Anisotropic flows and the shear viscosity of the QGP within a kinetic approach

S. Plumari, A. Puglisi, L. Guardo,
M. Ruggieri, F. Scardina, V. Greco
Transport approach at fixed $\eta/s$:

- Motivation
- How to fix locally $\eta/s \leftrightarrow \sigma(\theta), M, T$ - Chapman-Enkog approach.

$\eta/s$ and generation of $v_2$: from RHIC to LHC

$V_n$ from initial state fluctuations (preliminary)

Conclusions
Information from non-equilibrium: elliptic flow

\[ \lambda = (\sigma \rho)^{-1} \text{ or } \frac{\eta}{s} \text{ viscosity} \]

\[ c_s^2 = \frac{dP}{d\varepsilon}, \text{ EoS-lQCD} \]

\[ \varepsilon_x = \left\langle \frac{y^2 - x^2}{y^2 + x^2} \right\rangle \]

The \( v_2/\varepsilon \) measures efficiency in converting the eccentricity from Coordinate to Momentum space


Can be seen also as Fourier expansion

\[ \frac{dN}{dp_T d\phi} = \frac{dN}{dp_T} \left[ 1 + 2 v_2 \cos(2\phi) + 2 v_4 \cos(4\phi) + \ldots \right] \]

by symmetry \( v_n \) with \( n \) odd expected to be zero ... (but event by event fluctuations)

P. Romatschke, PRL99 (07)
Motivation for a kinetic approach:

\[ \left\{ p^\mu \partial_\mu + \left[ p_\nu F^{\mu\nu} + M \partial^\mu M \right] \partial_\mu^p \right\} f(x,p) = C_{22} + C_{23} + \ldots \]

- Starting from 1-body distribution function and not from \( T^{\mu\nu} \): possible to include \( f(x,p) \) out of equilibrium.
- It is not a gradient expansion in \( \eta/s \).
- Valid at intermediate \( p_T \) out of equilibrium.
- Valid at high \( \eta/s \) (cross over region): + self consistent kinetic freeze-out
- Include hadronization by coalescence + fragmentation.
\[ p^\mu \partial_\mu f(X, p) = C = C_{22} + C_{23} + \ldots \]

Collisions \[\varepsilon - 3p = 0, \quad \eta \neq 0\]

\[ C_{22} = \frac{1}{2E_1} \int \frac{d^3 p_2}{(2\pi)^3} \frac{d^3 p_1'}{2E_1'} \frac{d^3 p_2'}{2E_2'} f_1 f_2' \left| M_{1'2' \rightarrow 12} \right|^2 (2\pi)^4 \delta^{(4)}(p_1' + p_2' - p_1 - p_2) \]

For the numerical implementation of the collision integral we use the stochastic algorithm. (Z. Xu and C. Greiner, PRC 71 064901 (2005))

\[ P_{22} = \frac{\Delta N_{coll}^{2 \rightarrow 2}}{\Delta N_1 \Delta N_2} = v_{rel} \sigma_{22} \frac{\Delta t}{\Delta^3 x} \]

\[ \Delta t \rightarrow 0, \quad \Delta^3 x \rightarrow 0 \quad \text{right solution} \]
Do we really have the wanted shear viscosity $\eta$ with the relax. time approx.?

- Check $\eta$ with the Green-Kubo correlator
Extraction of the Shear Viscosity: Box calculation

Green – Kubo relation

\[
\eta = \frac{1}{T} \int_0^\infty dt \int_V d^3x \langle \pi^{xy}(x, t) \pi^{xy}(0, t) \rangle \\
\langle \pi^{xy}(\vec{x}, t) \pi^{xy}(\vec{0}, t) \rangle = \langle \pi^{xy}(0) \pi^{xy}(0) \rangle \cdot e^{-t/\tau}
\]

\[
\eta = \frac{V}{T} \langle \pi^{xy}(0) \pi^{xy}(0) \rangle \cdot \tau
\]
Extraction of the Shear Viscosity: Box calculation

**Green – Kubo relation**

\[
\eta = \frac{1}{T} \int_{0}^{\infty} dt \int_{V} d^{3}x \left\langle \pi^{xy}(x,t) \pi^{xy}(0,t) \right\rangle
\]

\[
\left\langle \pi^{xy}(\vec{x},t) \pi^{xy}(\vec{0},t) \right\rangle = \left\langle \pi^{xy}(0) \pi^{xy}(0) \right\rangle \cdot e^{-t/\tau}
\]

\[
\eta = \frac{V}{T} \left\langle \pi^{xy}(0) \pi^{xy}(0) \right\rangle \cdot \tau
\]

\[
\eta = \frac{4}{15} \frac{eT}{V}
\]

Depends on macroscopical details: \( \tau(\sigma) \)

Depends on microscopical details:

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Extraction of the Shear Viscosity: Box calculation

**Relaxation Time Approximation**

\[
\eta = \frac{1}{15T} \int_0^\infty \frac{d^3 p}{(2\pi)^3} \frac{p^4}{E^2} \tau(E) f^{eq}(E)
\]

\[\tau^{-1}(E) = \rho \left( \sigma_{tot} v_{rel} \right)\]

\[\eta_{relax} = 0.8 \frac{T}{\sigma_{tot}} \quad \eta \sim \frac{1}{\sigma_{tot}}\]

Usual as Relax. Time Approx. - Israel Stewart

\[\eta_{relax}^{IS} = 0.8 \frac{T}{\sigma_{tr}} = 1.2 \frac{T}{\sigma_{tot}}\]


Isotropic cross section: massless case

- At 1st order of approx. in the Chapman-Enskog:

\[ [\eta]^{CE}_{1st} = 1.2 \frac{T}{\sigma_{tot}} \]

- successive approx. up to 16 order:

\[ [\eta]^{16th}_{CE} = 1.267 \frac{T}{\sigma_{tot}} \]


Kapusta, PRC(2010); Gavin NPA(1985); Molnar-Huovenin PRC(2009), G. Ferini PLB(2009), Khvorostukhin PRC (2010)....

S. Plumari et al., PRC86 (2012) 054902.
Extraction of the Shear Viscosity: Box calculation

\[ \eta_{\text{relax}}^\text{IS} / s = \frac{1}{15} \langle p \rangle \tau = \frac{1}{15} \frac{\langle p \rangle}{\sigma_{\text{tot}} f(a) \nu_{\text{rel}}} \rho \]

\[ \sigma_{\text{tr}} = \int d\Omega \sin^2(\theta_{\text{cm}}) \frac{d\sigma}{d\Omega_{\text{cm}}} = \sigma_{\text{tot}} f(a) \leq \frac{2}{3} \sigma_{\text{tot}} \]

For the standard pQCD-like cross section

\[ \frac{d\sigma}{d\Omega_{\text{cm}}} = \frac{9\pi \alpha_s^2}{2} \frac{1}{(q^2 + m_D^2)^2} (1 + \frac{m_D^2}{s}) \]

Employed also for non-isotropic cross section:

\[ m_D \text{ regulates the anisotropy of collision} \]

\[ m_D \rightarrow \infty \text{ we recover the isotropic limit} \]

\[ f(a) = 4a(1+a)[(2a+1)\ln(1+a^{-1})-2], \quad a = m_D^2 / s \]

1\textsuperscript{st} Chapman-Enskog approximation

\[ [\eta]_{1\text{st}} / s = \frac{1}{15} \langle p \rangle \tau = \frac{1}{15} \frac{\langle p \rangle}{\sigma_{\text{tot}} g(a) \rho} \]

\[ g(a) = \frac{1}{50} \int_0^\infty dy y^6 \left[(y^2 + \frac{1}{3}) K_3(2y) - y K_2(2y)\right] f(a), \quad a = \frac{m_D}{2T} \]

- CE and RTA can differ by a factor of 2
- Green-Kubo agree with CE (< 5%)


S. Plumari et al., PRC86 (2012) 054902.
• We know how to fix locally $\eta/s(T)$

• We have checked the Chapmann-Enskog:
  - $CE$ good already at I° order $\approx 5\%$ ($\approx 3\%$ at II° order)
  - $RTA$ even with $\sigma_{tr}$ severely underestimates $\eta$
Simulating a constant $\eta/s$

For the general case of anisotropic cross section and massless particles:

$$\eta(\vec{x},t)/s = \frac{1}{15} \langle p \rangle \tau_\eta$$

$$\sigma_{tot}^{\eta/s} = \frac{1}{15} \frac{\langle p \rangle}{g(m_D/2T)n} \frac{1}{\eta/s}$$

$\sigma$ is evaluated in such way to keep fixed the $\eta/s$ during the dynamics according the Chapman-Enskog equation. (similar to D. Molnar, arXiv:0806.0026[nucl-th] but our approach is local.)

Knudsen number

$$K = \frac{L}{\lambda} \rightarrow \frac{\tau}{\lambda}$$

Large K small $\eta/s$

$$K = \frac{1}{5} \frac{T_0 \tau_0}{\eta/s}$$

$$\frac{\eta}{s} = \frac{1}{5} \frac{T \cdot \lambda}{s}$$

In the limit of small $\eta/s$ (<0.16) and for small pT equivalent viscous hydro
\( \frac{\eta}{s} = \frac{1}{15} \langle p \rangle \tau_\eta \)

\( \tau_\eta = \frac{1}{\sigma_{tot} \ g(a) \rho} \)

- \( \frac{\eta}{s} \) is the physical parameter determining the \( v_2 \) at least up to \( p_T \) 1.5 - 2 GeV.
- microscopic details becomes important at higher \( p_T \).
Applying kinetic theory to A+A Collisions....

- Impact of $\eta/s(T)$ on the build-up of $v_2(p_T)$ vs. beam energy
Initial condition of our simulation

- **r-space**: standard Glauber model
- **p-space**: Boltzmann-Juttner $T_{\text{max}}=1.7-3.5 \ T_c$
- $[pT<2 \ \text{GeV} \] + \minijet \ [pT>2-3\ \text{GeV}]$

**Discarded in viscous hydro**

We fix maximum initial $T$ at RHIC 200 AGeV

$$T_{\text{max}0} = 340 \ \text{MeV}$$

$$T_0 \ \tau_0 = 1 \Rightarrow \tau_0 = 0.6 \ \text{fm/c}$$

**Typical hydro condition**

Then we scale it according to

\[
\frac{1}{\tau A_T} \frac{dN_{ch}}{d\eta} \propto T^3.
\]

<table>
<thead>
<tr>
<th>$\sqrt{s}$</th>
<th>62 GeV</th>
<th>200 GeV</th>
<th>2.76 TeV</th>
</tr>
</thead>
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<tr>
<td>$T_0$</td>
<td>290 MeV</td>
<td>340 MeV</td>
<td>590 MeV</td>
</tr>
<tr>
<td>$\tau_0$</td>
<td>0.7 fm/c</td>
<td>0.6 fm/c</td>
<td>0.3 fm/c</td>
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- **r-space**: standard Glauber model
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- $[p_T<2 \ GeV] + \text{minijet } [p_T>2-3\mathrm{GeV}]$
  - Discarded in viscous hydro

![Graphs showing data points and lines for LHC and RHIC experiments.](image-url)
The f.o. is the increase of $\eta/s$ in the cross-over region, with a smooth transition between the QGP and the hadronic phase, the collisions are switched off.

For the $v_2$ similar to cut-off at $\varepsilon_0 = 0.7 \text{ GeV/fm}^3$
RHIC:
- Like viscous hydro the data are close to $\eta/s=1/(4\pi) + \text{f.o.}$
- Sensitive reduction of the $v_2$ when the f.o. is included the effect is about 20%.
- $p_T < 2.5$ GeV good agreement with the experimental data.

LHC:
- $p_T < 2$ GeV like hydro data described with $\eta/s=1/(4\pi) + \text{f.o.}$
- Smaller effect on the reduction of the $v_2$ when the f.o. is included an effect of about 5%.
- Without the kinetic freezeout the effect of a constant $\eta/s=2(4\pi)^{-1}$ is to reduce the $v_2$ of 15%.
At LHC the contamination of mixed and hadronic phase becomes negligible.

Longer life time of QGP $\rightarrow v_2$ completely developed in the QGP phase (at least up to 3 GeV)
\[ \eta/s(T) \text{ around to a phase transition} \]

- Quantum mechanism
\[ \Delta E \cdot \Delta t \geq 1 \rightarrow \eta/s \approx \frac{1}{15} \langle p \rangle \cdot \tau > \frac{1}{15} \]

- AdS/CFT suggest a lower bound \( \eta/s = 1/(4\pi) \sim 0.08 \)

The QGP viscosity is close to this bound!

Do we have signature of a 'U' shape of \( \eta/s(T) \) for the QCD matter?

From pQCD:
\[ \eta/s \sim \frac{1}{g^4 \ln(1/g)} \Rightarrow \eta/s \sim 1 \]


Phase transition physics suggest a T dependence of $\eta/s$ also in the QGP phase

- LQCD: some results for quenched approx. large error bars
- Quasi-Particle models seem to suggest a $\eta/s \sim T^\alpha$ $\alpha \sim 1 - 1.5$.
- Chiral perturbation theory $\rightarrow$ Meson Gas
- Intermediate Energies – IE ($\mu_B > T$)

Phase transition physics suggest a $T$ dependence of $\eta/s$ also in the QGP phase

- LQCD: some results for quenched approx. large error bars
- Quasi-particle models seem to suggest a $\eta/s \sim T^\alpha$ $\alpha \sim 1 - 1.5$.

**Temperature dependent $\eta/s(T)$**

![Graph showing $\eta/s(T)$ versus $T/T_C$ with data points and fit functions.](chart.png)

Temperature dependent $\eta/s(T)$

- For $4\pi\eta/s=1$ during all the evolution of the fireball we get a discrepancy for the $v_2(p_T)$, in particular we observe a smaller $v_2(p_T)$ at LHC.
- Similar results for $\eta/s \propto T^2 \rightarrow$ a discrepancy about 20%.
- Notice only with RHIC $\rightarrow$ scaling for $4\pi\eta/s=1$ LHC data play a key role.

Plumari, Greco, Csernai, arXiv:1304.6566
Invariance of $v_2(p_T)$ in BES suggest that the system goes through a phase transition.
Hope: $v_n$, $n>3$ with an event-by-event analysis will put even stronger constraint
Implementation of local fluctuation under development

Similar results: R. A. Lacey et al., arXiv:1305.3341 [nucl-ex].

Plumari, Greco, Csernai, arXiv:1304.6566

Temperature dependent $\eta/s(T)$
What about Color Glass condensate initial state?

- Kinetic Theory with a $Q_s$ saturation scale
**Initial Conditions: Glasma**

The two nuclei could be described as two tiny disks of Color Glass Condensate (CGC)

**Saturation scale**

\[ Q_{sat}^2(s) \propto \alpha_s(Q^2) \frac{xg(x, Q^2)}{\pi R^2} A^{1/3} \]

At RHIC \( Q_s^2 \sim 1-2 \text{ GeV}^2 \)
At LHC \( Q_s^2 \sim 2-5 \text{ GeV}^2 \)

The production of particle HIC is controlled by the \( Q_s \)

Increasing of gluon density as \( x = p_T/\sqrt{s} \) decreases


**Reviews**

McLerran, 2011
Iancu, 2009
McLerran, 2009
Lappi, 2010
Gelis, 2010
Fukushima, 2011
Initial Conditions: fKLN

\[
\frac{dN_g}{d^2x \, dy} \propto \int \frac{d^2p_T}{p_T^2} \int_0^{p_T} d^2k_T \alpha_s(Q^2) \times \phi_A \left( x_A, \frac{(p_T + k_T)^2}{4}; x_\perp \right) \\
\times \phi_B \left( x_B, \frac{(p_T - k_T)^2}{4}; x_\perp \right)
\]

\[
\phi_A(x_1, k_T^2; x_\perp) = \frac{\kappa Q_s^2}{\alpha_s(Q_s^2)} \left[ \frac{\theta(Q_s - k_T)}{Q_s^2 + \Lambda^2} + \frac{\theta(k_T - Q_s)}{k_T^2 + \Lambda^2} \right]
\]

Saturation effects built in the $\phi$.

Universal saturation scale, in agreement with:
- Drescher and Nara, PRC75, 034905 (2007)
- Hirano and Nara, PRC79, 064904 (2009)

\[
Q_{s,A}^2(x, x_\perp) \propto Q_s^2 T_A(x_\perp) x^{-\lambda}
\]
V2 from fKLN in viscous hydro

1) r-space from KLN (larger $\varepsilon_x$)
2) p-space thermal at $t_0 \approx 0.6$ fm/c - we call it fKLN-Th


Glauber

$\eta/s \approx \frac{1}{4\pi}$

CGC

$\eta/s \approx \frac{2}{4\pi}$
1) r-space from KLN (larger $\varepsilon_x$)
2) p-space thermal at $t_0 \approx 0.6$ fm/c - we call it fKLN-Th

Larger $\varepsilon_x$ - $\Rightarrow$ higher $\eta/s$
to get the same $v_2(p_T)$

Glauber: \[ \eta/s \approx \frac{1}{4\pi} \]
CGR: \[ \eta/s \approx \frac{2}{4\pi} \]

Uncertainty on initial conditions implies uncertainty of a factor 2 on $\eta/s$

Heinz et al., PRC 83, 054910 (2011)
Using kinetic theory at finite $\eta/s$ we can implement full fKLN (x & p space) - $\varepsilon_x=0.34$, $Q_s=1.44$ GeV

fKLN only in $x$ space (like in Hydro) $\varepsilon_x=0.34$, $Q_s=0$

Glauber in $x$ and thermal in $p$ $\varepsilon_x=0.289$, $Q_s=0$

M. Ruggieri et al., 1303.3178 [nucl-th]

- Thermalization in less than 1 fm/c, in agreement with Greiner et al., NPA806, 287 (2008).

- Not so surprising: $\eta/s$ is small -> large effective scattering rate -> fast thermalization.

$$\sigma_{tot} = \frac{\langle p \rangle}{\rho \ g(a)} \frac{1}{\eta/s}$$
Implementing fKLN pT distribution

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Semi-quantitative agreement with Florkowski PRD88 (2013) 034028.

our approach is 3+1D no relaxation time approximation but no field

≈ pQCD
When implementing KLN and Glauber like in Hydro we get the same of Hydro

When implementing full KLN we get close to the data with $4\pi\eta/s = 1$: larger $\varepsilon_x$ compensated by $Q_s$ saturation scale (non-equilibrium distribution)
At LHC the larger saturation $Q_s \ (\approx 2.4 \text{ GeV})$ scale makes the effect larger:
- $4\pi\eta/s=2$ not sufficient to get close to the data for Th-KLN
- $4\pi\eta/s=1$ it is enough if one implements both $x$ & $p$
Next step –
To include the Initial State Fluctuations
(Preliminary results)
Characterization of the initial profile in terms of Fourier coefficients

\[ \epsilon_n = \frac{\langle r_\perp^n \cos[n(\phi - \Phi_n)] \rangle}{\langle r_\perp^n \rangle} \]
\[ \Phi_n = \frac{1}{n} \arctan \frac{\langle r_\perp^n \sin(n\phi) \rangle}{\langle r_\perp^n \cos(n\phi) \rangle} \]
\[ r_\perp = \sqrt{x^2 + y^2}, \quad \phi = \arctan(y/x) \]

G-Y. Qin, H. Petersen, S.A. Bass and B. Muller, PRC82,064903 (2010).
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\[ r_\perp = \sqrt{x^2 + y^2}, \quad \phi = \arctan \left( \frac{y}{x} \right) \]

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v_2 and v_3 linearly correlated to the corresponding eccentricities \( \varepsilon_2 \) and \( \varepsilon_3 \) respectively.

- \( v_4 \) and \( \varepsilon_4 \) weak correlated similar to hydro calculations:

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Like in viscous hydro the data of $v_n(p_T)$ at RHIC energies are described with $4\pi\eta/s=1$. 

Conclusions

Enhancement of $\eta/s(T)$ in the cross-over region affect differently the expanding QGP from RHIC to LHC.

At LHC nearly all the $v_2$ from the QGP phase.

The scaling of $v_2(p_T)$ from Beam Energy Scan indicate a 'U' shape of $\eta/s(T)$ this would be a signature of $\eta/s(T)$ behavior typical of a phase transition.

For $4\pi\eta/s=1$ $v_2$ and $v_3$ are linearly correlated to the corresponding eccentricities $\varepsilon_2$ and $\varepsilon_3$. While $v_4$ and $\varepsilon_4$ are weakly correlated similar to hydro calculation. (More detailed study is going on)

Outlook

To study the role of $\eta/s(T)$ on the $v_n$ and their correlation on the initial eccentricities $\varepsilon_n$.

To study the effect of different initial condition (glasma) on $v_n \leftrightarrow \varepsilon_n$ correlation.
Effect of $\eta/s(T)$ in Hydro: Niemi et al.

Niemi et al., PRL 106 (2011).

$$T^\mu{}^\nu = T_{eq}^\mu{}^\nu + \delta T^\mu{}^\nu \leftarrow f_{eq} + \delta f$$

**Grad ansantz**

R. Lacey et al., PRC82

$$\delta f = \frac{\pi^\mu{}^\nu p_\mu p_\nu}{(\epsilon + p)T^2} f_{eq} \approx \frac{\eta}{3\pi s\tau T^2} f_{eq}$$

- This implies that the $\eta$ is in Relaxation Time Approximation
- Hydro is valid up to $p_T \sim 3$ GeV

D. Molnar, HQ2010
Extraction of the Shear Viscosity: Box calculation

Isotropic cross section: massive case

Massive case is relevant in quasi-particle models where $M(T)$. Good agreement with CE 1st order for isotropic cross section and massive particles.

1st Chapman-Enskog approximation

\[
[\eta]_{1st} = 10T \left[ \frac{K_3(z)}{K_2(z)} \right]^2 \frac{1}{c_{00}} = g(m_D, T) \frac{T}{\sigma_{tot}}
\]

\[
c_{00} = 16 \left[ \omega_2^{(2) - z^{-1}} \omega_1^{(2)} + (3z^2)^{-1} \omega_0^{(2)} \right]
\]

\[
\omega_i^{(s)} = \frac{2\pi z^2}{[K_2(z)]^2} \int_1^\infty dy (y^2 - 1)^3 y^i K_j(2z y) \int_0^\pi d\theta \sin \theta \frac{d\sigma}{d\Omega} (1 - \cos^s \theta)
\]

\[
[\eta]_{1st} = f(z) \frac{T}{\sigma_{tot}}
\]

\[
f(z) = \frac{15}{16} \frac{z^4 K_3^2(z)}{(15z^2 + 2) K_2(2z) + (3z^3 + 49z) K_3(2z)}
\]

S. Plumari et al., arXiv:1208.0481 [nucl-th].


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- Not so surprising: $\eta/s$ is small $\rightarrow$ large effective scattering rate $\rightarrow$ fast thermalization.

\[ \sigma_{tot} = \frac{\langle p \rangle}{\rho g(a) \eta/s} \]
We see that when non-equilibrium distribution is implemented in the initial stage ($\approx 1$ fm/c), $v_2$ grows slowly respect to thermal one.
Finite masses and EoS

\[ p^\mu \partial_\mu f(x, p) = C_{22} \]

\[ M \neq 0 \quad \text{and} \quad \epsilon - 3p \neq 0 \]

\[ C_s^2 \leq \frac{1}{3} \]