Spin ice dynamics :
generic vertex models

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Plan & summary

- **Brief introduction** to classical frustrated magnetism.

  $2d$ **spin-ice samples and the 16 vertex model**.

  Exact results for the **statics** of the 6 and 8 vertex models with integrable systems methods. Very little is known for the **dynamics**.

- **Our work :**

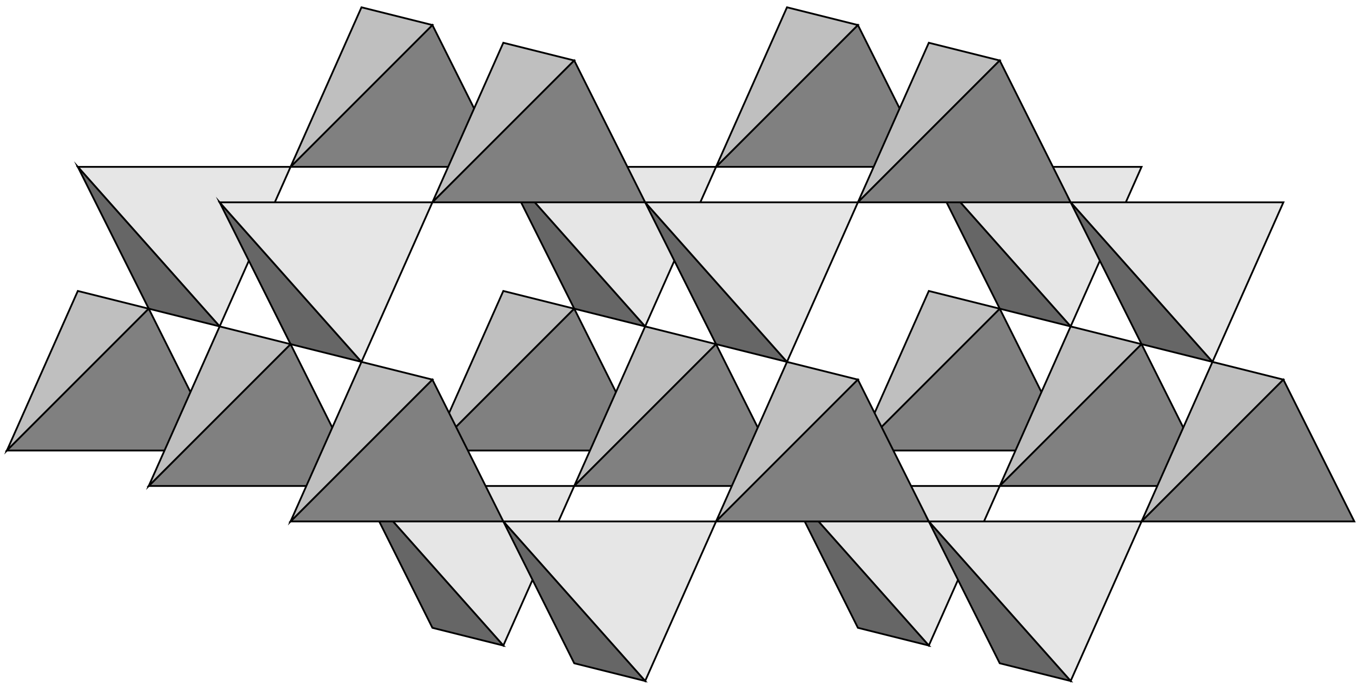
  Phase diagram of the generic model. Monte Carlo and Bethe-Peierls.

  Stochastic dissipative dynamics after quenches into the D, AF and FM phases. Metastability & growth of order in the AF and FM phases Monte Carlo simulations & dynamic scaling.

  Explanation of measurements in **as-grown artificial spin ice**.

Natural spin-ice

$3d$: the pyrochlore lattice

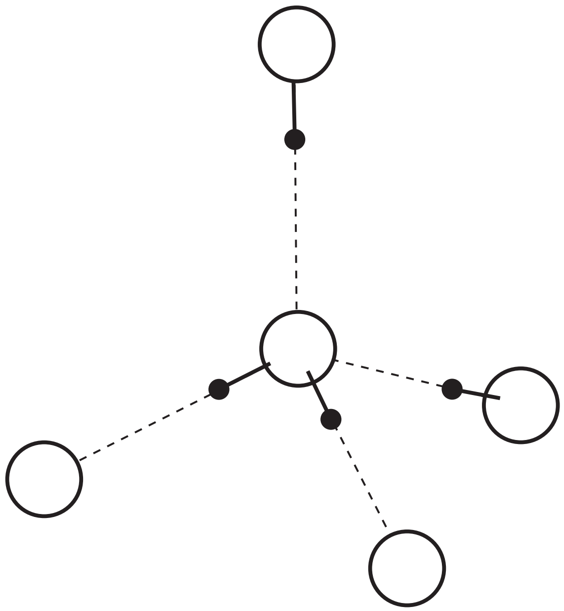


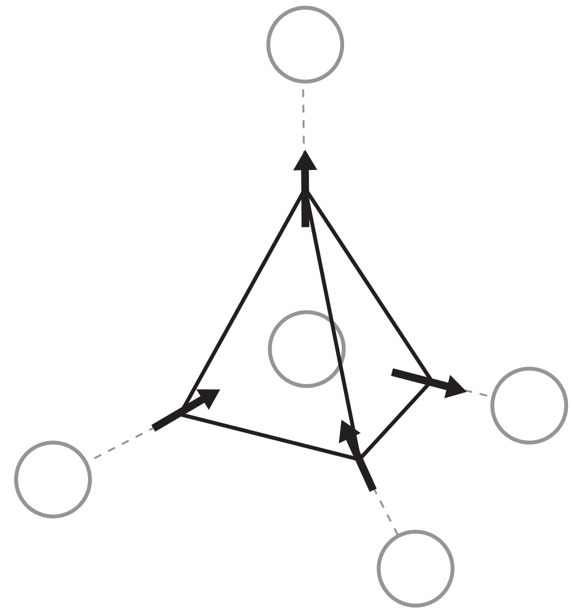
Coordination four lattice of corner linked tetahedra. The rare earth ions occupy the vertices of the tetrahedra ; **e.g. Dy$_2$ Ti$_2$ O$_7$**

Harris, Bramwell, McMorrow, Zeiske & Godfrey 97

Single unit

Water-ice and spin-ice



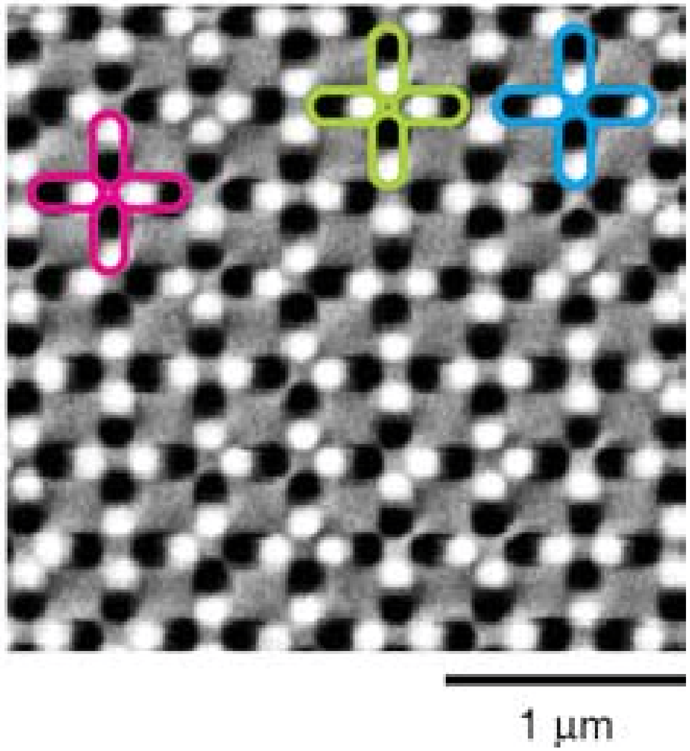


Water-ice : coordination four lattice. **Bernal & Fowler** rules, two H near and two far away from each O.

Spin-ice : four (Ising) spins on each tetrahedron forced to point along the axes that join the centers of two neighboring units (Ising anisotropy). Interactions imply the two-in two-out ice rule.

Artificial spin-ice

Bidimensional square lattice of elongated magnets



Bidimensional square lattice

Dipoles on the edges

Long-range interactions

16 possible vertices

Experimental conditions in this fig. :

vertices w/ two-in & two-out arrows

with staggered AF order

are much more numerous

AF 3in-1out FM

Wang *et al* **06, Nisoli** *et al* **10, Morgan** *et al* **12**

Square lattice artificial spin-ice

Local energy approximation $\Rightarrow 2d$ 16 vertex model

Just the interactions between dipoles attached to a vertex are added.

Dipole-dipole interactions. Dipoles are modeled as two opposite charges. Each vertex is made of 8 charges, 4 close to the center, 2 away from it. The energy of a vertex is the electrostatic energy of the eight charge configuration. With a convenient normalization, dependence on the lattice spacing ℓ :

$$\epsilon_{AF} = \epsilon_5 = \epsilon_6 = (-2\sqrt{2}+1)/\ell \qquad \epsilon_{FM} = \epsilon_1 = \dots = \epsilon_4 = -1/\ell$$

$$\epsilon_e = \epsilon_9 = \dots \epsilon_{16} = 0 \qquad \epsilon_d = \epsilon_7 = \epsilon_8 = (4\sqrt{2}+2)/\ell$$

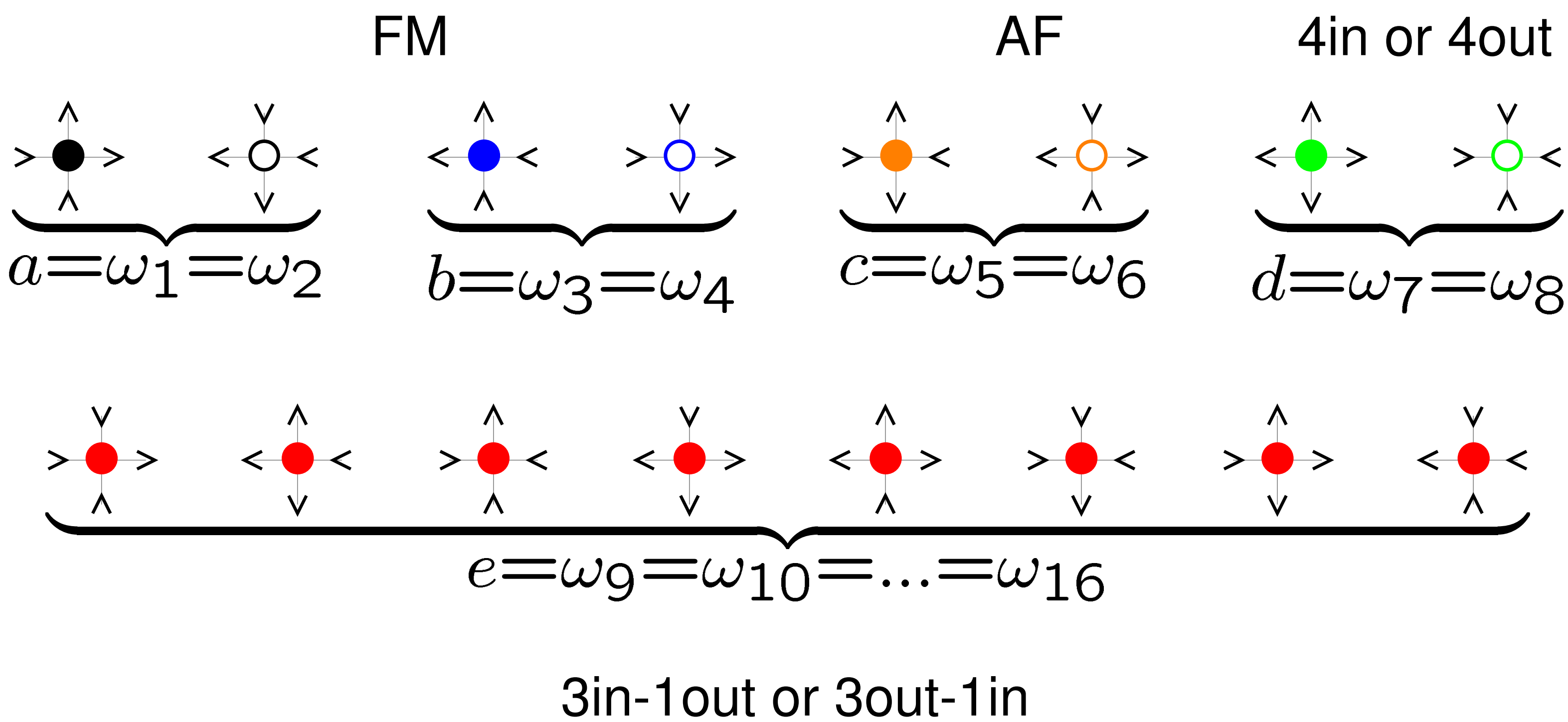
$$\boxed{\epsilon_{AF} < \epsilon_{FM} < \epsilon_e < \epsilon_d}$$

Nisoli *et al* 10

Energy could be tuned differently by adding fields, vertical off-sets, etc.

The $2d$ 16 vertex model

with 3-in 1-out vertices : non-integrable system



(Un-normalized) statistical weight of a vertex $\omega_k = e^{-\beta\epsilon_k}$.

In the model a, b, c, d, e are free parameters (usually, c is the scale).

In the experiments ϵ_k are fixed and β is the control parameter.

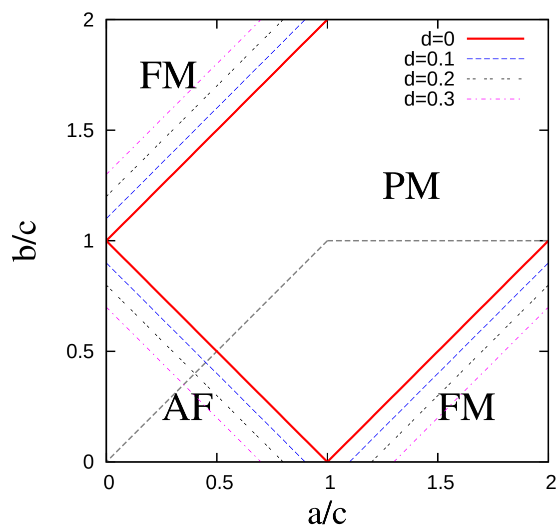
The vertex energies ϵ_k are estimated as explained above.

Static properties

What did we know ?

- **6 and 8 vertex models.**

  Integrable systems techniques (transfer matrix + Bethe Ansatz), mappings to many physical (e.g. quantum spin chains) and mathematical problems.



Phase diagram

critical exponents

ground state entropy

boundary conditions

etc.

Lieb 67 ; Baxter *Exactly solved models in statistical mechanics* **82**

- **16 vertex model.**

  Integrability is lost. Not much interest so far.

Static properties

What did we do ?

- Equilibrium simulations with finite-size scaling analysis.
  - Continuous time Monte Carlo.

    e.g. focus on the **AF-PM transition** ; cfr. experimental data.

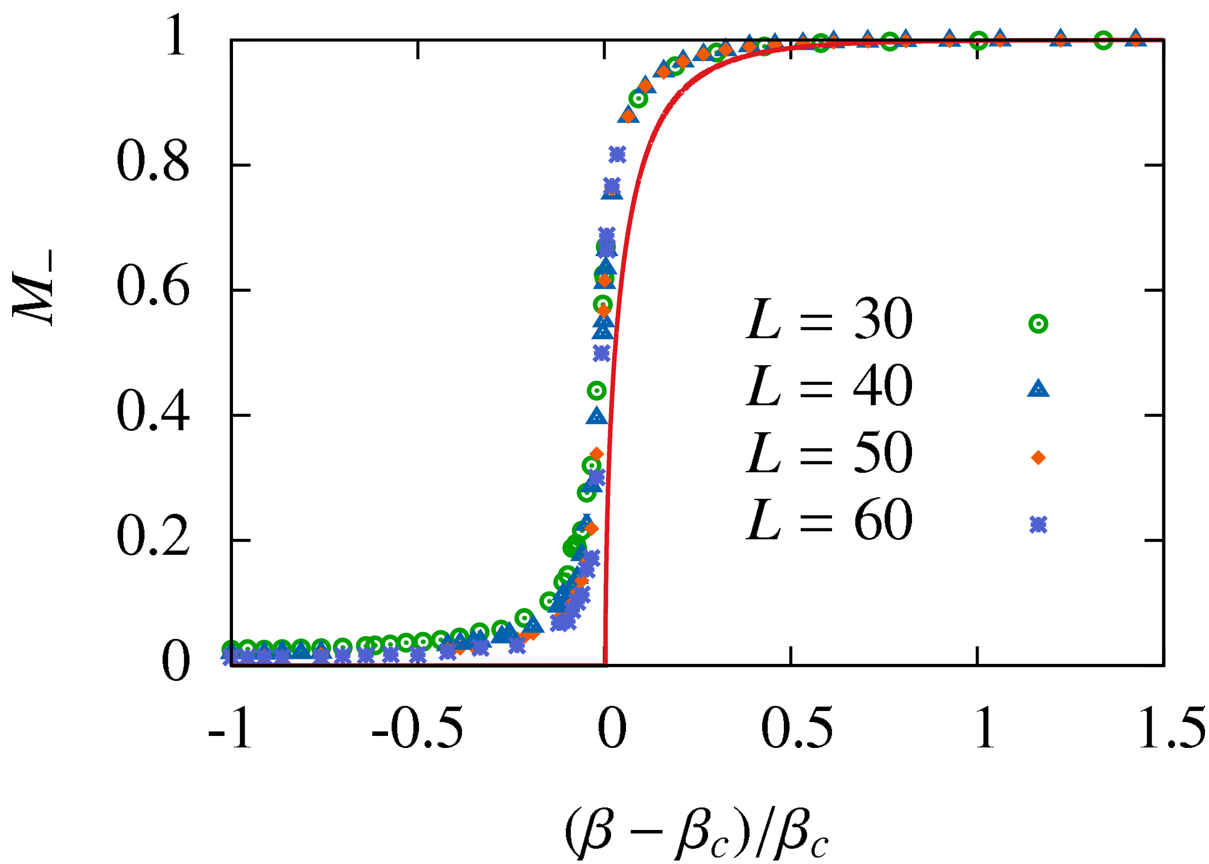
    AF order parameter : $\boxed{M_- = \frac{1}{2}\left(\langle|m_-^x|\rangle + \langle|m_-^y|\rangle\right)}$

    with $m_-^{x,y}$ the staggered magnetization along the x and y axes.
  - Finite-time relaxation $\boxed{M_-(t) \simeq t^{-\beta/(\nu z_c)}}$
- Cavity Bethe-Peierls mean-field approximation.
  - The model is defined on a tree of single vertices or 4-site plaquettes

Equilibrium CTMC

Magnetization across the PM-AF transition

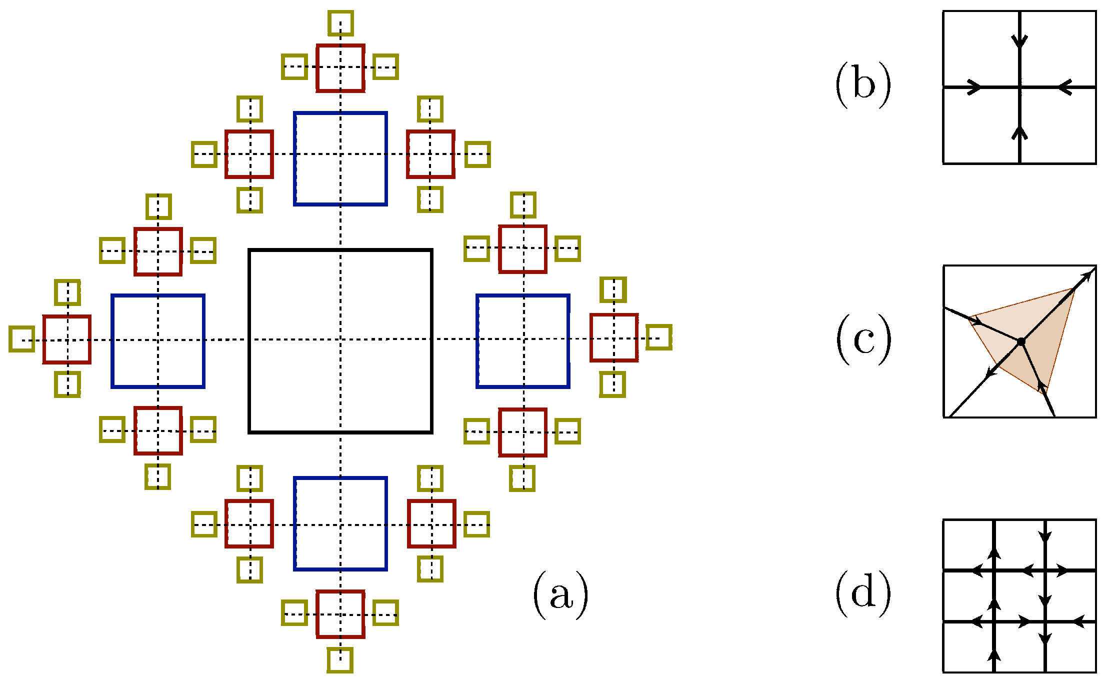
Vertex energies set to the values explained above.



Solid red line from the Bethe-Peierls calculation.

Equilibrium analytic

Bethe-Peierls or cavity method



Join an L-rooted tree from the left ; an U-rooted tree from above ;

an R-rooted tree from the right and a D-rooted tree from below.

Foini, Levis, Tarzia & LFC 12

is it a powerful technique ?

in, e.g., the 6 vertex model

With a tree in which the unit is a **vertex** we find the PM, FM, and AF phases.

$s_{PM} = \ln[(a+b+c)/(2c)]$

Pauling's entropy $s_{PM} = \ln 3/2 \sim 0.405$ at the spin-ice point $a = b = c$.

Location and 1st order transition between the PM and FM phases. ✔

Location ✔ but *1st order* PM-AF transition. ✘

no fluctuations in the frozen FM phase. ✔

no fluctuations in the AF phase. ✘

With a **four site plaquette** as a unit we find the PM, FM, and AF phases.

A more complicated expression for $s_{PM}(a,b,c)$ that yields

$s_{PM} \simeq 0.418$ closer to Lieb's entropy $s_{PM} \simeq 0.431$ at the spin-ice point.

Location and 1st order transition between the PM and FM phases. ✔

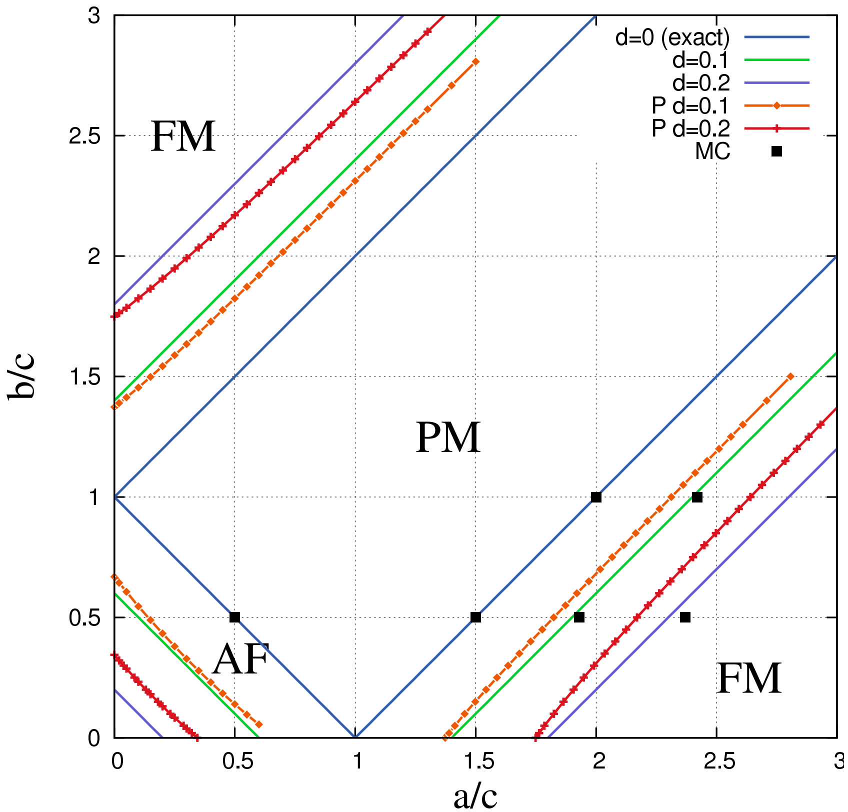
Location ✔ but *2nd order* (should be BKT) PM-AF transition. ✘

fluctuations in the AF phase and frozen FM phase. ✔

Static properties

Equilibrium phase diagram 16 vertex model

- MC simulations & cavity Bethe-Peierls method



Phase diagram

critical exponents

ground state entropy

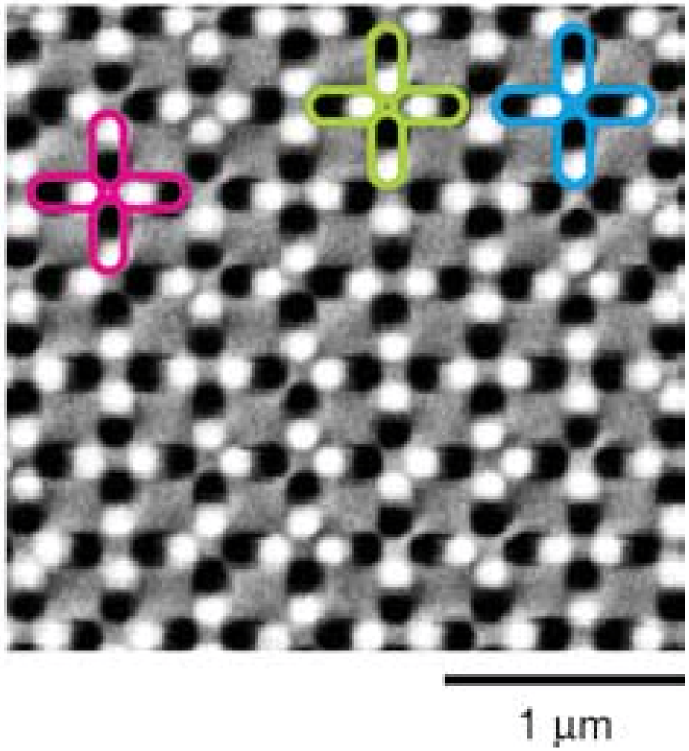
equilibrium fluctuations

etc.

Foini, Levis, Tarzia & LFC 12

Artificial spin-ice

Bidimensional square lattice of elongated magnets



Bidimensional square lattice

Magnetic material poured on edges

Magnets flip while they are small

& freeze when they reach some size

(analogy w/granular matter)

Magnetic force microscopy

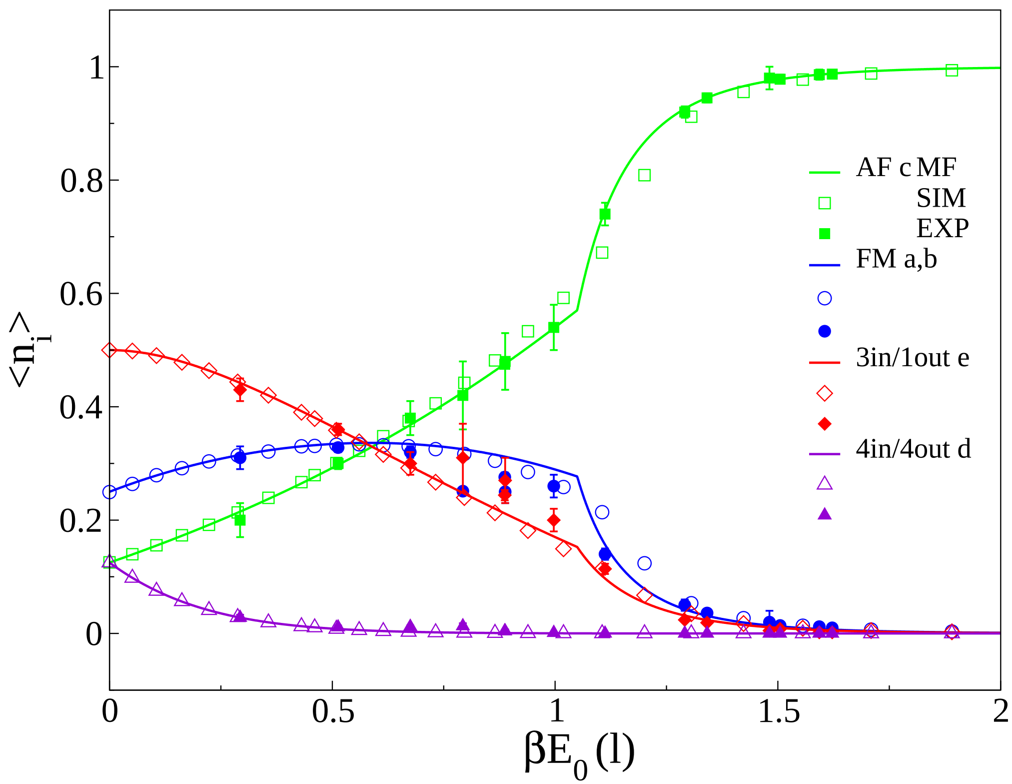
Images : vertex configurations

AF 3in-1out FM

Morgan *et al* **12** (UK collaboration)

Vertex density

Across the PM-AF transition – numerical, analytic and exp. data



PM - AF transition

AF vertices

FM vertices

3in-1out 3out-1in e-vert.

4in or 4out d-vertices

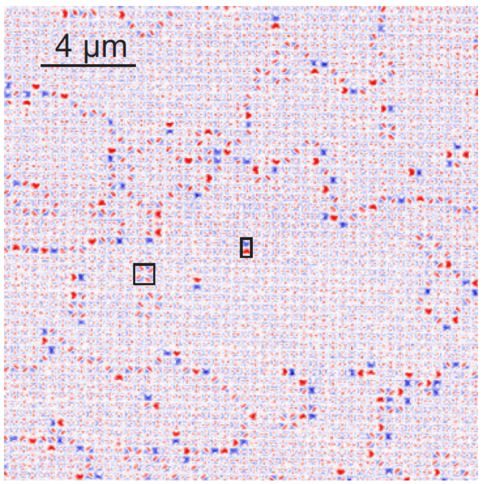
Each set of vertical points, $\beta E_0(\ell)$ value, corresponds to a different sample (varying lattice spacing ℓ or the compound). $1/\beta$ is the working temperature.

Levis, LFC, Foini & Tarzia 13 ; Experimental data courtesy of **Morgan** *et al.* **12**

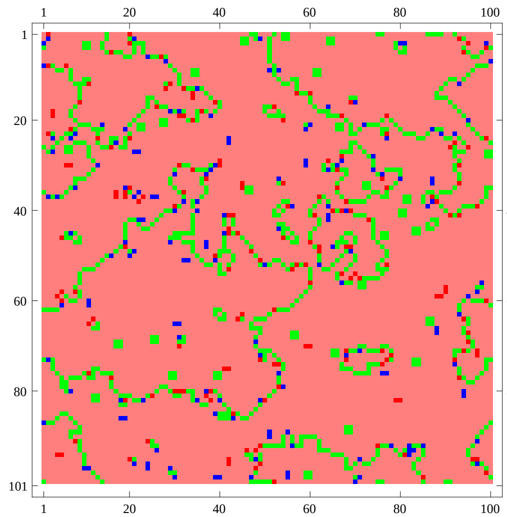
Artificial spin-ice

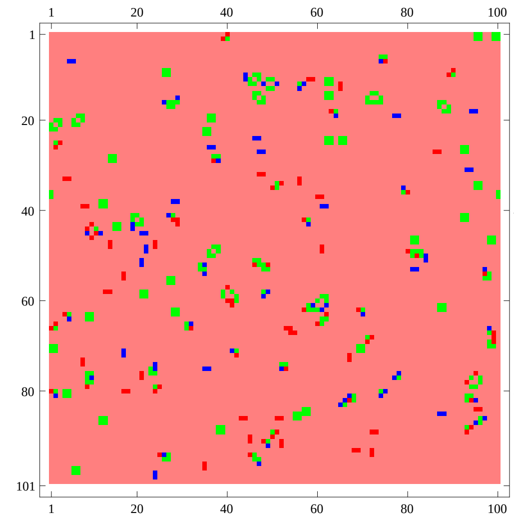
As-grown samples : in equilibrium at β or not ?

Magnetic force microscopy



Simulations





$t_1 \quad < \quad t_2$

Out of equilibrium — In equilibrium

A statistical and geometric analysis of domain walls should be done to conclude, especially for samples close to the transition.

Research project with F. Romà

Quench dynamics

Setting

- Take an initial condition in equilibrium at a_0, b_0, c_0, d_0, e_0.

  We used $a_0 = b_0 = c_0 = d_0 = e_0 = 1$ that corresponds to $\boxed{T_0 \to \infty}$

- We evolve it with a set of parameters a, b, c, d, e in the phases PM, FM, AF : an infinitely rapid quench at $t = 0$.

- We use **stochastic dynamics**.

  We update the vertices with the usual heat-bath rule,

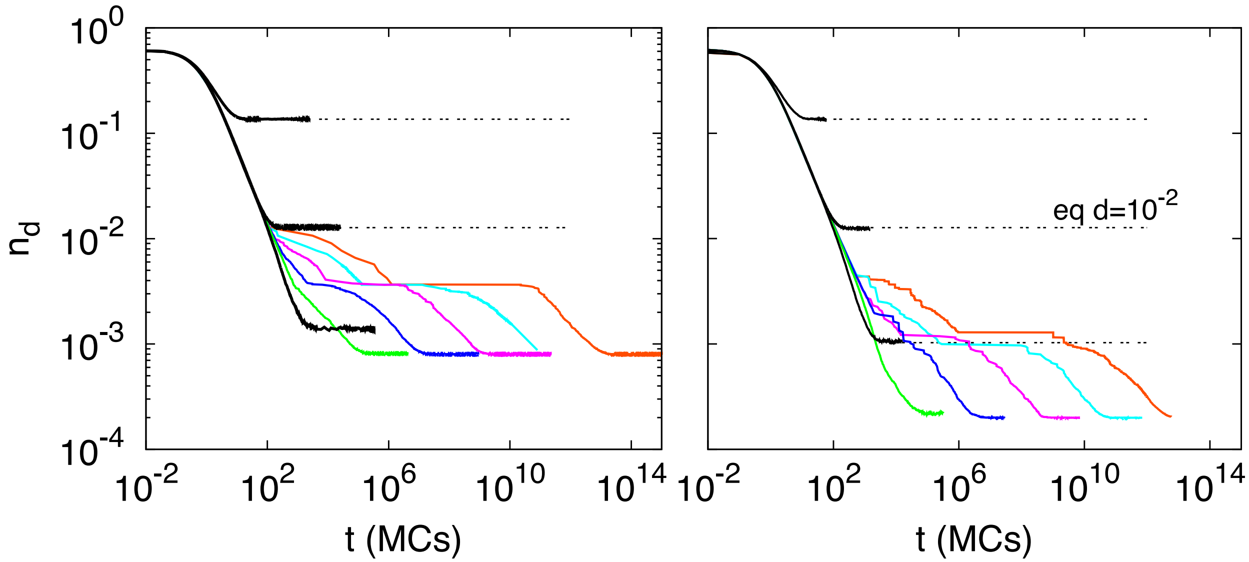
  we implement a **continuous time MC algorithm** to reach long time scales.

  Relevant dynamics experimentally (contrary to loop updates used to study equilibrium in the 8 vertex model)

Levis & LFC 11, 13

Dynamics in the PM phase

MeDensity of defects, $n_d = \#$defects$/\#$vertices



Relevant experimental sizes $L = 50$ $L = 100$

$a = b = c,\ d/c = e/c = 10^{-1}, 10^{-2}, \dots, 10^{-8}$ from left to right.

For $e = d \gtrsim 10^{-4}c$ the density of defects reaches its equilibrium value.

For $e = d \lesssim 10^{-4}c$ the density of defects gets blocked at $n_d \approx 10/L^2$.

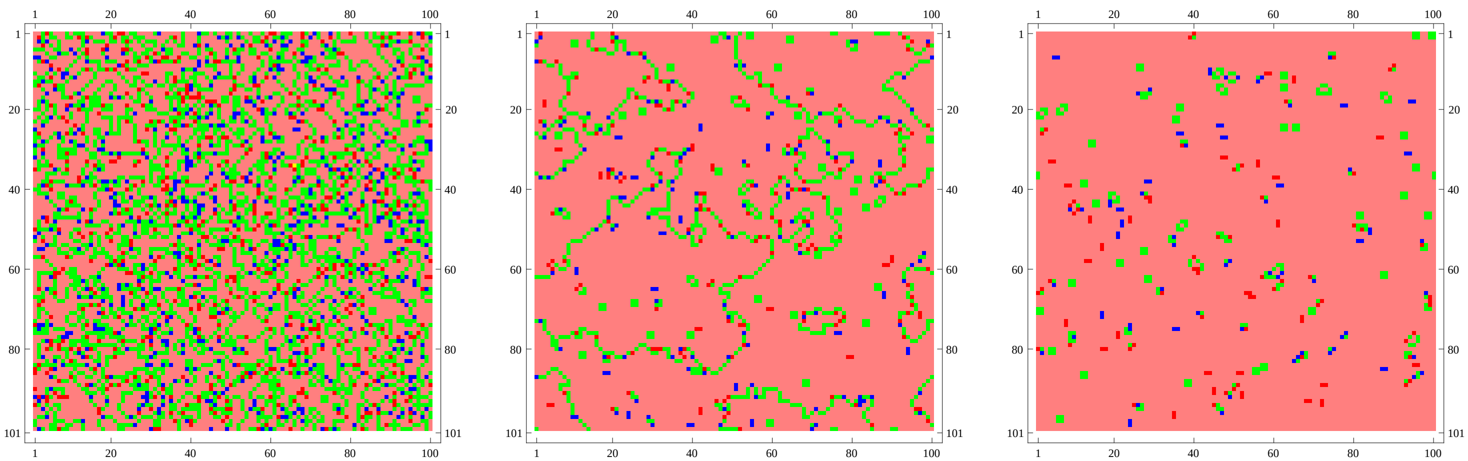
It eventually approaches the final value $n_d \approx 2/L^2$ indep. of bc ; rough estimate for t_{eq} from reaction-diffusion arguments.

Dynamics in the AF phase

Snapshots

Color code. Orange background : AF order of two kinds ; green FM vertices, red-blue defects.

Initial state | coarsening state | equilibrium state



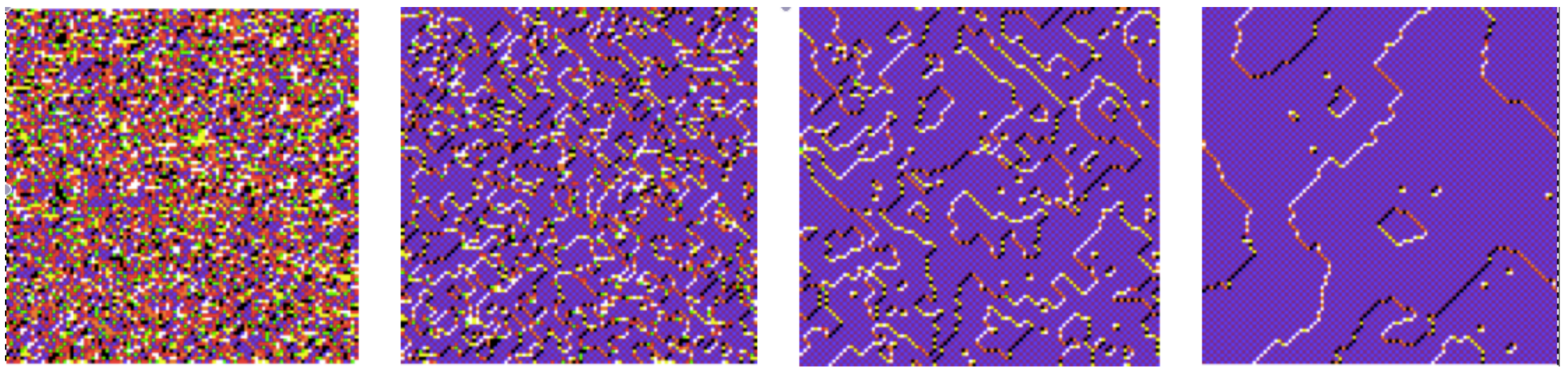
Isotropic growth of AF order for this choice of parameters

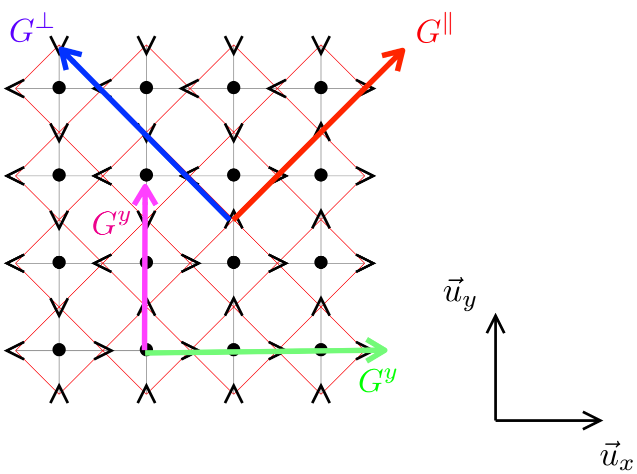
$\boxed{c \gg a = b}$ AF vertices are energetically preferred ;

there is no imposed anisotropy.

Dynamics in the AF phase

Snapshots, correlation functions & growing length



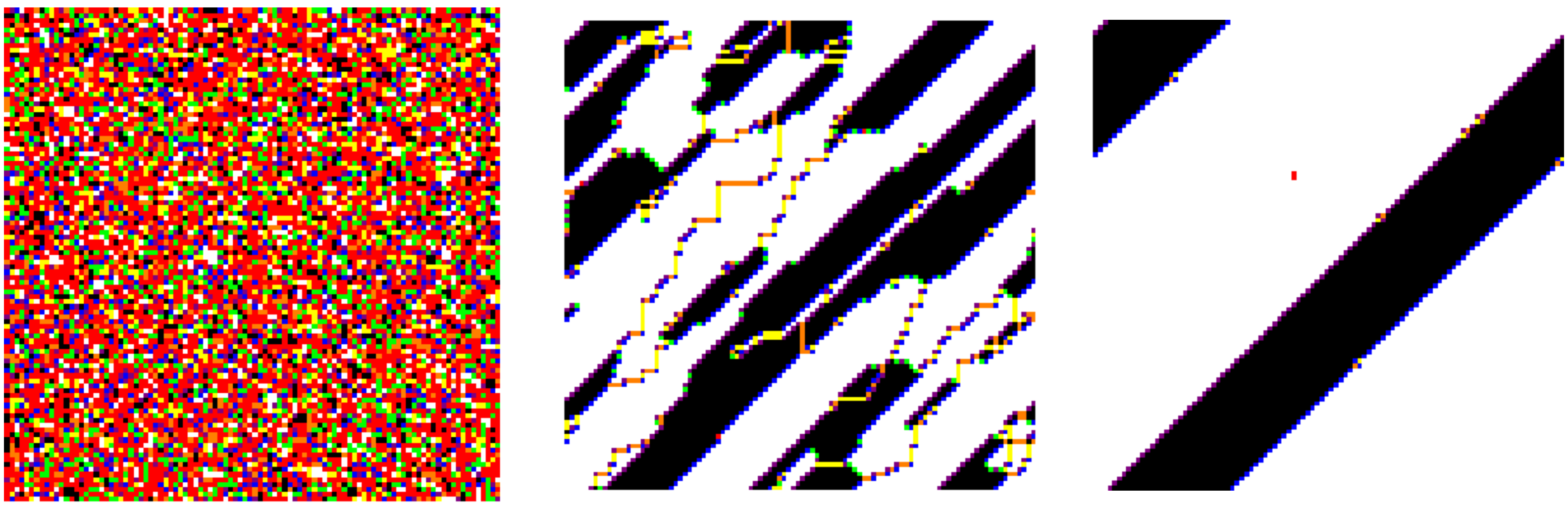


Scaling of correlation functions along the $\parallel$ and $\perp$ directions

$$\boxed{L(t) \simeq t^{1/2}}$$

Dynamics in the FM phase

Snapshots

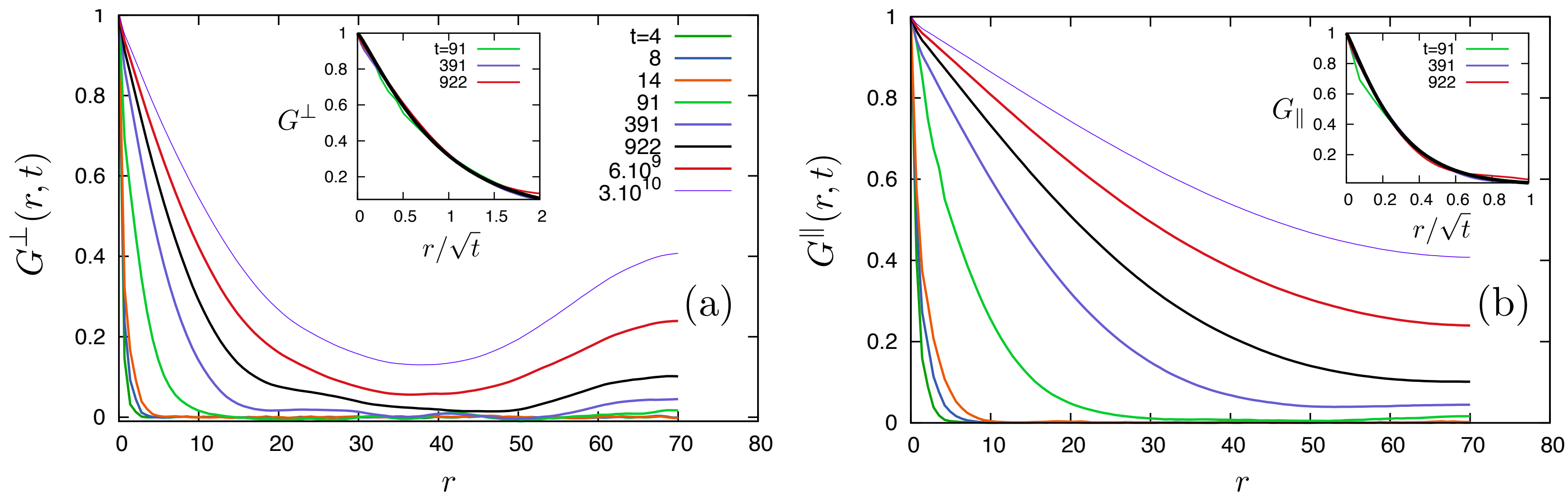


Growth of stripes

Quench to a large a value : black & white vertices energetically favored.

Dynamics in the FM phase

Dynamic scaling and growing lengths



$$G^{\perp}(r,t),\ G^{\parallel}(r,t) \simeq F_{\parallel,\perp}(r/L(t))$$

Stretched exponential $F(x) = e^{-(x/w)^v}$ with $v_{\parallel} \simeq v_{\perp} \simeq 0.15$ and $\neq w_{\parallel,\perp}$

the same growing length $\boxed{L_{\parallel}(t),\ L_{\perp}(t) \simeq t^{1/2}}$

until a band crosses the sample, then a different mechanism.

Summary

Classical frustrated magnetism ; spin-ice in two dimensions.

— The $2d$ **16 vertex model** : a problem with analytic, numeric and experimental interest. **Cfr. artificial spin-ice**

- Beyond integrable systems' methods to describe the static properties.

  — Some results of the Bethe-Peierls approximation are exact, others are at least extremely accurate. **Analytic challenge**

- Slow coarsening (or near critical in PM) dynamics.

  Stripes of growing ferromagnetic order in the **FM** phase, isotropic **AF** growth for $a = b$, with the same growing length and scaling functions but different parameters ;

  $L_{\parallel}^{\rm FM}(t) \simeq L_{\perp}^{\rm FM}(t) \simeq L^{\rm AF}(t) \simeq t^{1/2}$ **Analytically ?**

  Dynamics blocked in striped states later.

Equilibrium : the tree vs 2d

16 vertex model

- The cavity method can deal with the generic vertex model.

  More complicated recursion relations, more cases to be considered, but no further difficulties.

- The transition lines do not get parallelly translated with respect to the ones of the 6-vertex model. **?**

  They are all of 2nd order. ✔

  They are remarkably close to the numerical values in $2d$. ✔

  The exponents : on the tree they are mean-field, in $2d$ **?** In progress.

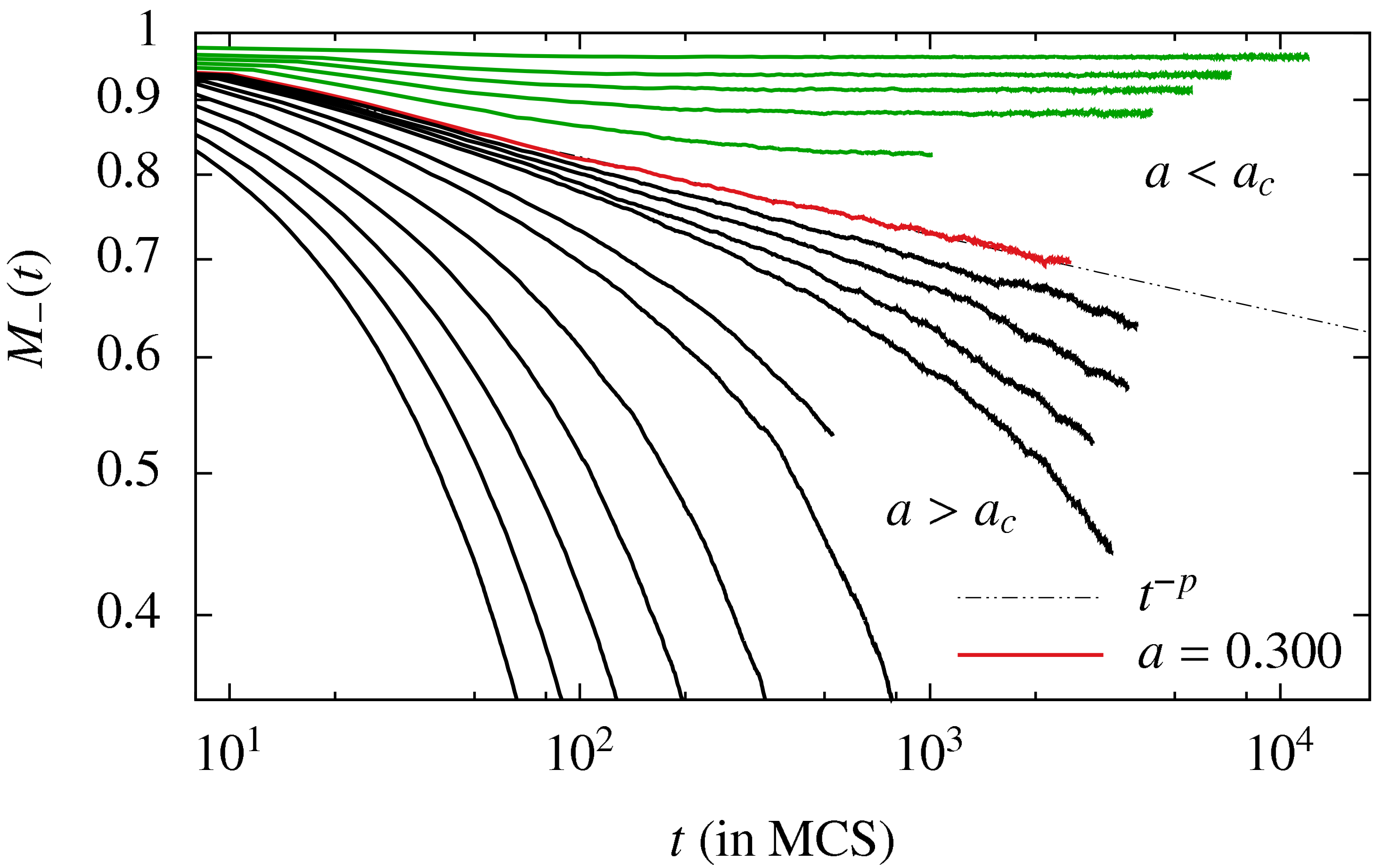
- MF expression for Δ_{16} In $2d$ **?**

- The quantum Ising chain for the 16 vertex model should include new terms. In progress.

Foini, Levis, Tarzia & LFC 12

Finite time relaxation

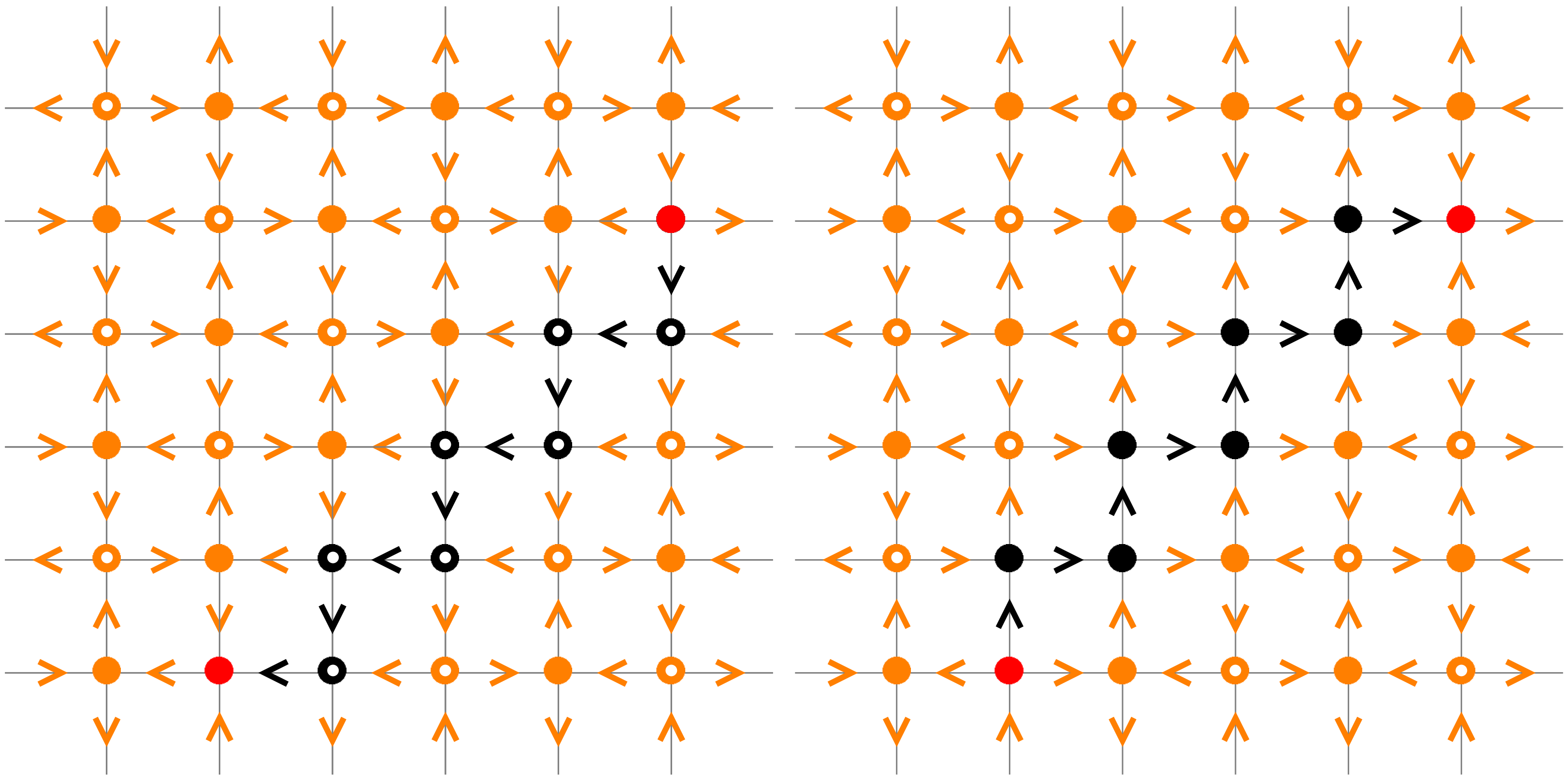
Magnetization across the PM-AF transition



$a_c = e^{-\beta_c e_1} \simeq 0.3$ with $e_1 = 0.45$ $\Rightarrow$ $\beta_c = 2.67 \pm 0.02$

Fluctuations

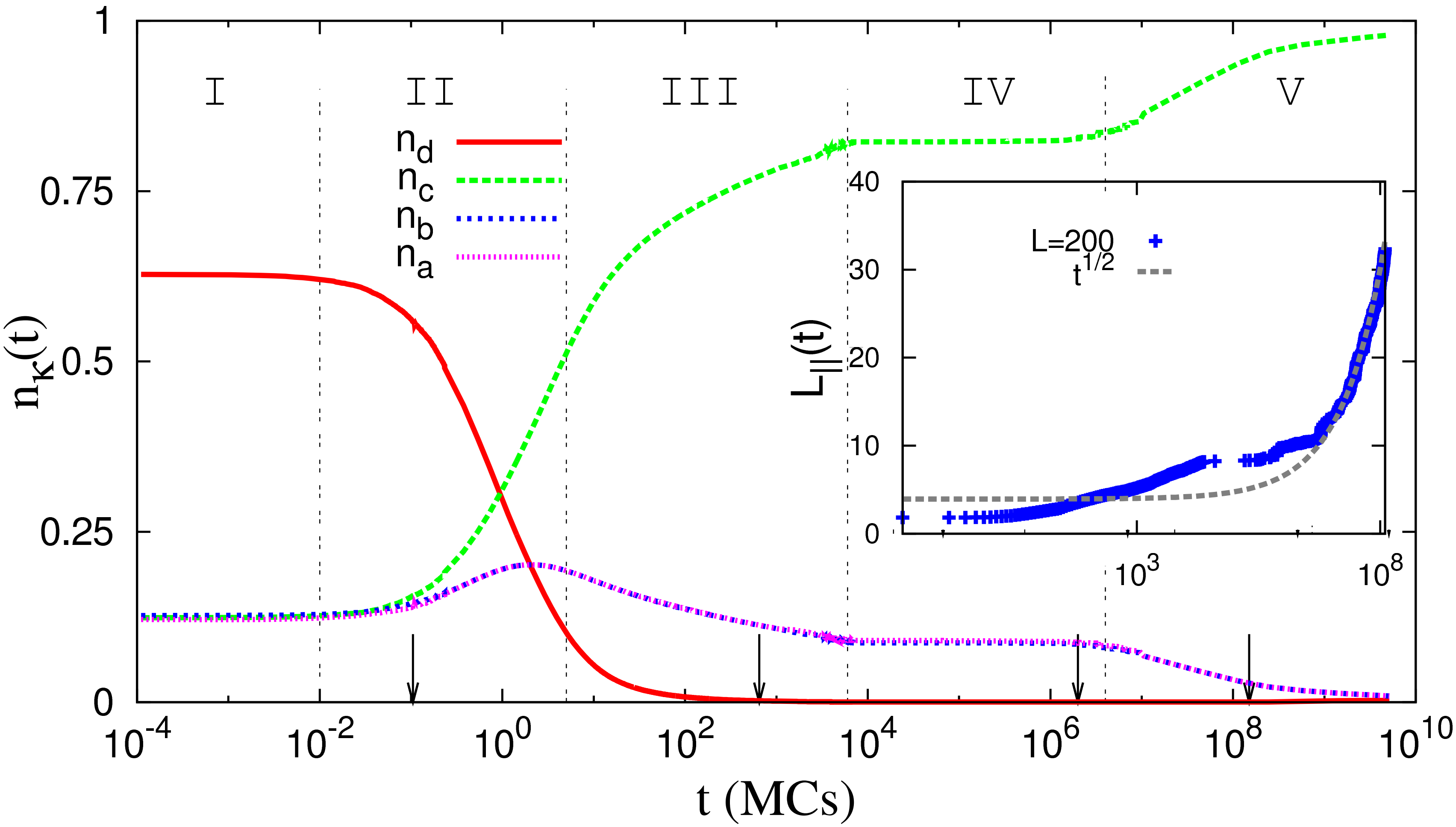
Sketch



The probability of such fluctuations can be estimated with the Bethe-Peierls calculation on a tree of four-site plaquettes !

Dynamics in the AF phase

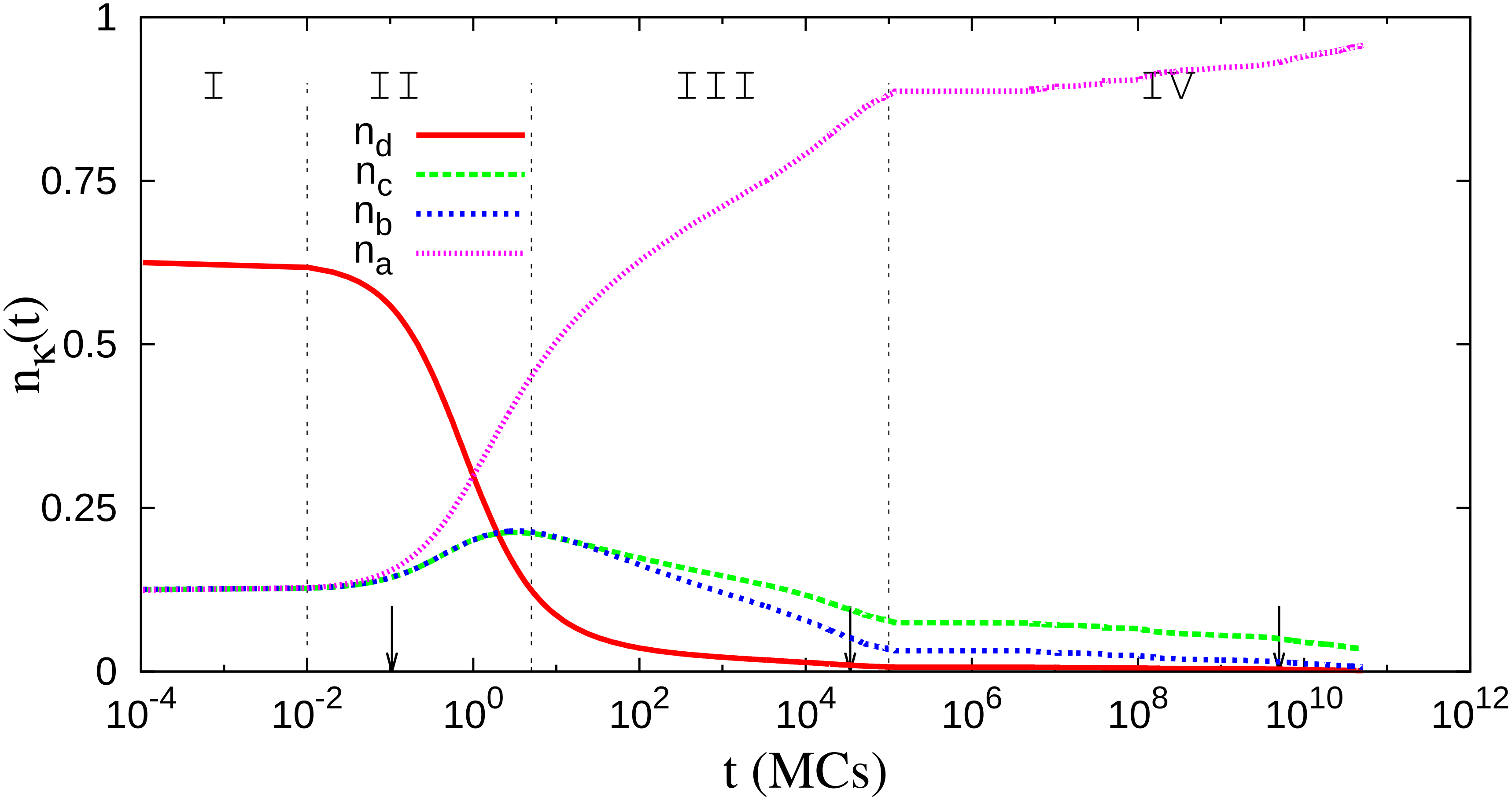
Density of defects & growing length ($d = e$ here)



Isotropic growth of AF order with $\boxed{L(t) \simeq t^{1/2}}$

Dynamics in the FM phase

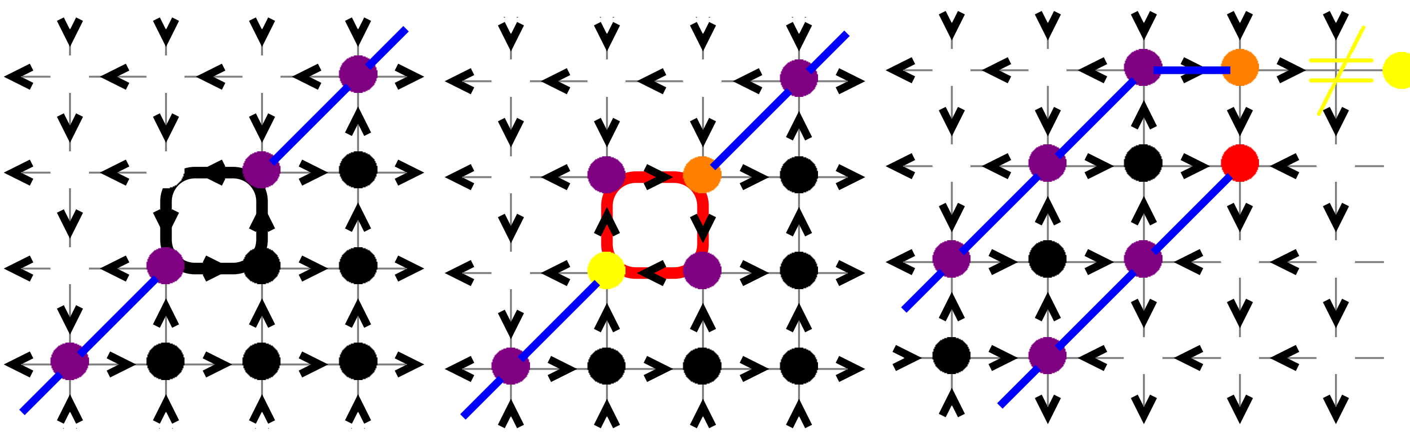
Density of defects ($d = e$ here)



Four regimes

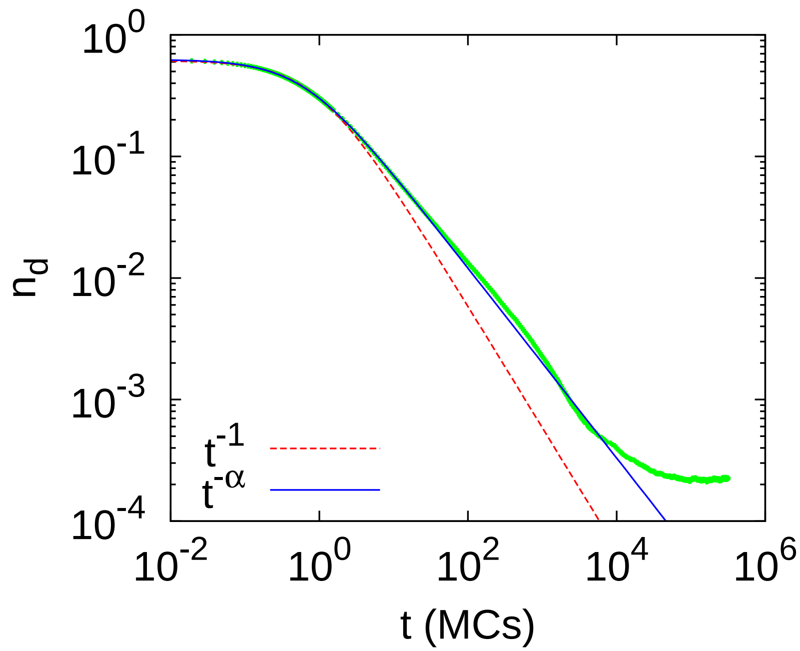
Dynamics in the FM phase

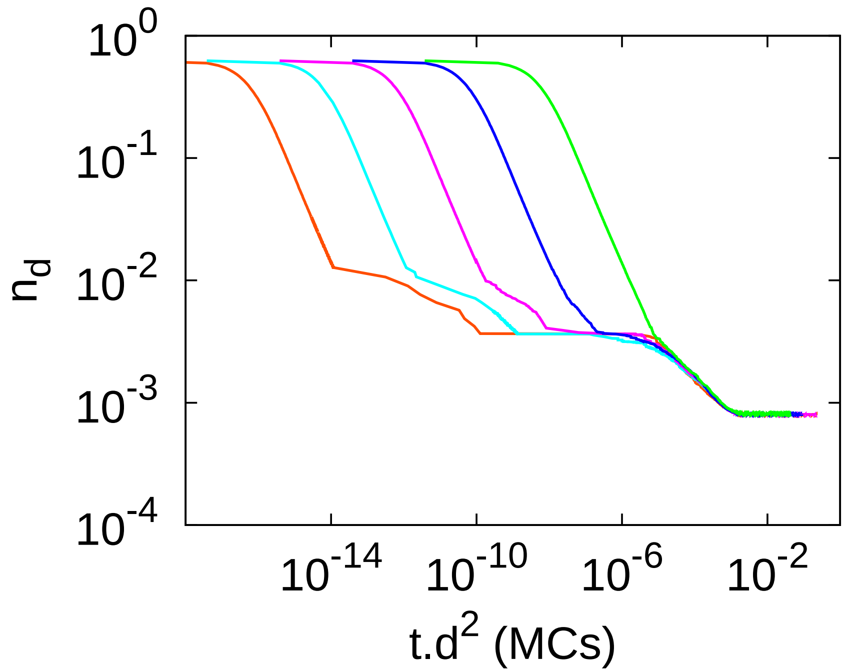
Some elementary moves



Dynamics in the D phase

Density of defects





Short-time decay $t^{-0.78}$

Different from MF approximation

to reaction - diffusion model t^{-1}.

$n_d \simeq f(td^2)$

Scaling below the plateau.