Dynamics of Non-Gaussianity in Heavy Ion Collisions

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MK, Asakawa, PRC85, 021901C(2012); PRC86, 024904(2012)
MK, Asakawa, Ono, arXiv:1307.2978

NFQCD, YITP, Kyoto, 27/Nov./2013
Beam-Energy Scan

- Hadrons
- Color SC
Beam-Energy Scan

Hadrons

high

beam energy

low

Color SC

Grand Canonical Ensemble

Au+Au

00-05%

05-10%

10-20%

20-30%

30-40%

40-60%

60-80%

Cleymans

Andronic

STAR Preliminary

STAR 2012
Fluctuations

- Fluctuations reflect properties of matter.
- Enhancement near the critical point
  - Stephanov, Rajagopal, Shuryak (’98); Hatta, Stephanov (’02); Stephanov (’09); …
- Ratios between cumulants of conserved charges
  - Asakawa, Heinz, Muller (’00); Jeon, Koch (’00); Ejiri, Karsch, Redlich (’06)
- Signs of higher order cumulants
  - Asakawa, Ejiri, MK (’09); Friman, et al. (’11); Stephanov (’11)
Conserved Charges: Theoretical Advantage

- Definite definition for operators
  - as a Noether current
  - calculable on any theory
  - ex: on the lattice
Conserved Charges: Theoretical Advantage

- Definite definition for operators
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  ex: on the lattice

- Simple thermodynamic relations

\[ \langle \delta N_{c}^{n} \rangle = \frac{1}{VT^{n-1}} \frac{\partial^{n} \Omega}{\partial \mu_{c}^{n}} \]

- Intuitive interpretation for the behaviors of cumulants

ex: \[ \langle \delta N_{B}^{3} \rangle = \frac{1}{VT^{2}} \frac{\partial \langle \delta N_{B}^{2} \rangle}{\partial \mu_{B}} \]

Asakawa, Ejiri, MK, 2009
Conserved Charge Fluctuations

\[
\frac{\langle \delta N_B^4 \rangle_c}{\langle \delta N_B^2 \rangle}
\]

Cumulants of \( N_B \) and \( N_Q \) are **suppressed** at high \( T \).

Asakawa, Heinz, Muller, 2000; Jeon, Koch, 2000; Ejiri, Karsch, Redlich, 2006; Asakawa, Ejiri, MK, 2009; Friman, et al., 2011; Stephanov, 2011
Proton # Cumulants @ STAR-BES

\[ \frac{C_4}{C_2} \]

\[ \frac{C_3}{C_1} = \frac{C_3/C_2}{\text{Poissonian}} \]

Something interesting??

CAUTION!
proton number ≠ baryon number

MK, Asakawa, 2011;2012
Charge Fluctuation @ LHC

ALICE, PRL110, 152301 (2013)

D-measure

\[ D = 4 \frac{\langle \delta N_Q^2 \rangle}{\langle N_Q^+ + N_Q^- \rangle} \]

- \( D \sim 3-4 \) Hadronic
- \( D \sim 1-1.5 \) Quark

significant suppression from hadronic value at LHC energy!

\( \langle \delta N_Q^2 \rangle \) is not equilibrated at freeze-out at LHC energy!
$\Delta \eta$ Dependence @ ALICE

- 0-5%
- 20-30%
- 40-50%

Function $\text{Erf}\left(\frac{\Delta \eta}{\sqrt{8} \sigma_f}\right)$

Extrapolation

Rapidity window

ALICE
PRL 2013
Variation of a conserved charge is achieved only through diffusion. The larger $\Delta \eta$, the slower diffusion.
$\Delta \eta$ Dependence @ ALICE

$\Delta \eta$ dependences of fluctuation observables encode history of the hot medium!
Cumulants: HIC@RHIC vs Lattice

parameter window constrained by lattice

HotQCD, LATTICE2013

fluctuations “exp + lattice”

μ/T

discrepancy

particle abundance (chem. freezeout T)
$\langle \delta N_B^2 \rangle$ and $\langle \delta N_p^2 \rangle$ @ LHC?

$\langle \delta N_Q^2 \rangle$, $\langle \delta N_B^2 \rangle$, $\langle \delta N_p^2 \rangle$
should have different $\Delta \eta$ dependence.

Baryon # cumulants are experimentally observable! MK, Asakawa, 2011;2012
$\langle \delta N_B^2 \rangle$ and $\langle \delta N^2 \rangle @ LHC$?

$\langle \delta N_Q^2 \rangle, \langle \delta N_B^2 \rangle, \langle \delta N_p^2 \rangle$ should have different $\Delta \eta$ dependence.

Baryon # cumulants are experimentally observable! MK, Asakawa, 2011;2012
How does $\langle \delta N_Q^4 \rangle_c$ behave as a function of $\Delta \eta$?

- suppression
- enhancement
Baryon vs Proton Number Fluctuations

MK, Asakawa, PRC85,021901C(2012); PRC86, 024904(2012)

\[
\frac{\langle \delta N_B^n \rangle_c}{\langle \delta N_B^m \rangle_c} \neq \frac{\langle \delta N_p^n \rangle_c}{\langle \delta N_p^m \rangle_c}
\]

\[
\langle \delta N_B^n \rangle_c \text{ are experimentally observable}
\]
Nucleons have two isospin states.
Nucleon Isospin as Two Sides of a Coin

Nucleons have two isospin states.

Coins have two sides.

MK, Asakawa, 2012
Slot Machine Analogy

\[ P_x(N) = \text{coin} + \text{tokens} \rightarrow N \]

\[ P_y(N) \rightarrow N \]
Extreme Examples

Fixed # of coins

Constant probabilities

\[ N \]
Reconstructing Total Coin Number

\[ P_{\text{\$100}}(N_{\text{\$100}}) = \sum P_{\text{\$100}}(N_{\text{\$100}}) B_{1/2}(N_{\text{\$100}}; N_{\text{\$100}}) \]

\[ B_p(k; N) = p^k (1 - p)^{N-k} \binom{N}{k} : \text{binomial distr. func.} \]
Nucleons in Hadronic Phase

hadronize
chem. f.o.
kinetic f.o.

10~20fm

$$m_\pi \simeq T \ll m_N - \mu_N$$

$$n_N \ll 1$$
- rare NN collisions
- no quantum corr.

$$n_N \ll n_\pi$$
- many pions

\[\begin{array}{ll}
p, \bar{p} & \text{mesons} \\
n, \bar{n} & \text{baryons} \\
\Delta(1232) & \end{array}\]
Isospin of baryons can vary after chemical freezeout via charge exchange reactions mediated by $\Delta(1232)$:

\[ p, n \xrightarrow{\pi} \Delta(1232) \xrightarrow{\pi} p, n \]

\[ I = \frac{3}{2} \]

\[ \Gamma \simeq 1.8 \text{ [fm]} \]

![Graph showing cross section](image)
$\Delta(1232)$

Cross sections of $\rho$

\[
\begin{align*}
p + \pi^+ & \rightarrow \Delta^{++} \rightarrow p + \pi^+ \\
p + \pi^0 & \rightarrow \Delta^+ \rightarrow p + \pi^0 \\
n + \pi^+ & \rightarrow \Delta^+ \rightarrow n + \pi^+ \\
p + \pi^- & \rightarrow \Delta^0 \rightarrow p + \pi^- \\
n + \pi^0 & \rightarrow \Delta^0 \rightarrow n + \pi^0 \\
n + \pi^- & \rightarrow \Delta^- \rightarrow n + \pi^-
\end{align*}
\]

Decay rates of $\Delta$
Δ(1232)

cross sections of $p$

$\begin{align*}
\Delta^+ & \quad \Delta^+ \rightarrow p + \pi^+ \\
\Delta^0 & \quad \Delta^0 \rightarrow p + \pi^- \\
\Delta^- & \quad \Delta^- \rightarrow n + \pi^- \\
\Delta^+ & \quad \Delta^+ \rightarrow p + \pi^0 \\
\Delta^0 & \quad \Delta^0 \rightarrow n + \pi^0 \\
\Delta^- & \quad \Delta^- \rightarrow n + \pi^- \\
\end{align*}$

decay rates of Δ

$p + \pi \rightarrow \Delta^{+,0}$

$\rightarrow p : n$

$= 5 : 4$
$$\Delta(1232)$$

**Cross sections of p**

$$p + \pi^+ \rightarrow \Delta^{++} \rightarrow p + \pi^+$$

$$p + \pi^0 \rightarrow \Delta^+ \rightarrow p + \pi^0$$

$$n + \pi^+ \rightarrow \Delta^+ \rightarrow n + \pi^+$$

$$p + \pi^- \rightarrow \Delta^0 \rightarrow p + \pi^-$$

$$n + \pi^0 \rightarrow \Delta^0 \rightarrow n + \pi^0$$

$$n + \pi^- \rightarrow \Delta^- \rightarrow n + \pi^-$$

**Decay rates of Δ**

$$p + \pi \rightarrow \Delta^{+,0} \rightarrow p : n$$

$$= 5 : 4$$

**Lifetime to create Δ⁺ or Δ⁰**

$$\tau^{-1} = \int \frac{d^3 k_\pi}{(2\pi)^3} \sigma(E_{cm}) v_\pi n(E_\pi)$$

$$\simeq 20[\text{fm}]$$

**BW for σ & thermal pion**

$$T[\text{MeV}]$$
(1) $N_B^{(\text{net})} = N_B - N_{\bar{B}}$ deviates from the equilibrium value.

(2) Boltzmann (Poisson) distribution for $N_B, N_{\bar{B}}$.

\[ 2\langle (\delta N_p^{(\text{net})})^2 \rangle = \frac{1}{2} \langle (\delta N_B^{(\text{net})})^2 \rangle + \frac{1}{2} \langle (\delta N_B^{(\text{net})})^2 \rangle_{\text{free}} \]
\[ 2\langle (\delta N_p^{(\text{net})})^3 \rangle = \frac{1}{4} \langle (\delta N_B^{(\text{net})})^3 \rangle + \frac{3}{4} \langle (\delta N_B^{(\text{net})})^3 \rangle_{\text{free}} \]
\[ 2\langle (\delta N_p^{(\text{net})})^4 \rangle_c = \frac{1}{8} \langle (\delta N_B^{(\text{net})})^4 \rangle_c + \cdots \]

Difference from Poisson (thermal) distribution is suppressed in proton number fluctuations.
Difference btw Baryon and Proton Numbers

\( (1) \quad N_B^{(\text{net})} = N_B - N_{\bar{B}} \) deviates from the equilibrium value.

\( (2) \) Boltzmann (Poisson) distribution for \( N_B, N_{\bar{B}} \).

\[
2\langle (\delta N_p^{(\text{net})})^2 \rangle = \frac{1}{2} \langle (\delta N_B^{(\text{net})})^2 \rangle + \frac{1}{2} \langle (\delta N_B^{(\text{net})})^2 \rangle_{\text{free}}
\]

\[
2\langle (\delta N_p^{(\text{net})})^3 \rangle = \frac{1}{4} \langle (\delta N_B^{(\text{net})})^3 \rangle + \frac{3}{4} \langle (\delta N_B^{(\text{net})})^3 \rangle_{\text{free}} + \cdots
\]

Genuine info. Poissonian noise

from Poisson (thermal) distribution suppressed in proton number fluctuations.
Time Evolution of Higher Order Cumulants

MK, Asakawa, Ono, arXiv:1307.2978
How does \( \langle \delta N_Q^4 \rangle_c \) behave as a function of \( \Delta \eta \)? 

- suppression 
- enhancement
Hydrodynamic Fluctuations

Stochastic diffusion equation

\[ \partial_\tau n = D \partial_\eta^2 n + \partial_\eta \xi(\eta, \tau) \]

Fluctuation of \( n \) is Gaussian in equilibrium

Markov (white noise) + continuity

Gaussian noise

cf) Gardiner, “Stochastic Methods”

Landau, Lifshitz, Statistical Mechanics II
Kapusta, Muller, Stephanov, 2012
Stephanov, Shuryak, 2001
How to Introduce Non-Gaussianity?

**Stochastic diffusion equation**

\[ \partial_\tau n = D \partial_\eta^2 n + \partial_\eta \xi(\eta, \tau) \]

- Choices to introduce non-Gaussianity in equil.:
  - \( n \) dependence of diffusion constant \( D(n) \)
  - colored noise
  - discretization of \( n \)
How to Introduce Non-Gaussianity?

**Stochastic diffusion equation**

\[ \partial_\tau n = D \partial_\eta^2 n + \partial_\eta \xi(\eta, \tau) \]

- Choices to introduce non-Gaussianity in equil.:
  - \( n \) dependence of diffusion constant \( D(n) \)
  - Colored noise
  - Discretization of \( n \) \( \rightarrow \) our choice

**Remark:** Fluctuations measured in HIC are almost Poissonian.
Diffusion Master Equation

Divide spatial coordinate into discrete cells

\[ \cdots \quad n_{x-1} \quad n_x \quad n_{x+1} \quad n_{x+2} \quad \cdots \]

Probability \[ P(n) \]
Divide spatial coordinate into discrete cells

Master Equation for $P(n)$

$$\frac{\partial}{\partial t} P(n) = \gamma \sum_x [(n_x + 1) \{P(n + e_x - e_{x+1}) + P(n + e_x - e_{x-1})\} - 2n_x P(n)]$$

Solve the DME exactly, and take $a \rightarrow 0$ limit

No approx., ex. van Kampen’s system size expansion
Baryons in Hadronic Phase

Baryons behave like Brownian pollens in water
Prepare 2 species of (non-interacting) particles

\[ \bar{Q}(\tau) = \int_0^{\Delta \eta} d\eta \left( n_1(\eta, \tau) - n_2(\eta, \tau) \right) \]

Let us investigate \( \langle \bar{Q}^2 \rangle_c \) and \( \langle \bar{Q}^4 \rangle_c \) at freezeout time \( t \)
Solution of DME in a $a \to 0$ Limit

1st order (deterministic) $\langle n \rangle$
- consistent with diffusion equation with $D = \gamma a^2$
- Continuum limit with fixed $D = \gamma a^2$

2nd order $\langle \delta n^2 \rangle$
- consistent with stochastic diffusion eq.
  (for sufficiently smooth initial conditions)

Shuryak, Stephanov, 2001

Nontrivial results for non-Gaussian fluctuations
Time Evolution in Hadronic Phase

Hadronization (initial condition)

- Boost invariance / infinitely long system
- Local equilibration / local correlation

\[
\langle \vec{Q}^2 \rangle_c, \langle \vec{Q}^4 \rangle_c, \langle \vec{Q}^2 Q_{(tot)} \rangle_c, \langle Q_{(tot)}^2 \rangle_c
\]

- Suppression owing to local charge conservation
- Strongly dependent on hadronization mechanism
Time Evolution in Hadronic Phase

Hadronization (initial condition)

- Boost invariance / infinitely long system
- Local equilibration / local correlation

\[
\langle \bar{Q}^2 \rangle_c, \langle \bar{Q}^4 \rangle_c, \langle \bar{Q}^2 Q_{(tot)} \rangle_c, \langle Q_{(tot)}^2 \rangle_c
\]

suppression owing to local charge conservation

strongly dependent on hadronization mechanism

Freezeout
In recombination model,

\[ N_B^{(\text{net})} = 0 \]
\[ N_B^{(\text{tot})} = 4 \]

\[ N_B^{(\text{net})} = 0 \]
\[ N_B^{(\text{tot})} = 0 \]

\( N_B^{(\text{tot})} \) can fluctuate, while \( N_B^{(\text{net})} \) does not.
Initial fluctuations:
\[ \langle \bar{Q}^2 \rangle_c = \langle \bar{Q}^4 \rangle_c = \langle \bar{Q}^2 Q_{(tot)} \rangle_c = 0 \]

\[ c = \frac{\langle Q_{(net)}^2 \rangle_c}{\langle Q_{(tot)} \rangle_c} \]

parameter sensitive to hadronization
Dependence at Freezeout

Initial fluctuations:

\[
\langle \bar{Q}^2 \rangle_c = \langle \bar{Q}^4 \rangle_c = \langle \bar{Q}^2 Q_{\text{tot}} \rangle_c = 0.5 \langle Q_{\text{tot}} \rangle
\]

\[
c = \frac{\langle Q_{\text{net}}^2 \rangle_c}{\langle Q_{\text{tot}} \rangle_c} \quad \frac{\langle Q_{\text{net}}^4 \rangle_c}{\langle Q_{\text{tot}} \rangle}
\]

Parameter sensitive to hadronization.
\langle \delta N_Q^4 \rangle @ LHC

Assumptions
\begin{itemize}
  \item boost invariant system
  \item small fluctuations of CC at hadronization
  \item short correlation in hadronic stage
\end{itemize}

4th-order cumulant will be suppressed at LHC energy!

\[ \Delta \eta \text{ dependences encode various information on the dynamics of HIC!} \]
The ratio $\frac{\langle \delta N_p^4 \rangle_c}{\langle \delta N_p^2 \rangle}$ decreases as $\Delta \eta$ becomes larger at RHIC energy.
OUR SUGGESTIONS:

- Plot $\langle \delta N^2 \rangle$ and $\langle \delta N^4 \rangle$ separately
- Plot baryon number cumulants
Global Charge Conservation

Solve SDE or DME in a finite volume

- Effect of GCC can be read off from $\Delta \eta$ dependence.
- No GCC effect in ALICE experiments!

Effect of a boundary appears only in the range of diffusion length

\[ \frac{\langle \delta N^2 \rangle(\eta)}{\langle \delta N^2 \rangle(0)} \]

\[ \frac{\Delta \eta}{2\sqrt{D\tau}} \]
Fluctuations in HIC are nonthermal!

Physical meanings of fluctuation obs. in experiments.

Plenty of physics in $\Delta \eta$ dependences of various cumulants

$\langle N_Q^2 \rangle_c$, $\langle N_B^2 \rangle_c$, $\langle N_Q^4 \rangle_c$, $\langle N_B^4 \rangle_c$, $\langle N_{ch}^2 \rangle_c$, ...
Summary

Fluctuations in HIC are nonthermal!

- Plenty of physics in $\Delta \eta$ dependences of various cumulants
  - $\langle N_Q^2 \rangle_c$, $\langle N_B^2 \rangle_c$, $\langle N_Q^4 \rangle_c$, $\langle N_B^4 \rangle_c$
  - $\langle N_{ch}^2 \rangle_c$, $\cdots$

- Physical meanings of fluctuation obs. in experiments.

- Diagnosing dynamics of HIC
  - history of hot medium
  - mechanism of hadronization
  - diffusion constant

Search of QCD Phase Structure in HIC
Open Questions & Future Work

- Why the primordial fluctuations are observed only at LHC, and not RHIC?
- Extract more information on each stage of fireballs using fluctuations

- Model refinement
  - Including the effects of nonzero correlation length / relaxation time
  - Global charge conservation

- Non Poissonian system ← interaction of particles
Chemical Reaction 1

$$X \xrightarrow{k_1 \quad k_2} A$$

$x$: # of $X$
$a$: # of $A$ (fixed)

Master eq.:

$$\frac{\partial}{\partial t} P(x, t) = k_2 a P(x - 1, t) + k_1 (x + 1) P(x + 1, t) - (k_1 x + k_2 a) P(x, t)$$

Cumulants with fixed initial condition $P(x, 0) = \delta_{x,N_0}$

$$\langle x(t) \rangle = N_0 e^{-k_1 t} + N_{eq}(1 - e^{-k_1 t})$$

$$\langle \delta x(t)^2 \rangle = N_0 (e^{-k_1 t} - e^{-2k_1 t}) + N_{eq}(1 - e^{-k_1 t})$$

$$\langle \delta x(t)^3 \rangle = N_0 (e^{-k_1 t} - 3e^{-2k_1 t} + 2e^{-3k_1 t}) + N_{eq}(1 - e^{-k_1 t})$$
Chemical Reaction 2

\[ X \xrightarrow{k_1} A \xleftarrow{k_2} \]

\[ N_0 = N_{eq} \]

\[
\begin{align*}
\langle x(t) \rangle &= N_{eq} \\
\langle \delta x(t)^2 \rangle &= N_{eq}(1 - e^{-2k_1 t}) \\
\langle \delta x(t)^3 \rangle &= N_{eq}(1 - 3e^{-2k_1 t} + 2e^{-3k_1 t})
\end{align*}
\]

Higher-order cumulants grow slower.
Time Evolution in HIC

Quark-Gluon Plasma

Hadronization

Freezeout

\[ \langle \Delta N^2 \rangle \]

\[ \Delta \eta \]

\[ \chi_{\text{HAD}} \]
\[ \chi_{\text{QGP}} \]

\[ \Delta \eta \]

\[ \chi_{\text{HAD}} \]
\[ \chi_{\text{QGP}} \]

\[ \Delta \eta \]
Hydrodynamic Fluctuations

Diffusion equation

\[ \partial_\tau n = D \partial^2_\eta n \]

Stochastic diffusion equation

\[ \partial_\tau n = D \partial^2_\eta n + \partial_\eta \xi(\eta, \tau) \]

Stochastic Force
determined by fluctuation-dissipation relation

Landau, Lifshitz, Statistical Mechanics II
Kapusta, Muller, Stephanov, 2012
$$\Delta \eta$$ Dependence

- Initial condition: $$\langle \delta n(\eta_1, 0) \delta n(\eta_2, 0) \rangle = \sigma_2 \delta(\eta_1 - \eta_2)$$
- Translational invariance

$$Q(\tau) = \int_0^{\Delta \eta} d\eta n(\eta, \tau)$$

$$\langle \delta Q(\tau)^2 \rangle = \sigma_2 F_2(X) + \chi_2(1 - F_2(X))$$

Shuryak, Stephanov, 2001
It is **impossible** to directly extend the theory of hydro fluctuations to treat higher orders.

- No a priori extension of FD relations to higher orders
- **Theorem**
  - Markov process + continuous variable
  - ➔ Gaussian random force

*cf* Gardiner, “Stochastic Methods”
Event-by-Event Analysis @ HIC

Fluctuations can be measured by e-by-e analysis in HIC.

STAR, PRL105 (2010)

\[ \langle \delta N^2 \rangle, \langle \delta N^3 \rangle, \langle \delta N^4 \rangle_{c}, \ldots \]
Non-Gaussianity

fluctuations (correlations)

\[ \langle \delta n_1 \delta n_2 \rangle, \langle \delta n_1 \delta n_2 \delta n_3 \rangle, \langle \delta n_1 \delta n_2 \delta n_3 \delta n_4 \rangle_c, \ldots \]

→ Non-Gaussianity

PLANCK: statistics insufficient to see non-Gaussianity…(2013)