Factorization of density matrix elements of higher spin chains at $T > 0$

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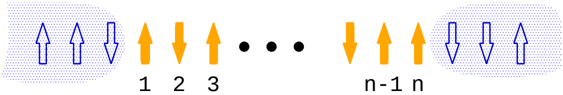
Based on Collaboration with
A. Klümper, F. Göhmann, D. Nawrath, A. Seel ...

Reduced Density Matrix Elements

A measure of correlations in finite segments of quantum spin chains.

•(reduced) Density Matrix Elements (DME)



$$D_n := \langle E_1 \otimes E_2 \cdots \otimes E_n \rangle$$

$$(D_n)^{\beta_1,\beta_2,\cdots,\beta_n}_{\alpha_1,\alpha_2,\cdots,\alpha_n} := \langle E^{\alpha_1}_{\beta_1} E^{\alpha_2}_{\beta_2} \cdots E^{\alpha_n}_{\beta_n} \rangle \qquad (E^{\alpha}_{\beta})^i_j := \delta_{\alpha,i}\delta_{\beta,j}$$

can be used to evaluate

- short correlation functions
- entanglement entropy

Success story: $S = \frac{1}{2}$ at $T = 0$

- Multiple integral formula for $D_n(\xi_1, \cdots, \xi_n)$
  - Vertex Operator approach (Jimbo et al.(1992-))
  - $q-$KZ approach (Jimbo and Miwa (1994))
  - QISM : Solving inverse problems (Maillet et al (2000-))
- Factorize multiple integrals into sums of products of single integrals "by hand"
  - Boos Korepin Smirnov (2001-)
  - Sato Shiroishi Takahashi (2005) ($n = 8$)

Conjecture (Boos-Korepin)

Correlation functions at $T = 0$ for $S = \frac{1}{2}$ XXX model are described by $(\ln 2)$ and Riemann's ζ functions with odd arguments.

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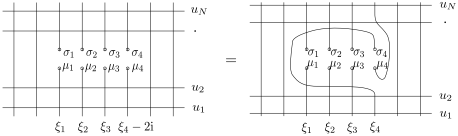
reduced $q-$KZ equation uncovers the mystery
(Boos et al (2004-))



Solution(Exponential formula)

- contains a transcendental fcn ω
- contains "Fermions"

It explains factorization, appearance of $\zeta(2k+1)$ and so on. (M. Jimbo's talk in this conference)

Status of DME $S = \frac{1}{2}, T > 0$

DME at $T > 0$ (Göhmann et al., JPA 36 (2005))

- Algebraic part : parallel to $T = 0$ case.
- Trotter limit : NLIE $\mathfrak{a}$ (Kluemper et al. (1991), Destri-de Vega , (1995))
- integrations contain "Fermi" (spinon) distribution functions $\mathfrak{A}(:= 1 + \mathfrak{a})$.

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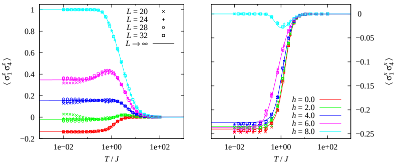

Main Problem Today

Problem

Consider the integrable isotropic $S = 1$ chain with L sites.

$$H = \frac{J}{4}\sum_{j=1}^{L}[\vec{S}_{j-1}\cdot\vec{S}_j - (\vec{S}_{j-1}\cdot\vec{S}_j)^2]$$

Evaluate Density Matrix Elements D_n in $L \to \infty$ or its “inhomogenous” generalization $D_n(\xi_1, \cdots, \xi_n)$, at any T s in a " factorized" form.

Aim of research

Common belief is..

- $S = \frac{1}{2}$ is fundamental. For example, $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$.
- Description of $S > \frac{1}{2}$ is mere a modification (at least for integrable cases)

What I believe is..

- Description of $S > \frac{1}{2}$ using $S = \frac{1}{2}$ is sometimes flawed.
- Each higher spins (composite particles) needs its own description of the Hilbert space.
- Natural description may offer an efficient formalism in numerics.

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Main problem Again

To be concrete , for $S = 1$

- multiple integral formula at $T = 0$ ✓
  - VO (Bougourzi et al., Konno, Idzumi)
  - QISM (Kitanine, Deguchi-Matsui)
- multiple integral formula at $T > 0$?
- factorization at $T \geq 0$?
- exponential formula at $T > 0$?

QTM

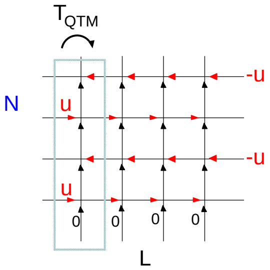
Main tool : QTM framework

M. Suzuki (1985), M. Suzuki and Inoue (1987),

Koma(1987), J.S. et al (1990), Klümper (1992)

Map d-D quantum to $d+1$-D classical. $(d=1)$

$$Z_{\rm 1Dquantum}(\beta, L) = Z_{\rm 2D}(N, L)$$
$$= {\rm tr} T_{\rm QTM}(u)^L$$
$$u = -\frac{\beta}{N}$$



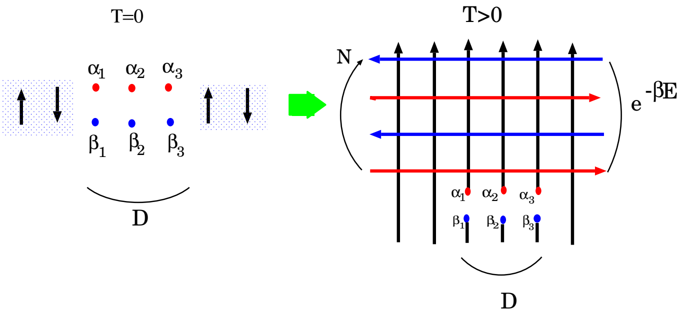
Theorem (M.Suzuki)

Only the largest eigenvalue of $T_{\rm QTM}$ contributes.

Neither summation nor variation necessary

DME in QTM formulation at $T > 0$

In QTM framework you do not have to solve inverse problem (Göhmann et al., JPA 36)



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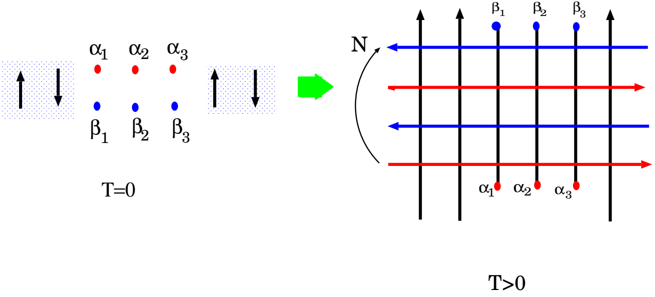
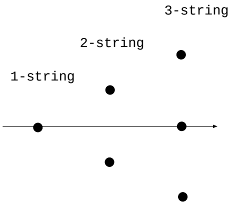


$$\Big(D\Big)^{\alpha_1,\cdots,\alpha_n}_{\beta_1,\cdots,\beta_n}(\xi_1,\cdots,\xi_n) = \frac{\langle\Psi|T^{\alpha_1}_{\beta_1}(\xi_1)\cdots T^{\alpha_n}_{\beta_n}(\xi_n)|\Psi\rangle}{\langle\Psi|T_{\mathrm{QTM}}(\xi_1)\cdots T_{\mathrm{QTM}}(\xi_n)|\Psi\rangle}$$ $\Rightarrow$ parallel to $T = 0$!

Bulk thermodynamics of $S > \frac{1}{2}$

Although Bethe ansatz roots characterizes highest weight states (Gaudin)...

composite states = strings
∞/∞=highly singular



Numerics (Alcaraz et al '88)

- Ground state = 2S string
- Excited states= very complicated

Better not to deal with BAE roots directly..

Other descriptions?

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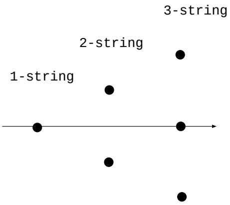

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Conjecture(Reshetikhin '91)

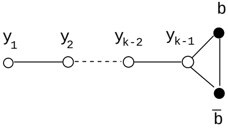
$\mathcal{H}_{\text{spin}\,S} = \mathcal{H}_{\text{spinon}} \oplus \mathcal{H}_{\text{RSOS}_k}$

Thermodynamics (JS '99): consists of two pieces.

- "RSOS" pieces. $(1 \leq j \leq k-1)$
  $y_j, Y_j (:= 1 + y_j)$
- Spinon pieces
  $\mathfrak{b}, \mathfrak{B} (:= 1 + \mathfrak{b})$

They are nice objects, as

1. Good analyticity
2. They satisfy functional relations (Klümper-Pearce transf)
3. The relations among nice objects lead to NLIE.
4. NLIE yields bulk quantities (specific heat..)



multiple integral formula $S = 1, T \geq 0$

Not dealing with BAE roots directly

- Good: no need to deal with singular objects
- No good: problem with DME

The algebraic part of calculation of DME goes parallel to $S = \frac{1}{2}$ case:

$$\langle T^{\alpha_1}_{\beta_1}(\xi_1) \cdots T^{\alpha_n}_{\beta_n}(\xi_n) \rangle \sim \sum_{\mathrm{BAEroots}\{\mu_j\} \cup \mathrm{others}} \mathcal{S}(\{\mu_j\})$$

- Zeros of Q ($=$ BAE roots $\{\mu_j\}$) are not encoded in $\mathfrak{B}, Y_j!$, $\mathfrak{B}(\mu_j) \neq 0$
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Still we can

- adopt narrower contours separated in the upper and lower half planes
- impose "subtle relations" among these contours
- introduce one more auxiliary function $\mathfrak{f}, \mathfrak{F} := 1 + \mathfrak{f}$

Theorem (Göhmann et al (2010))

$S = 1$ DME at $T > 0$ has the following multiple integral formula

$$D^{\alpha_1,\dots,\alpha_m}_{\beta_1,\dots,\beta_m}(\xi) = \frac{2^{-m-n_+(\alpha)-n_-(\beta)}}{\prod_{1\le j<k\le m}(\xi_k-\xi_j)^2[(\xi_k-\xi_j)^2+4]}$$

$$\left[\prod_{j=1}^{p}\int_{\mathcal{C}}\frac{\mathrm{d}\lambda_j}{2\pi\mathrm{i}}F_{z_j}(\lambda_j)\right]\left[\prod_{j=p+1}^{2m}\int_{\overline{\mathcal{C}}}\frac{\mathrm{d}\lambda_j}{2\pi\mathrm{i}}\overline{F}_{z_j}(\lambda_j)\right]\frac{\det_{2m}\Theta^{(p)}_{j,k}}{\prod_{1\le j<k\le 2m}(\lambda_j-\lambda_k-2\mathrm{i})}$$

Factorization?

Still, improvement necessary..

- multiple integrals : too complicated to factorize into single loop integrals
- If $\mathfrak{B}(\mu), Y$ already describe physics, only they should appear

Take other routes to find factorized expressions

- fusion of (already factorized) spin $\frac{1}{2}$ DME
- use difference equations of $q-$ KZ type at discrete points. (Aufgebauer et al (2012) $S = \frac{1}{2}$)

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fusion of DME

The idea is trivially simple.

- evaluate D_{2m} of $S = \frac{1}{2}$
- replace $\omega_{\alpha,q}$ of $S = \frac{1}{2}$ to $\omega_{\alpha,q}$ of $S = 1$
- proper combinations of D_{2m} give D_m of $S = 1$ after proper normalization

The actual calculation is simple, not elegant but elephant.

$S = 1, m = 3$ result

conevient to present $S = 1$ DME using $SU(2)$ invariant projector

$$D_m^{S=1}(\xi_1, \cdots, \xi_m) = \sum_{\alpha=1}^{N_m} \rho_\alpha^{S=1}(\xi_1, \cdots, \xi_m) P_\alpha^{S=1}$$

$N_2 = 3, N_3 = 15...$

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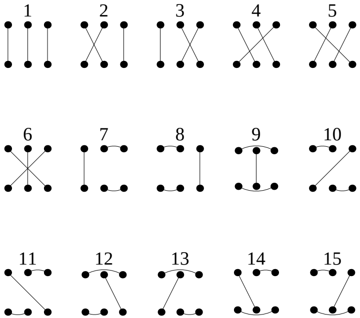
example of projectors for $S = 1$ $m = 3$



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factorized solution ($\xi^\pm := \xi \pm i$)

$$\begin{aligned}
\rho_1^{S=1}(\xi_1, \xi_2, \xi_3) &= \frac{1}{27} + \frac{1}{N(\xi_1)N(\xi_2)N(\xi_3)} \Big(c_1^{(1)} \omega(\xi_1^-, \xi_2^-) + c_2^{(1)} \omega(\xi_1^-, \xi_1^+) + c_3^{(1)} \omega(\xi_1^-, \xi_2^+) \\
&+ c_1^{(2)} \omega(\xi_1^-, \xi_1^+)\omega(\xi_2^-, \xi_3^-) + c_2^{(2)} \omega(\xi_1^-, \xi_2^-)\omega(\xi_2^+, \xi_3^-) + c_3^{(2)} \omega(\xi_1^-, \xi_1^+)\omega(\xi_2^-, \xi_3^+) \\
&+ c_4^{(2)} \omega(\xi_1^+, \xi_3^-)\omega(\xi_2^-, \xi_3^+) + c_5^{(2)} \omega(\xi_1^-, \xi_2^-)\omega(\xi_1^+, \xi_3^+) + c_6^{(2)} \omega(\xi_2^-, \xi_3^+)\omega(\xi_2^+, \xi_3^-) \\
&+ c_7^{(2)} \omega(\xi_1^-, \xi_1^+)\omega(\xi_2^-, \xi_2^+) + c_8^{(2)} \omega(\xi_2^-, \xi_3^-)\omega(\xi_2^+, \xi_3^+) + c_1^{(3)} \omega(\xi_1^-, \xi_1^+)\omega(\xi_2^-, \xi_3^+)\omega(\xi_2^+, \xi_3^-) \\
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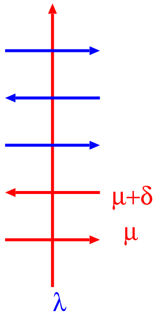
&+ \text{permutations and negation} \Big)
\end{aligned}$$

- $N(\xi)$ comes form normalization
  $N(\xi) = \frac{3}{4} + \frac{\omega(\xi^-, \xi^+)}{2}$.
- $c_j^{(a)}$ are known rational functions of $\xi_k^{\pm}$.
- ω is $(S = \frac{1}{2}) \times (S = 1)$ object

$$\omega(\lambda, \mu) \sim \frac{d}{d\delta} \ln \Lambda^{[1]}(\lambda, \mu)|_{\delta=0}$$

can be obtained only from $\mathfrak{B}, Y$: no need of $\mathfrak{F}$



Almost what we want

homogeneous and $T = 0$ limit of $S = 1$ result

One can take

- zero T limit

$$\omega^{S=1}_{T=0}(\lambda,\mu) = \omega^{S=\frac{1}{2}}_{T=0}(\lambda,\mu) + \frac{(\lambda-\mu)^2+4}{8}\,\frac{\pi(\lambda-\mu)}{2\sin\frac{\pi}{2}(\lambda-\mu)}$$

- homogeneous limit $\xi_j \to 0$

All $\rho^{S=1}_{\alpha}$ are given by rational numbers and $\pi^2, \pi^4 \ldots$,
example

$$8\rho^{S=1}_{1} = \frac{1879}{432} - \frac{3497}{1350}\pi^2 + \frac{53}{135}\pi^4 - \frac{11296}{637875}\pi^6$$

Conjecture (Klümper et al 2013)

The correlation functions at $T = 0$ for the integrable $S = 1$ (integer) spin chain of XXX-type are described by Riemann's ζ functions with even arguments.

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Still unsatisfactory..

- $\omega(\lambda, \mu)$ is $(S = \frac{1}{2}) \times (S = 1)$ object.
- for $S = \frac{1}{2}$, homogeneous limit= $\lim_{\lambda,\mu \to 0} \omega(\lambda, \mu)$
- for $S = 1$, homogeneous limit=$\lim_{\lambda,\mu\to 0}\omega(\lambda + i, \mu - i)$: needs singular object,

$$\omega(\lambda, \mu) - \frac{1}{2} = -\frac{(\lambda - \mu)^2 + 4}{2i} \frac{d}{d\delta} \ln \Lambda^{[1]}(\lambda, \mu)|_{\delta=0}$$

Question: Any proper $(S = 1) \times (S = 1)$ object, free from singular objects?

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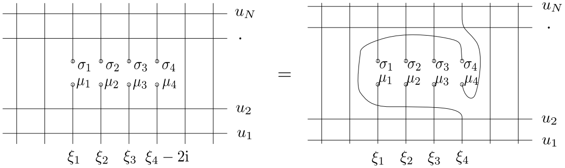
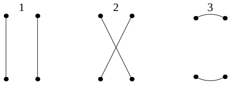
The difference equation at discrete points (Aufgebauer et al (2012)) gives a hint.



$$\xi_4 \in \{u_1, \cdots, u_N\}$$

difference equation at discrete points

Concentrate on $m = 2$: $D_2(\xi_1, \xi_2) = \sum_{\alpha=1}^{3} \rho_\alpha^{S=1}(\xi_1, \xi_2) P_\alpha$



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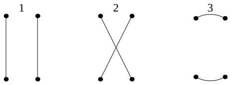

Difference equations

$$\begin{pmatrix} \rho_1(\xi_1 - 2\mathrm{i}, \xi_2) \\ \rho_2(\xi_1 - 2\mathrm{i}, \xi_2) \\ \rho_3(\xi_1 - 2\mathrm{i}, \xi_2) \end{pmatrix} = L(\xi_1 - \xi_2) \cdot \begin{pmatrix} \rho_1(\xi_1, \xi_2) \\ \rho_2(\xi_1, \xi_2) \\ \rho_3(\xi_1, \xi_2) \end{pmatrix}$$

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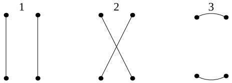

Change of variables

$$\begin{pmatrix} \rho_1(\xi_1,\xi_2) \\ \rho_2(\xi_1,\xi_2) \\ \rho_3(\xi_1,\xi_2) \end{pmatrix} = \begin{pmatrix} \frac{5\xi^2+36}{45(\xi^2+4)} & -\frac{\xi^2}{30(\xi^2+4)} & \frac{\xi^2+6}{15(\xi^2+4)} \\ -\frac{64}{45(\xi^2+4)} & \frac{3\xi^2-20}{60(\xi^2+4)} & -\frac{3\xi^2+28}{30(\xi^2+4)} \\ \frac{16}{45(\xi^2+4)} & \frac{3\xi^2+20}{60(\xi^2+4)} & -\frac{3\xi^2+8}{30(\xi^2+4)} \end{pmatrix} \begin{pmatrix} 1 \\ G(\xi_1,\xi_2) \\ H(\xi_1,\xi_2) \end{pmatrix}$$

$\xi = \xi_1 - \xi_2$

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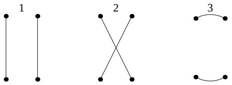


Much simpler difference equation

$$\begin{pmatrix} 1 \\ \bar{G} \\ \bar{H} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{\xi(\xi-6\mathrm{i})}{(\xi-2\mathrm{i})(\xi+4\mathrm{i})} & 0 \\ -\frac{256i(\xi-\mathrm{i})}{3(\xi+2\mathrm{i})(\xi-2\mathrm{i})^2(\xi+4\mathrm{i})} & -\frac{\xi(\xi-6\mathrm{i})(\xi^2-2\mathrm{i}\xi-4)}{(\xi-2\mathrm{i})^2(\xi+2\mathrm{i})(\xi+4\mathrm{i})} & \frac{\xi^2(\xi-6\mathrm{i})(\xi-4\mathrm{i})}{(\xi-2\mathrm{i})^2(\xi+2\mathrm{i})(\xi+4\mathrm{i})} \end{pmatrix} \begin{pmatrix} 1 \\ G \\ H \end{pmatrix}$$

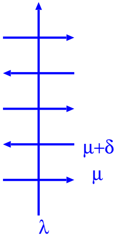
$$\bar{G} = G(\xi_1 - 2i, \xi_2)$$

$$G(\lambda,\mu) \sim (\Omega(\lambda-i,\mu-i)+\Omega(\lambda-i,\mu+i)+\Omega(\lambda+i,\mu-i)+\Omega(\lambda+i,\mu+i))$$

where $\Omega(\lambda,\mu) = 2i\frac{\omega(\lambda,\mu)+1/2)}{(\lambda-\mu)^2+4}$.

- G is expressed by a $(S=1)\times(S=1)$ object.
- homogeneous limit is in the physical strip of $\Lambda^{[2]}(\lambda,\mu)$.
- H satisfies difference eq whose source term is G. Thus H is also proper $(S=1)\times(S=1)$ object.

$$G(\lambda,\mu) \sim \frac{d}{d\delta}\ln\Lambda^{[2]}(\lambda,\mu)|_{\delta=0}$$



The simplicit of $m = 2$ resultat $T = 0$ can be understood from

$$G(\lambda, \mu) \to 0$$

$$H(\lambda, \mu) \to \frac{1}{\sinh^2 \frac{\pi}{2}(\lambda - \mu)}$$

Summary and Future problems

Our question was,
can we play the same game for $S > \frac{1}{2}$?

- multiple integral formula at $T = 0$ ✓
- multiple integral formula at $T > 0$ ✓
- factorization at $T \geq 0$ ✓
- magnetic field?
- XXZ, XYZ?
- exponential formula at $T > 0$?
- Mixed spin chains?
- scaling limit: space of operators in SUSYsG?

Thank you for your attention.