{\substack{j,k=1\\ j\neq k}}^N (z_j - z_k)\prod_{j=1}^N z_j^{-N} = \delta_{\lambda\mu}$$

the world of free fermion (random matrix)


Quantum Integrable models
Stochastic Integrable models
Totally asymmetric simple exclusion process
Integrable lattice models
Integrable five vertex models
Geometric representation theory
Grothendieck polynomials
Cauchy identity
Orthogonality

**Quantum Integrable models**

**Stochastic Integrable models**

**Totally asymmetric simple exclusion process**

# This is not the whole story!

**Integrable lattice models**

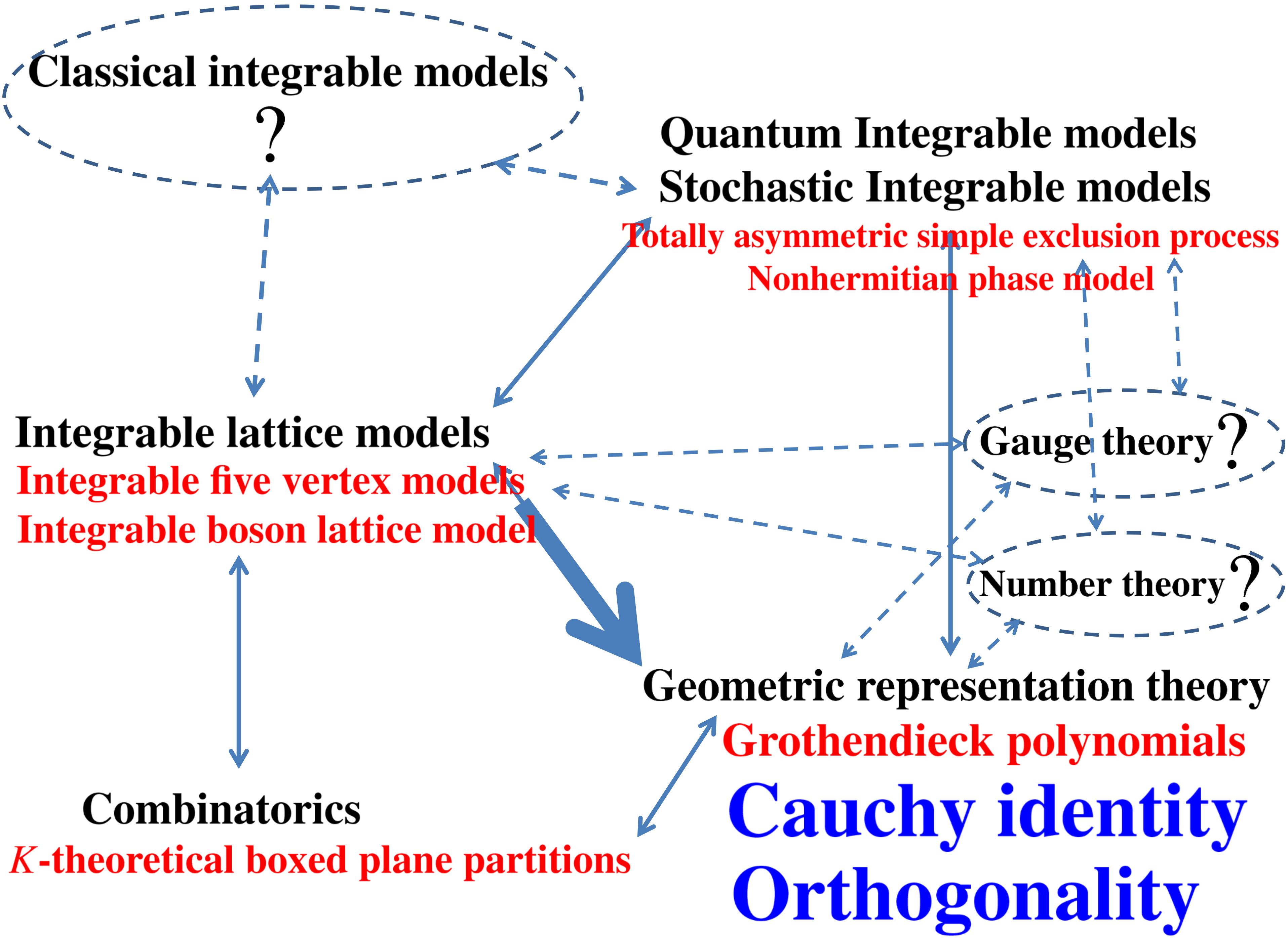
**Integrable five vertex models**

**Geometric representation theory**

**Grothendieck polynomials**

## **Cauchy identity**

## **Orthogonality**


Classical integrable models
?
Quantum Integrable models
Stochastic Integrable models
Totally asymmetric simple exclusion process
Nonhermitian phase model
Integrable lattice models
Integrable five vertex models
Integrable boson lattice model
Gauge theory ?
Number theory ?
Geometric representation theory
Grothendieck polynomials
Combinatorics
K-theoretical boxed plane partitions
Cauchy identity
Orthogonality

# Conclusions and Perspectives

- **Integrable five vertex models** $\longleftrightarrow$ **Grothendieck polynomials**

$\left\{ \begin{array}{l} \textbf{Cauchy identity} \\ \textbf{Orthogonality} \\ \quad\cdots\cdots \end{array} \right.$

naturally follows from the correspondence

- integrable boson models, boxed plane partitions (M-Sakai)
- nonequilibrium dynamics of long-range interaction model (Arita-M-Sakai)
- classical integrable structure?

  $\beta = 0$ KP hierarchy, Toda lattice (Foda-Wheeler-Zuparic,Takasaki,$\cdots$)
- integrable five vertex models provide a natural framework to study the Grothendieck ring

  $\longrightarrow$ extend to the quantum Grothendieck ring

  $\beta = 0$ quantum cohomology ring (Korff-Stroppel)

  (cf. Gerasimov, Nekrasov, Shatashvili)