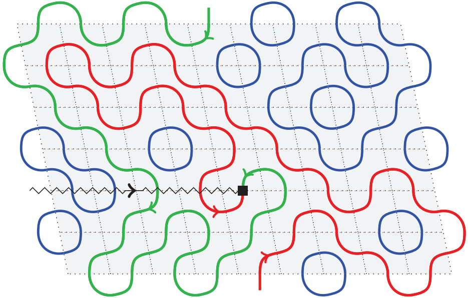
Discrete Holomorphicity and Quantum Affine Algebras



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MSP, Kyoto, Aug 3rd, 2013

Plan

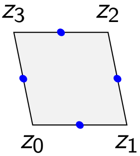
Ref: Y. Ikhlef, R.W., M. Wheeler, P. Zinn-Justin: Discrete Holomorphicity and Quantized Affine Algebras, J. Phys.A 46 (2013) 265205, arxiv:1302.4649

What is Discrete Holomorphicity?

- Λ a planar graph in $\mathbb{R}^2$, embedded in complex plane.
  Let f be a complex-valued fn defined at midpoint of edges

- f said to be DH if it obeys lattice version of $\oint f(z)dz = 0$ around any cycle.

  Around elementary plaquette, we use:
  $f(z_{01})(z_1 - z_0) + f(z_{12})(z_2 - z_1) + f(z_{23})(z_3 - z_2) + f(z_{30})(z_0 - z_3) = 0$



$z_{ij} = (z_i + z_j)/2$

- Can be written for this cycle as

$$\frac{f(z_{23}) - f(z_{01})}{z_2 - z_1} = \frac{f(z_{12}) - f(z_{30})}{z_1 - z_0},$$

a discrete Cauchy-Riemann reln

What is use of DH in SM/CFT?

- For review see [S. Smirnov, Proc. ICM 2006, 2010]
- DH observables used in proof of long-standing conjectures on conformal invariance of scaling limit, e.g.,
    - planar Ising model [S. Smirnov, C. Hongler, D. Chelkak . . . , 2001-]
    - percolation on honeycomb lattice - Cardy's crossing formula and reln to SLE(6) [S. Smirnov: 2001]

Relation to Integrability

- DH seems also to be related to integrability [Riva & Cardy 07, Cardy & Ikhlef 09, Ikhlef 12, Alam & Batchelor 12, de Gier et al13]
- e.g. parafermions of dilute $O(n)$ loop model are DH precisely in the case when loop weights obey a linear relation whose solution corresponds to a solution of Yang-Baxter relation.
- How to interprete linear relation for R implying YB?

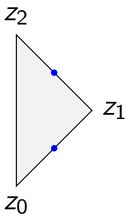
  Natural to assume that $R\Delta(x) = \Delta(x)R$ for a quantum group is behind this.

  i.e. DH observables should be understood in terms of quantum group generators [Bernard & Fendley have publicly made this point].

Our Key Results

- Dense/dilute $0(n)$ PFs are essentially non-local quantum group currents for $U_q(A_1^{(1)})/U_q(A_2^{(2)})$
- DH of these currents just comes from $R\Delta(x) = \Delta(x)R$
- Currents of boundary (co-ideal) subalgebra gives rise to observables that have discrete boundary conditions of form

$$\mathrm{Re}\big(\Psi(z_{01})(z_1 - z_0) + \Psi(z_{12})(z_2 - z_1)\big) = 0$$



Non-local quantum group currents in vertex models

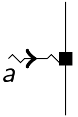
- Following Bernard and Felder [1991] we consider a set of elements $\{J_a, \Theta_a{}^b, \widehat{\Theta}^a{}_b\}$, $a, b = 1, 2, \ldots, n$, of a Hopf algebra U.

$$\text{Properties:} \quad \Theta_a{}^b \widehat{\Theta}^c{}_b = \delta_{a,c} \quad \text{and} \quad \widehat{\Theta}^b{}_a \Theta_b{}^c = \delta_{a,c}$$

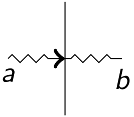
- Co-product Δ and antipode S are (with summation convention):

$$\Delta(J_a) = J_a \otimes 1 + \Theta_a{}^b \otimes J_b \qquad S(J_a) = -\widehat{\Theta}^b{}_a J_b$$
$$\Delta(\Theta_a{}^b) = \Theta_a{}^c \otimes \Theta_c{}^b \qquad S(\Theta_a{}^b) = \widehat{\Theta}^b{}_a$$
$$\Delta(\widehat{\Theta}^a{}_b) = \widehat{\Theta}^a{}_c \otimes \widehat{\Theta}^c{}_b \qquad S(\widehat{\Theta}^a{}_b) = \Theta_b{}^a.$$

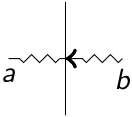
- Acting on rep of U, we represent as

$J_a =$ 

,

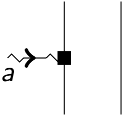
$\Theta_a{}^b =$ 

,

$\widehat{\Theta}^a{}_b =$ 

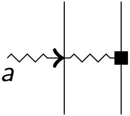
- Coproducts pictures are:

$\Delta(J_a) \quad = \quad$



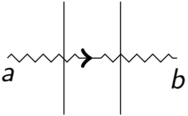
$J_a \otimes 1$

$+$



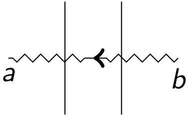
$\Theta_a{}^b \otimes J_b$

$\Delta(\Theta_a{}^b) \quad = \quad$



$\Theta_a{}^c \otimes \Theta_c{}^b$,

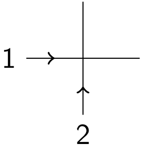
$\Delta(\widehat{\Theta}^a{}_b) =$



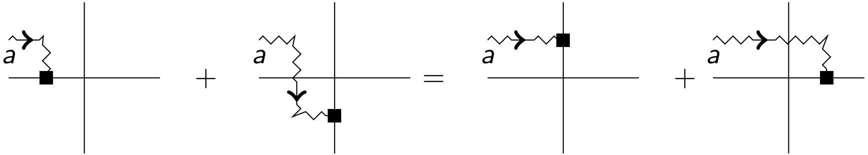
$\widehat{\Theta}^a{}_c \otimes \widehat{\Theta}^c{}_b$

and obvious extensions to $\Delta^{(N)}(x)$.

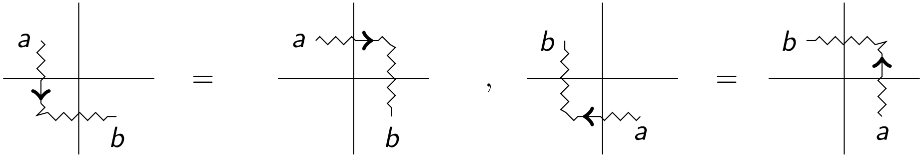
- With $R : V_1 \otimes V_2 \to V_2 \otimes V_1$



$R\Delta(x) = \Delta(x)R$ becomes:

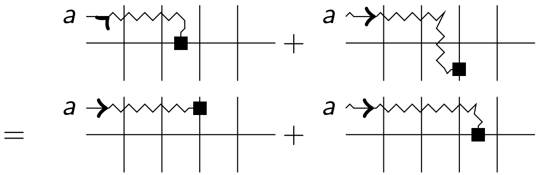


$$R(J_a \otimes 1) + R({\Theta_a}^b \otimes J_b) = (J_a \otimes 1)R + ({\Theta_a}^b \otimes J_b)R$$



$$({\Theta_a}^c \otimes {\Theta_c}^b) = ({\Theta_a}^c \otimes {\Theta_c}^b)R, \quad R(\widehat{\Theta}^b{}_c \otimes \widehat{\Theta}^c{}_a) = (\widehat{\Theta}^b{}_c \otimes \widehat{\Theta}^c{}_a)R$$

- For monodromy matrix, we have non-local currents



- Gives

$$j_a(x-\frac{1}{2},t)-j_a(x+\frac{1}{2},t)+j_a(x,t-\frac{1}{2})-j_a(x,t+\frac{1}{2})=0$$

when inserted into a correlation function.

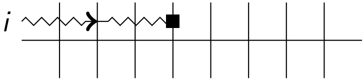
Quantum Affine Algebras

- Consider algebra U gen. by $e_i, f_i, t_i^{\pm 1}$ with standard relns and

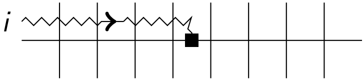
$$\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta(t_i) = t_i \otimes t_i$$

- Hence can consider currents:

$$e_i(x, t + \frac{1}{2}) \quad \sim$$



$$e_i(x + \frac{1}{2}, t) \quad \sim$$



- We consider two cases with $i \in \{0,1\}$ with irreps:
- $U_q(A_1^{(1)})$: 6-Vertex Model

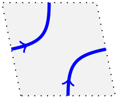
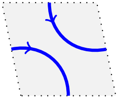
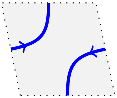
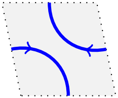
$$e_0 = z \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad t_0 = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}$$

- $U_q(A_2^{(2)})$: 19-Vertex Izergin-Korepin Model

$$e_0 = z^{1-\ell} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & q & 0 \end{pmatrix}, \quad t_0 = \begin{pmatrix} q^{-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^2 \end{pmatrix}$$

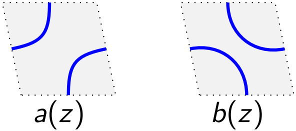
From vertex models to loop models - the $A_1^{(1)}$ dense case

- 6-vertex model $R(z = z_h/z_v) = \begin{pmatrix} A(z) & 0 & 0 & 0 \\ 0 & B(z) & C(z) & 0 \\ 0 & C(z) & B(z) & 0 \\ 0 & 0 & 0 & A(z) \end{pmatrix}$ can be written in dressed-loop picture as

$A(z) =$  , $B(z) =$  , $C(z) =$  $+$ 

plus reversed arrow cases.

- These can be rewriiten as appropriate loop weights $a(z) = qz - q^{-1}z^{-1}$, $b(z) = z - z^{-1}$:



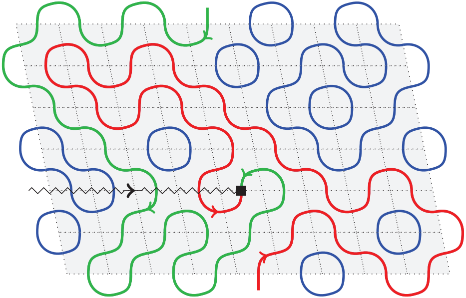
times additional factor $(-q)^{\frac{\delta}{2\pi}}$ from directed line turning through angle δ. Acute angle α given by $z = (-q)^{-\frac{\alpha}{\pi}}$.

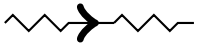
- Thus $A(z) = a(z)$, $B(z) = b(z)$, $C(z) = a(z)(-q)^{\frac{\alpha}{\pi}} + b(z)(-q)^{\frac{\alpha}{\pi}-1} = q - q^{-1}$.
- Partition fn becomes: $Z = \sum a^{N_a} b^{N_b} (-q - q^{-1})^{N_{loops}}$

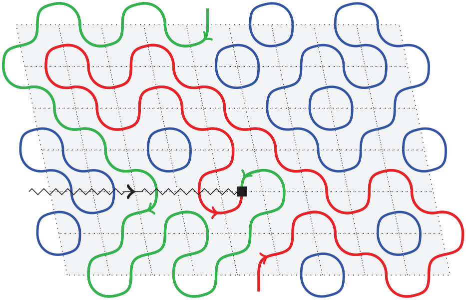
$e_0(x,t)$ in the loop picture - the $A_1^{(1)}$ dense case

- For $U_q(A_1^{(1)})$, we have $e_0 = z\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, so sends up arrow to down, or right arrow to left: Simple boundary conditions consistent with $\langle e_0(a,b)\rangle \neq 0$ are below, with a free line passing through (a,b) and attached to boundaries as shown:

$$\langle e_0(x,t+\tfrac{1}{2})\rangle = \frac{1}{Z}\sum$$



The tail  can be moved through loops on boundary.



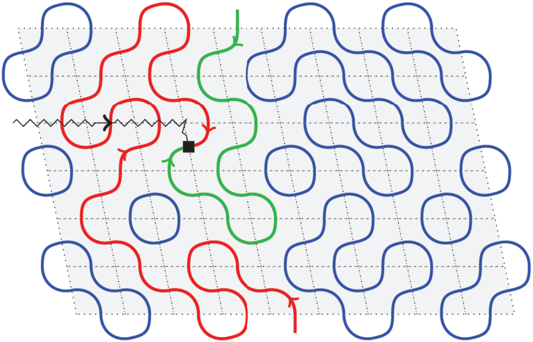
To express purely in terms of loop configuration C, consider angle turns of $\longrightarrow$ and $\longrightarrow$ and effects of $\rightsquigarrow$

- Both $\rightarrow$ and $\rightarrow$ have same angle turn $\theta(C) = \pi k(C)$, where $k(C) \in \mathbb{Z}$, equals 2 in example. Weight $= (-q)^{k(C)}$.
- No. down - no. up crossing of $\rightsquigarrow$ also $k(C)$. Weight $= q^{k(C)}$.

Hence $\langle e_0(x, t+\frac{1}{2})\rangle = \frac{z_v}{Z} \sum_{C|(x+\frac{1}{2},t)\in\gamma} W(C)(-q^2)^{\theta(C)/\pi}$.

- Similarly

$$\langle e_0(x+\frac{1}{2},t)\rangle = \frac{1}{Z}\sum$$



$$= \frac{z_h}{Z} q^{\alpha/\pi} \sum_{C|(x,t+\frac{1}{2})\in\gamma} W(C)(-q^2)^{\theta(C)/\pi} = \frac{z_v}{Z} e^{-i\alpha} \sum_{C|(x,t+\frac{1}{2})\in\gamma} W(C)(-q^2)^{\theta(C)}$$

- Defining non-local operator ϕ_0 on edges, by

$$\phi_0(x,t+\frac{1}{2}) = z_v^{-1} e_0(x,t+\frac{1}{2}), \quad \phi_0(x+\frac{1}{2},t) = z_v^{-1} e^{i\alpha} e_0(x+\frac{1}{2},t).$$

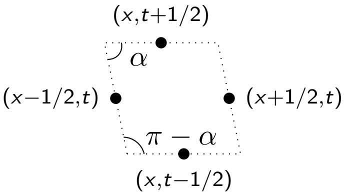
we have $\langle \phi_0(a,b)\rangle = \frac{1}{Z} \sum_{C|(a,b)\in\gamma} W(C)(-q^2)^{\theta(C)/\pi}$ and

$$e_0(x-1/2,t) + e_0(x,t-1/2) - e_0(x+1/2,t) - e_0(x,t+1/2) = 0\,.$$

becomes

$$\phi_0(x,t-1/2) + e^{i(\pi-\alpha)}\phi_0(x+1/2,t) - \phi_0(x,t+1/2) - e^{i(\pi-\alpha)}\phi_0(x-1/2,t) = 0$$

- ϕ_0 is the known parafermionic operator with DH around plaquette [Riva &Cardy 06, Smirnov 06]:



$$(x,t) \in \mathbb{Z}^2\,.$$

$e_1(x,t)$ in the loop picture - dense case

- A similar argument works for $e_1(x,t)$, but leads to a simpler DH variable. Defining a non-local operator ϕ_1 on edges, by

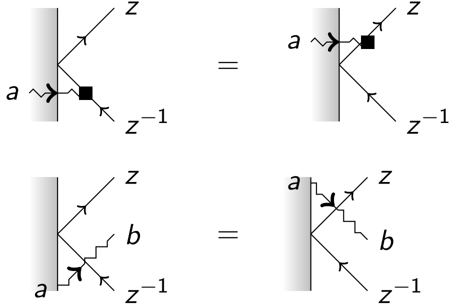
$$\phi_1(x+\frac{1}{2},t) = z_v^{-1} e_1(x+\frac{1}{2},t), \quad \phi_1(x,t+\frac{1}{2}) = z_v^{-1} e^{i\alpha} e_1(x,t+\frac{1}{2}).$$

we have $\langle \phi_1(a,b)\rangle = \frac{1}{Z}\sum_{C|(a,b)\in\gamma} W(C)e^{-i\theta(C)}$ which is DH as above.

- Note, if we define $\bar{e}_i = t_i f_i$, then we have $\Delta(\bar{e}_i) = \bar{e}_i \otimes 1 + t_i \otimes \bar{e}_i$ and the above argument can be repeated. We find corresponding anti-holomorphic observables.

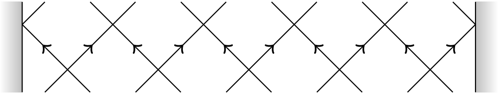
Interacting Boundaries

- To obtain integrable interacting boundary conditions, identify co-ideal subalgebra $B \subset U$, $\Delta(B) = B \otimes U$, and use Sklyanin formalism.
- For our $V(z)$ reps earlier, we have $K_L(z) : V(z^{-1}) \to V(z)$ and $K_L(z)\, x = x\, K_L(x)$, $x \in B$.
- If J_a, ${\Theta_a}^b \in B$, we have

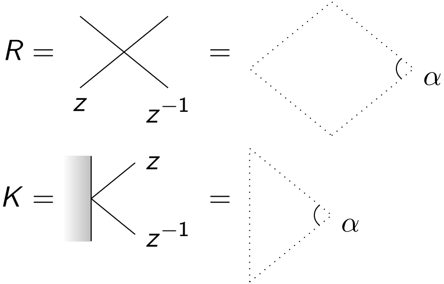


Towards the loop picture

- To make the change to the loop picture, we start from double row transfer matrix on diagonal (light-cone) lattice:



- Then consider loop picture on dual lattice:



etc

The $A_1^{(1)}$ case

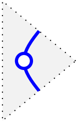
- Co-ideal sub-algebra B generated by

$$\{T_0\,,\,T_1\,,\,Q:=e_1+r\bar{e}_0\,,\,\bar{Q}:=\bar{e}_1+re_0\}\,,$$

  where r is a real parameter.
- $K_L(z)x=xK_L(z)$ gives:

$$K_L(z)=\begin{pmatrix} z+rz^{-1} & 0 \\ 0 & z^{-1}+rz \end{pmatrix}$$

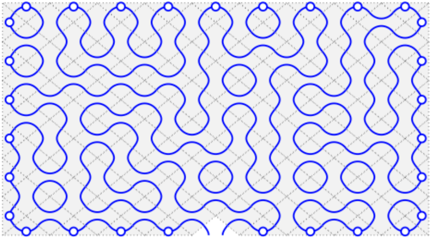
- In loop picture, becomes:



$$\sim(-q)^{\mp(\alpha-\beta)/2\pi}$$

  where β is a deficit angle - given by $(-q)^{-(\alpha-\beta)/\pi}=\frac{z+rz^{-1}}{z^{-1}+rz}$.

- For boundary conditions compatible with $\langle e_0(x,t)\rangle \neq 0$, can use:



- Then find (with $x+t=0$ mod(2)):

$$\langle e_0(x+1,t)\rangle = \frac{z^{-1}(-q)^{-\frac{1}{2}}}{Z} \sum_{C|(x+1,t)\in\gamma} W(C)(-q^2)^{\theta(C)/\pi}(-q)^{n\beta/2\pi}$$

$$\langle e_0(x,t)\rangle = \frac{z^{-1}(-q)^{-\frac{1}{2}}e^{-i\alpha}}{Z} \sum_{C|(x,t)\in\gamma} W(C)(-q^2)^{\theta(C)/\pi}(-q)^{n\beta/2}$$

$n =$ no. times left path touches boundary minus no. times right path touches boundary

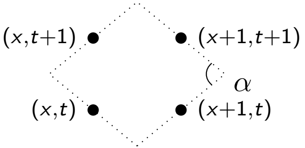
- Bulk comm relns for modified:

$$\langle \phi_0(a,b) \rangle = \frac{1}{Z} \sum_{C|(a,b)\in\gamma} W(C)(-q^2)^{\theta(C)/\pi}(-q)^{n\beta/2\pi}$$

are

$$\phi_0(x,t) + e^{i\alpha}\phi_0(x+1,t) - \phi_0(x+1,t+1) - e^{i\alpha}\phi_0(x,t+1) = 0$$

- This is DH on light-cone lattice



Relation at the left boundary

- $Q = e_1 + r\bar{e}_0$ is conserved at left boundary:

$$e_1(1,t) + r\bar{e}_0(1,t) = e_1(1,t+1) + r\bar{e}_0(1,t+1)\,, \qquad t = 0 \pmod 2)$$

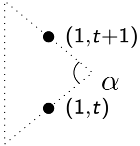
- which can be translated into

$$z^{-1}\phi_1(1,t) + rz\bar{\phi}_0(1,t) = e^{-i\alpha}z^{-1}\phi_1(1,t+1) + e^{i\alpha}rz\bar{\phi}_0(1,t+1)$$

plus conjugate relns from $\bar{Q}$.

- Defining $\psi := z^{-1}(\phi_1 + r\phi_0)$, we find

$$\mathrm{Re}\left[\psi(1,t) + e^{i(\pi-\alpha)}\psi(1,t+1)\right] = 0,$$



which is BC around plaquette linked to integrability by [Ikhlef 12; de Gier, Lee, Rasmussen 13].

The Continuum Limit

- When $|q| = 1$, theories have CFT continuum limits.
- Non-rigorous identification of fields obtained by Coulomb gas approach of Nienhuis [84] with

$$c = 1 - \frac{6(1-g^2)}{g}, \quad h_{r,s} = \frac{(r-gs)^2-(1-g)^2}{4g}, \quad g = 1 - 2\nu.$$

- We find:

Dense case: $\phi_0 \sim (h_{13}, 0), \phi_1 \sim (1, 0); \qquad q = -e^{2\pi i\nu}.$

Dilute case: $\phi_0 \sim (h_{12}, 0), \phi_1 \sim (1, 0); \qquad q^4 = -e^{2\pi i\nu}.$

Conclusions & Comments

- Parafermions come directly from quantum group currents
- Quantum group invariance leads to DH property
- Discrete integral boundary conditions understood similarly from boundary quantum groups
- Why is underlying connection between quasitriangular Hopf algebras and discrete calculus?
- All our results with exception of CFT limit seem to be true for generic q, including $-1 < q < 0$ massive regimes.

Appendix: Non-local operators in lattice models

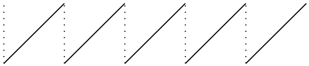
- Consider 1D Ising model in terms of transfer matrix:

  $V = \mathbb{C}^2$, $T : V \to V$, with $Z = \mathrm{Tr}_V(T^N)$.

- Could also write for lattice Λ with positions $x \in \{1, 2, \cdots, N\}$:

  $V(x) \cong V$, $V_\Lambda = \otimes_{x \in \Lambda} V(x)$, $T(x) : V(x) \to V(x+1)$,

  $B : V_\Lambda \to V_\Lambda$ with $B = \otimes_{x\in\Lambda} T(x)$, with $Z = \mathrm{Tr}_{V_\Lambda}(B)$

  Schematically: 

- A local operator $\sigma^z(x) : V(x) \to V(x)$ is then well defined and

$$\langle \sigma^z(n) \sigma^y(m) \rangle = \frac{1}{Z} \mathrm{Tr}_{V_\Lambda}(\sigma^z(n)\sigma^z(m)B).$$

  Can just be written as $\langle \sigma^z(n)\sigma^y(m)\rangle = \frac{1}{Z}\mathrm{Tr}_V(\sigma^z T^{m-n} \sigma^z T^{N-m+n})$.

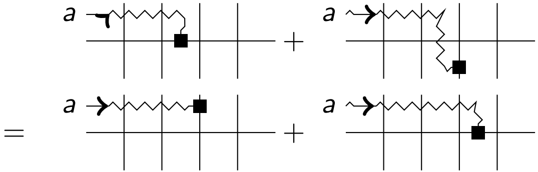
- Gen. formalism *is* useful for quasi-local operators in 2D [B&F 91]
- If edge mid-points $p \in \Lambda$, points $p^* \in \Lambda^*$:

$$
\begin{aligned}
V_\Lambda &= \otimes_{p\in\Lambda} V(p), \\
R(x,t) &: V(x-\frac{1}{2},t)\otimes V(x,t-\frac{1}{2}) \to V(x,t+\frac{1}{2})\otimes V(x+\frac{1}{2},t)
\end{aligned}
$$

$$B = \otimes_{p*\in\Lambda^*} R(p*) : V_\Lambda \to V_\Lambda, \quad Z = \mathrm{Tr}_{V_\Lambda}(B)$$

- Any operator acts as $\mathcal{O} : V_\Lambda \to V_\Lambda$ and $\langle\mathcal{O}\rangle = \frac{1}{Z}\mathrm{Tr}_{V_\Lambda}(\mathcal{O}B)$
- Local operator $\mathcal{O}(p)$ acts as identity on every edge except the one p.
- Quasi-local operator $\mathcal{O}(p)$ acts as identity except along a string of edges terminating in p.

- Thus we consider a quasi-local operator $j_a(p)$ associated with a node attached to the edge p and a tail labelled by a terminating at a fixed point on the left boundary.
- The operator relations



become $j_a(x-\frac{1}{2},t)-j_a(x+\frac{1}{2},t)+j_a(x,t-\frac{1}{2})-j_a(x,t+\frac{1}{2})$ when inserted into a correlation function.