Exact matrix product solution for the boundary-driven Lindblad XXZ chain

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joint work with D. Karevski (Nancy) and V. Popkov, (MPI Dresden)

- Boundary driven Lindblad XXZ chain
- Matrix product ansatz for the stationary density matrix
- Isotropic Lindblad-Heisenberg chain
- Conclusions

1. Boundary-driven XXZ Lindblad chain

Non-equilibrium behaviour of open quantum system:

- Experimentally accessible (quasi one-dimensional spin chain materials, artificially assembled nanomagnets)

- Theoretically challenging:

- Interplay of magnon excitations, magnetization currents with twisted boundary fields (➔ non-equilibrium stationary state)

- Fundamental problems

  - ➢ No density matrix exp(-βH)

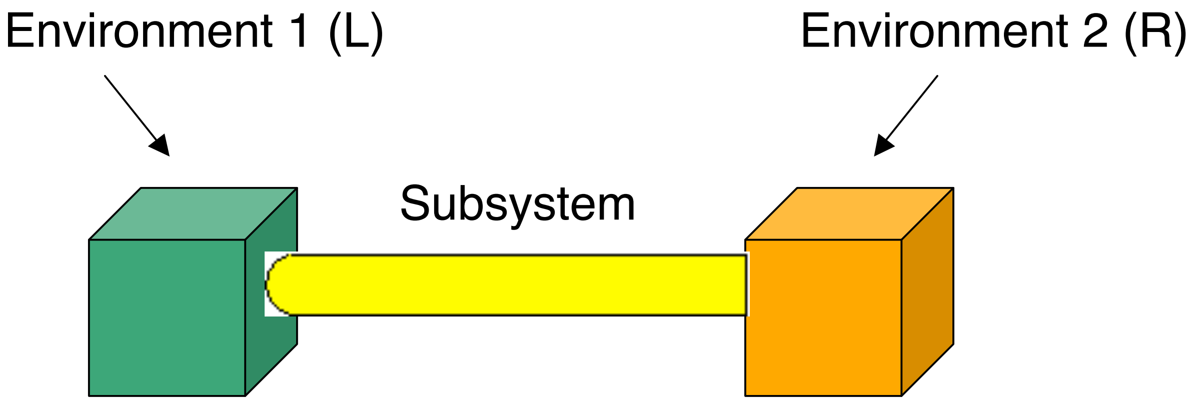
  - ➢ Non-linear response far from equilibrium

  - ➢ Interplay of bulk transport with boundary pumping

Lindblad equation for open quantum systems

Lindblad (1976): General time evolution equation of a quantum subsystem

E.g.



Total System: Hamiltonian $H_{tot} = H_L + H + H_R$

Subsystem: Quantum Hamiltonian H, reduced density matrix $\rho(t)$

Quantity of interest: Stationary density matrix $\rho^* = \lim_{t \to \infty} \rho(t) \neq \exp(-\beta H)$

Lindblad equation $$\frac{d}{dt}\rho = -i[H, \rho] + \mathcal{D}^L(\rho) + \mathcal{D}^R(\rho)$$

unitary part (subsystem), left dissipator, right dissipator

To preserve unitarity and normalization (Tr $\rho(t) = 1$):

$$\mathcal{D}^{L,R}(\rho) = D^{L,R}\rho D^{L,R\dagger} - 1/2\{\rho, D^{L,R\dagger}D^{L,R}\}$$

Boundary pumping: $\rho^* \neq \exp(-\beta H)$

Task: a) Choose H and $D^L(\rho) \neq D^R(\rho)$ appropriately for physical scenario

b) Find ρ^*

c) Compute observables

Lindblad equation for XXZ Heisenberg quantum chain

Anisotropic (XXZ) Heisenberg spin-1/2 quantum chain:

$$H = J\sum_k[\sigma_k^x\sigma_{k+1}^x + \sigma_k^y\sigma_{k+1}^y + \Delta(\sigma_k^z\sigma_{k+1}^z - \varepsilon_0)] + g_1^L + g_N^R$$

Exchange constant: J=1/2

Bulk interaction: $\Delta = (q + q^{-1})/2$, $\varepsilon_0 = 1$

Boundary fields: $\vec{f}^L \cdot \vec{\sigma} = f^L\sigma_u^z$ (left) $\vec{f}^R \cdot \vec{\sigma} = f^R\sigma_v^z$ (right)

Choose $\sigma^z_u = \sin\theta_L\,\sigma^y + \cos\theta_L\,\sigma^z$ and $\sigma^z_v = -\sin\theta_R\,\sigma^x + \cos\theta_R\,\sigma^z$

y-z plane x-z plane

Boundary pumping:

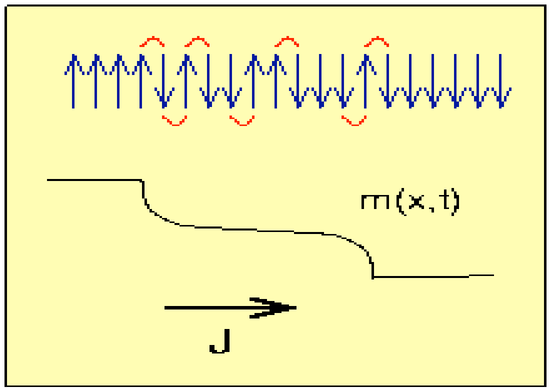
Consider Lindblad terms corresponding to complete polarization in the plane of the quantum boundary fields

$$D^L = \sqrt{\frac{\Gamma}{2}}(\sigma_1^x + i\cos\theta_L \sigma_1^y - i\sin\theta_L \sigma_1^z) \qquad \text{(y-z plane)}$$

$$D^R = \sqrt{\frac{\Gamma}{2}}(\cos\theta_R \sigma_N^x - i\sigma_N^y + \sin\theta_R \sigma_N^z) \qquad \text{(x-z plane)}$$

Stationary solution (without bulk dynamics): $\rho_{L,R} = (1 \pm \sigma^z_{u,v})/2)$

Bulk dynamics ==> Current, magnetization profile



?

2. Matrix product ansatz for the stationary density matrix

- Determine ρ from stationary Lindblad equation $i[H,\rho] = \mathcal{D}^L(\rho) + \mathcal{D}^R(\rho)$
- Write $\rho = SS^\dagger / \mathrm{Tr}(SS^\dagger)$, $S \in C^{2N}$
- Matrix product ansatz $$S = \langle \phi | \Omega^{\otimes N} | \psi \rangle$$

with 2x2 matrix $$\Omega = \begin{pmatrix} A_1 & A_+ \\ A_- & A_2 \end{pmatrix}$$

where $\langle \phi |$ and $| \psi \rangle$ are vectors in some space and A_i are matrices

A_i, $\langle \phi |$ and $| \psi \rangle$ have to determined such that stationary LE is satisfied!

Two steps: (1) bulk part for A_i, (2) boundary part for $\langle \phi |$ and $| \psi \rangle$

Solution of LE (bulk part)

Step 1: Introduce local divergence condition (different from Prosen 2011)

- remember $H = \Sigma_{k=1}^{N-1} h_{k,k+1} + g_1^L + g_N^R$

  with 4x4 matrix $h = [\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y + \Delta (\sigma^z \otimes \sigma^z - 1)]/2$

  and 2x2 boundary matrices $g^L = f^L \sigma^z{}_u$, $g^R = f^R \sigma^z{}_v$

- introduce 2x2 matrix $\Xi = \begin{pmatrix} E_1 & E_+ \\ E_- & E_2 \end{pmatrix}$

  with non-commutative auxiliary matrices E_i

- require $[h, \Omega \otimes \Omega] = \Xi \otimes \Omega - \Omega \otimes \Xi$ (local divergence condition)

==> 16 quadratic equations for the 8 matrices A_i, E_i

$$\begin{pmatrix} 0 & \Delta A_1 A_+ - A_+ A_1 & \Delta A_+ A_1 - A_1 A_+ & 0 \\ -\Delta A_1 A_- + A_- A_1 & -[A_+ , A_-] & -[A_1 , A_2] & -\Delta A_+ A_2 + A_2 A_+ \\ -\Delta A_- A_1 + A_1 A_- & [A_1 , A_2] & [A_+ , A_-] & -\Delta A_2 A_+ + A_+ A_2 \\ 0 & \Delta A_- A_2 - A_2 A_- & \Delta A_2 A_- - A_- A_2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} E_1 A_1 - A_1 E_1 & E_1 A_+ - A_1 E_+ & E_+ A_1 - A_+ E_1 & E_+ A_+ - A_+ E_+ \\ E_1 A_- - A_1 E_- & E_1 A_2 - A_1 E_2 & E_+ A_- - A_+ E_- & E_+ A_2 - A_+ E_2 \\ E_- A_1 - A_- E_1 & E_- A_+ - A_- E_+ & E_2 A_1 - A_2 E_1 & E_2 A_+ - A_2 E_+ \\ E_- A_- - A_- E_- & E_- A_2 - A_- E_2 & E_2 A_- - A_2 E_- & E_2 A_2 - A_2 E_2 \end{pmatrix}$$

4 Commutation relations: $0 = [E_i, A_i]$

8 relations with q-commutators, e.g., $\Delta A_1 A_+ - A_+ A_1 = E_1 A_+ - A_1 E_+$

4 relations with commutators, e.g. $[A_+ , A_-] = E_2 A_1 - A_2 E_1$

- Solution of all 16 equations in terms of only three matrices $A_\pm$, Q with relations

$$[A_+ , A_-] = - (q-q^{-1}) (b\bar{b}\, Q - c\bar{c}\, Q^{-1})$$

$$QA_\pm = q^{\pm 1} A_\pm Q$$

$$Q\, Q^{-1} = Q^{-1} Q = 1$$

by setting ($b,\bar{b},c,\bar{c}$ arbitrary)

$A_1 = b\, Q + c\, Q^{-1}$, $A_2 = \bar{b}\, Q + \bar{c}\, Q^{-1}$ (diagonal part of Ω)

$E_\pm = 0$

$E_1 = (q-q^{-1})/2\, (b\, Q - c\, Q^{-1})$, $E_2 = -(q-q^{-1})/2\, (\bar{b}\, Q - \bar{c}\, Q^{-1})$ (diagonal part of Ξ)

- Relations define Ω and Ξ in terms of $A_\pm$, Q

  - Proof by straightforward computation

  - $\Xi + \kappa\Omega$ is also a solution

Relation to quantum algebra $U_q[SU(2)]$

Use parametrization

$$b = \frac{\alpha}{q - q^{-1}} \frac{\nu}{\lambda}, \qquad \bar{b} = \frac{\alpha}{q - q^{-1}} \frac{1}{\lambda\nu},$$

$$c = -\frac{\alpha}{q - q^{-1}} \mu\lambda, \qquad \bar{c} = -\frac{\alpha}{q - q^{-1}} \frac{\lambda}{\mu}$$

and define $A_\pm = i\alpha S_\pm$, $Q = \lambda\, q^{Sz}$

==> Defining relations for $U_q[SU(2)]$

$$[S_+, S_-] = \frac{q^{2S_z} - q^{-2S_z}}{q - q^{-1}}$$

$$q^{S_z} S_\pm = q^{\pm 1} S_\pm q^{S_z}.$$

- Matrix product ansatz with $U_q[SU(2)]$ generators!
- Symmetry of bulk Hamiltonian (without boundary fields)

Representation theory

Define $[x]_q = (q^x - q^{-x})/(q - q^{-1})$

- Finite-dimensional irreps not of interest

-Infinite-dimensional representation (with complex parameter p)

$$S_z = \sum_{k=0}^{\infty} (p-k)|k\rangle\langle k|,$$

$$S_+ = \sum_{k=0}^{\infty} [k+1]_q |k\rangle\langle k+1|,$$

$$S_- = \sum_{k=0}^{\infty} [2p-k]_q |k+1\rangle\langle k|$$

==> Explicit form of Ω!

Solution of LE (boundary part)

Step 2: Condition on boundary vectors

- remember $H = \sum_{k=1}^{N-1} h_{k,k+1} + g_1^L + g_N^R$

  with 4x4 matrix $h = [\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y + \Delta\,(\sigma^z \otimes \sigma^z - 1)]/2$

  and 2x2 boundary matrices $g^L = f^L \sigma^z{}_u$, $g^R = f^R \sigma^z{}_v$

- define 2x2 matrix $\Phi = [g,\Omega]$ and introduce $\Upsilon_k := \Omega^{\otimes(k-1)} \otimes \Upsilon \otimes \Omega^{\otimes(N-k)}$

==> local divergence condition implies

$$[H, \Omega^{\otimes N}] = \Phi_1{}^L + \Xi_1 + \Phi_N{}^R - \Xi_N$$

(reduction of infinitesimal unitary part of evolution to boundary terms)

Also Lindblad operator has only boundary parts:

==> split stationary LE into two boundary equations

$$\mathcal{D}^L(SS^\dagger) = i(\Phi_1^L + \Xi_1)S^\dagger - iS(\Phi_1^{L^\dagger} + \Xi_1^\dagger),$$

$$\mathcal{D}^R(SS^\dagger) = i(\Phi_N^R - \Xi_N)S^\dagger - iS(\Phi_N^{R\dagger} - \Xi_N^\dagger),$$

Define $A_0 = (A_1+A_2)/2$, $A_z = (A_1-A_2)/2$, Make decomposition

- left boundary: $S = \langle\phi| \, [A_0 + A_z\sigma^z + A_+\sigma^+ + A_-\sigma^-] \otimes \Omega^{\otimes(N-1)} \, |\psi\rangle$

- right boundary: $S = \langle\phi| \, \Omega^{\otimes(N-1)} \otimes [A_0 + A_z\sigma^z + A_+\sigma^+ + A_-\sigma^-] \, |\psi\rangle$

(likewise $S^\dagger$)

==> Two separate sets of equations for action of A_i on boundary vectors

==> Complete construction of ρ with some constraints on parameters

3. Isotropic Lindblad-Heisenberg chain

- Isotropic Heisenberg chain: Δ=1 (q=1)
- SU(2) symmetric (only bulk Hamiltonian, not boundary fields, not Lindblad terms)
- For convenience: $\alpha = \lambda = 1$, $\mu = \nu = i$
- $[x]_1 = x$, limits $q \to 1$ in representation well-defined

$$\Omega = i \begin{pmatrix} S^z & S_+ \\ S_- & -S^z \end{pmatrix}, \quad \Xi = i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

or in vector notation $\vec{S} = (S_x, S_y, S_z), \quad \vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$

$$\Omega = i\vec{S} \cdot \vec{\sigma}, \quad \Xi = i\mathbb{1}$$

Solution of boundary equations:

- Key idea: Introduce coherent states

$$\langle\phi| := \sum_{n=0}^{\infty} \frac{\phi^n}{n!}\langle 0|(S_+)^n = \sum_{n=0}^{\infty} \phi^n|\langle n|,$$

$$|\psi\rangle := \sum_{n=0}^{\infty} \frac{\psi^n}{n!}(S_-)^n|0\rangle = \sum_{n=0}^{\infty} \psi^n \binom{2p}{n}|n\rangle$$

- SU(2) commutation relations:

$$\langle\phi|S_z = \langle\phi|(p - \phi S_+), \qquad S_z|\psi\rangle = (p - \psi S_-)|\psi\rangle,$$

$$\langle\phi|S_- = \phi\langle\phi|(2p - \phi S_+) \qquad S_+|\psi\rangle = \psi(2p - \psi S_-)|\psi\rangle$$

==> Action of S_z, S_- reduced to action of S_+! (right boundary: S_z, S_+ to S_+)

- Left Lindblad operator can be obtained from complete polarization along z-axis by unitary transformation $U = \exp(i\theta_L \sigma^x/2)$

==> new basis $\Omega(\theta_L) = i[S_z(\theta_L)\sigma_u^z + S_+(\theta_L)\sigma_u^+ + S_-(\theta_L)\sigma_u^-]$

with

$$S_z(\theta_L) = S_z \cos\theta_L + i \sin\theta_L \frac{S_+ - S_-}{2},$$
$$S_+(\theta_L) = \frac{S_+ + S_-}{2} + \cos\theta_L \frac{S_+ - S_-}{2} + iS_z \sin\theta_L$$
$$S_-(\theta_L) = \frac{S_+ + S_-}{2} - \cos\theta_L \frac{S_+ - S_-}{2} - iS_z \sin\theta_L$$

- new left boundary equations require: $\langle\phi|S_-(\theta_L) = 0, \qquad \langle\phi|S_z(\theta_L) = p\langle\phi|$

- Solution:

$$\phi = \tan(\theta_L/2), \qquad p = \frac{i}{\Gamma - 2if^L}$$

Proof: Coherent state relations and solution for ϕ lead to

$$S = \langle\phi|\Omega^{\otimes N}|\psi\rangle = ip\sigma_u^z \otimes \tilde{S} + \sigma_u^+ \otimes W$$

where $\tilde{S} = \langle\phi| \Omega^{\otimes(N-1)} |\psi\rangle$, $W = i \langle\phi| [S_+ + S^-] \Omega^{\otimes(N-1)} |\psi\rangle$

live on space for N-1 sites

$$\Longrightarrow \quad SS^\dagger = |p|^2 \mathbb{1} \otimes \tilde{S}S^\dagger - ip\sigma_u^- \otimes \tilde{S}W^\dagger - (ip)^*\sigma_u^+ \otimes W\tilde{S}^\dagger + \sigma_u^+\sigma_u^- \otimes WW^\dagger.$$

Left Lindblad:

$$\mathcal{D}^L(SS^\dagger) = 2\Gamma|p|^2\sigma_u^z \otimes \tilde{S}\tilde{S}^\dagger + \Gamma ip\sigma_u^- \otimes \tilde{S}W^\dagger + \Gamma(ip)^*\sigma_u^+ \otimes W\tilde{S}^\dagger.$$

Left Hamiltonian:

$$i[H, SS^\dagger]\,\|_{\text{Left}} = -[ip + (ip)^*]\sigma_u^z \otimes \tilde{S}\tilde{S}^\dagger - \sigma_u^- \otimes [1 - 2if^L(ip)]\tilde{S}W^\dagger - \sigma_u^+ \otimes [1 + 2if^L(ip)^*]W\tilde{S}^\dagger$$

} equal with condition on p

Treatment of right boundary similar:

- Lindblad operator can be obtained from complete polarization along (-z)-axis by unitary transformation $U = \exp(i\theta_R \sigma^y/2)$

- new right boundary equations require: $S_+(\theta_R)|\psi\rangle = 0$

- Solution: $\psi = -\tan(\theta_R/2)$, $f^L = -f^R$

==> Complete explicit construction of ρ for isotropic case

Remark: For anisotropic case and no quantum boundary fields relation between representation parameter p and Lindblad coupling strength Γ reads

$$2\Gamma = i\,(q^p + q^{-p}) / [p]_q$$

Currents and magnetization profiles:

Local conservation law for local magnetization:

$$\frac{d}{dt}\ \sigma_n^\alpha = j_{n-1}^\alpha - j_n^\alpha \qquad \text{for } \alpha = x,y,z$$

with currents $j_n^\alpha = 2 \sum_{\beta,\gamma} \varepsilon_{\alpha\beta\gamma}\ \sigma_n^\beta\, \sigma_{n+1}^\gamma =$

==> Stationary case: $< j_n^\alpha > = j^\alpha \ \ \forall\, n$

➢ Untwisted model $\theta_L = \theta_R = 0$ [Prosen 2011]:

- $< \sigma_n^x > = < \sigma_n^y > = 0\ \forall n$ (flat magnetization profiles for x and y component
- $j^x = j^y = 0$

Proof: z-Parity symmetry $U_z = (\sigma^z)^{\otimes N}$ of density matrix: $U_z \rho U_z = \rho$

==> $< \sigma_n^b > = \mathrm{Tr}\,(\sigma_n^b\, \rho) = \mathrm{Tr}\,(\sigma_n^b\, U_z \rho U_z) = \mathrm{Tr}\,(U_z \sigma_n^b U_z\, \rho) = - < \sigma_n^b >$ for b=x,y

and similar for j^x, j^y

- <u>Twisted case:</u> $< \sigma_n^\alpha > \neq 0, \quad j^\alpha \neq 0 \; \forall \, \alpha$

All components have non-zero expectation!

- $< \sigma_n^z > = - < \sigma_{N+1-n}^z > \; \forall \, n$
- $j^x = - j^y$

Proof: Key idea: Consider instead of parity another symmetry U of ρ

Specifically, for $\theta_R = - \theta_L = \pi/2$

$U = U_x V R$

with Space reflection R: $n \rightarrow N + 1 - n$,

x-y Rotation of spins $V = \mathrm{diag}(1,i)^{\otimes N}$

4. Conclusions

- Matrix product construction of stationary density matrix for boundary driven XXZ-Lindblad-chain using local-divergence condition
- Quadratic matrix algebra
- Relation with bulk symmetry, but not boundary terms
- Non-trivial magnetization profiles and non-vanishing magnetization current for all spin components even in isotropic case with general boundary twist

Open problems:

- Extension to other quantum systems with nearest-neighbour interaction
- Relationship with bulk symmetry and (possibly) full integrability
- Dynamical matrix product ansatz