Dark Solitons in the 1D Bose gas

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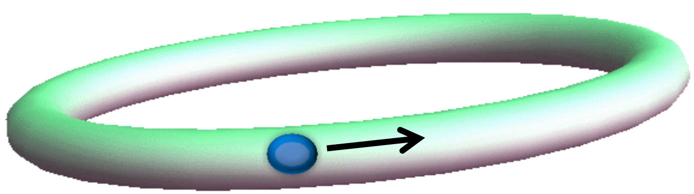
Lieb-Liniger model

Lieb-Liniger model

$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{1 \le j < k \le N} \delta(x_j - x_k)$$

1D Bose gas with delta function potential（PBC）



Toy model

⇓

Realized in experiments

- Realization of the Tonks–Girardeau gas
  Kinoshita, et al., Science **305**, 1125 (2004)
- Observation of Out-of-equilibriumm dynamics
  Kinoshita, et al., Nature **440**, 900 (2006).

Lieb-Liniger model

$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{1 \le j < k \le N} \delta(x_j - x_k)$$

- Second quantization

L: System size

$$[\Psi(x), \Psi^\dagger(y)] = \delta(x-y), \quad [\Psi(x), \Psi(y)] = [\Psi^\dagger(x), \Psi^\dagger(y)] = 0$$

$$\Psi(x)|0\rangle = 0, \quad \langle 0|\Psi^\dagger(x) = 0, \quad \langle 0|0\rangle = 1$$

$$H = \int_0^L dx \left[\partial_x \Psi^\dagger(x) \partial_x \Psi(x) + c\Psi^\dagger(x)\Psi^\dagger(x)\Psi(x)\Psi(x)\right]$$

Exactly solved by Bethe ansatz

Lieb-Liniger model

$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{1 \le j < k \le N} \delta(x_j - x_k)$$

- Parameters

***L*: System size**

***N*: Number of particle**

***c*: coupling constant**

$n = N / L$: density

$\gamma = c/n$

Thermodynamics is characterized by a single parameter γ

Lieb-Linger model
and
Bethe ansatz

Bethe ansatz

Quasi momentum : $k_1, k_2, \cdots, k_N$

Bethe ansatz equation

$$e^{\mathrm{i}k_j L} = \prod_{\ell \neq j}^{N} \frac{k_j - k_\ell + \mathrm{i}c}{k_j - k_\ell - \mathrm{i}c}, \qquad j = 1, 2, \cdots, N$$

Bethe wave function

$\boxed{\text{if } k_j = k_\ell \text{ then } \varphi = 0}$

$$\varphi(x_1, \cdots, x_N) = \sum_{\sigma \in S_N}^{N!} A_\sigma \exp\left[\mathrm{i} \sum_{j=1}^{N} k_{\sigma_j} x_j\right]$$

$$A_\sigma = (-1)^\sigma \prod_{j>\ell}^{N} \left[k_{\sigma_j} - k_{\sigma_\ell} - \mathrm{i}c\, \mathrm{sign}(x_j - x_\ell)\right]$$

Quasi momentum : $k_1, k_2, \cdots, k_N$

Bethe ansatz equation

$$e^{\mathrm{i}k_j L} = \prod_{\ell \neq j}^{N} \frac{k_j - k_\ell + \mathrm{i}c}{k_j - k_\ell - \mathrm{i}c}, \qquad j = 1, 2, \cdots, N$$

Take logarithm

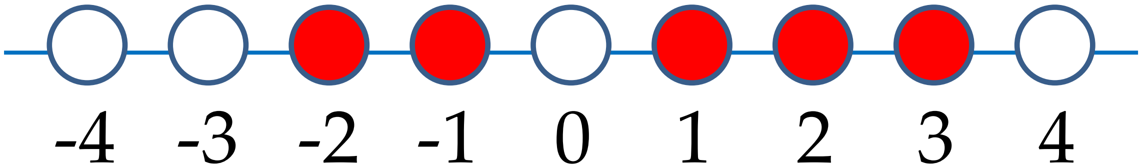
$$k_j L + \sum_{\ell=1}^{N} 2 \tan^{-1} \frac{k_j - k_\ell}{c} = 2\pi I_j, \qquad j = 1, 2, \cdots, N$$

I_j : Bethe quantum number

Easily solved numerically

Bethe ansatz

- Specify Bethe quantum number



ex) $\boldsymbol{N}$ **= 5**

$$\{\boldsymbol{I_j}\} = \{\mathbf{-2, -1, 1, 2, 3}\}$$

- Solve Bethe ansatz equation

$$k_j L + \sum_{\ell=1}^{N} 2\tan^{-1}\frac{k_j - k_\ell}{c} = 2\pi I_j$$

cf. $\boldsymbol{c} \to \infty$

$$k_j = \frac{2\pi}{L} I_j$$

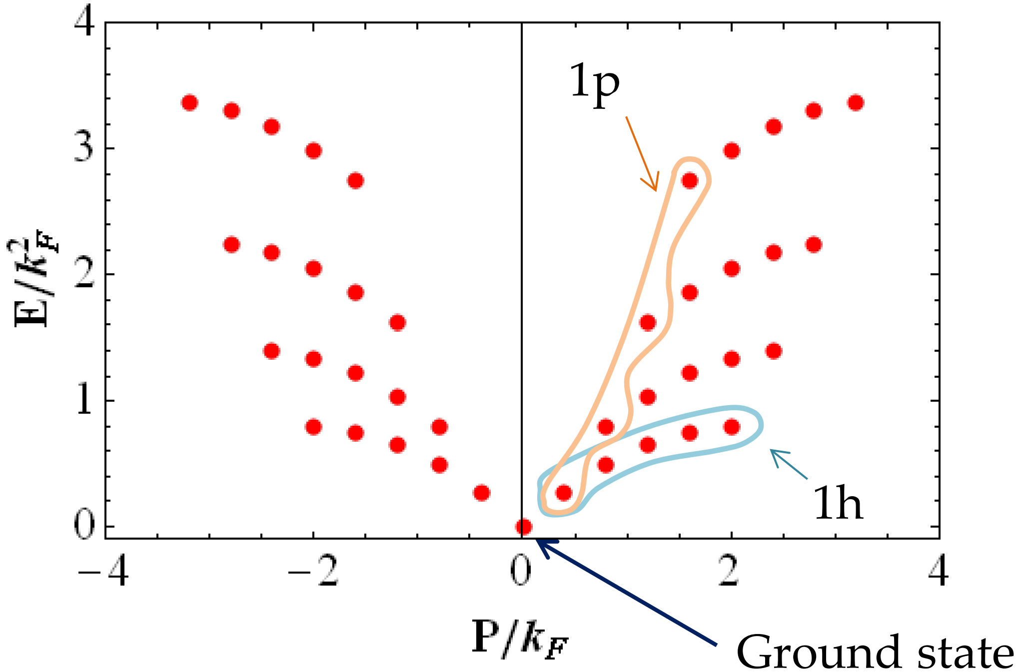
- Energy and momentum eigenvalues

$$E = \sum_{j=1}^{N} k_j^2, \quad P = \sum_{j=1}^{N} k_j$$

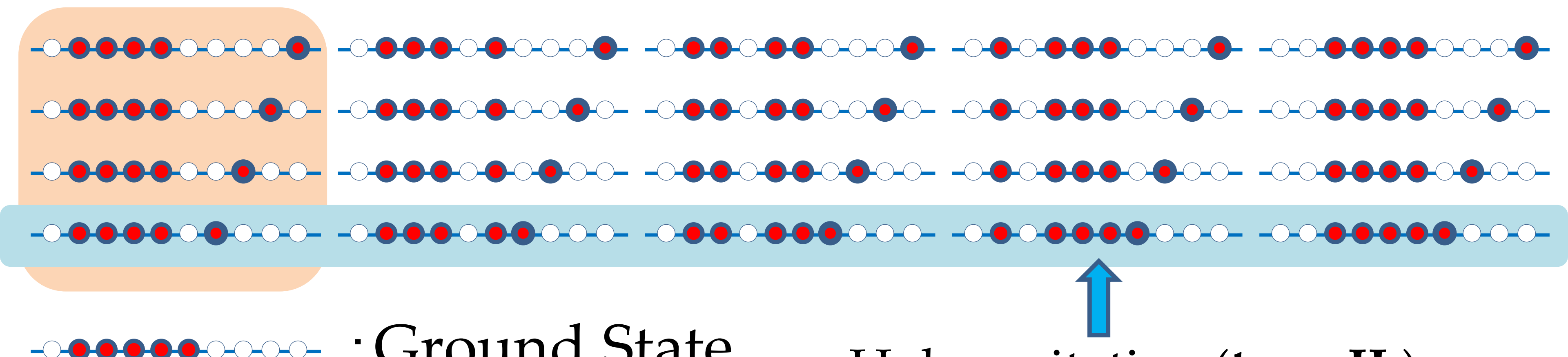
One-to-one correspondence between $\{I_j\}$ and $\{k_j\}$

1-p 1-h excitation

ex) $N = 5,\ \gamma = 1$



Particle excitation (**type I**)



: Ground State

(-2,-1,0,1,2)

Hole excitation (**type II**)

Determinant formula

Vector $|k\rangle$ Labeled by $k = (k_1, k_2, \cdots, k_N)$

Norm

$$\boxed{\langle k|k\rangle = c^N \left(\prod_{j>\ell}^{N} \frac{k_{j\ell}^2 + c^2}{k_{j\ell}^2} \right) \det{}_N G(k)}$$

$$k_{j\ell} := k_j - k_\ell$$

Gaudin Matrix :

$$G(k)_{j\ell} = \delta_{j\ell} \left[L + \sum_{\ell=1}^{N} K(k_{j\ell}) \right] - K(k_{j\ell})$$

$$K(k) = \frac{2c}{k^2 + c^2}$$

M. Gaudin, ``La fonction d'onde de Bethe'', Masson (Paris) (1983);
V. E. Korepin, Commun. Math. Phys. **86**, 391 (1982).

Form factor

$$\langle k'|\hat{\psi}^\dagger\hat{\psi}|k\rangle = \mathrm{i}^N (P_k - P'_k)\left(\prod_{j,\ell=1}^{N}\frac{k_j - k_\ell + \mathrm{i}c}{k'_j - k_\ell}\right)\det{}_{N-1}U$$

$$\langle k'|\hat{\psi}|k\rangle = -\frac{\mathrm{i}^N}{\sqrt{c}}\frac{\prod_{j,\ell=1}^{N}(k_j - k_\ell + \mathrm{i}c)}{\prod_{j=1}^{N-1}\prod_{\ell=1}^{N}(k'_j - k_\ell)}\det{}_{N-1}U$$

Slavnov (1989),
Caux, Calabrese, Slavnov (2002),
Kojima, Korepin, Slavnov (1997)

$$P_k = \sum_{j=1}^{N} k_j$$

$$U_{j\ell} = \delta_{j\ell}\frac{V_j^+ - V_j^-}{\mathrm{i}} + \frac{\prod_a (k'_a - k_j)}{\prod_{a\neq j}^{N}(k_a - k_j)}(K(k_j - k_\ell) - K(k_N - k_\ell))$$

$$V_j^\pm = \frac{\prod_a (k'_a - k_j \pm \mathrm{i}c)}{\prod_{a=1}^{N}(k_a - k_j \pm \mathrm{i}c)}$$

Nonlinear Schrödinger equation
and
Soliton

Nonlinear Schrödinger eq. and Soliton

2^{nd} quantized Hamiltonian

$$\mathcal{H} = \int_0^L dx [\partial_x \hat{\psi}^\dagger \partial_x \hat{\psi} + c \hat{\psi}^\dagger \hat{\psi}^\dagger \hat{\psi} \hat{\psi}]$$

Eq. of motion $i\partial_t \hat{\psi} = [\hat{\psi}, \mathcal{H}]$

$$i\partial_t \hat{\psi} = -\partial_x^2 \hat{\psi} + 2c \hat{\psi}^\dagger \hat{\psi} \hat{\psi}$$

Exactly diagonalized by Quantum inverse scattering method
= Algebraic Bethe ansatz

L.A. Takhtajan, L.D. Faddeev (1979)

Field operator $\rightarrow$ *c*-number: $\hat{\psi}(x,t) \rightarrow \phi(x,t)$

$$i\partial_t \phi = -\partial_x^2 \phi + 2c|\phi|^2 \phi$$

Inverse scattering method
$\Rightarrow$ Soliton solution

V.E. Zakharov, A.B. Shabat (1972)

Soliton solution

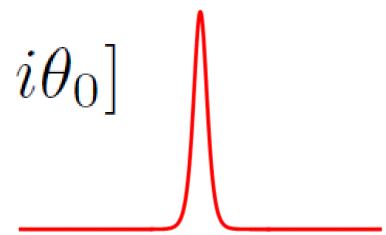
Nonlinear Schrödinger equation: $i\partial_t \phi = -\partial_x^2 \phi + 2c|\phi|^2 \phi$

$c < 0$: Bright Soliton

$4\eta = |c|(N+1)$

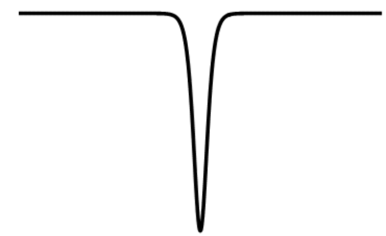
V.E. Zakharov,
A.B. Shabat (1972)

$$\phi(x,t) = 2|c|^{-1/2}\eta \exp[4i\eta^2 t - iv^2 t/4 + ivx/2 + i\theta_0]$$
$$\times \operatorname{sech}[2\eta(x - x_0 - vt)]$$



$c > 0$: Dark Soliton

$\beta = 1 - (v/v_c)^2$

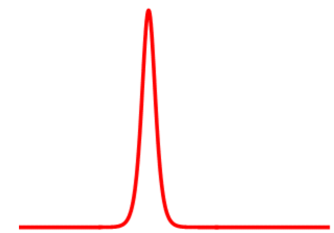


$$\phi(x,t) = n^{1/2}\left\{1 - \beta \operatorname{sech}^2\left[(\beta n c)^{1/2}(x - vt)\right]\right\}^{1/2}$$
$$\times \exp\left\{\pm i \sin^{-1}\left(\frac{\beta^{1/2}\tanh\left[(\beta n c)^{1/2}(x - vt)\right]}{\left\{1 - \beta \operatorname{sech}^2\left[(\beta n c)^{1/2}(x - vt)\right]\right\}^{1/2}}\right)\right\}$$

Tsuzuki (1970)

QFT and Soliton

$c < 0$: Bright Soliton



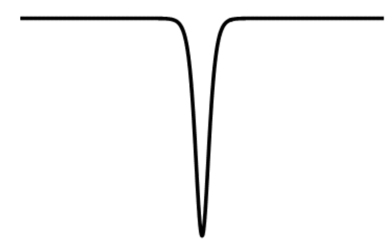
$|N, P\rangle$: N-particle bound state with total momentum P

Localized wave packet: $|N, X, t\rangle = \int \frac{\mathrm{d}P}{2\pi} e^{-iPX} e^{i(P^2/N)t} |N, P, t\rangle$

$$\lim_{N\to\infty} \langle N, X, t|\hat{\psi}(x)|N+1, X, t\rangle = \phi(x, t)|_{v=0, x_0=X}$$

C.R. Nohl, Ann. Phys. **96**, 234 (1976),
M. Wadati, M.Sakagami, J. Phys. Soc. Jpn. **53**, 1933 (1984),
M. Wadati, A. Kuniba, T. Konishi, J. Phys. Soc. Jpn. **54**, 1710 (1985),
M. Wadati, A. Kuniba, J. Phys. Soc. Jpn. **55**, 76 (1986).

$c > 0$: Dark Soliton



It is shown that type II excitation and dark soliton have the same dispersion relation.

M. Ishikawa, H. Takayama, J. Phys. Soc. Jpn. **49**, 1242 (1980).

Construction of the Quantum Localized State
and
Classical-Quantum Correspondence

Construction of the Quantum Localized State

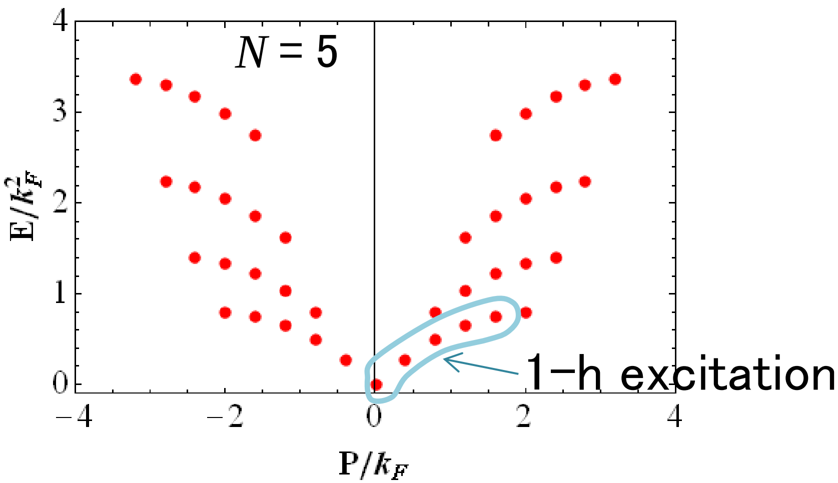
JS, R. Kanamoto, E. Kaminishi, T. Deguchi, PRL **108**, 110401 (2012).

$$|X\rangle = \frac{1}{\sqrt{N}} \sum_{p=0}^{N-1} \exp(-2\pi i p q/N)\, |P\rangle$$

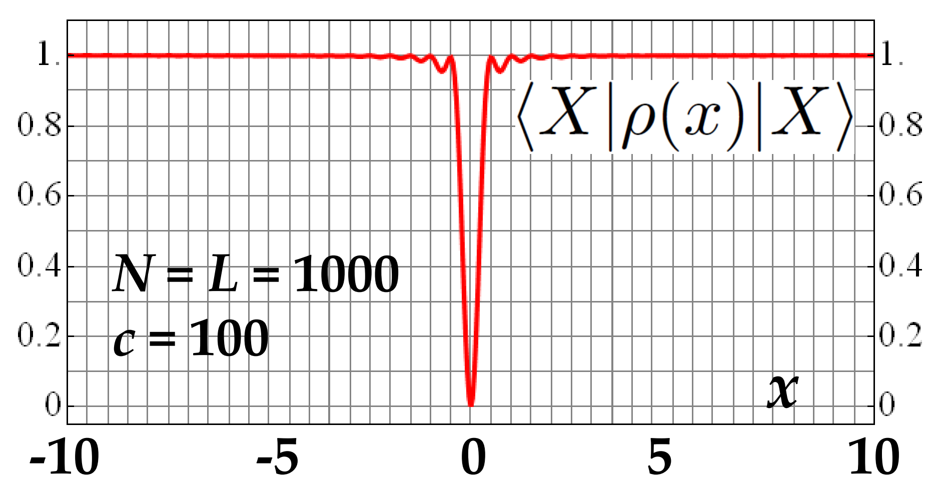
$$p, q \in \{0, 1, 2, \cdots, N-1\}$$

$|P\rangle$: Hole excitation with total momentum $P = 2\pi p/L$

$|X\rangle$: Wave packet localized at $X = qL/N + L/2$

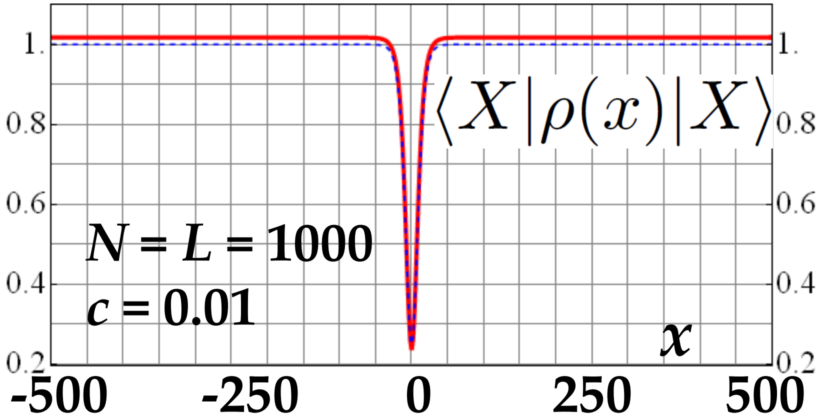


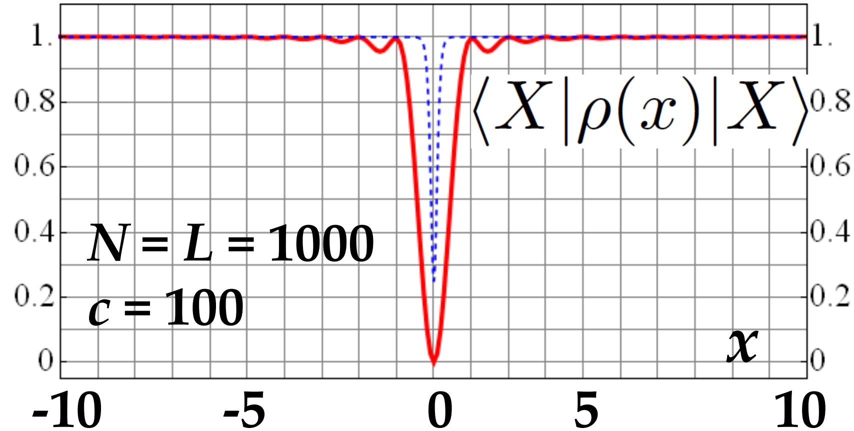
Density operator : $\rho(x) = \hat{\psi}^\dagger(x)\hat{\psi}(x)$



Type II Excitation & Dark Soliton: Amplitude

Density operator : $\rho(x) = \hat{\psi}^{\dagger}(x)\hat{\psi}(x)$





$\langle X|\rho(x)|X\rangle$

$$|\phi(x)|^2 = n\left(1 - \beta \mathrm{sech}^2\left[(\beta n c)^{1/2} x\right]\right)$$

$$\beta = 1 - (v/v_c)^2 = 3/4$$

$n = 1$

Density profile of |X> ⇔ Squared amplitude of dark soliton

Type II Excitation & Dark Soliton: Phase

Dark soliton is described by Elliptic function under PBC

$$\mathrm{Arg}\left[\phi(x)\right] = vx/2 - \frac{1}{K}\sqrt{\frac{g_{\mathrm{sn}}h_{\mathrm{sn}}}{2f_{\mathrm{sn}}}}\Pi\left(-\frac{2mK^2}{f_{\mathrm{sn}}};\frac{2Kx}{L}|m\right)$$

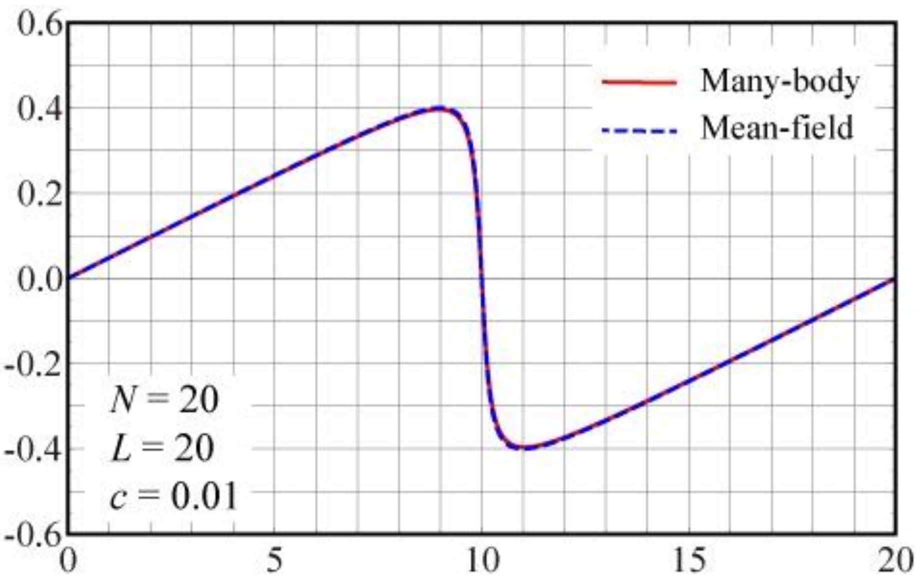
$$f_{\mathrm{sn}} = \frac{cL^2}{4\pi} - 2K^2 + 2KE$$
$$g_{\mathrm{sn}} = f_{\mathrm{sn}} + 2K^2$$
$$h_{\mathrm{sn}} = f_{\mathrm{sn}} + 2mK^2$$

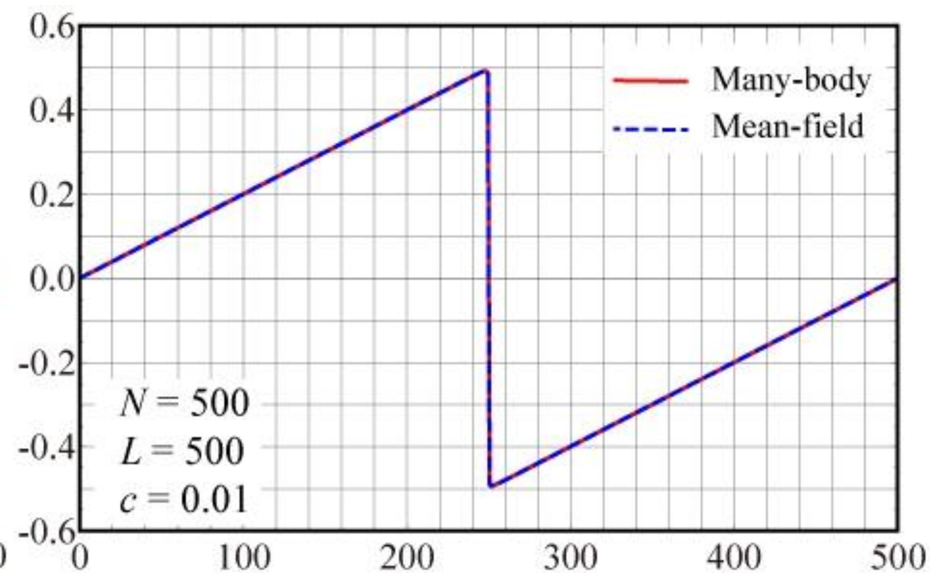
***K*, *E*, Π**: complete elliptic integral

R. Kanamoto, L.D. Carr, M. Ueda, Phys. Rev. A **79**, 063616 (2009).

$\mathrm{Arg}\left[\langle N-1, X|\hat{\psi}(x)|N, X\rangle\right]/\pi$

$\mathrm{Arg}\left[\phi(x)\right]/\pi$





Completely overlapping

Dynamics of
Quantum Wave Packet

Time evolution of the quantum wave packet

$$|X\rangle = \frac{1}{\sqrt{N}} \sum_{p=0}^{N-1} \exp(-2\pi ipq/N)\,|P\rangle$$

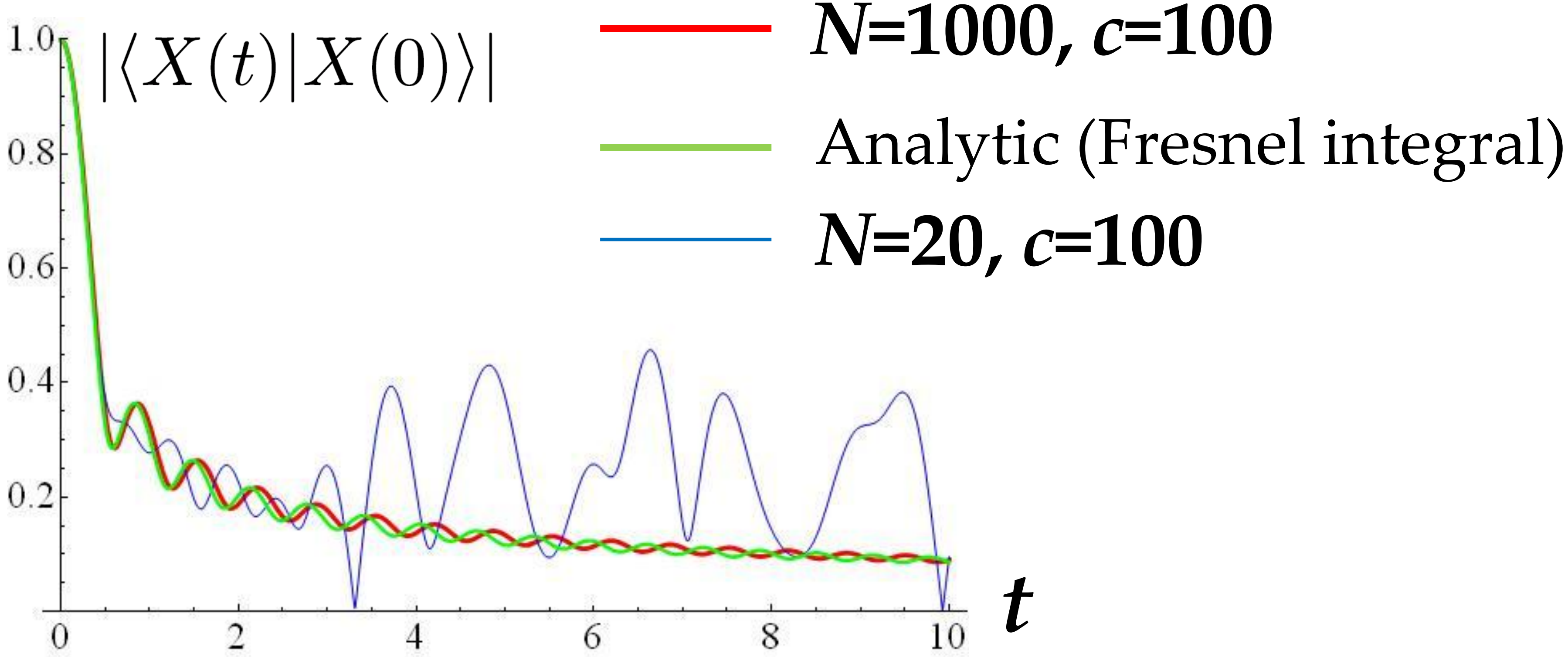
$$\begin{aligned} |X(t)\rangle &= \exp(-\mathrm{i}\mathcal{H}t)|X\rangle \\ &= \frac{1}{\sqrt{N}} \sum_{p=0}^{N-1} \exp(-2\pi \mathrm{i}pq/N)\exp(-\mathrm{i}E_p t)\,|P\rangle \end{aligned}$$

E_p is exactly obtained by Bethe ansatz

⇩

Time evolution is exactly calculated

Loschmidt echo



$$c \to \infty, N \to \infty$$

$$|\langle X(t)|X(0)\rangle| \longrightarrow \frac{1}{n\sqrt{2\pi t}}\sqrt{C(n\sqrt{2\pi t})^2 + S(n\sqrt{2\pi t})^2} \quad \boxed{\sim t^{-1/2}}$$

Fresnel integral : $C(x) = \int_0^x \cos\left(\frac{\pi}{2}s^2\right) \mathrm{d}s, \quad S(x) = \int_0^x \sin\left(\frac{\pi}{2}s^2\right) \mathrm{d}s$

Time evolution of the density operator

$$\rho(x) = \hat{\psi}^\dagger(x)\hat{\psi}(x)$$

$$\langle X(t)|\rho(x)|X(t)\rangle = \sum_{p,p'=0}^{N-1} e^{2\pi i(p-p')q/N} \times e^{i(P-P')x - i(E_p - E_{p'})t}\langle P'|\rho(0)|P\rangle$$

Determinant Formula

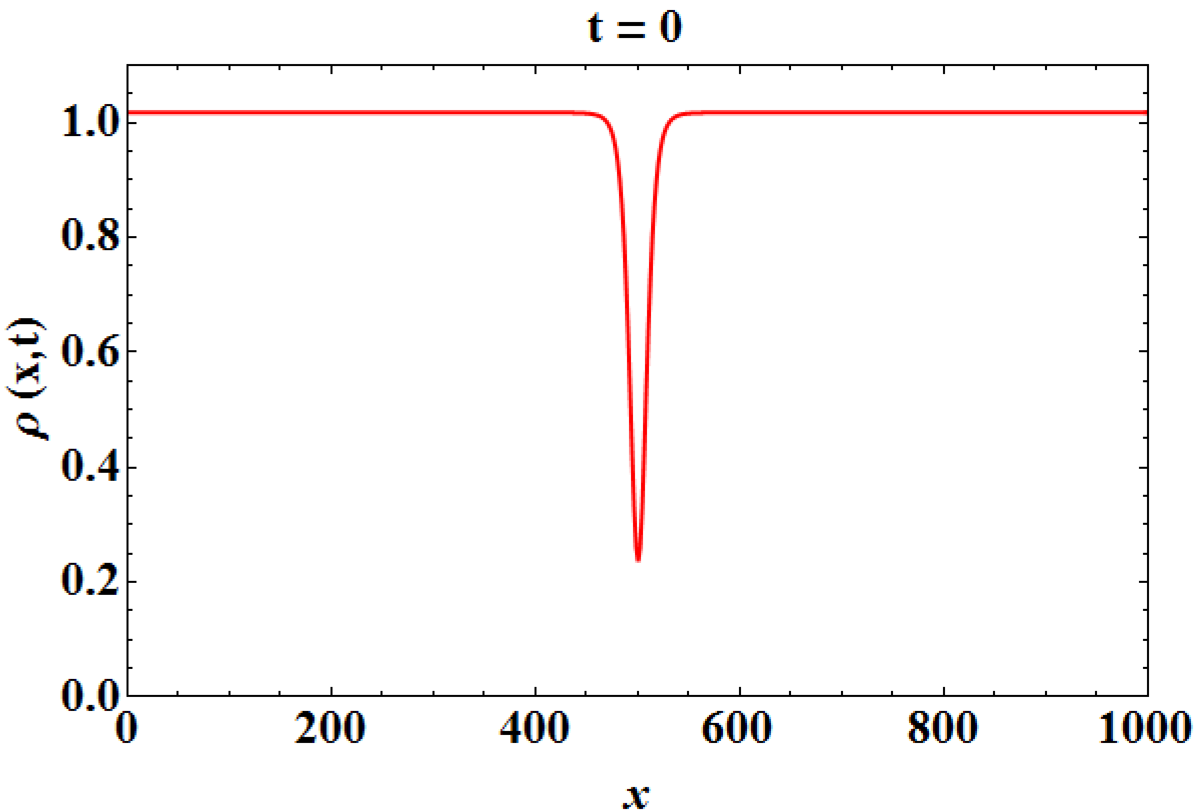
$$\langle P'|\rho(0)|P\rangle = i^N(P-P')\left(\prod_{j,\ell=1}^{N}\frac{k_j - k_\ell + ic}{k'_j - k_\ell}\right)\det_{N-1} U(k,k')$$

$$U(k,k')_{j,\ell} = 2\delta_{j\ell}\mathrm{Im}\left[\prod_{a=1}^{N}\frac{k'_a - k_j + ic}{k_a - k_j + ic}\right] + \frac{\prod_{a=1}^{N}(k'_a - k_j)}{\prod_{a\neq j}^{N}(k_a - k_j)}\left(K(k_j - k_\ell) - K(k_N - k_\ell)\right)$$

$$K(k) = 2c/(k^2 + c^2)$$

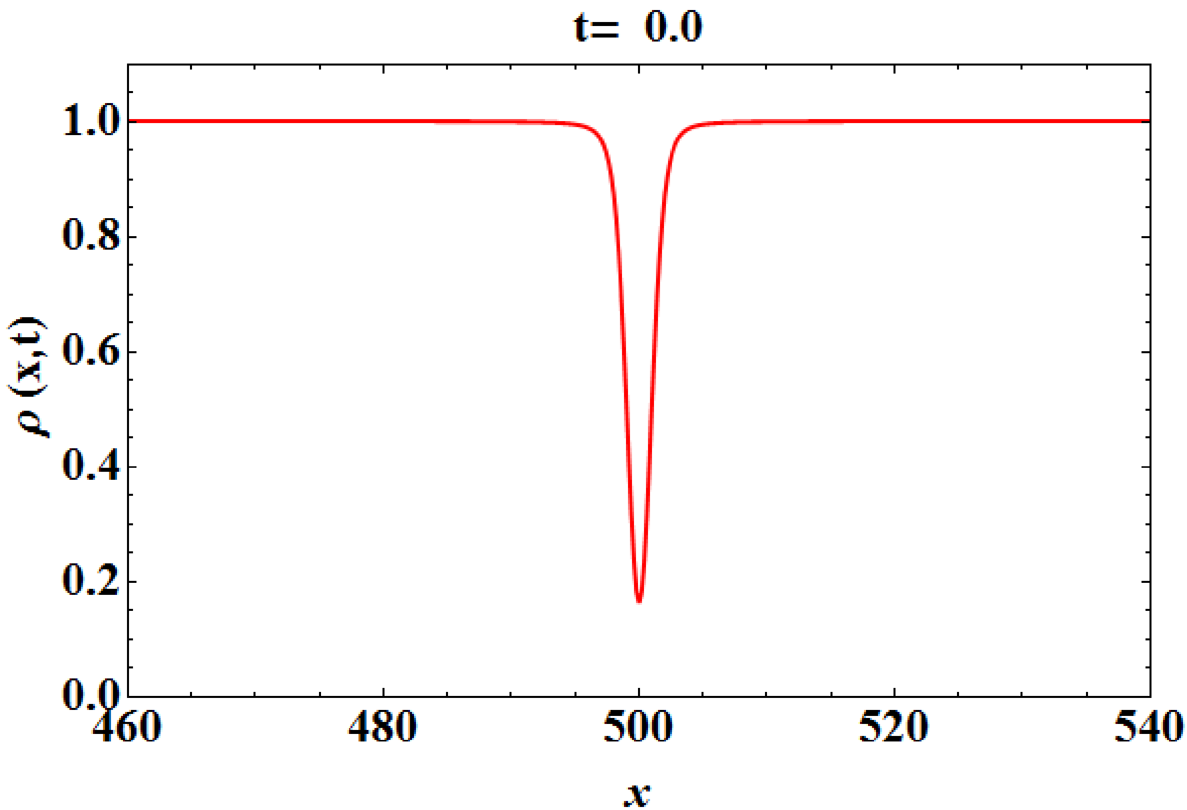
Animation of the Dynamics

$N = 1000$



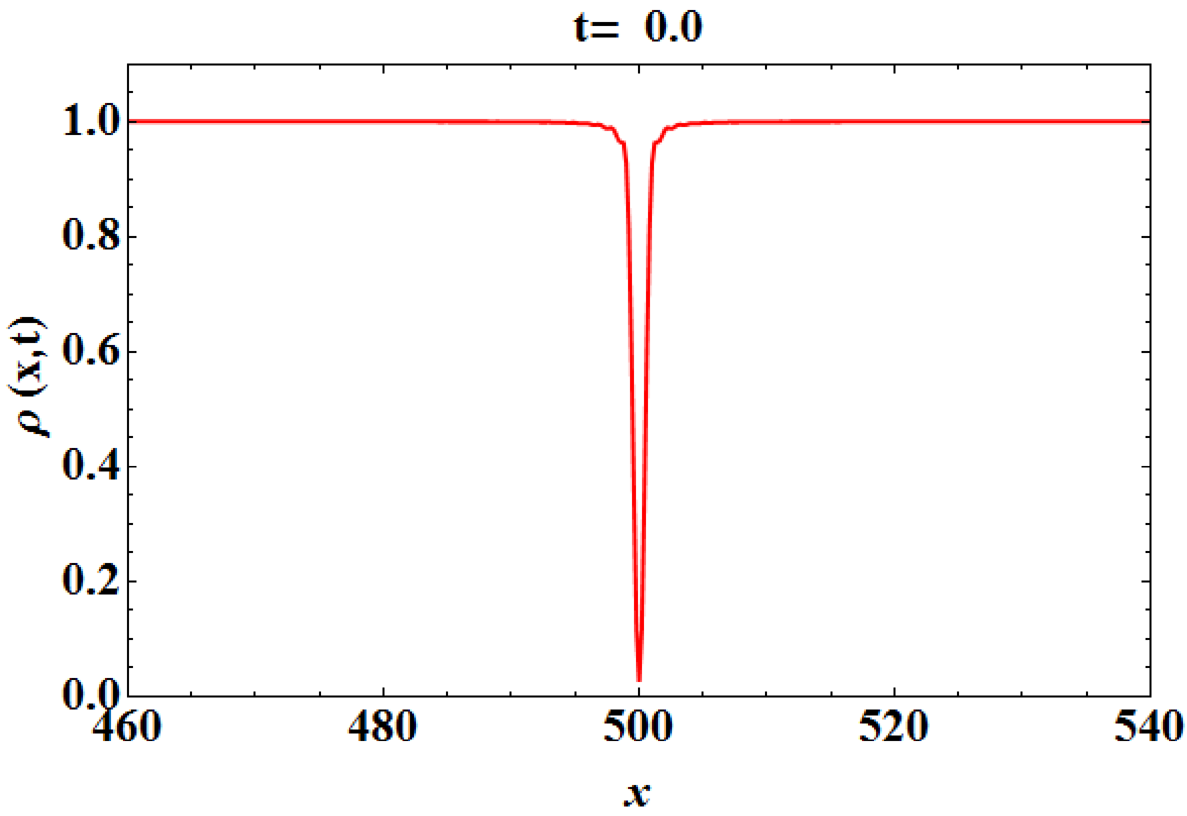
$c = 0.01$

$N = 1000$



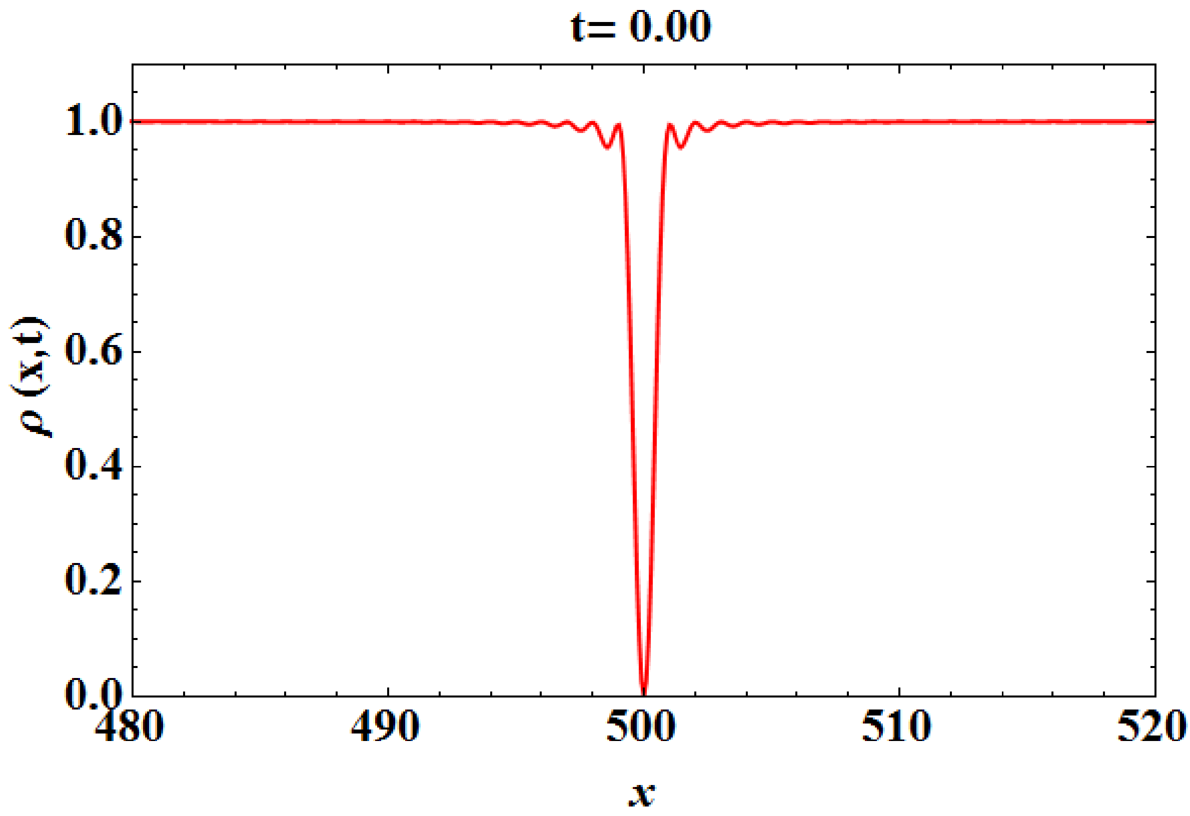
$c = 1$

$N = 1000$



$c = 10$

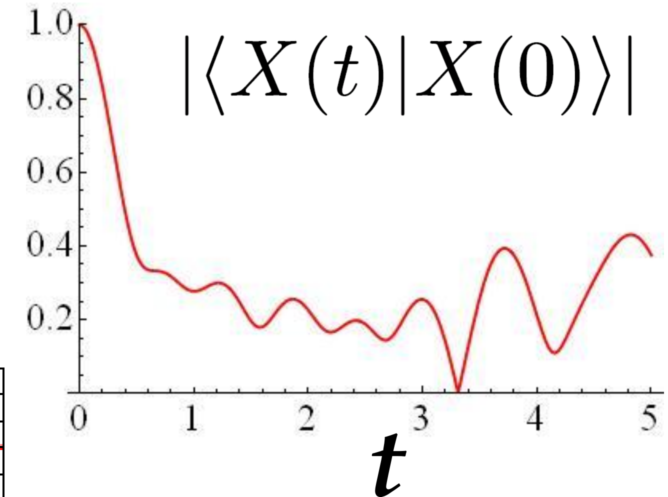
$N = 1000$

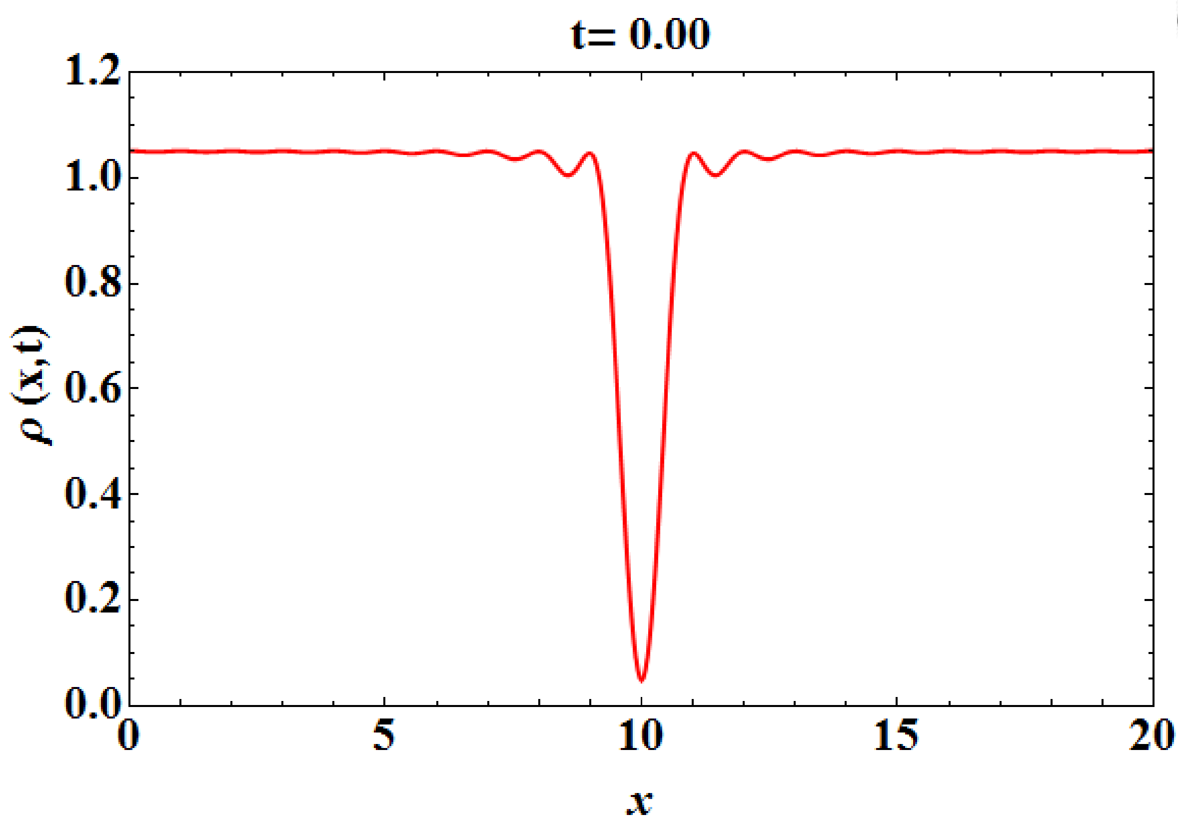


$c = 100$

$N = 20$, $c = 100$

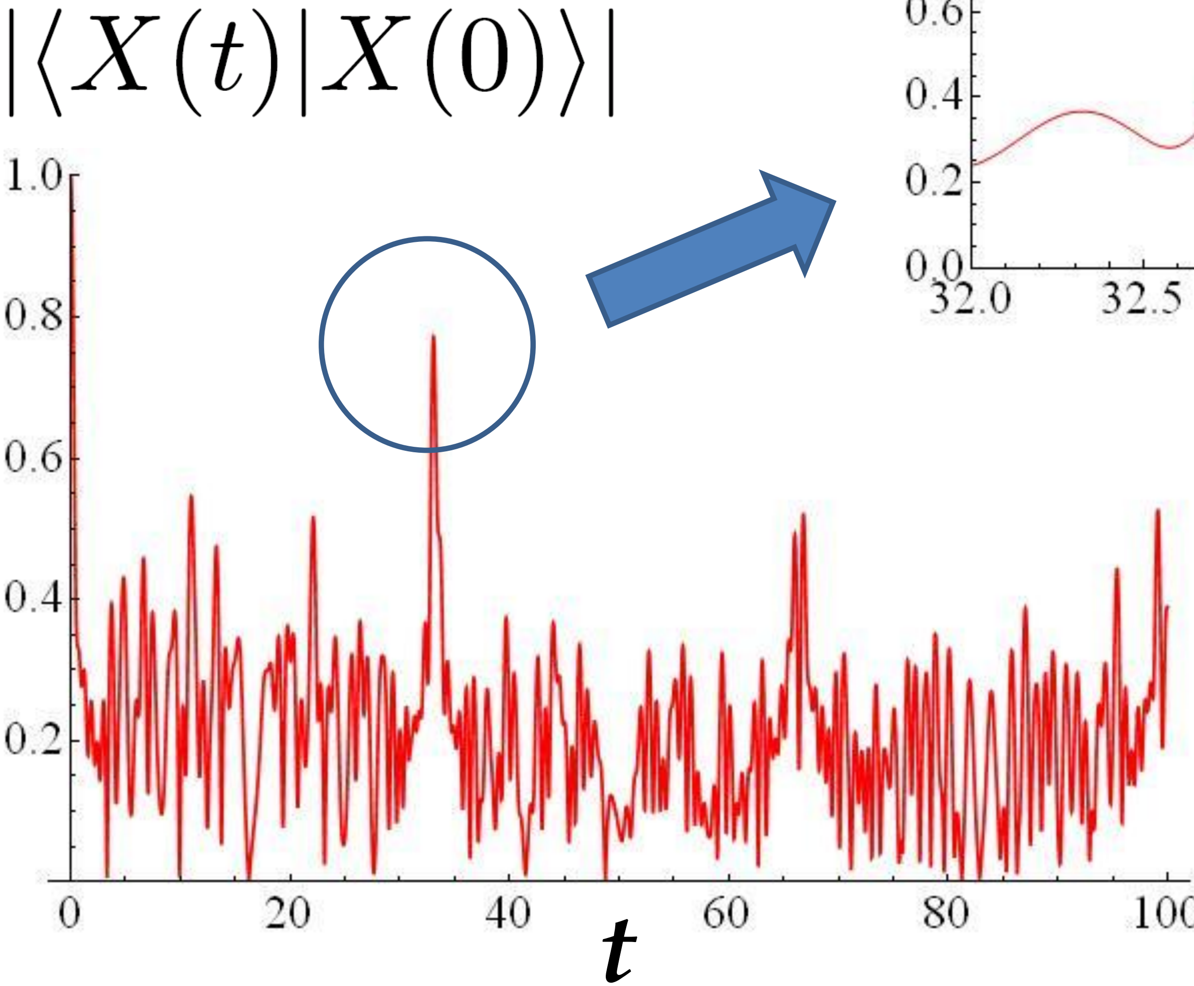
Short time Dynamics

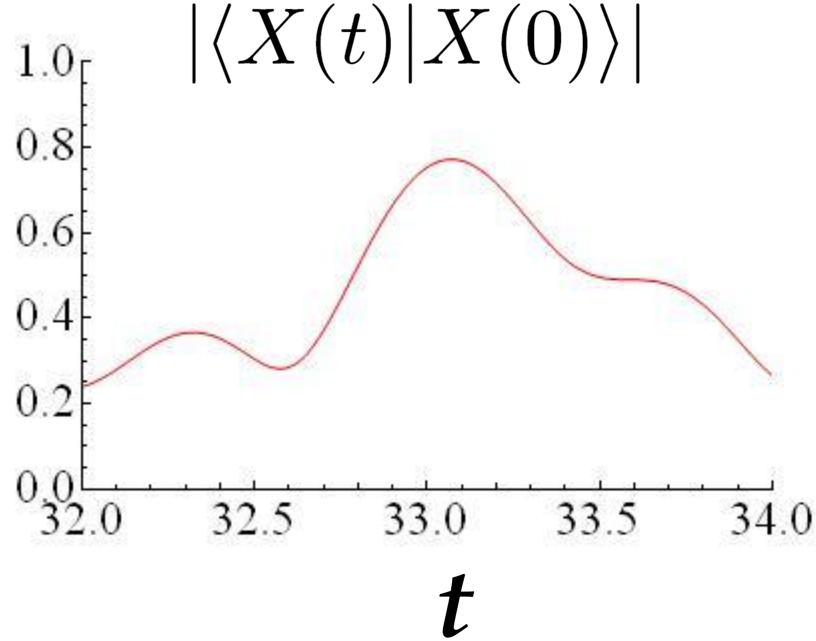




$N = 20$, $c = 100$

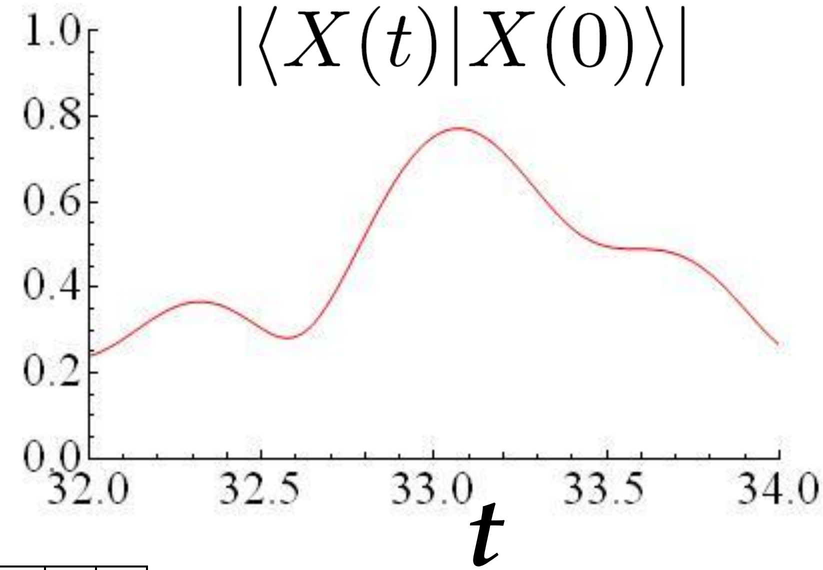
Long time Dynamics



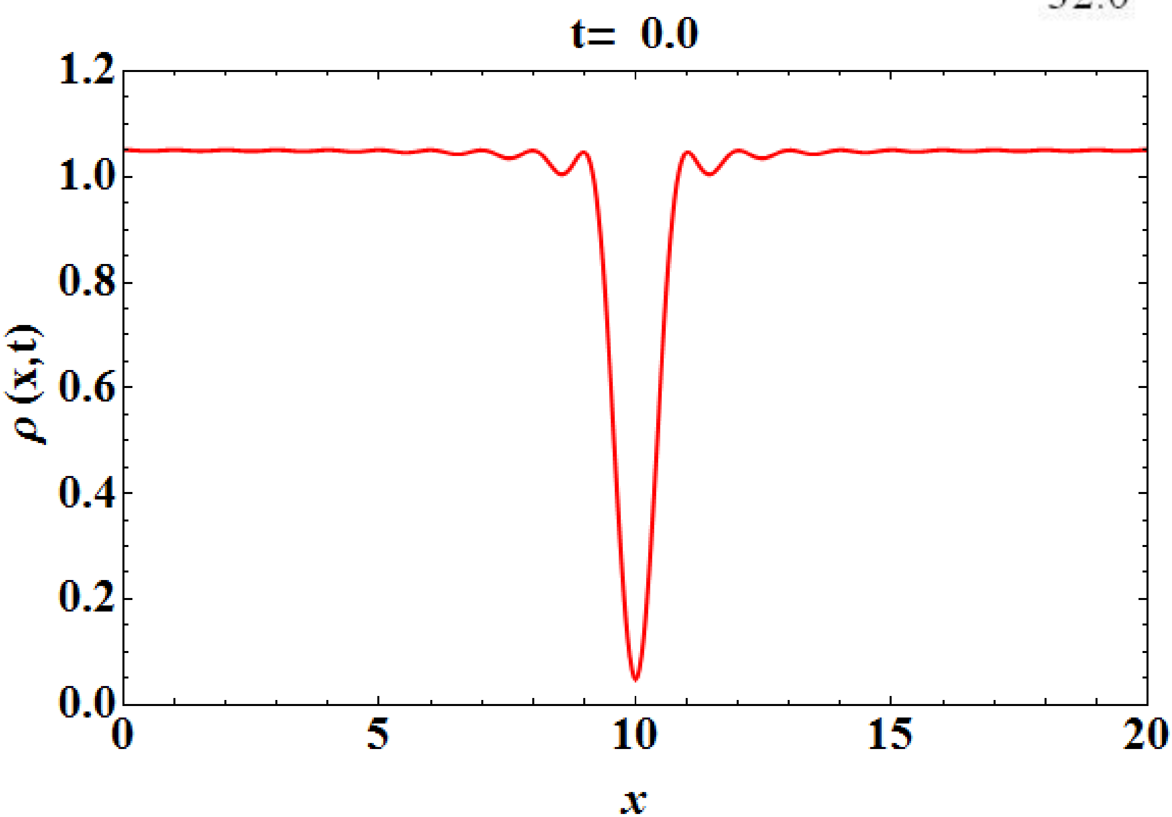


Recurrence at
$t \sim 33$?

$N = 20$, $c = 100$

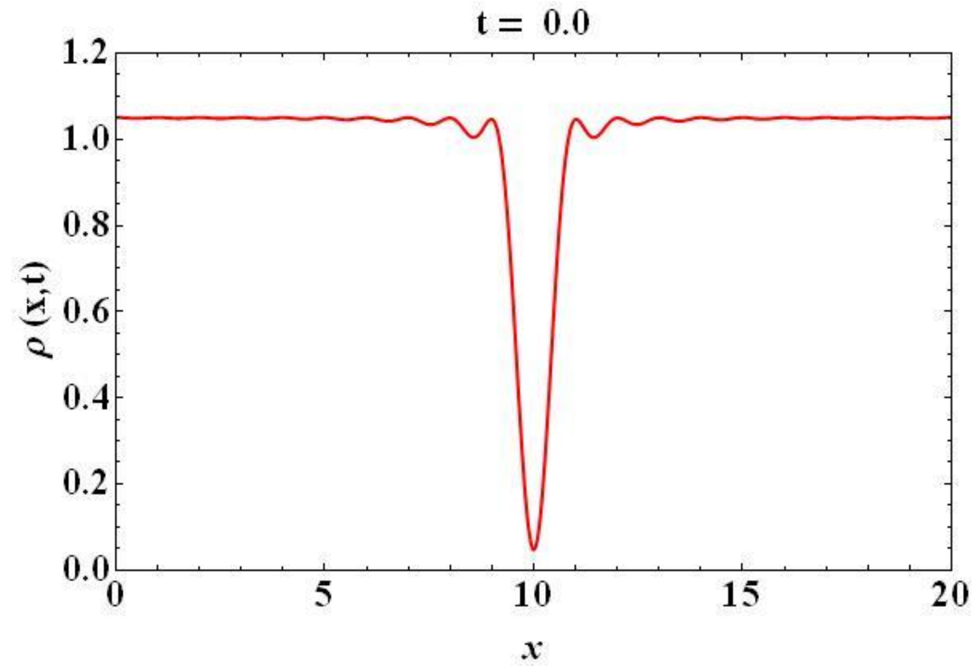


Long time Dynamics

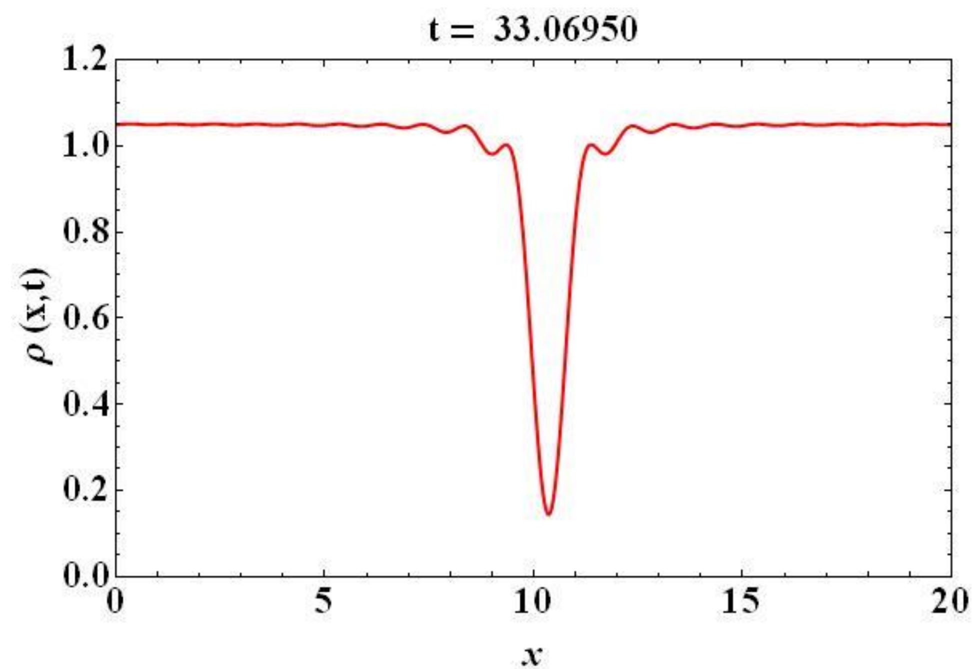


Recurrence at
$t \sim 33$?

Snap shots at
$t = 0$ and $t \sim 33$



We observe
recurrence phenomena
Kaminishi, et. al., arXive:1305.3412



Summary

1. We construct quantum wave packets by the discrete Fourier transform of the type II excitations.

2. Density profile of the quantum wave packet coincides with the squared amplitude of dark soliton.

3. Phase of the matrix element of the field operator between quantum wave packet coincides with that of dark soliton.

4. We can exactly calculate time evolution of the quantum wave packet.