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Multi-component KPZ* Equations in One Dimension

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joint work with

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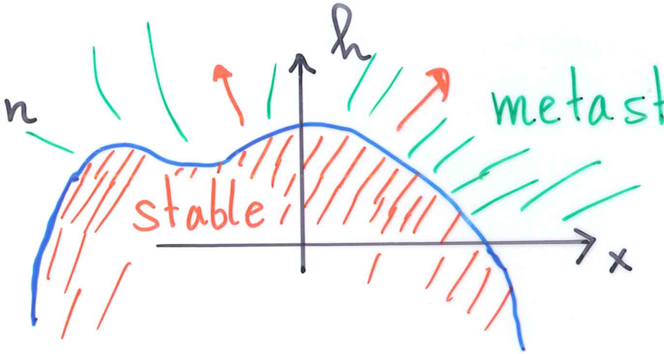
* Kardar, Parisi, Zhang (1986)

0.) What is it, why study?

one-component KPZ, height $h(x,t)$

$$\partial_t h = (\partial_x h)^2 + \partial_x^2 h + \xi \qquad \xi(x,t) \text{ space-time white noise}$$

- interface motion



see: Takeuchi, Sano 2011

- stochastic integrable system

multi-component h_α, $\alpha = 1, \dots, n$ usually $n = 2, 3$

→ normal modes

$$\boxed{\partial_t h_\alpha = -c_\alpha \partial_x h_\alpha - \langle \partial_x \vec{h}, G^\alpha \partial_x \vec{h} \rangle + \partial_x^2 (D\vec{h})_\alpha + (B\vec{\xi})_\alpha}$$

$n \times n$ matrices B, D, G^α, $BB^T = 2D > 0$

early 1990ies — Ertaş, Kardar 1992

multi-component KPZ models

- dynamic roughening of directed lines (dislocation, vortex,..)
- sedimenting crystals, colloidal suspensions
- stochastic lattice gases $\Leftarrow$ to be discussed
- magnetohydrodynamics

$\Rightarrow$ numerical work on scaling exponents, invariant measures

$\Rightarrow$ G^α, c_α are model-specific my motivation ‖ nonlinear fluctuating hydrodynamics ‖

$$\phi_\alpha = \partial_x h_\alpha \quad \Big\| \quad \partial_t \phi_\alpha + \partial_x \left(c_\alpha \phi_\alpha + \langle \vec{\phi}, G^\alpha \vec{\phi} \rangle - \partial_x (D\phi)_\alpha + (B\xi)_\alpha \right) = 0$$

normal modes

system of hyperbolic conservation laws

→ with noise + dissipation ←

examples: 1D fluids (classical / quantum), anharmonic chains, ...

plan of talk

1. AHR model ← 2-component lattice gas
2. nonlinear fluctuating hydrodynamics
3. MC simulations
4. mode-coupling theory

1.) working example AHR model Arndt, Heinzel, Rittenberg 1998

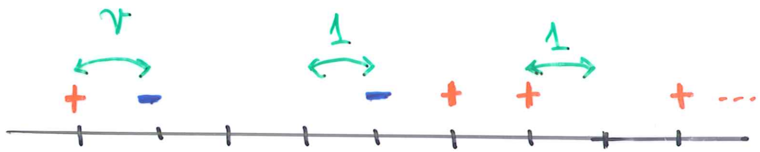
phase diagram

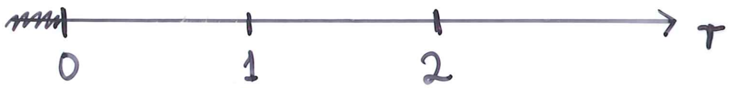
$d = 1$ lattice $\mathbb{Z}$

$+1$ right movers TASEP rate 1

-1 left movers TASEP rate 1

interaction $+\,- \rightsquigarrow -\,+$ rate $r > 0$





invariant measures

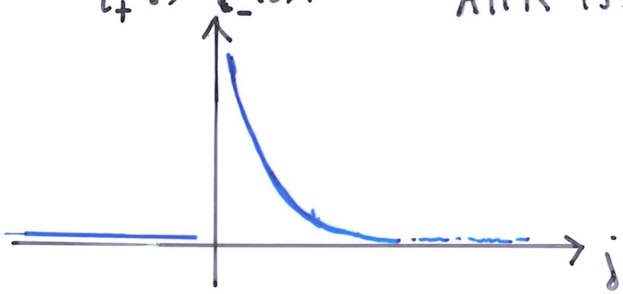
density ρ_+, ρ_- $\quad 0 \leq \rho_+ + \rho_- \leq 1$

computable via MPA

AHR 1998, Rajewski, Sasamoto, Speer 2000

exponential mixing

one-sided correlations



- hydrodynamic limit Euler scale

$r = 2$ Fritz, Tóth 2004 Leroux system

HERE steady state time correlations

fix ρ_+, ρ_- stationary process $\eta_\alpha(j,t)$ $\alpha = \pm 1$

$$\Rightarrow \quad S_{\alpha\beta}(j,t) = \langle \eta_\alpha(j,t)\, \eta_\beta(0,0) \rangle - \rho_\alpha \rho_\beta$$

correlator
2×2 matrix

recall 1-component $S(j,t)$

$$\sum_{j \in \mathbb{Z}} S(j,t) = \sum_{j \in \mathbb{Z}} S(j,0) = \chi$$

$$\sum_{j \in \mathbb{Z}} \frac{1}{\chi} S(j,t)\, j = ct$$

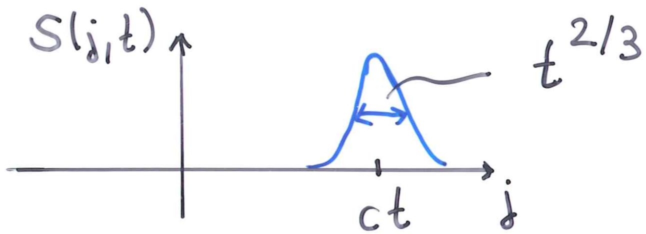
"sound velocity"

scaling $\quad S(j,t) = \varkappa \, (\lambda_B t)^{-2/3} f_{KPZ}\left((\lambda_B t)^{-2/3} (j - ct)\right)$

steady state current $\quad j(\rho)$

$\rightsquigarrow \quad c = j'(\rho) \quad , \quad \lambda_B = \sqrt{2}\,|j''(\rho)| > 0$

$\int f_{KPZ}(x)\,dx = 1$, $f_{KPZ}(x) = f_{KPZ}(-x)$, $\int f_{KPZ}(x)\,x^2\,dx \cong 0.5$, $f_{KPZ}(x) \cong e^{-0.3|x|^3}$



|| What about AHR model? ||

2.) nonlinear fluctuating hydrodynamics $\Big\}$ it is a guess

n component lattice gas $\eta_\alpha(j,t)$ $\alpha = 1,\dots,n$

space-time stationary invariant measure $\langle \cdot \rangle_{\vec\rho}$, $\langle \eta_\alpha(j,t)\rangle = \rho_\alpha$

CORRELATOR

$$S_{\alpha\beta}(j,t) = \langle \eta_\alpha(j,t)\,\eta_\beta(0,0)\rangle^{c}_{\vec\rho}$$

static susceptibility $C = \sum_j S(j,0)$ $C = C^T$

$n \times n$ matrices

sum rules $\sum_j S(j,t) = C$

$$\sum_j j\, S(j,t) = t\, A C \qquad A_{\alpha\beta}(\vec\rho) = \partial_{\rho_\beta} J^\alpha(\vec\rho)$$

J^α = steady current of component α

conservation law $\quad AC = CA^T \quad \Rightarrow$ A has real eigenvalues
+ stationarity

$S(j,t)$ propagation of small perturbation

lowest order Euler equations $\qquad$ fields $u_\alpha(x,t)$

$$\partial_t u_\alpha + \partial_x J^\alpha(\vec{u}) = 0$$

genuinely nonlinear

linearized $\qquad \rho_\alpha + u_\alpha$

$$\partial_t u_\alpha + \partial_x (Au)_\alpha = 0$$

transformation to normal modes $\qquad \phi = Ru$

$$RAR^{-1} = \mathrm{diag}(c_1, \dots, c_n) \, , \qquad RCR^T = 1$$

eigenvalues $\qquad\qquad$ orthogonal w.r.t. $\langle \cdot \rangle_{\vec{\rho}}$

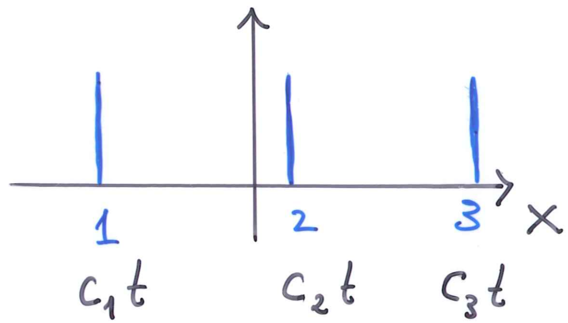
R is "unique"

$\Longrightarrow$ ϕ_α travels with velocity c_α

to lowest order $\qquad\qquad\qquad$ $\times$ continuum for j

$$(RSR^T)_{\alpha\beta}(j,t) \simeq \delta_{\alpha\beta}\,\delta(x-c_\alpha t)$$

broadening? correlations?



- sound cone
- Lieb Robinson propagation bounds

?

fluctuations add noise and dissipation

$$\partial_t \phi_\alpha + \partial_x \left(c_\alpha \phi_\alpha - \partial_x (D\vec{\phi})_\alpha + (B\vec{\xi})_\alpha \right) = 0 \qquad BB^T = 2D$$

$\xi_\alpha(x,t)$ Gaussian white noise

- diffusive $\Leftarrow$ would be ok on $\mathbb{Z}^d$, $d \geq 3$
- $n=1$ superdiffusive

$\Rightarrow$ GUESS: expand Euler up to second order

Hessians $H^\alpha_{\beta\gamma} = \partial_{\rho_\beta} \partial_{\rho_\gamma} J^\alpha(\vec{\rho})$

$$\partial_t u_\alpha + \partial_x \left(A u_\alpha + \tfrac{1}{2} \langle u, H^\alpha u \rangle \right) = 0$$

normal modes

$$\partial_t \phi_\alpha + \partial_x \left(c_\alpha \phi_\alpha + \langle \phi, G^\alpha \phi \rangle - \partial_x (D\phi)_\alpha + (B\vec{\xi})_\alpha \right) = 0$$

multi-component KPZ

coupling constants $\qquad G^{\alpha} = \sum_{\alpha'=1}^{n} R_{\alpha\alpha'} \left(R^{-1T} H^{\alpha'} R^{-1} \right)$

$n = 1 \qquad \partial_t \phi_1 + \partial_x \left(c_1 \phi_1 + G^1_{11} \phi_1^2 - D \partial_x \phi_1 + B \xi_1 \right) = 0$ $\qquad$ ok!

replica solution: Imamura, Sasamoto 2012
TASEP, PNG

problem of invariant measures

- naive spatial discretization

$n = 1$ $\qquad$ mean zero $\qquad$ $\phi(x,t)$ stationary $\qquad$ $x \mapsto \phi(x,t)$ is white noise

$n > 1$ $\qquad$ $G = 0$ $\qquad$ $x \mapsto \phi_\alpha(x,t)$ is white noise

general G $\qquad$ requires $\qquad$ $G^{\alpha}_{\beta\gamma} \underset{\substack{\uparrow \\ \text{trivial}}}{=} G^{\alpha}_{\gamma\beta} = G^{\gamma}_{\beta\alpha}$!!

central observation

$\{c_\alpha\}$ distinct, modes separate as t

leads to "independence"

example $n=2$

$$\partial_t \phi_1 + \partial_x \left(c_1 \phi_1 + G^1_{11} \phi_1^2 + G^1_{12} \phi_1 \phi_2 + G^2_{22} \phi_2^2 + \dots \right) = 0$$

no overlap with ϕ_1

sub-leading

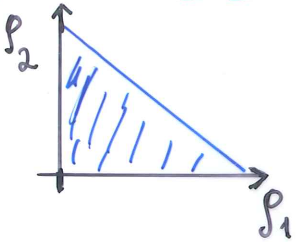
Conjecture: $c_\alpha \neq c_\beta$ for $\beta \neq \alpha$ and $[\![G^\alpha_{\alpha\alpha} \neq 0]\!]$

Then

$$\underbrace{(R S R^T)_{\alpha\beta}(j,t)}_{\text{normal mode}} \cong \delta_{\alpha\beta} (\lambda t)^{-2/3} f_{KPZ}\left((\lambda t)^{-2/3}(j - c_\alpha t)\right)$$

3.) AHR Monte Carlo simulations

$n = 2$ $\qquad \rho_1, \rho_2$



$G^1_{11} \neq 0 \, , \quad G^2_{22} \neq 0$

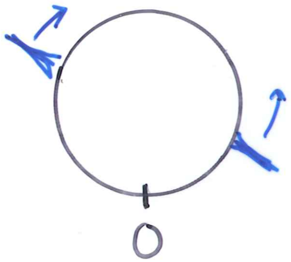
BUT $G^1_{22} = 0 \, , \quad G^2_{11} = 0$

$\rho_1 = \rho_2 \, , \quad \rho_1 = 0.25$

$T = 0.8 \, , \; 1.1 \, , \; 2$

$N = 400$

$c_1 \, , \; c_2$



$|c_2 - c_1| t \leq N/2$

study also collision

non-universal coefficients

$c_1, c_2 \, , \quad R \quad , \quad G^1_{11} \, , \; G^2_{22}$ are computed by MPA

normal mode

- initial state grand canonical $r = 2$ generate i.i.d

  $r \neq 2$ algorithm based on MPA

- maximal time 300 MCS

- average over 10^8 runs

- $(R^t S R)(j, t)$ - not diagonal for short t

  - diagonal for long t

  - good fit with f_{KPZ}

    + nonuniversal coefficients

Why is the fit so good?

|| difficulties for classical anharmonic chains ||

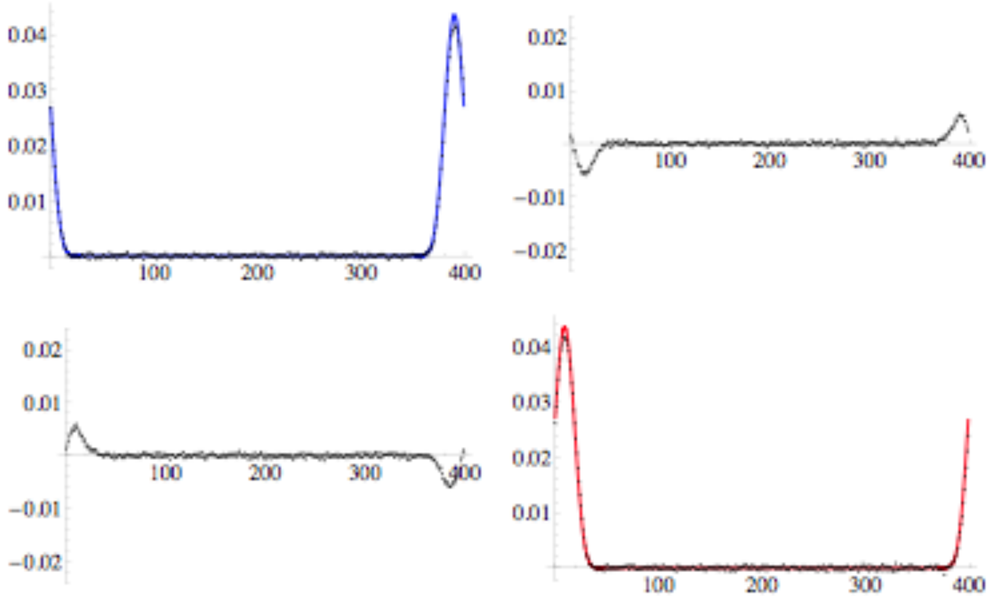
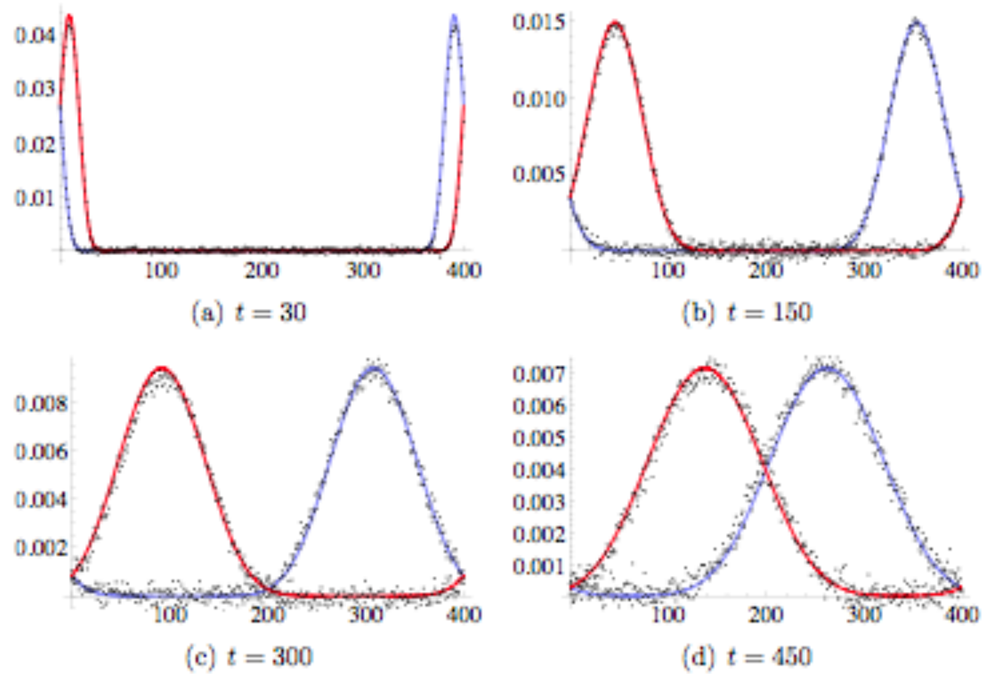


Figure 1: Plot of the matrix elements $S^\sharp$ for small time, $t = 30$, parameters $\xi = 0.5$, $\alpha = 10/9$, and 20×10^6 MC runs. There is a signal in $S^\sharp_{12}$ and $S^\sharp_{21}$ distinct from the background noise, in the region where $S^\sharp_{11}$ and $S^\sharp_{22}$ still overlap.



(a) $t = 30$ (b) $t = 150$

(c) $t = 300$ (d) $t = 450$

Figure 2: Plot of the matrix elements $S^{\sharp}_{11}$ (blue) and $S^{\sharp}_{22}$ (red) for different times and parameters $\xi = 1.5$, $\alpha = 10/9$, with an average over 20×10^6 MC runs.

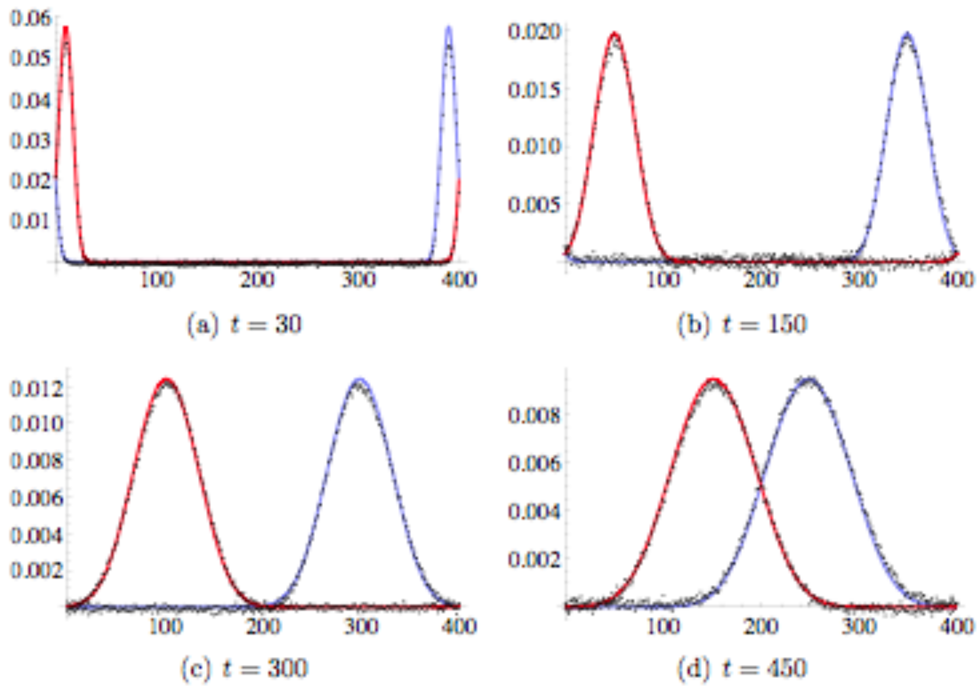


Figure 3: Plot of the matrix elements S^{ξ}_{11} (blue) and S^{ξ}_{22} (red) for different times and parameters $\xi = 1.5$, $\alpha = 2/3$, with an average over 20×10^6 MC runs for (a),(b), and 100×10^6 MC runs for (c),(d).

4.) Mode coupling theory

→ normal modes
$$S^{\#}_{\alpha\beta}(x,t) = \langle \phi_\alpha(x,t)\, \phi_\beta(0,0) \rangle$$
$$S^{\#}(x,0) = \mathbb{1}\, \delta(x)$$

- approximate equation for $S^{\#}$

$$\partial_t S^{\#}_{\alpha\beta}(x,t) = -c_\alpha \partial_x S^{\#}_{\alpha\beta}(x,t) + \sum_{\alpha'} D_{\alpha\alpha'} \partial_x^2 S^{\#}_{\alpha'\beta}(x,t)$$
$$+ \int_0^t ds \int dy \sum_{\alpha'} \partial_y^2 \underbrace{M_{\alpha\alpha'}(y,s)}_{\text{memory kernel}} S^{\#}_{\alpha'\beta}(x-y,t-s)$$

$$M_{\alpha\alpha'}(x,t) = 2 \sum_{\substack{\beta',\beta'' \\ \gamma',\gamma''}} G^{\alpha}_{\beta'\gamma'}\, G^{\alpha'}_{\beta''\gamma''}\, S^{\#}_{\beta'\beta''}(x,t)\, S^{\#}_{\gamma'\gamma''}(x,t)$$

diagonal approximation $\qquad S^{\#}_{\alpha\beta}(x,t) = \delta_{\alpha\beta}\, f_\alpha(x,t)$

$$\partial_t f_\alpha = -c_\alpha \partial_x f_\alpha + D_\alpha \partial_x^2 f_\alpha + \int_0^t ds \int dy\, \partial_y^2 \underbrace{M_\alpha(y,s)}_{\text{memory kernel}}\, f_\alpha(x-y,t-s)$$

$$M_\alpha(x,t) = 2\sum_{\beta,\gamma} \left(G^\alpha_{\beta\gamma}\right)^2 f_\beta(x,t)\, f_\gamma(x,t)$$

- f_α moves with velocity c_α $\rightsquigarrow$ $f_\beta f_\gamma = 0$ unless $\beta = \gamma$

  $f_\beta f_\beta$ moves with velocity c_β $\rightsquigarrow$ slowly decaying correction

$\rightsquigarrow$ leading $G^\alpha_{\alpha\alpha}$

sub-leading $G^\alpha_{\beta\beta}$, $\beta \neq \alpha$, $(= 0$ for AHR$)$

5.) Outlook

theory can be applied to anharmonic chains

$n = 3$ (compression, momentum, energy)

special features $\alpha = 0, \pm 1$

$c_0 = 0$ $c_1 = c$, $c_{-1} = -c$ c sound velocity

normal modes ± 1 sound modes symmetry

0 heat mode

$G^1_{11} = -G^{-1}_{-1-1} \neq 0$ • sound: KPZ scaling

$G^0_{00} = 0$ always • heat: non-KPZ scaling

• Levy $\alpha = \frac{5}{3}$, $\hat{f}_0(k) = e^{-|k|^{5/3}}$ mode-coupling approximation since 1998: molecular dynamics simulations - many

- large scale

comparison ??

related work for 1D fluids: Henk van Beijeren 2012