2D Ising Model:
Near-Critical Scaling Limit and Magnetization Critical Exponent *

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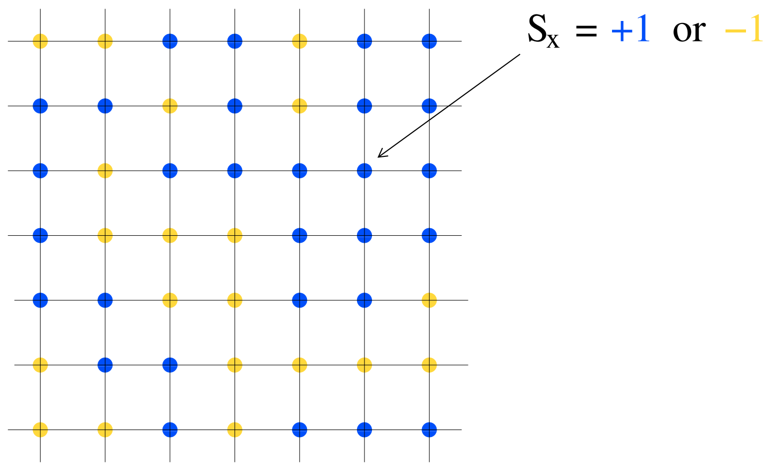
*Based on joint work with Federico Camia and Christophe Garban.

Ising Model on $\mathbb{Z}^2$



Probability $\propto$ exp $\left(\beta \sum_{\{x,y\}} S_x S_y + h \sum_x S_x\right)$

Spins: $S_x, S_y = \pm 1$
Edges: $e = \{x, y\}$ $(||x - y|| = 1)$

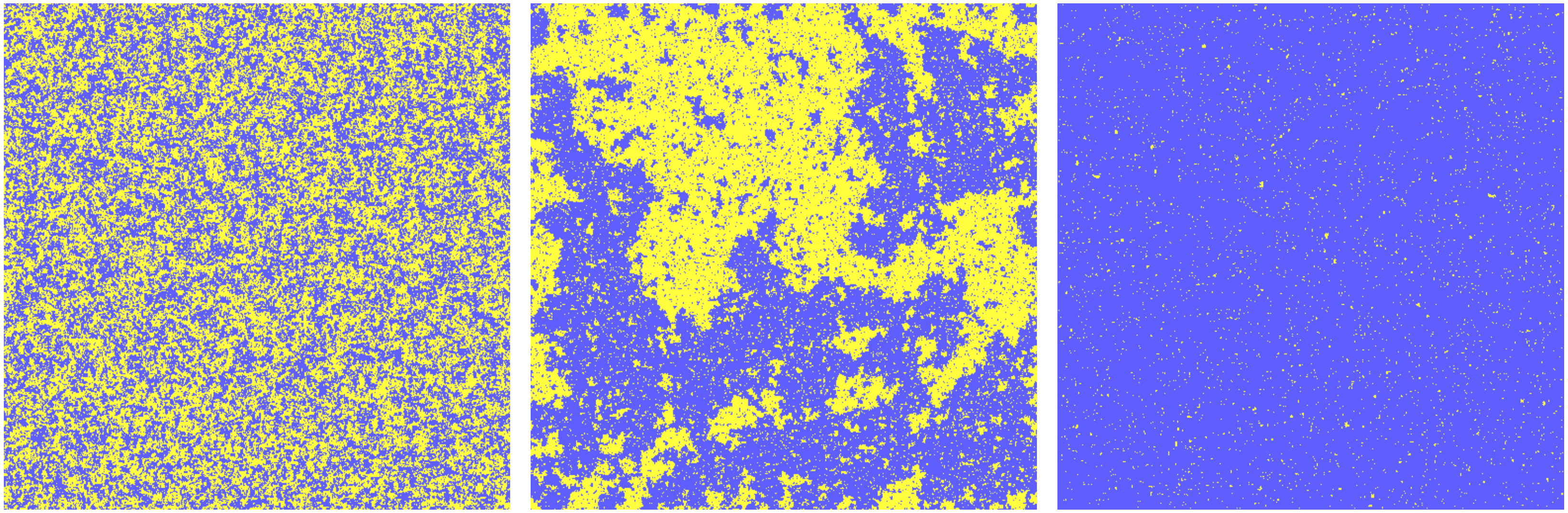
Continuum scaling limit: replace $\mathbb{Z}^2$ by $a\,\mathbb{Z}^2$ and let $a \to 0$.

Ising Model in a Finite Domain

$$\mathbb{P}_L^{\beta,h}(S) := \frac{1}{Z_{L,\beta,h}} e^{-\beta\, E_L(S) + h\, M_L(S)}$$

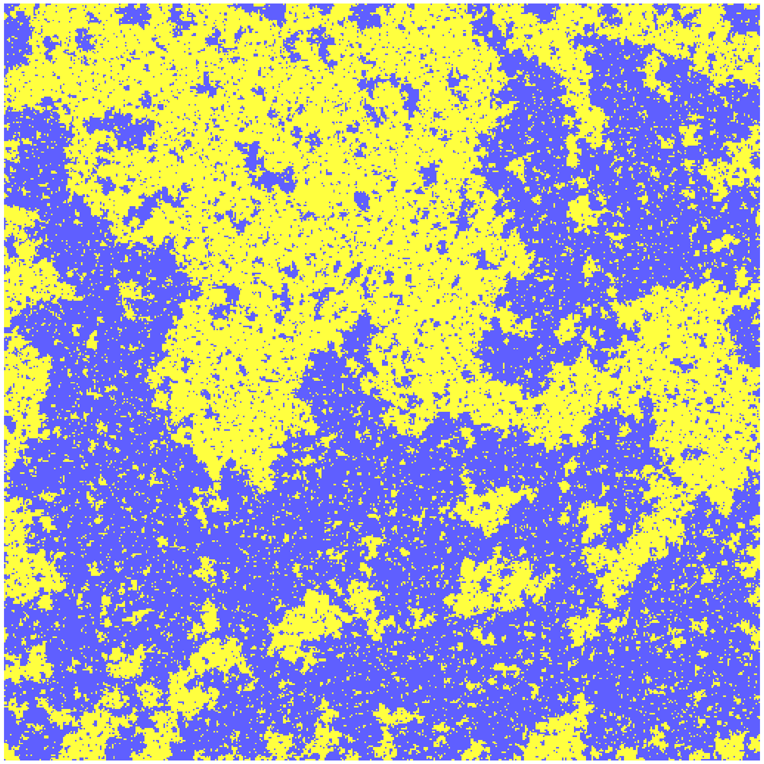
$$\begin{cases} \Lambda_L := [-L, L]^2 \cap \mathbb{Z}^2 & \text{domain} \\ E_L(S) := -\sum_{\{x,y\}} S_x S_y & \text{interaction energy} \\ M_L(S) := \sum_{x \in \Lambda_L} S_x & \text{total magnetization in } \Lambda_L \\ Z_{L,\beta,h} := \sum_S e^{-\beta\, E_L(S) + h\, M_L(S)} & \text{partition function} \end{cases}$$

$h = 0$ Case: The Three Regimes



Evidence of a phase transition.

The Critical Point: $\beta = \beta_c = \frac{1}{2}\log(1+\sqrt{2})$



Thermodynamic Limit

$h > 0$ or $\beta \leq \beta_c \Rightarrow \mathbb{P}_L^{\beta,h}$ has a unique infinite volume limit as $L \to \infty$:

$$\mathbb{P}_L^{\beta,h} \overset{L\to\infty}{\longrightarrow} \mathbb{P}^{\beta,h}$$

$\langle \cdot \rangle_{\beta,h}$ denotes expectation with respect to $\mathbb{P}^{\beta,h}$

The Magnetization Exponent

(F. Camia, C. Garban, C.M.N.; arXiv:1205.6612)

Theorem. Consider the Ising model on $\mathbb{Z}^2$ at β_c with a positive external magnetic field $h > 0$, then *

$$\langle S_0 \rangle_{\beta_c,h} \asymp h^{\frac{1}{15}} .$$

* $f(a) \asymp g(a)$ as $a \searrow 0$ means that $f(a)/g(a)$ is bounded away from 0 and ∞.

Critical Exponents

$$\begin{cases} \text{Heat capacity:} & C(T) \sim |T - T_c|^{-\alpha} \\ \text{Order parameter:} & M(T) \sim |T - T_c|^{b} \\ \text{Susceptibility:} & \chi(T) \sim |T - T_c|^{-\gamma} \\ \text{Equation of state } (T = T_c): & M(h) \sim h^{1/\delta} \end{cases}$$

2D Ising Critical Exponents

Onsager's solution shows that

- susceptibility has logarithmic divergence $\Rightarrow \alpha = 0$
- $b = 1/8$ (Yang)

Scaling theory predicts

- correlation length at T_c: $\xi(h) \sim h^{-8/15}$

Scaling Laws

$$\begin{cases} \text{Rushbrooke:} & \alpha + 2b + \gamma = 2 \\ \text{Widom:} & \gamma = b(\delta - 1) \end{cases}$$

$$\Downarrow$$

$$\delta = \frac{2 - \alpha - b}{b} \overset{\text{2D Ising}}{=} 15$$

Proof of the Exponent Theorem

Lower bound: Use Ising ghost spin representation + standard percolation arguments; tools: FKG + RSW for FK percolation.

Upper bound: Combine GHS inequality with first and second moment bounds for the magnetization; tools: GHS + FKG + RSW for FK percolation.

RSW for Ising-FK proved by Duminil-Copin, Hongler, Nolin (2011).

GHS Inequality

Theorem [Griffith, Hurst, Sherman, 1970]. Let $\langle\cdot\rangle$ denote expectation with respect to $\mathbb{P}_L^{\beta,h}$ ($h \geq 0$). Then, for any vertices $x, y, z \in \Lambda_L$,

$$\langle S_x S_y S_z\rangle - \Big(\langle S_x\rangle \langle S_y S_z\rangle + \langle S_y\rangle \langle S_x S_z\rangle + \langle S_z\rangle \langle S_x S_y\rangle\Big) + 2\langle S_x\rangle\langle S_y\rangle\langle S_z\rangle \leq 0\,.$$

Corollary. The GHS inequality implies that

$$\partial_h^3 \log(Z_{L,\beta,h}) \leq 0\,.$$

Magnetization

$$\langle S_0 \rangle_{\beta_c,h} = \frac{1}{|\Lambda_L|} \langle M_L \rangle_{\beta_c,h} \le \frac{1}{|\Lambda_L|} \langle M_L \rangle_{\beta_c,h,+} \text{ (+ b.c. on } \Lambda_L\text{)}$$

$$\langle M_L \rangle_{\beta_c,h,+} = \frac{\langle M_L \, e^{hM_L} \rangle_{\beta_c,0,+}}{\langle e^{hM_L} \rangle_{\beta_c,0,+}} = \frac{\frac{\partial}{\partial h} \langle e^{hM_L} \rangle_{\beta_c,0,+}}{\langle e^{hM_L} \rangle_{\beta_c,0,+}}$$

Consequences of GHS

$$\text{GHS} \Rightarrow \quad \frac{\partial^3}{\partial h^3} \log\left(\sum_S e^{-\beta_c E_L(S) + h M_L(S)}\right) \quad \leq 0$$

$$\Leftrightarrow \quad \frac{\partial^3}{\partial h^3} \log\left(\frac{\sum e^{-\beta_c E_L + h M_L}}{\sum e^{-\beta_c E_L}}\right) \quad \leq 0$$

$$\Leftrightarrow \quad \frac{\partial^2}{\partial h^2}\left(\frac{\frac{\partial}{\partial h}\langle e^{hM_L}\rangle_{\beta_c,0,+}}{\langle e^{hM_L}\rangle_{\beta_c,0,+}}\right) = \frac{\partial^2}{\partial h^2}\langle M_L\rangle_{\beta_c,h,+} \quad \leq 0$$

Let $F(h) \equiv F_L(h) := \frac{\frac{\partial}{\partial h}\langle e^{hM_L}\rangle_{\beta_c,0,+}}{\langle e^{hM_L}\rangle_{\beta_c,0,+}} = \langle M_L\rangle_{\beta_c,h,+}$, then

$$\begin{aligned} F(h) &\leq F(0) + h\,F'(0) \\ &= \langle M_L\rangle_{\beta_c,0,+} + h\left(\langle M_L^2\rangle_{\beta_c,0,+} - \langle M_L\rangle^2_{\beta_c,0,+}\right) \end{aligned}$$

Magnetization Bounds

Theorem [T.T. Wu, 1966]. There exists an explicit constant $c > 0$ such that as $n \to \infty$

$$\rho(n) := \langle S_{(0,0)} S_{(n,n)} \rangle_{\beta_c,0} \sim c\, n^{-1/4}\,.$$

Proposition [F. Camia, C. Garban, C.M.N.]. There is a universal constant $C > 0$ such that for L sufficiently large, one has

(i) $\langle M_L \rangle_{\beta_c,0,+} \le C\, L^2 \rho(L)^{1/2}$,

(ii) $\langle M_L^2 \rangle_{\beta_c,0,+} \le C\, L^4\, \rho(L)$.

Upper Bound

$$\langle S_0 \rangle_{\beta_c,h} \le \frac{1}{L^2} \langle M_L \rangle_{\beta_c,h,+} \le C\Big(L^{15/8} + h\, L^{15/4}\Big)/L^2$$

(optimize in $L = L(h)$) $\Leftrightarrow$ (choose $L(h) \asymp h^{-8/15} \sim \xi(h)$)

$$\begin{aligned} \langle S_0 \rangle_{\beta_c,h} &\le O(1)\, \frac{1}{L(h)^2} L(h)^{15/8} = O(1)\, L(h)^{-1/8} \\ &\le O(1)\, h^{1/15} \end{aligned}$$

Scaling Limit: $\mathbb{Z}^2$ replaced by $a\mathbb{Z}^2$; $a \to 0$

Approach 1: Boundaries of $\left\{ \begin{array}{l} \text{Spin} \\ \text{FK} \end{array} \right\}$ clusters as conformal loop ensembles in plane related to Schramm-Loewner Evolution (SLE_κ)

with $\kappa = \left\{ \begin{array}{l} 3 \\ 16/3 \end{array} \right\}$ (Schramm, Smirnov)

Approach 2 (*today*): Random (Euclidean) field,

$$\Phi^a(z) = \Theta_a \sum_{x \in a\mathbb{Z}^2} S_x \delta_x$$

Heuristics for Magnetization Field

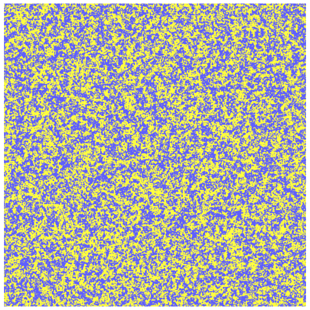
$$\langle S_x S_y \rangle^T_{\beta,h} \equiv \text{Cov}_{\beta,h}(S_x, S_y) \sim e^{-||x-y||/\xi(\beta,h)} \text{ as } ||x-y|| \to \infty$$

1. $\beta < \beta_c$ fixed ($h = 0$) with $\xi(\beta) < \infty$: Φ^0 trivial (i.e., Gaussian white noise) by some CLT.

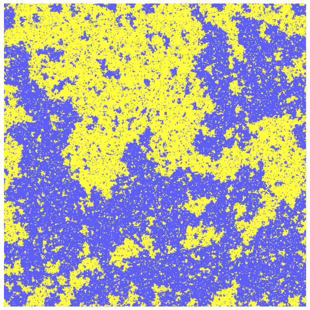
2. $\beta = \beta_c$ ($h = 0$), $\xi(\beta_c) = \infty$: Φ^0 massless;

3. $\beta = \beta(a) \uparrow \beta_c$ ($h = 0$) or $\beta = \beta_c$, $h(a) \downarrow 0$ s. t. $a\,\xi(a) \to 1/m \in (0, \infty)$ as $a \to 0$: "near-critical" Φ^0 is massive.

Continuum Scaling Limits for the Magnetization ($h = 0$)



High temperature: $\frac{1}{\sqrt{1/a^2}} \sum_{x \in \text{square}} S_x \xrightarrow{a \to 0} M \sim$ Normal dist.



Critical temperature: classical CLT does not hold.

Scaling Limit at β_c and $h = 0$

In the scaling limit ($a \to 0$) one hopes that

$$a^{15/8} \sum_{x \in \text{square}} S_x \xrightarrow{a \to 0} \int_{[0,1]^2} \Phi(z) dz$$

for some magnetization field $\Phi = \Phi^0$.

Φ should describe the fluctuations of the magnetization around its mean ($= 0$).

However, Φ cannot be a function.

Critical Magnetization Field

(F. Camia, C. Garban, C.M.N.; arXiv:1205.6610)

$$\Phi^a := a^{15/8} \sum_{x \in a\mathbb{Z}^2} S_x \delta_x$$

Critical scaling limit: $\beta = \beta_c$, $h = 0$, $a \to 0$

$\Phi^a \to$ random generalized function Φ^0: massless field (power-law decay of correlations).

The limiting magnetization field is not Gaussian:

$$\log \mathbb{P}(\Phi^0([0,1]^2) > x) \overset{x \to \infty}{\sim} -c\, x^{16}$$

Conformal Covariance

(F. Camia, C. Garban, C.M.N.; arXiv:1205.6610)

The magnetization field $\Phi = \Phi^0$ exists as a random generalized function and is conformally covariant:

If f is a conformal map,

$$\text{“}\Phi(f(z)) \stackrel{dist.}{=} \left|f'(z)\right|^{-1/8} \Phi(z)\text{”} \,.$$

E.g., for a scale transformation $f(z) = \alpha z$ ($\alpha > 0$),

$$\int_{[-\alpha L, \alpha L]} \Phi(z)\, dz \stackrel{dist.}{=} \alpha^{15/8} \int_{[-L,L]} \Phi(z)\, dz \,.$$

$h \to 0$ **Near-Critical Field**

(F.C., C.G, C.M.N.; arXiv:1307.3926)

(**Why?**: Borthwick-Garibaldi, 2011; McCoy-Maillard, 2012)

Near-critical (off-critical) scaling limit:
$\beta = \beta_c$, $a \to 0$, $h \to 0$, $ha^{-15/8} \to \lambda \in (0, \infty)$.

Heuristics: choose $h = \lambda a^{15/8}$ and note that

$$\xi(h) = \xi(\lambda a^{15/8}) \sim (a^{15/8})^{-8/15} = 1 \, .$$

Limit yields one-parameter (λ) family of fields [in progress: massive; i.e., exponential decay of correlations].

Heuristics: multiply zero-field measure by "$\exp(\lambda \int_{\mathbb{R}^2} \Phi^0(z)dz)$;" exponential decay based on FK percolation props. of critical Φ^0.