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Gapped Ground State Phases of Quantum Lattice Systems[1]

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based on joint work with

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Outline

- What is a gapped ground state phase?
- Automorphic equivalence
- Example: the AKLT model
- Symmety protected phases
- Particle-like elementary excitations
- Concluding remarks: locality and its implications

What is a quantum ground state phase?

By phase, here we mean a set of models with qualitatively similar behavior. E.g., a g.s. ψ_0 of one model could evolve to a g.s. ψ_1 of another model in the same phase by some physically acceptable dynamics and in finite time. For finite systems such a dynamics is provided by a quasi-local unitary U_Λ.

When we take the thermodynamic limit

$$\lim_{\Lambda \uparrow \Gamma} U_\Lambda^* A U_\Lambda = \alpha(A), \quad A \in \mathcal{A}_{\Lambda_0},$$

this dynamics converges to an automorphism of the algebra of observables.

To make this more precise, we need some notation.

Ground states of quantum spin models

By quantum spin system we mean quantum systems of the following type:

- (finite) collection of quantum systems labeled by $x \in \Lambda$, each with a finite-dimensional Hilbert space of states $\mathcal{H}_x$. E.g., a spin of magnitude $S = 1/2, 1, 3/2, \ldots$ would have $\mathcal{H}_x = \mathbb{C}^2, \mathbb{C}^3, \mathbb{C}^4, \ldots$.
- The Hilbert space describing the total system is the tensor product
$$\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x.$$
with a tensor product basis $|\{\alpha_x\}\rangle = \bigotimes_{x \in \Lambda} |\alpha_x\rangle$

- The algebra of observables of the composite system is

$$\mathcal{A}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{B}(\mathcal{H}_x) = \mathcal{B}(\mathcal{H}_\Lambda).$$

  If $X \subset \Lambda$, we have $\mathcal{A}_X \subset \mathcal{A}_\Lambda$, by identifying $A \in \mathcal{A}_X$ with $A \otimes \mathbb{1}_{\Lambda\setminus X} \in \mathcal{A}_\Lambda$. Then

$$\mathcal{A} = \overline{\bigcup_\Lambda \mathcal{A}_\Lambda}^{\|\cdot\|}$$

A common choice for Λ's are finite subsets of a graph Γ (often called the 'lattice'). E.g., if $\Gamma = \mathbb{Z}^\nu$, we may consider Λ of the form $[1, L]^\nu$ or $[-N, N]^\nu$.

Interactions, Dynamics, Ground States

The Hamiltonian $H_\Lambda = H_\Lambda^* \in \mathcal{A}_\Lambda$ is defined in terms of an interaction Φ: for any finite set X, $\Phi(X) = \Phi(X)^* \in \mathcal{A}_X$, and

$$H_\Lambda = \sum_{X \subset \Lambda} \Phi(X)$$

For finite-range interactions, $\Phi(X) = 0$ if $\operatorname{diam} X \geq R$.
Heisenberg Dynamics: $A(t) = \tau_t^\Lambda(A)$ is defined by

$$\tau_t^\Lambda(A) = e^{itH_\Lambda} A e^{-itH_\Lambda}$$

For finite systems, ground states are simply eigenvectors of H_Λ belonging to its smallest eigenvalue (sometimes several 'small eigenvalues').

The quasi-locality property is expressed as follows: there exists a rapidly decreasing function $F(d)$, such that for observables A supported in a set $X \subset \Gamma$, there exists $A_d \in \mathcal{A}_{X_d}$ such that

$$\|\alpha(A) - A_d\| \leq \|A\| F(d)$$

where $X_d \subset \Gamma$ is all sites of distance $\leq d$ to X.

α is the time evolution for a given unit of time.

For a short-range real dynamics we would have something of the form

$$\|\tau_t(A) - A_d\| \leq \|A\| F(d - v|t|)$$

where v is often referred to as the Lieb-Robinson velocity.

For $X, Y \subset \Lambda$, s.t., $X \cap Y = \emptyset$, $A \in \mathcal{A}_X, B \in \mathcal{A}_Y$, $AB - BA = [A, B] = 0$: observables with disjoint supports commute. Conversely, if $A \in \mathcal{A}_\Lambda$ satisfies

$$[A, B] = 0, \quad \text{for all } B \in \mathcal{A}_Y$$

then $Y \cap \operatorname{supp} A = \emptyset$.
So, one can find the support of A by looking which B it commutes with.
A more general statement is true: if the commutators are uniformly small for $B \in A_Y$, then A is close to $\mathcal{A}_{\Lambda \setminus Y}$.

Lemma

Let $A \in \mathcal{A}_\Lambda$, $\epsilon \geq 0$, *and* $Y \subset \Lambda$ *be such that*

$$\|[A, B]\| \leq \epsilon\|B\|, \quad \textit{for all } B \in \mathcal{A}_Y \tag{1}$$

then there exists $A' \in A_{\Lambda \setminus Y}$ *such that*

$$\|A' \otimes \mathbb{1} - A\| \leq \epsilon$$

$\Rightarrow$ we can investigate $\operatorname{supp} \tau_t^\Lambda(A)$ by estimating $[\tau_t^\Lambda(A), B]$ for $B \in \mathcal{A}_Y$. This is what Lieb-Robinson bounds are all about.

Lieb-Robinson bounds

Theorem (Lieb-Robinson 1972, Hastings-Koma 2006, N-Sims 2006, N-Ogata-Sims 2006)

Let $F : [0, \infty) \to (0, \infty)$ *be a suitable non-increasing function such that the interaction* Φ *satisfies*

$$\|\Phi\|_F = \sup_{x \neq y} F(d(x,y))^{-1} \sum_{X \ni x,y} \|\Phi(X)\| < \infty$$

Then, $\exists$ *constants* C *and* v *(depending only on* F, $\|\Phi\|_F$, *and the lattice dimension, s. t.* $\forall$ $A \in \mathcal{A}_X$ *and* $B \in \mathcal{A}_Y$,

$$\left\|[\tau_t^\Lambda(A), B]\right\| \leq C\|A\| \, \|B\| \, \min(|X|, |Y|) e^{v|t|} F(d(X,Y)).$$

where $d(X, Y)$ *is the distance between* X *and* Y.

Suppose Φ_0 and Φ_1 are two interactions for two models on lattices Γ.

Each has its set $\mathcal{S}_i$, $i = 0, 1$, of ground states in the thermodynamic limit. I.e., for $\omega \in \mathcal{S}_i$, there exists

$$\psi_{\Lambda_n} \text{ g.s. of } H_{\Lambda_n} = \sum_{X \subset \Lambda_n} \Phi_i(X),$$

for a sequence of $\Lambda_n \in \Gamma$ such that

$$\omega(A) = \lim_{n \to \infty} \langle \psi_{\Lambda_n}, A \psi_{\Lambda_n} \rangle.$$

If the two models are in the same phase, we have a suitably local automorphism α such that

$$\mathcal{S}_1 = \mathcal{S}_0 \circ \alpha$$

This means that for any state $\omega_1 \in \mathcal{S}_1$, there exists a state $\omega_0 \in \mathcal{S}_0$, such that the expectation value of any observable A in ω_1 can be obtained by computing the expectation of $\alpha(A)$ in ω_0:

$$\omega_1(A) = \omega_0(\alpha(A)).$$

The quasi-local character of α guarantees that the support of $\alpha(A)$ need not be much larger than the support of A in order to have this identity with small error.

Where do such quasi-local automorphisms α come form?

Fix some lattice of interest, Γ and a sequence $\Lambda_n \uparrow \Gamma$. Let $\Phi_s, 0, \leq s \leq 1$, be a differentiable family of short-range interactions for a quantum spin system on Γ.
Assume that for some $a, M > 0$, the interactions Φ_s satisfy

$$\sup_{x,y\in\Gamma} e^{ad(x,y)} \sum_{\substack{X\subset\Gamma \\ x,y\in X}} \|\Phi_s(X)\| + |X|\|\partial_s\Phi_s(X)\| \leq M.$$

E.g,

$$\Phi_s = \Phi_0 + s\Psi$$

with both Φ_0 and Ψ finite-range and uniformly bounded.
Let $\Lambda_n \subset \Gamma$, $\Lambda_n \to \Gamma$, be a sequence of finite volumes, satisfying suitable regularity conditions and suppose that the spectral gap above the ground state (or a low-energy interval) of

$$H_{\Lambda_n}(s) = \sum_{X\subset\Lambda_n} \Phi_s(X)$$

is uniformly bounded below by $\gamma > 0$.

Theorem (Bachmann, Michalakis, N, Sims (2012))

Under the assumptions of above, there exist a co-cycle of automorphisms $\alpha_{s,t}$ of the algebra of observables such that $\mathcal{S}(s) = \mathcal{S}(0) \circ \alpha_{s,0}$, for $s \in [0,1]$.
The automorphisms $\alpha_{s,t}$ can be constructed as the thermodynamic limit of the s-dependent "time" evolution for an interaction $\Omega(X,s)$, which decays almost exponentially.

Concretely, the action of the quasi-local transformations $\alpha_s = \alpha_{s,0}$ on observables is given by

$$\alpha_s(A) = \lim_{n\to\infty} V_n^*(s) A V_n(s)$$

where $V_n(s)$ solves a Schrödinger equation:

$$\frac{d}{ds} V_n(s) = i D_n(s) V_n(s), \quad V_n(0) = \mathbb{1},$$

with $D_n(s) = \sum_{X\subset\Lambda_n} \Omega(X,s)$.

The $\alpha_{t,s}$ satisfy a Lieb-Robinson bound of the form

$$\|[\alpha_{t,s}(A), B]\| \leq \|A\|\|B\| \min(|X|, |Y|) e^{C|t-s|} F(d(X, Y)),$$

where $A \in \mathcal{A}_X, B \in \mathcal{A}_Y$, $d(X, Y)$ is the distance between X and Y. $F(d)$ can be chosen of the form

$$F(d) = Ce^{-b\frac{d}{(\log d)^2}}.$$

with $b \sim \gamma/v$, where γ and v are bounds for the gap and the Lieb-Robinson velocity of the interactions Φ_s, i.e., $b \sim a\gamma M^{-1}$.

The AKLT model (Affleck-Kennedy-Lieb-Tasaki, 1987)

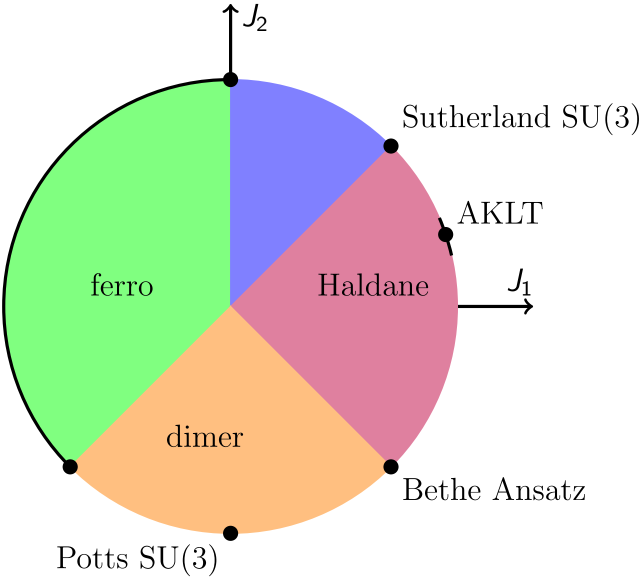
Antiferromagnetic spin-1 chain: $[1, L] \subset \mathbb{Z}$, $\mathcal{H}_x = \mathbb{C}^3$,

$$H_{[1,L]} = \sum_{x=1}^{L} \left(\frac{1}{3}\mathbb{1} + \frac{1}{2}\mathbf{S}_x \cdot \mathbf{S}_{x+1} + \frac{1}{6}(\mathbf{S}_x \cdot \mathbf{S}_{x+1})^2 \right) = \sum_{x=1}^{L} P^{(2)}_{x,x+1}$$

The ground state space of $H_{[1,L]}$ is 4-dimensional for all $L \geq 2$. In the limit of the infinite chain, the ground state is unique, has a finite correlation length, and there is a non-vanishing gap in the spectrum above the ground state (Haldane phase).

Theorem (Bachmann-N, CMP 2013, to appear)

There exists a curve of uniformly gapped Hamiltonians with nearest neighbor interaction $s \mapsto \Phi_s$ such that Φ_0 is the AKLT interaction and Φ_1 defines a model with a unique ground state of the infinite chain that is a product state.



$$H = \sum_x J_1 \mathbf{S}_x \cdot \mathbf{S}_{x+1} + J_2 (\mathbf{S}_x \cdot \mathbf{S}_{x+1})^2$$

Symmetry protected phases in 1 dimension

For a given system with λ-dependent G-symmetric interactions, we would like to find criteria to recognize that the system at λ_0 is in a different gapped phase than at λ_1, meaning that the gap above the ground state necessarily closes for at least one intermediate value of λ.

This is the same problem as before but restricted to a class of models with a given symmetry group (and representation) G.

Our goal is to find invariants, i.e., computable and, in principle, observable quantities that can be different at λ_0 and λ_1, only if the model is in a different ground state phase.

The case $G = SU(2)$ and the Excess Spin

Models to keep in mind: antiferromagnetic chains in the Haldane phase and generalizations. Unique ground state with a spectral gap and an unbroken continuous symmetry.

Let S_x^i, $i = 1, 2, 3$, $x \in \mathbb{Z}$, denote the ith component of the spin at site x. Claim: one can define

$$\sum_{x=1}^{+\infty} S_x^i,$$

as s.a. operators on the GNS space of the ground state and they generate a representation of $SU(2)$ that is characteristic of the gapped ground state phase.

We can prove the existence of these excess spin operators for two classes of models (Bachmann-N, arXiv:1307.0716):

1) models with a random loop representation;

2) models with a matrix product ground state (MPS).

Frustration-free chains with $SU(2)$ invariant MPS ground states

$$H = \sum_x h_{x,x+1}$$

Ground state is defined in terms of an isometry V, which intertwines two representations of $SU(2)$:

$$Vu_g = (U_g \otimes u_g)V, \quad g \in SU(2).$$

E.g., in the AKLT chain U_g is the spin-1 representation and u_g is the spin-$1/2$ representation of $SU(2)$, corresponding to the well-known spin $1/2$ degrees of freedom at the edges.

Outline of the argument

(i) First consider the model on the half-infinite chain. The space of ground states transforms as u_g under the action of $SU(2)$. We call this the edge representation. We prove that, in general, along a curve of models with a non-vanishing gap, the edge representation is constant.

(ii) On the infinite chain, we show that the excess spin operators are well-defined.

(iii) Observe that on the subspace of the GNS Hilbert space of the infinite-chain ground state consisting of the ground state of the Hamiltonian of the half-infinite chain, acts as (an infinite number of copies of) u_g.

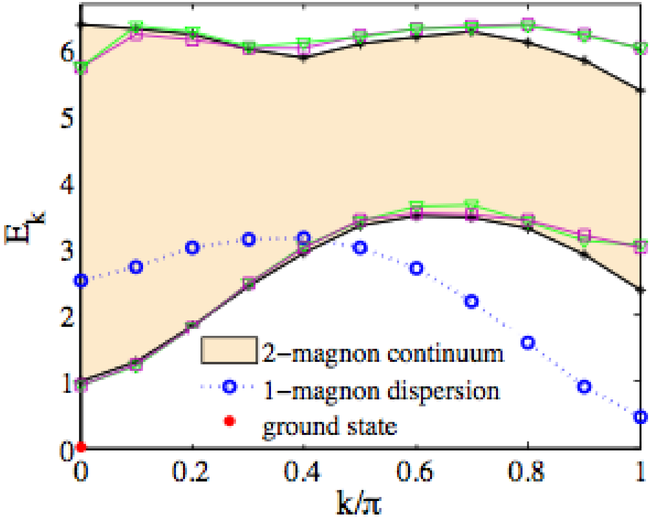
This is also shows that u_g is experimentally observable.

Elementary excitations

The current interest in gapped ground state phases is motivated by the potential applications of topologically ordered phases to quantum information processing, in particular the nature of elementary excitations (anyons) in systems with topological order.

As a first step, we looked at the localized nature of the excitations corresponding to isolated branches in the spectrum (‘particles’).

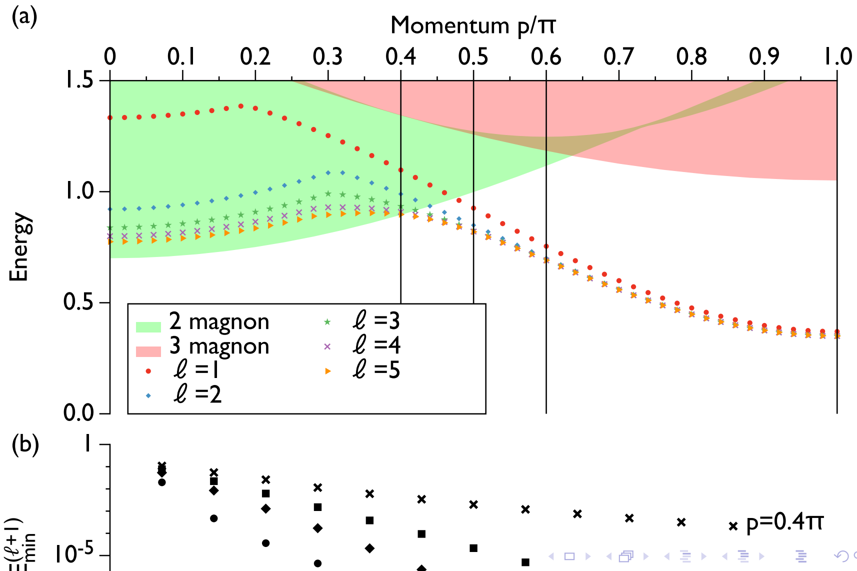
Such excitations occur, e.g., the spin-1 Heisenberg antiferromagnetic chain.



AF Heisenberg chain spectrum. From: Zheng-Xin Liu, Yi Zhou, Tai-Kai Ng, arXiv:1307.4958

AKLT chain

The excitation spectrum of the AKLT chain looks similar:



Assume that at quasi-momentum p we have a gap $\geq \delta > 0$ between E_p and the higher eigenvalues of the Hamiltonian and the same quasi-momentum, uniformly in the size of the system.

The general result is that, under a technical condition, the eigenvectors belonging E_p, are of the form

$$\psi_p = \psi(A_p) = \sum_x e^{ipx} T_x(A_p)\Omega$$

where Ω is the ground state, T_x denotes translation by x and A_p is a quasi-local observable. More precisely:

Theorem (Haegeman, Michalakis, N, Osborne, Schuch, Verstraete, PRL, to appear)

There exists a constants $v > 0$ and $n \geq 1$, such that for $\ell \geq \ell_0$, there exists $A_p^{(\ell)} \in \mathcal{A}_{B_\ell}$ such that

$$|\langle \psi_p, \psi(A_p^{(\ell)}) \rangle| \geq 1 - c\ell^n e^{-\delta\ell/v}.$$

Concluding comments: implications of locality

The fundamental theories of physics, all relativistic quantum field theories (QFT) as well as all standard Hamiltonian models in quantum statistical mechanics (QSM), have a locality property reflecting the nature of physical space.

In QFT this is due to the finite speed of light and Poincaré invariance, and is usually expressed by the commutation of observables with space-like separated supports.

In QSM there is a corresponding of finite speed of propagation property that can be proved if the particle interactions are of short (or at least not too long) range: Lieb-Robinson bounds.

Wightman Axioms (R. Haag, *Local Quantum Physics*, 1992).

For (x, t) and $(y, s) \in \mathbb{R}^3 \times \mathbb{R}$ are space-like separated if $\|x - y\| > c|t - s|$, and two regions X and Y are space-like separated if all points in X are space-like separated from all points in Y.

The smeared fields are operators on a Hilbert space defined by

$$\psi(f) = \int f(r)\psi(r)dr$$

Where f is a test function. The locality property is expressed by the causality axiom: for f and g with space-like separated supports we have (in the bosonic case)

$$[\psi(f), \psi(g)] = 0.$$

Consequences of Locality for QFT

Together with the other Wightman axioms, causality implies a number of fundamental properties:

- A mass gap implies exponential clustering (Araki, Hepp, Ruelle (1962), Fredenhagen (1985))
- The Spin-Statistics Theorem
- Additivity of the Energy-Momentum Spectrum: If (p_1, E_1) and (p_2, E_2) are in the spectrum of the (Momentum,Energy) operator, then so is $(p_1 + p_2, E_1 + E_2)$.
- Borchers classes: a field ψ' on the same Hilbert space as ψ that commutes with ψ at space-like separation, is 'equivalent' in the sense that the S-matrix is the same.
- Particles

Consequences of Locality for Lattice Systems

- A spectral gap above the ground states implies exponential decay of correlations. (N. Sims, 2006, Hastings-Koma 2006).
- Existence of thermodynamic limit of the dynamics.
- Local Perturbations Perturb Locally
- Automorphic Equivalence within gapped phases ($\sim$ Borchers classes). Quantum Phase Transitions between different equivalence classes.
- Area Law for the entanglement entropy
- Particle-like spectrum of excitations.

Concluding Remarks

- Non-relativistic quantum many-body system have a locality property similar to relativistic quantum field theories.
- This locality property can be exploited much in the same way as one can combine causality in QFT with other properties to derive important general properties.
- The ground state problem of one-dimensional spin systems is universal
- We are close to a comprehensive picture gapped ground state phases in one dimension, but in two (and more) dimensions many questions remain open (work in progress with Bachmann, Hamza, and Young.)