Weakly self-avoiding walk in dimension four

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Abstract

We report on recent and ongoing work on the continuous-time weakly self-avoiding walk on the 4-dimensional integer lattice, with focus on a proof that the susceptibility diverges at the critical point with a logarithmic correction to mean-field scaling. The proof is based on a rigorous renormalisation group analysis of a supersymmetric field theory representation of the weakly self-avoiding walk.

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Self-avoiding walk

Discrete-time model: Let $\mathcal{S}_n(x)$ be the set of $\omega : \{0, 1, \dots, n\} \to \mathbb{Z}^d$ with:
$\omega(0) = 0$, $\omega(n) = x$, $|\omega(i+1) - \omega(i)| = 1$, and $\omega(i) \neq \omega(j)$ for all $i \neq j$.
Let $\mathcal{S}_n = \cup_{x \in \mathbb{Z}^d} \mathcal{S}_n(x)$.

Let $c_n(x) = |\mathcal{S}_n(x)|$. Let $c_n = \sum_x c_n(x) = |\mathcal{S}_n|$. Easy: $c_n^{1/n} \to \mu$.
Declare all walks in $\mathcal{S}_n$ to be equally likely: each has probability c_n^{-1}.

Two-point function: $G_z(x) = \sum_{n=0}^{\infty} c_n(x) z^n$, radius of convergence $z_c = \mu^{-1}$.

Predicted asymptotic behaviour:

$$c_n \sim A\mu^n n^{\gamma-1}, \quad \mathbb{E}_n|\omega(n)|^2 \sim Dn^{2\nu}, \quad G_{z_c}(x) \sim C|x|^{-(d-2+\eta)},$$

with universal critical exponents γ, ν, η obeying $\gamma = (2 - \eta)\nu$.

Dimensions other than $d = 4$

Theorem. (Brydges–Spencer (1985); Hara–Slade (1992); Hara (2008)...)
For $d \geq 5$,

$$c_n \sim A\mu^n, \quad \mathbb{E}_n|\omega(n)|^2 \sim Dn, \quad G_{z_c}(x) \sim c|x|^{-(d-2)}, \quad \frac{1}{\sqrt{Dn}}\omega(\lfloor nt \rfloor) \Rightarrow B_t.$$

Proof uses lace expansion, requires $d > 4$.

$d = 2$. Prediction: $\gamma = \frac{43}{32}$, $\nu = \frac{3}{4}$, $\eta = \frac{5}{24}$,
Nienhuis (1982); Lawler–Schramm-Werner (2004) — connection with $\mathrm{SLE}_{8/3}$.

$d = 3$. Numerical: $\gamma \approx 1.16$, $\nu \approx 0.588$, $\eta \approx 0.031$.
E.g., Clisby (2011): $\nu = 0.587597(7)$.

Theorem. (lower: Madras 2012, upper: Duminil-Copin–Hammond 2012)

$$\tfrac{1}{6}n^{4/3d} \leq \mathbb{E}_n|\omega(n)|^2 \leq o(n^2), \qquad \text{so } \nu \geq 2/(3d).$$

Not proved for $d = 2, 3, 4$: $\mathbb{E}_n|\omega(n)|^2 \leq O(n^{2-\epsilon})$, i.e., that $\nu < 1$.

Predictions for $d = 4$

Prediction is that upper critical dimension is 4, and asymptotic behaviour for $\mathbb{Z}^4$ has log corrections (e.g., Brézin, Le Guillou, Zinn-Justin 1973):

$$c_n \sim A\mu^n (\log n)^{1/4}, \quad \mathbb{E}_n|\omega(n)|^2 \sim Dn(\log n)^{1/4}, \quad G_{z_c}(x) \sim c|x|^{-2}.$$

The susceptibility and correlation length are defined by:

$$\chi(z) = \sum_{n=0}^{\infty} c_n z^n, \qquad \frac{1}{\xi(z)} = -\lim_{n\to\infty} \frac{1}{n} \log G_z(ne_1).$$

For these the prediction is:

$$\chi(z) \sim \frac{A'|\log(1 - z/z_c)|^{1/4}}{1 - z/z_c}, \qquad \xi(z) \sim \frac{D'|\log(1 - z/z_c)|^{1/8}}{(1 - z/z_c)^{1/2}} \qquad \text{as } z \uparrow z_c.$$

Universality hypothesis.

Continuous-time weakly self-avoiding walk

A.k.a. discrete Edwards model.

Let E_0 denote the expectation for continuous-time nearest-neighbour simple random walk $X(t)$ on $\mathbb{Z}^d$ started from 0 (steps at events of rate-$2d$ Poisson process).

Let $L_{u,T} = \int_0^T \mathbb{1}_{X(s)=u} ds$ and

$$I(T) = \int_0^T \int_0^T \mathbb{1}_{X(s)=X(t)} ds\, dt = \sum_{u \in \mathbb{Z}^d} L_{u,T}^2.$$

Let $g \in (0,\infty)$, $\nu \in (-\infty,\infty)$. The *two-point function* is

$$G_{g,\nu}(x) = \int_0^\infty E_0 \left(e^{-gI(T)}\ \mathbb{1}_{X(T)=x} \right) e^{-\nu T} dT$$

(compare $\sum_n c_n(x) z^n$).

Subadditivity $\Rightarrow$ $\exists \nu_c(g)$ s.t. *susceptibility* $\chi_g(\nu) = \sum_{x \in \mathbb{Z}^d} G_{g,\nu}(x)$ obeys

$$\chi_g(\nu) < \infty \qquad (\nu > \nu_c(g)),$$

$$\chi_g(\nu) = \infty \qquad (\nu < \nu_c(g)).$$

Main results

Theorem 1 (Bauerschmidt–Brydges–Slade 2013+). Let $d = 4$. There exists $g_0 > 0$ such that for $0 < g \le g_0$, as $t \downarrow 0$,

$$\chi_g(\nu_c(1+t)) \sim \frac{A(\log|t|)^{1/4}}{t}.$$

Theorem 2 (Brydges–Slade 2011, 2013+). Let $d \ge 4$. There exists $g_0 > 0$ such that for $0 < g \le g_0$, as $|x| \to \infty$,

$$G_{g,\nu_c}(x) \sim \frac{c}{|x|^{d-2}}.$$

Related results:

- weakly SAW on 4-dimensional hierarchical lattice (replacement of $\mathbb{Z}^4$ by a recursive structure well-suited to RG): Brydges–Evans–Imbrie (1992); Brydges–Imbrie (2003); and with different RG method Ohno (2013+).
- 4-dimensional ϕ^4 field theory: Gawędzki–Kupiainen (1985), Feldman–Magnen–Rivasseau–Sénéor (1987), Hara–Tasaki (1987).

Bubble diagram and role of $d = 4$

Let Δ denote the discrete Laplacian on $\mathbb{Z}^d$, i.e., $\Delta\phi_x = \sum_{y:|y-x|=1}(\phi_y - \phi_x)$.
Let

$$C_{m^2}(x) = \int_0^\infty E_0(\mathbb{1}_{X(T)=x})e^{-m^2T}dT = (-\Delta + m^2)^{-1}_{0x}.$$

Let X, Y be independent continuous-time simple random walks started from $0 \in \mathbb{Z}^d$.
The simple random walk bubble diagram is

$$\mathsf{B}_{m^2} = \sum_{x\in\mathbb{Z}^d}(C_{m^2}(x))^2 = \int_0^\infty E_{0,0}(\mathbb{1}_{X(T)=Y(S)})e^{-m^2S}e^{-m^2T}dSdT,$$

and the expected mutual intersection time is

$$\mathsf{B}_0 = \int_0^\infty E_{0,0}(\mathbb{1}_{X(T)=Y(S)})dSdT.$$

Direct calculation shows $d = 4$ is critical: as $m^2 \downarrow 0$,

$$\mathsf{B}_{m^2} \sim \begin{cases} cm^{-(d-4)} & d < 4 \\ c|\log m| & d = 4 \\ c & d > 4. \end{cases}$$

Bubble diagram and role of $d = 4$

For $d \geq 5$ and use of the lace expansion an essential feature is $\mathsf{B}_0 < \infty$.

For $d = 4$, the logarithmic divergence $\mathsf{B}_{m^2} \sim c|\log m|$ is the source of the logarithmic corrections to scaling for the 4-d SAW.

Comparison of WSAW and SRW

Our strategy is to determine an effective approximation of the WSAW two-point function by the two-point function of a renormalised SRW:

$$G_{g,\nu}(x) \approx (1+z_0)G_{0,m^2}(x) \qquad \text{with } m^2 \downarrow 0 \text{ as } \nu \downarrow \nu_c.$$

In physics terminology:

- m is the renormalised mass (or physical mass),
- $1+z_0$ is the field strength renormalisation.

We use a rigorous RG method to construct $z_0 = z_0(g,\nu)$ and $m^2 = m^2(g,\nu)$ such that

$$\chi_g(\nu) = (1+z_0)\chi_0(m^2) = (1+z_0)m^{-2}$$

with, as $t \downarrow 0$,

$$z_0(g,\nu_c(1+t)) \to \text{const}, \qquad m^2(g,\nu_c(1+t)) \sim \text{const}\frac{t}{|\log t|^{1/4}}.$$

Finite-volume approximation

Fix $g > 0$. Given a (large) positive integer L, let Λ_N be the torus $\mathbb{Z}^d / L^N \mathbb{Z}^d$. Finite-volume two-point function is defined by

$$G_{N,\nu}(x) = \int_0^\infty E_0^N \left(e^{-gI(T)} \mathbb{1}_{X(T)=x} \right) e^{-\nu T} dT,$$

with E_0^N the expectation for the continuous-time simple random walk on Λ_N. Let $\chi_N(\nu) = \sum_{x \in \Lambda_N} G_{N,\nu}(x)$ denote the susceptibility on Λ_N.

Easy:

$$\lim_{N \to \infty} \chi_N(\nu) = \chi(\nu) \in [0, \infty] \qquad (\nu \in \mathbb{R}),$$

$$\lim_{N \to \infty} \chi_N'(\nu) = \chi'(\nu) \qquad (\nu > \nu_c).$$

We work in finite volume, maintaining sufficient control to take the limit.

Gaussian expectation and super-expectation

Let $\phi : \Lambda \to \mathbb{C}$, with complex conjugate $\bar{\phi}$, and let $C = (-\Delta + m^2)^{-1}$.
The standard Gaussian expectation is

$$E_C F(\bar{\phi}, \phi) = Z_C^{-1} \int_{\mathbb{C}^\Lambda} e^{-\bar{\phi} C^{-1} \phi} F(\bar{\phi}, \phi) d\bar{\phi} d\phi.$$

The super-expectation is (differentials anti-commute)

$$\mathbb{E}_C F(\bar{\phi}, \phi, d\bar{\phi}, d\phi) = \int_{\mathbb{C}^\Lambda} e^{-\bar{\phi} C^{-1} \phi - \frac{1}{2\pi i} d\bar{\phi} C^{-1} d\phi} F(\bar{\phi}, \phi, d\bar{\phi}, d\phi).$$

Then

$$\mathbb{E}_C F(\bar{\phi}, \phi) = E_C F(\bar{\phi}, \phi), \quad \text{so in particular } \mathbb{E}_C \bar{\phi}_0 \phi_x = E_C \bar{\phi}_0 \phi_x = C_{0x}.$$

Much of the standard theory of Gaussian integration carries over to this setting, with beautiful properties, e.g., for a function of $\tau = (\tau_x)$ with $\tau_x = \bar{\phi}_x \phi_x + \frac{1}{2\pi i} d\bar{\phi}_x d\phi_x$,

$$\mathbb{E}_C F(\tau) = F(0).$$

Functional integral representation

Let

$$\tau_x = \phi_x \bar\phi_x + \tfrac{1}{2\pi i} d\phi_x d\bar\phi_x,$$
$$\tau_{\Delta,x} = \frac{1}{2}\Big(\phi_x(-\Delta\bar\phi)_x + \tfrac{1}{2\pi i} d\phi_x(-\Delta d\bar\phi)_x + \text{c.c.}\Big),$$

Theorem.

$$G_{N,\nu}(x) = \int_0^\infty E_0^N \left(e^{-gI(T)} \mathbb{1}_{X(T)=x}\right) e^{-\nu T} dT$$
$$= \int_{\mathbb{C}^{\Lambda_N}} e^{-\sum_{u\in\Lambda}(g\tau_u^2 + \nu\tau_u + \tau_{\Delta,u})} \bar\phi_0 \phi_x.$$

RHS is the two-point function of a supersymmetric field theory with boson field $(\phi, \bar\phi)$ and fermion field $(d\phi, d\bar\phi)$.

(Parisi–Sourlas '80; McKane '80; Dynkin '83; Le Jan '87; Brydges–Imbrie '03; Brydges–Imbrie–Slade '09).

Renormalised parameters and Gaussian approximation

Let $z_0 > -1$ and $m^2 > 0$. Change of variables $\phi_x \mapsto \sqrt{1+z_0}\phi_x$ in the integral representation gives

$$G_{g,\nu}(x) = (1+z_0)\mathbb{E}_C(e^{-V_0}\bar{\phi}_0\phi_x)$$

where $\mathbb{E}_C$ denotes Gaussian super-expectation with covariance

$$C = (-\Delta + m^2)^{-1},$$

and

$$V_0 = \sum_{u\in\Lambda}(g_0\tau_u^2 + \nu_0\tau_u + z_0\tau_{\Delta,u})$$

$$g_0 = g(1+z_0)^2, \quad \nu_0 = (1+z_0)\nu - m^2.$$

Thus the two-point function is the two-point function of a perturbation (by e^{-V_0}) of a supersymmetric Gaussian field.

Now we study $\mathbb{E}_C(e^{-V_0}\bar{\phi}_0\phi_x)$ and forget about the walks.

Objective

Given m^2, g_0, ν_0, z_0, define $C = (-\Delta + m^2)^{-1}$, $V_0 = \sum_{u \in \Lambda} (g_0 \tau_u^2 + \nu_0 \tau_u + z_0 \tau_{\Delta,u})$,

$$\hat{\chi}_N = \hat{\chi}_N(g_0, \nu_0, z_0, m^2) = \sum_{x \in \Lambda} \mathbb{E}_C(e^{-V_0} \bar{\phi}_0 \phi_x), \qquad \hat{\chi} = \lim_{N \to \infty} \hat{\chi}_N.$$

Objective: choose z_0, ν_0 depending on g_0, m^2 such that

$$\hat{\chi} = \frac{1}{m^2}, \qquad \frac{\partial \hat{\chi}}{\partial \nu_0} \sim -c_{g_0} \frac{1}{m^4} \frac{1}{\mathsf{B}_{m^2}^{1/4}}.$$

This suffices because after some implicit function theory it allows $\nu_c(g)$ to be identified and gives

$$\frac{\partial \chi}{\partial \nu} \sim -C_g \chi^2 (\log \chi)^{1/4} \quad (\nu \downarrow \nu_c)$$

which implies that

$$\chi(\nu_c(1+t)) \sim c t^{-1} (|\log t|)^{1/4}.$$

So our focus now is on $\hat{\chi}_N$.

Laplace transformation

Omit conjugates for simpler formulas. Let $Z_0(\phi) = e^{-V_0}$. Given $f : \Lambda \to \mathbb{C}$, let

$$\Gamma(f) = \mathbb{E}_C(e^{(\phi,f)} Z_0(\phi)) = e^{(f,Cf)} \mathbb{E}_C(Z_0(\phi + Cf)) \equiv e^{(f,Cf)} Z_N(Cf)$$

(by completing the square). Then with $f \equiv 1$ (so $Cf = (-\Delta + m^2)^{-1} f = m^{-2}$),

$$\begin{aligned} \hat{\chi}_N = \sum_{x \in \Lambda} \mathbb{E}_C(Z_0(\phi)\phi_0\phi_x) &= \frac{1}{|\Lambda_N|} D^2\Gamma(0; f, f) \\ &= \frac{1}{|\Lambda_N|}(f, Cf) + \frac{1}{|\Lambda_N|} D^2 Z_N(0; Cf, Cf) \\ &= \frac{1}{m^2} + \frac{1}{|\Lambda_N|} D^2 Z_N(0; Cf, Cf). \end{aligned}$$

Want to show in particular that, given m^2, g_0, with well chosen z_0, ν_0, the last term goes to zero as $N \to \infty$. So we study $Z_N(\phi)$.

Need for multi-scale analysis

Naive attempt via cumulant expansion:

$$\mathbb{E}_C e^{-V_0} \approx \exp\left[-\mathbb{E}_C V_0 + \frac{1}{2}\mathbb{E}_C(V_0; V_0) - \cdots\right]$$

fails, e.g., a contribution to the second term on RHS is

$$\nu_0^2 \sum_{x,y\in\Lambda} C(x,y)^2 \sim \nu_0^2 |\Lambda| \mathsf{B}_{m^2},$$

and it becomes worse at higher order, $(\mathsf{B}_{m^2})^2$, etc. Terms are exploding.

The *renormalisation group* method (Wilson, . . .) proposes an approach to solve this problem at the level of theoretical physics via a multi-scale analysis:
Perform the integration by progressively taking into account increasingly large scales.

We do this in a mathematically rigorous manner.

Convolution integrals and progressive integration

Recall that a random variable $X \sim N(0, \sigma_1^2 + \sigma_2^2)$ has the same distribution as $X_1 + X_2$ where $X_1 \sim N(0, \sigma_1^2)$ and $X_2 \sim N(0, \sigma_2^2)$ are independent. In particular,

$$E_{\sigma_2^2 + \sigma_1^2} f(X) = E_{\sigma_2^2} \left(E_{\sigma_1^2}(f(X_1 + X_2)|X_2) \right).$$

This finds expression for $\mathbb{E}_C$ via:

$$\mathbb{E}_{C_2+C_1} F = \mathbb{E}_{C_2} \circ \mathbb{E}_{C_1} \theta F,$$

where

$$(\theta F)(\phi, \xi, d\phi, d\xi) = F(\phi + \xi, d\phi + d\xi),$$

$\mathbb{E}_{C_1}$ integrates out ξ and $d\xi$, leaving ϕ and $d\phi$ fixed, $\mathbb{E}_{C_2}$ integrates out ϕ and $d\phi$.

More generally,

$$\mathbb{E}_{C_N + \cdots + C_1} \theta = \mathbb{E}_{C_N} \theta \circ \cdots \circ \mathbb{E}_{C_2} \theta \circ \mathbb{E}_{C_1} \theta.$$

Finite-range decomposition of covariance

Theorem (Brydges–Guadagni–Mitter '04, Bauerschmidt '13).
Let $d = 4$ and let $C = (-\Delta_\Lambda + m^2)^{-1}$ with $\Lambda = \mathbb{Z}^d / L^N \mathbb{Z}^d$.
There exist positive definite $C_1, \ldots, C_N$ such that:

- $C = \sum_{j=1}^N C_j$
- $C_j(x,y) = 0$ if $|x-y| \geq \frac{1}{2} L^j$
- for $j = 1, \ldots, N-1$, $|\nabla_x^\alpha \nabla_y^\alpha C_j(x,y)| \leq O(L^{-(2+2|\alpha|_1)j})$.

Progressive integration with this covariance decomposition gives

$$Z_N(\phi) = \mathbb{E}_C(Z_0(\phi' + \phi)) = \mathbb{E}_{C_N}\theta \circ \cdots \circ \mathbb{E}_{C_2}\theta \circ \mathbb{E}_{C_1}\theta Z_0.$$

Thus we study the mapping

$$Z_j \mapsto Z_{j+1} = \mathbb{E}_{C_{j+1}}\theta Z_j$$

and for this we need good coordinates to describe the mapping.

Relevant, marginal, irrelevant directions

The covariance estimates suggest that under $\mathbb{E}_{C_{j+1}}$:

- a typical field $\phi_x \approx [C_{j+1;x,x}]^{1/2} \approx L^{-j}$,
- this field is approximately constant over distance L^j.

Thus, for a block B of side L^j,

$$\sum_{x \in B} |\phi_x|^p \approx |B| L^{-jp} = L^{j(4-p)}.$$

The RHS is *relevant* for $p < 4$, *marginal* for $p = 4$, *irrelevant* for $p > 4$.

Taking symmetries and derivatives into account, the relevant and marginal monomials are:

τ (relevant), τ_Δ (marginal), τ^2 (marginal).

The RG map

Up to an error that must be controlled, seek approximation $Z_j \approx e^{-V_j(\Lambda)}$, with

$$V_j(\Lambda) = \sum_{u\in\Lambda}(g_j\tau_u^2 + \nu_j\tau_u + z_j\tau_{\Delta,u}),$$

and write $\mu_j = L^{2j}\nu_j$.
The error in the approximation is described by a family of forms $K_j = (K_j(X))$:

$$Z_j = \sum_{X\in\mathcal{P}_j(\Lambda)} e^{-V_j(\Lambda\setminus X)}K_j(X).$$

Then

$$Z_j \text{ is characterised by } (g_j, \mu_j, z_j, K_j).$$

The main effort: to devise an appropriate Banach space whose norm measures the size of K_j, and calculate how the coupling constants in V_j should evolve with j in such a way that K_j remains small.

The RG map is the description of the dynamical system $Z_j \mapsto Z_{j+1} = \mathbb{E}_{C_{j+1}}Z_j$ via

$$\mathrm{RG} : (g_j, z_j, \mu_j, K_j) \mapsto (g_{j+1}, z_{j+1}, \mu_{j+1}, K_{j+1}).$$

Flow of coupling constants

We compute V_{j+1} accurately to second order in the coupling constants, estimate higher-order errors, and prove that K_j contracts. In particular,

$$g_{j+1} = g_j - \beta_j g_j^2 + \cdots \qquad \text{(marginal)}$$

$$z_{j+1} = z_j + \cdots \qquad \text{(marginal)}$$

$$\mu_{j+1} = L^2 \left(1 - \frac{1}{4}\beta_j g_j\right) \mu_j + \cdots \qquad \text{(relevant)}$$

The important coefficient β_j is related to the bubble diagram:

$$\sum_{j=1}^{\infty} \beta_j = 8\mathsf{B}_{m^2}.$$

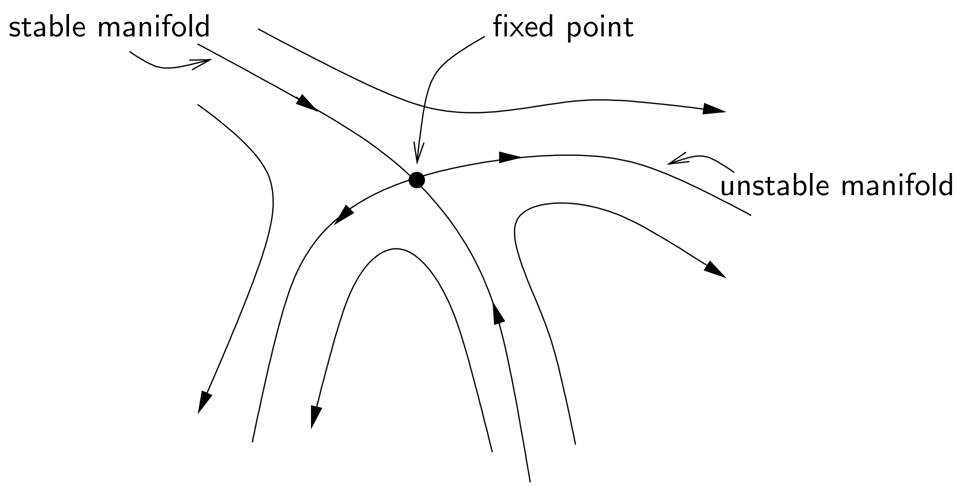
Phase portrait

For each $m^2 \geq 0$, study the dynamical system:

$$\mathrm{RG} : (g_j, z_j, \mu_j, K_j) \mapsto (g_{j+1}, z_{j+1}, \mu_{j+1}, K_{j+1}),$$

Fixed point: $\mathrm{RG}(0,0,0,0) = (0,0,0,0) =$ free field $=$ simple random walk.

Phase portrait of dynamical system near a hyperbolic fixed point:



Difficulty: Fixed point is not hyperbolic, but picture remains true.

Susceptibility

On the stable manifold (choose z_0, ν_0 depending on g_0, m^2), (V_N, K_N) is bounded, and

$$Z_N(\phi) = e^{-V_N(\phi)} + K_N(\phi) \approx e^{-V_N(\phi)}.$$

Thus, with $Cf = m^{-2}$ (constant),

$$\begin{aligned}
\hat{\chi} &= \frac{1}{m^2} + \lim_{N\to\infty} \frac{1}{|\Lambda_N|} D^2 Z_N(0; Cf, Cf) \\
&= \frac{1}{m^2} + \lim_{N\to\infty} \frac{1}{|\Lambda_N|} D^2 e^{-V_N}(0; Cf, Cf) \\
&= \frac{1}{m^2} - \lim_{N\to\infty} 2\nu_N \frac{1}{m^4} \\
&= \frac{1}{m^2},
\end{aligned}$$

since $\nu_N = L^{-2N}\mu_N \to 0$.

Logarithmic correction to susceptibility

Study derivative with respect to ν_0 along stable flow:

$$\frac{\partial \hat{\chi}}{\partial \nu_0} = \lim_{N\to\infty} \frac{1}{|\Lambda_N|} \frac{\partial}{\partial \nu_0} D^2 e^{-V_N(\phi)}(0; Cf, Cf) = -2\frac{1}{m^4} \lim_{N\to\infty} L^{-2N} \frac{\partial \mu_N}{\partial \nu_0}.$$

Use in particular that

$$g_{j+1} = g_j - \beta_j g_j^2 + \cdots$$
$$\mu_{j+1} = L^2 \left(1 - \frac{1}{4}\beta_j g_j\right) \mu_j + \cdots ,$$

with $\sum_j \beta_j = 8\mathsf{B}_{m^2}$ to conclude that

$$g_N \to \text{const}\frac{1}{\mathsf{B}_{m^2}}, \qquad \frac{\partial \mu_N}{\partial \nu_0} \sim L^{2N} g_N^{1/4}$$

and hence the desired result:

$$\frac{\partial \hat{\chi}}{\partial \nu_0} \sim -\text{const}\frac{1}{m^4}\left(\frac{1}{\mathsf{B}_{m^2}}\right)^{1/4} \sim -\text{const}\frac{1}{m^4}\left(\frac{1}{-\log m^2}\right)^{1/4}.$$

Outlook

Some other problems that could be attempted with this method:

1. Similar results for WSAW with nearest-neighbour attraction.
   (In preparation Bauerschmidt–Brydges–Slade.)
2. Logarithmic correction for two mutually interacting continuous-time 4-d WSAWs.
   (In preparation Bauerschmidt–Tomberg–Slade: 2-watermelon and 2-star.)
3. Logarithmic correction to correlation length for $d = 4$.
4. Logarithmic corrections to fixed-T quantities (mean-square displacement) for $d = 4$. Solved on 4-d hierarchical lattice by Brydges–Imbrie 2003.
5. Similar results for the particular model of *discrete*-time *strictly* SAW on $\mathbb{Z}^4$ with arbitrary steps (x, y) with weight $(-\frac{1}{\varepsilon}\Delta + 1)^{-1}_{xy}$ and $\varepsilon \ll 1$.
6. 4-d N-component ϕ^4 field theory. Solved for $N = 1$ by Gawędzki–Kupiainen and Hara–Tasaki 1980's.