Predictability in Nonequilibrium Discrete Spin Dynamics

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Dynamical Evolution of Ising Model Following a Deep Quench

Consider the stochastic process $\sigma^t = \sigma^t(\omega)$ with

$$\sigma^t \in \{-1,+1\}^{Z^d}$$

corresponding to the zero-temperature limit of Glauber dynamics for an Ising model with Hamiltonian

$$\mathcal{H} = -\sum_{\|x-y\|=1} J_{xy}\,\sigma_x \sigma_y$$

We are particularly interested in σ^0's chosen from a symmetric Bernoulli product measure P_{σ^0} .

The continuous time dynamics are given by independent, rate-1 Poisson processes at each *x* when a spin flip ($\sigma_x^{t+0} = -\sigma_x^{t-0}$) is *considered*. If the change in energy

$$\mathcal{H}_x(\sigma) = 2 \sum_{y:\|x-y\|=1} J_{xy}\,\sigma_x \sigma_y$$

is negative (or zero or positive) then the flip is done with probability 1 (or ½ or 0). We denote by P_ω the probability distribution on the realizations ω of the dynamics and by $P_{\sigma^0,\omega} = P_{\sigma^0} \times P_\omega$ the joint distribution of the σ^0's and ω's.

In physics, the time evolution of such a model is known as *coarsening, phase separation,* or *spinodal decomposition.*

http://webphysics.davidson.edu/applets/ising/default.html

Two questions

1) For a.e. σ^0 and ω, does $\sigma^\infty(\sigma^0, \omega)$ exist? (Or equivalently, for every *x* does $\sigma^t_x(\sigma^0, \omega)$ flip only finitely many times?)

2) As *t* gets large, to what extent does $\sigma^t(\sigma^0, \omega)$ depend on σ^0 (``nature'') and to what extent on $\omega$ (``nurture'')?

Phrasing (2) more precisely depends on the answer to (1).

We will consider two kinds of models:

- the homogeneous ferromagnet where $J_{xy}=+1$ for all $\{x,y\}$.
- disordered models where a realization *J* of the J_{xy}'s is chosen from the independent product measure P_J of some probability measure on the real line.

Simplest case: $d = 1$

Theorem (Arratia `83, Cox-Griffeath `86): For the $d = 1$ homogeneous ferromagnet, $\sigma^{\infty}_x(\sigma^0, \omega)$ does not exist for a.e. σ_0 and ω and every x.

Proof: The joint distribution $P_{\sigma^0,\omega}$ is translation-invariant and translation-ergodic. Define A_x^+ (A_x^-) to be the event (in the space of (σ^0,ω)'s) that $\sigma_x^{\infty}(\sigma^0,\omega)$ exists and equals +1 (-1); denote the respective indicator functions as I_x^+ (I_x^-). By translation-invariance and symmetry under $\sigma^0 \to -\sigma^0$, it follows that for all x, $P_{\sigma^0,\omega}(A_x^+) = P_{\sigma^0,\omega}(A_x^-) = p$ with $0 \le p \le ½$. So, by translation-ergodicity,

$$\lim_{N\to\infty}(1/N)\sum_{x=1}^{N} I_x^+(\sigma^0,\omega) = \lim_{N\to\infty}(1/N)\sum_{x=1}^{N} I_x^-(\sigma^0,\omega) = p$$

for a.e. σ^0 and ω.

R. Arratia, *Ann. Prob.* **11**, 706-713 (1983); J.T. Cox and D. Griffeath, *Ann. Prob.* **14**, 347-370 (1986).

Proof (continued): Suppose now that p > 0. Then for some $x < x'$, $\sigma_x^{\infty}=+1$ and $\sigma_{x'}^{\infty}=-1$ with strictly positive probability, and so for some t', $\sigma_x^t=+1$ and $\sigma_{x'}^t=-1$ for all $t \geq t'$. But for this to be true requires (at least) the following: denote by *S'* the set of spin configurations on **Z** such that $\sigma_x = +1$ and $\sigma_{x'} = +1$. Then one needs the transition probabilities of the Markov process σ^t to satisfy

$$\inf_{\sigma \in S'} P_\omega(\sigma^{t+1} \notin S' \mid \sigma^t = \sigma) = 0.$$

But this is not so, since for any such σ, we would end up with $\sigma_x^{t+1} = -1$ if during the time interval [t,t+1] the Poisson clock at *x'* does not ring while those at *x'*-1, *x'*-2, …, *x* each ring exactly once and in the correct order (and all relevant coin tosses are favorable).

What about higher dimensions?

Theorem (NNS '00): In the d = 2 homogeneous ferromagnet, for a.e. σ^0 and ω and for every x in $\mathbf{Z}^2$, $\sigma_x^t(\omega)$ flips infinitely often.

Higher dimensions: remains open. Older numerical work (Stauffer '94) suggests that every spin flips infinitely often for dimensions 3 and 4, but a positive fraction (possibly equal to 1) of spins flips only finitely often for $d \geq 5$.

We'll return to the question of predictability in homogeneous Ising ferromagnets, but first we'll look at the behavior of $\sigma^\infty(\sigma^0,\omega)$ for *disordered* Ising models.

S. Nanda, C.M. Newman, and D.L. Stein, pp. 183—194, in *On Dobrushin's Way (from Probability Theory to Statistical Physics)*, eds. R. Minlos, S. Shlosman, and Y. Suhov, Amer. Math. Soc. Trans. (2) 198 (2000).

D. Stauffer, *J. Phys. A* **27**, 5029—5032 (1994).

In some respects, this case is *simpler* than the homogeneous one.

Recall that the J_{xy}'s are chosen from the independent product measure P_J of some probability measure on the real line. Let μ denote this measure.

Theorem (NNS '00): If μ has finite mean, then for a.e. J, σ^0 and ω, and for every x, there are only finitely many flips of σ_x^t that result in a *nonzero* energy change.

It follows that a spin lattice in any dimension with continuous coupling disorder having finite mean (e.g., Gaussian) has a limiting spin configuration at all sites.

But the result holds not only for systems with continuous coupling disorder. It holds also for discrete distributions and even homogeneous models where each site has an odd number of nearest neighbors (e.g., hexagonal lattice in 2D).

Will provide proof in one dimension.

Proof (*1D* only): Consider a chain of spins with couplings $J_{x,x+1}$ chosen from a continuous distribution (which in *1D* need not have finite mean). Consider the doubly infinite sequences x_n of sites where $|J_{x_n,x_n+1}|$ is a *strict* local maximum and y_n in the interval (x_n, x_{n+1}) where $|J_{y_n,y_n+1}|$ is a *strict* local minimum:

$$\left|J_{x_n,x_n+1}\right| > \left|J_{x_n-1,x_n}\right|, \left|J_{x_n+1,x_n+2}\right|$$

$$\left|J_{y_n,y_n+1}\right| < \left|J_{y_n-1,y_n}\right|, \left|J_{y_n+1,y_n+2}\right|$$

That is, the coupling magnitudes are strictly increasing from y_{n-1} to x_n and strictly decreasing from x_n to y_n.

Now notice that the coupling $|J_{x_n,x_n+1}|$ is a ``bully''; once it's satisfied (i.e., $J_{x_n,x_n+1}\sigma^0_{x_n}\sigma^0_{x_n+1} > 0$, the values of σ_{x_n} and σ_{x_n+1} can never change thereafter, regardless of what's happening next to them. For all other spins in $\{y_{n-1}+1, y_{n-1}+2, \ldots, y_n\}$, σ_y^∞ exists and its value is determined so that $J_{x,y}\sigma_x^\infty\sigma_y^\infty > 0$ for x and y=x+1 in that interval.

In other words, there is a *cascade of influence* to either side of $\{x_n, x_n+1\}$ until $y_{n-1}+1$ and y_n, respectively, are reached.

Predictability

Define ``order parameter'' $q_D = \lim_{t\to\infty} q^t$, where

$$q^t = \lim_{L\to\infty}(2L+1)^{-d} \sum_{x\in\Lambda_L} (\langle \sigma_x \rangle_t)^2 = E_{J,\sigma^0}(\langle \sigma_x \rangle_t)^2$$

Theorem (NNS '00): For the one-dimensional spin chain with continuous coupling disorder, q_D=½.

Proof: Choose the origin as a typical point of **Z** and define X=X(*J*) to be the x_n such that that 0 lies in the interval $\{y_{n-1}+1, y_{n-1}+2, \ldots, y_n\}$. Then σ_0^∞ is completely determined by (*J* and) σ^0 if $J_{X,X+1}\sigma_X^0\sigma_{X+1}^0 > 0$ (so that $\langle\sigma_0\rangle^\infty = +1$ or -1) and otherwise is completely determined by ω (so that $\langle\sigma_0\rangle^\infty = 0$). Thus q_D is the probability that $\sigma_X^0 \sigma_{X+1}^0 = \text{sgn}(J_{X,X+1})$, which is ½.

How does one define and study predictability in systems where σ^∞ does *not* exist?

NS '99: Consider the *dynamically averaged* measure κ_t; that is, the distribution of σ^t over dynamical realizations ω for fixed *J* and σ^0. Two possibilities were conjectured:

- Even though σ^t has no limit σ^∞ for a.e. *J*, σ^0 and ω, κ^t *does* have a limit κ^∞.

- κ^t does not converge as $t \to \infty$. (This has been proved to occur for some systems; see Fontes, Isopi, and Newman, *Prob. Theory Rel. Fields* **115**, 417-443 (1999).)

We refer to the first as ``weak local nonequilibration (weak LNE)", and to the second as ``chaotic time dependence (CTD)" .

C.M. Newman and D.L. Stein, *J. Stat. Phys.* **94**, 709-722 (1999).

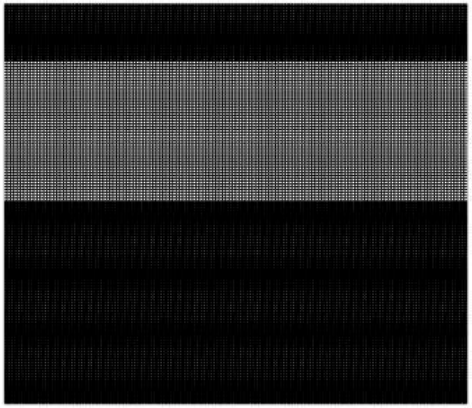
Numerical work

J. Ye, J. Machta, C.M. Newman and D.L. Stein, arXiv 1305.3667: simulations on L x L square lattice with

$$E = -\sum_{|x-y|=1} S_x S_y$$

Have to use finite-size scaling approach.

``Stripe states'' occur roughly 1/3 of the time (V. Spirin, P.L. Krapivsky, and S. Redner, *Phys. Rev. E* **63**, 036118 (2000)).



Also P.M.C. de Oliveira, CMN, V. Sidoravicious, and DLS, *J. Phys. A* **39**, 6841-6849 (2006).

To distinguish the effects of nature vs. nurture, we simulated a *pair* of Ising lattices with identical initial conditions (i.e., ``twins'') (*cf.* damage spreading).

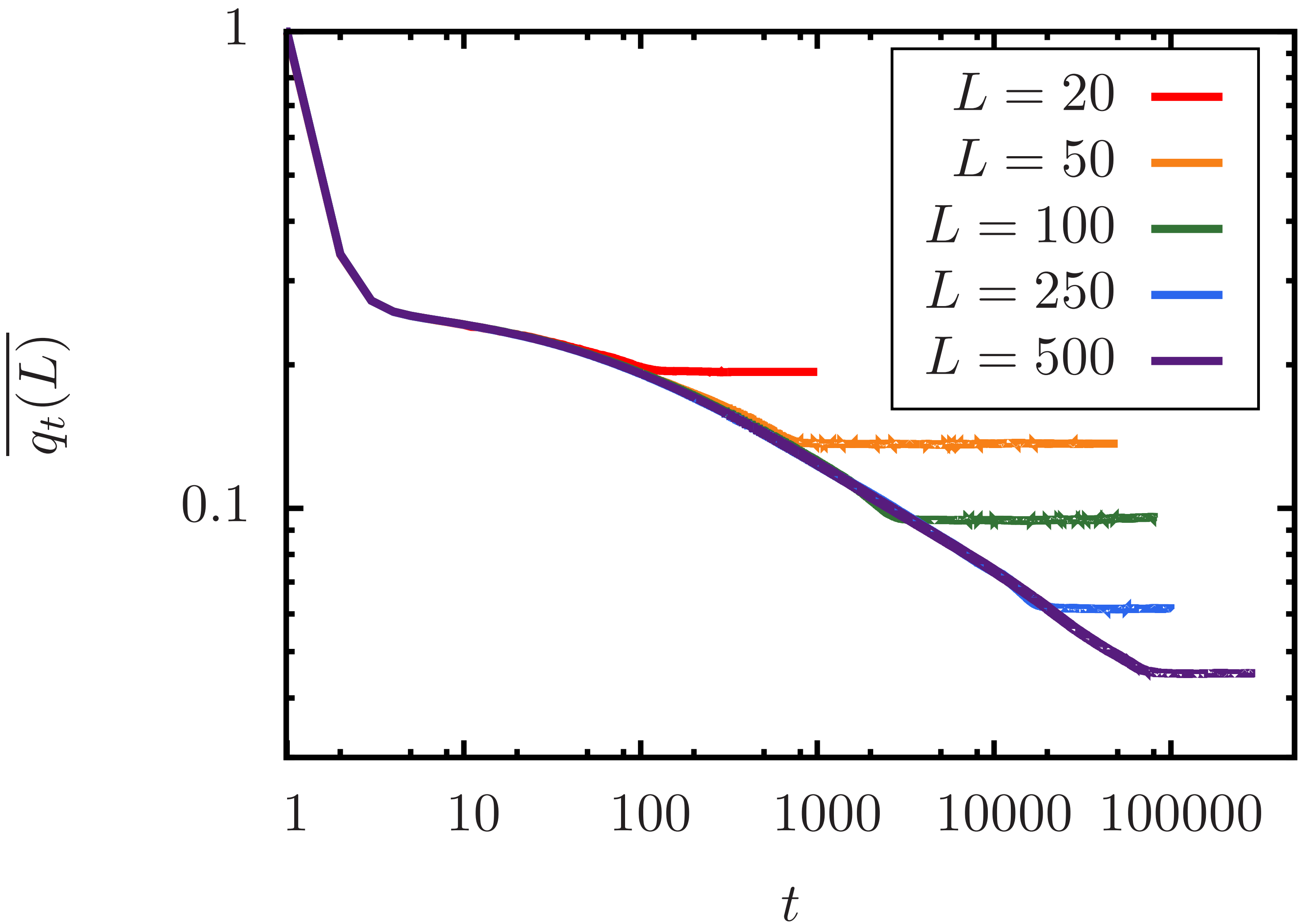
Examine the overlap q between a pair of twins at time t:

$$q_t(L) = \frac{1}{N}\sum_{i=1}^{L^2} S_i^1(t) S_i^2(t)$$

We are interested in the time evolution of the mean $\overline{q}_t$ and its final value $\overline{q}_\infty$ when the twins have reached absorbing states.

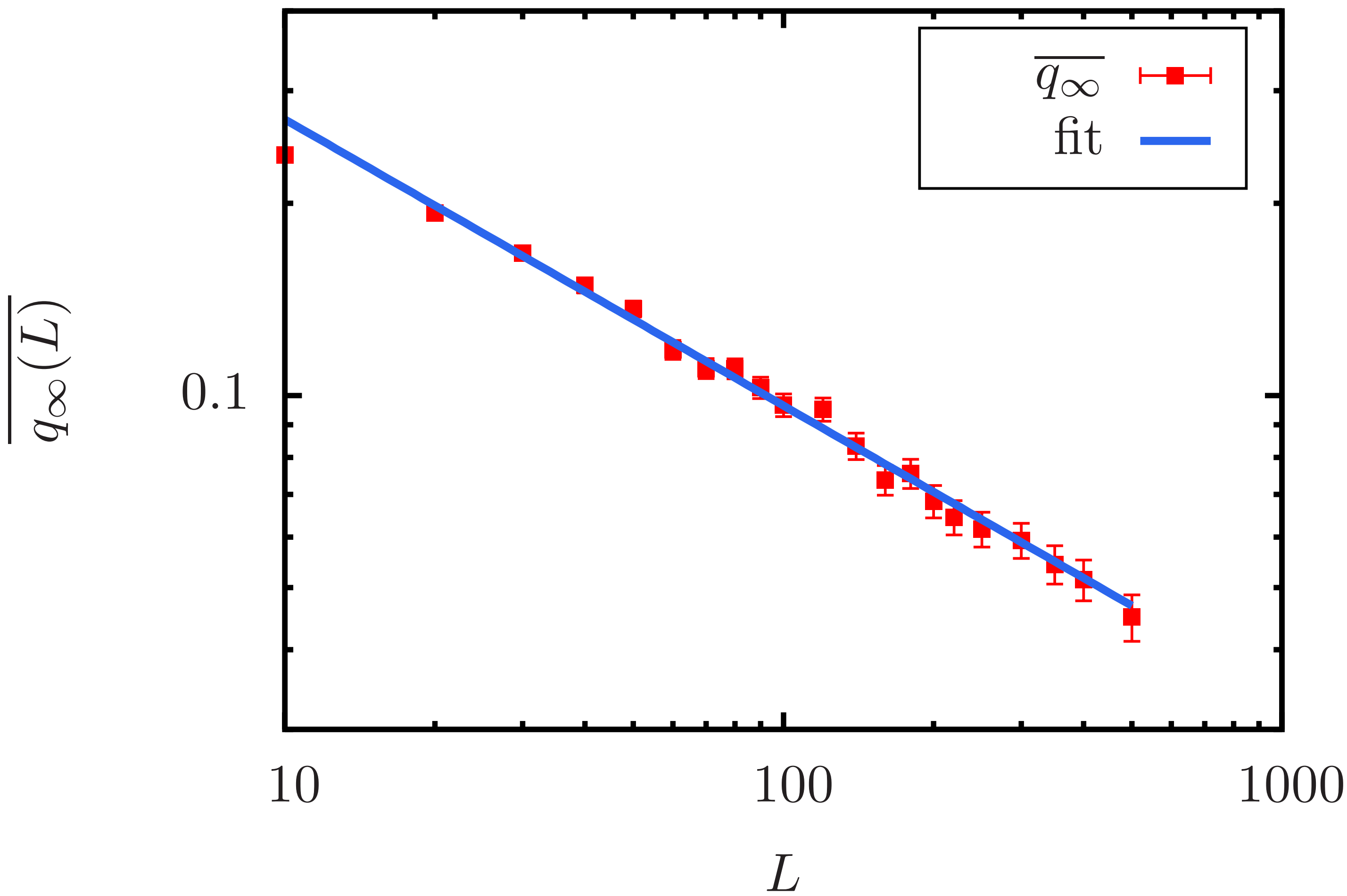
Looked at 21 lattice sizes from L = 10 to L = 500. For each size studied 30,000 independent twin pairs out to their absorbing states (or almost there).

Look at $\overline{q_t(L)}$ vs. *t* for several *L*.



The plateau value decreases from small to large *L*. A power law fit of the form $\overline{q_t} = dt^{-\theta_h}$ for the largest sizes gives a ``heritability exponent” θ_h=0.22±0.02.

Next look at $\overline{q_\infty(L)}$ vs. L for sizes 10 to 500.



The solid line is the best power law fit for size 20 to 500 and corresponds to

$$\overline{q_\infty(L)} \approx L^{-0.46}$$

Finite size scaling ansatz

Use the fact that during coarsening, the typical domain size grows as $t^{1/z}$, with $z = 2$ for zero-temperature Glauber dynamics in the 2D ferromagnet.

Postulate the finite-size scaling form $\overline{q_t(L)} \approx t^{-\theta_h} f(\frac{t^{1/z}}{L})$, where the function f(x) is expected to behave as

$$f(x) \approx \begin{cases} 1 & for\ x << 1 \\ x^{z\theta_h} & for\ x >> 1 \end{cases}$$

So the t->∞ behavior is $\overline{q_\infty(L)} \approx L^{-z\theta_h}$, giving $b = z\theta_h = 2\theta_h$.

A.J. Bray, *Adv. Phys.* **43**, 357-459 (1994).

Summary so far

- For finite *L*, there are limiting absorbing states and overlaps $\overline{q_\infty(L)} \approx aL^{-b}$ with *b* = 0.46 ± 0.02.

- $\overline{q_t(L)}$ appears to approach $\overline{q_t} = dt^{-\theta_h}$ as $L \to \infty$ with $\theta_h = 0.22 \pm 0.02$.

- A finite size scaling analysis suggests that $b = 2\theta_h$, consistent with our numerical results.

- Since $\theta_h > 0$, the *2D* Ising model displays weak LNE. But given the smallness of θ_h, information about the initial state decays slowly.

Heritability and Persistence

Persistence: in the context of phase ordering kinetics, persistence is defined as the fraction of spins that have *not* flipped up to time *t*. This quantity is found to decay as a power law with exponent θ_p called the *persistence exponent*.

Numerical simulations on the *2D* Ising model yield $\theta_p = 0.22$ (Stauffer, *J. Phys. A* **27**, 5029-5032 (1994)) and $\theta_p = 0.209 \pm 0.002$ (Jain, *Phys. Rev. E* **59**, R2493-R2495 (1999)). Within error bars of our $\theta_h = 0.22 \pm 0.02$.

Moreover, our exponent $b = 0.46 \pm 0.02$ describing the finite size decay of heritability can be directly compared to the finite-size persistence exponent $\theta_{Ising} = 0.45 \pm 0.01$ (Majumdar and Sire, *Phys. Rev. Lett.* **77**, 1420-1423 (1996)).

So, is there something deeper going on?

Heritability and Persistence (continued)

In *1D* one can compute analytically both the persistence exponent and the heritability exponent. One finds there that $\theta_p = 3/8$, but $\theta_h = ½$ and b = 1.

Currently looking at ferromagnets and disordered models in higher dimensions, and also Potts models.

Stay tuned …


SPIN GLASSES
AND COMPLEXITY
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