Commonalities between Edwards ensemble and glasses

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Workshop:
Physics of glassy and granular materials
July 16-19, Kyoto

Random packings of hard spheres

Physics

Granular matter

Random close packing (RCP)

Bernal packings (1960)

Glasses

1. Edwards ensemble for grains and glass theory

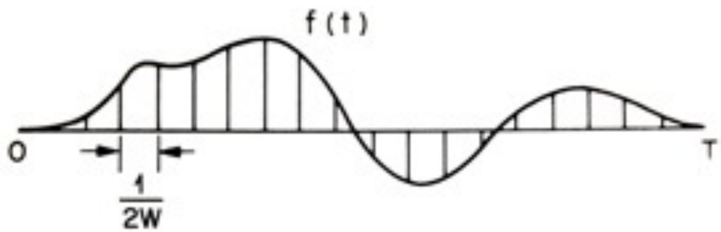
Volume fraction

Force distribution

Mathematics

Information theory

Shannon (1948)



Signals → High dimensional spheres

2. High-dimensional packings

Applications

3. Non-spherical shapes

Theoretical approach I: Statistical mechanics (Edwards' ensemble)

Edwards and Oakeshott, Physica A (1989)

Constraint optimization problem

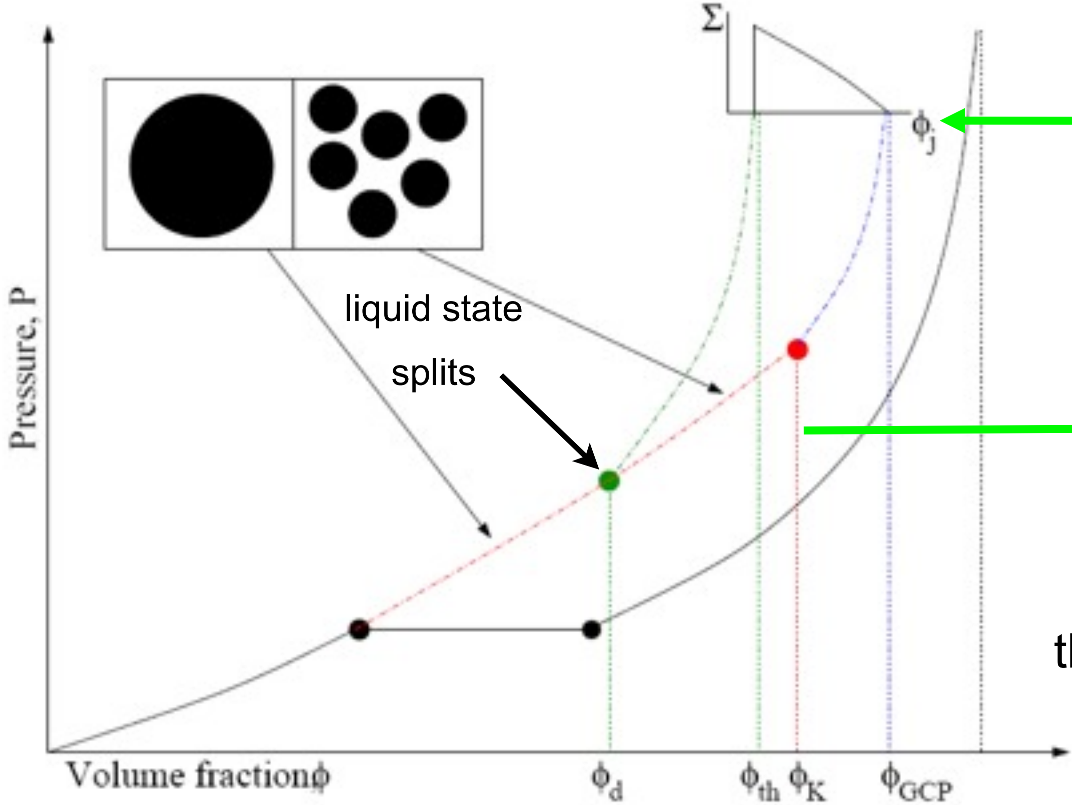
$$\mathcal{Z}(X,T) = \int \exp[-\mathcal{W}(\vec{x})/X] \exp[-\mathcal{H}(\vec{x},\vec{f})/T] \mathcal{D}\vec{x}\mathcal{D}\vec{f}$$

Minimize volume (X=0) with constraint of force balance (T=0) and non-overlaping.

OPTIMIZATION	STATISTICAL PHYS	EDWARDS
instance	sample	packing
cost function	energy	volume
optimal configuration	ground state	RCP at X=0
minimal cost	ground state energy	minimal volume

Theoretical approach II: Mean field theory of jammed hard-sphere (remnant of RSB solution from replica theory)

Parisi and Zamponi (2010)



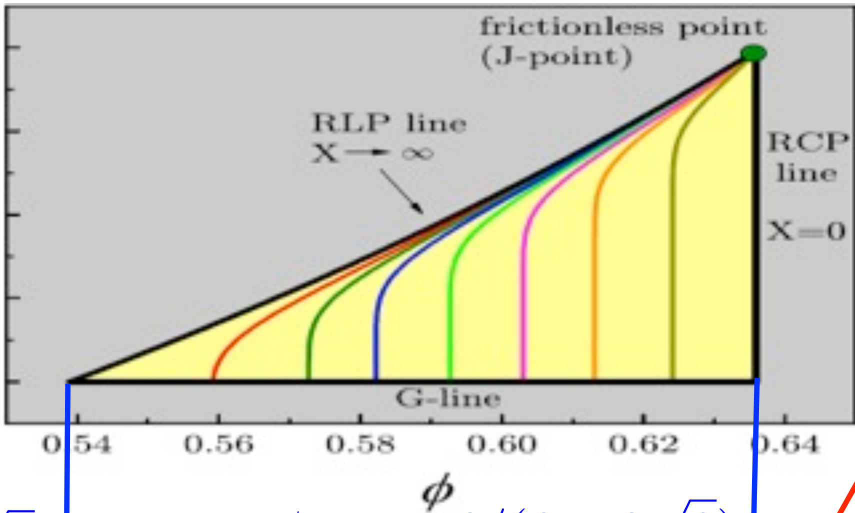
Replica theory: jammed states are the infinite pressure limit of metastable hard sphere glasses

- Approach jamming from the liquid phase.
- Predict a range of RCP densities $[\phi_{\rm th}, \phi_{\rm GCP}] \approx [0.64, 0.68]$
- Mean field theory based on RSB solution in the glass phase.

(un)Commonalities between Edwards ensemble and RT: 3d

8 densities in
0.64 ± 0.04
6%

$\max\{S\}$



$\phi_{\rm RLP} = 4/(4+2\sqrt{3})$

$\phi_{\rm edw} = 6/(6+2\sqrt{3})$

$\phi_{\rm onset}$

$\phi_{\rm th} = 0.64$

$\phi_{\rm RCP} = \phi_J$

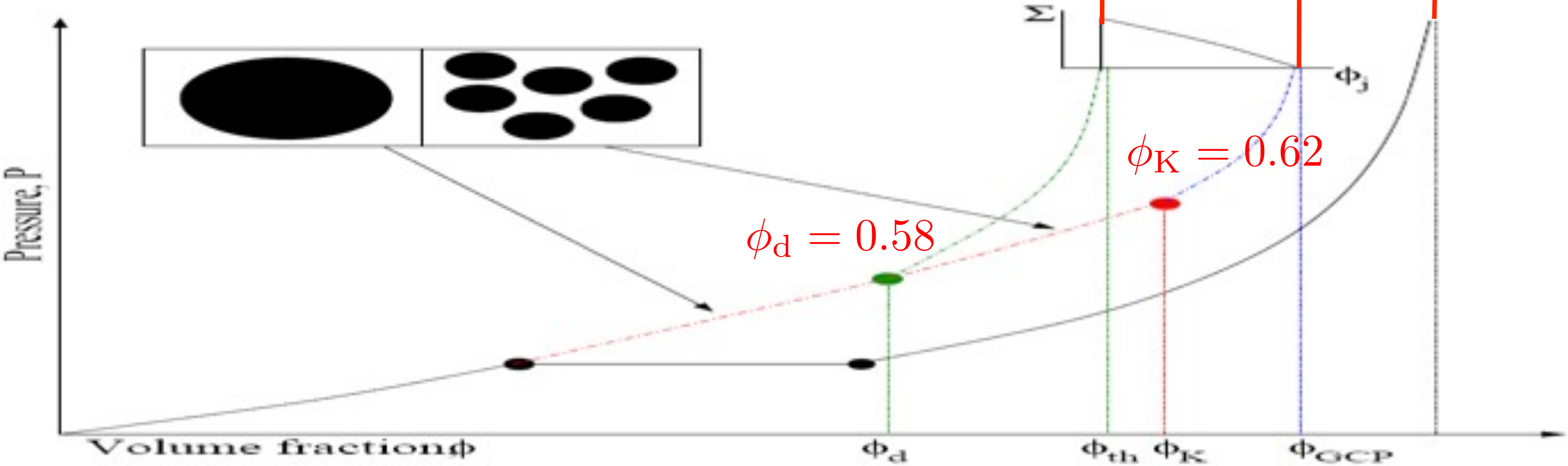
$\phi_{\rm MRJ}$

$\phi_{\rm c}$

$\phi_{\rm GCP} = 0.68$

$\phi_{\rm fcc} = 0.74$

$\max\{\Sigma + s\} \quad s \to 0$



$\phi_{\rm K} = 0.62$

$\phi_{\rm d} = 0.58$

Very difficult in practice: very small range for 3d equal-size spheres

3d **4d**

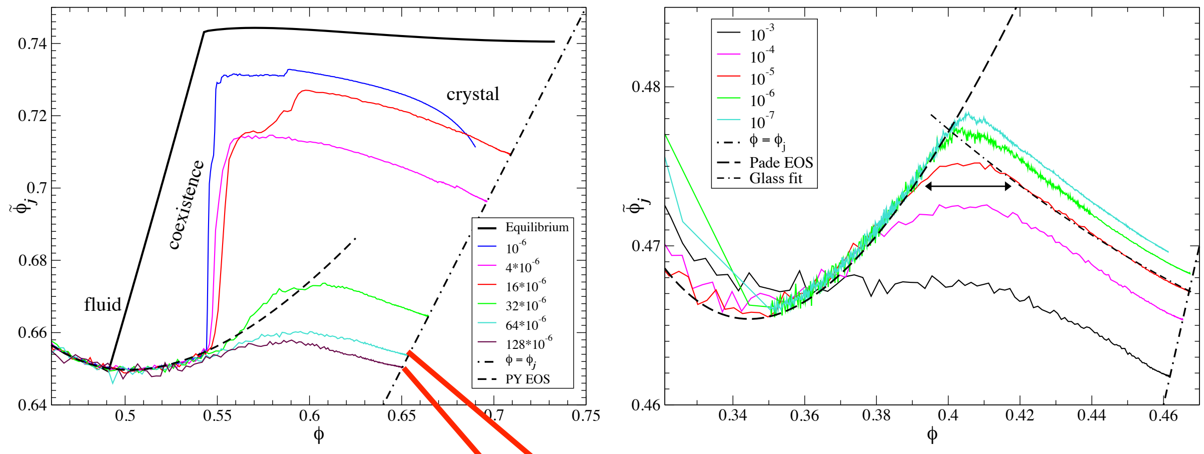
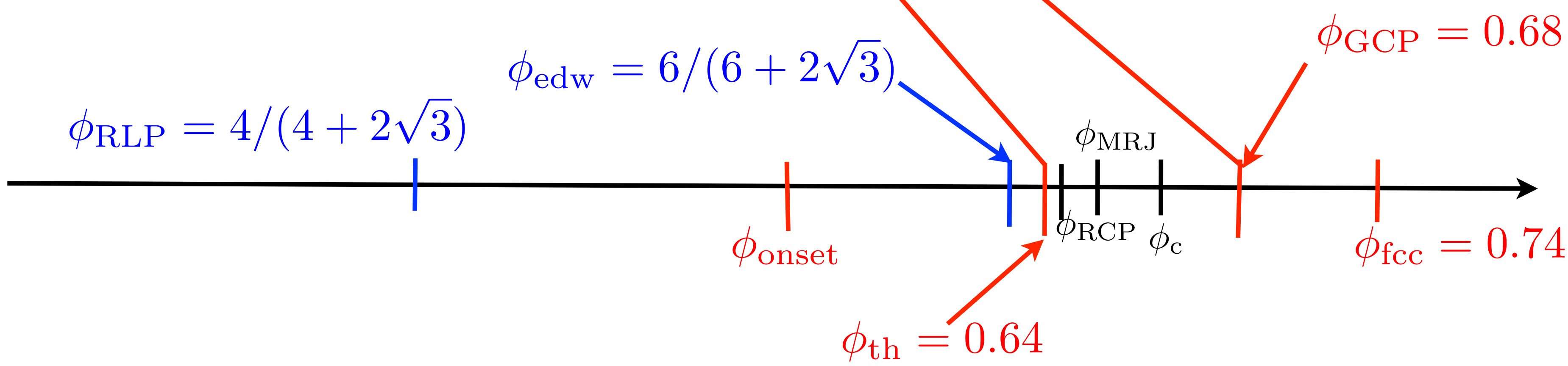


FIG. 1 (From (Skoge *et al.*, 2006)) Evolution of the pressure during compression at rate γ in $d = 3$ (left) and $d = 4$ (right). The density φ is increased at rate γ and the reduced pressure $p(\varphi) = \beta P/\rho$ is measured during the process. See (Skoge *et al.*, 2006) for details. The quantity $\widetilde{\varphi}_j(\varphi) = \frac{p(\varphi)\varphi}{p(\varphi)-d}$ is plotted as a function of φ. If the system jams at density φ_j, $p \to \infty$ and $\widetilde{\varphi}_j \to \varphi_j$. Thus the final jamming density is the point where $\widetilde{\varphi}_j(\varphi)$ intersects the dot-dashed line $\widetilde{\varphi}_j = \varphi$. (Left) The dotted line is the liquid (Percus-Yevick) equation of state. The curves at high γ follow the liquid branch at low density; when they leave it, the pressure increases faster and diverges at φ_j. The curves for lower γ show first a drop in the pressure, which signals crystallization. (Right) All the curves follow the liquid equation of state (obtained from Eq.(9) of (Bishop and Whitlock, 2005)) and leave it at a density that depends on γ. In this case no crystallization is observed. For $\gamma = 10^{-5}$ the dot-dashed line is a fit to the high-density part of the pressure (glass branch). The arrow marks the region where the pressure crosses over from the liquid to the glass branch.



1. Full solution: Constraint optimization problem

$$\mathcal{Z}(X,T) = \int \exp[-\mathcal{W}(\vec{x})/X] \exp[-\mathcal{H}(\vec{x},\vec{f})/T] \mathcal{D}\vec{x}\mathcal{D}\vec{f}$$

T=0 and X=0 optimization problem

2. Approximation: Decouple forces from geometry.

$$\mathcal{Z}(X,T) = \int \exp[-\mathcal{W}(\vec{x})/X]\mathcal{D}\vec{x} \times \int \exp[-\mathcal{H}(\vec{f})/T]\mathcal{D}\vec{f}$$

3. Edwards for volume ensemble + Isostaticity

$$\mathcal{Z}(X,Z) = \int_Z^6 \exp\left[-\frac{w(z)}{X}\right] g(z)dz$$

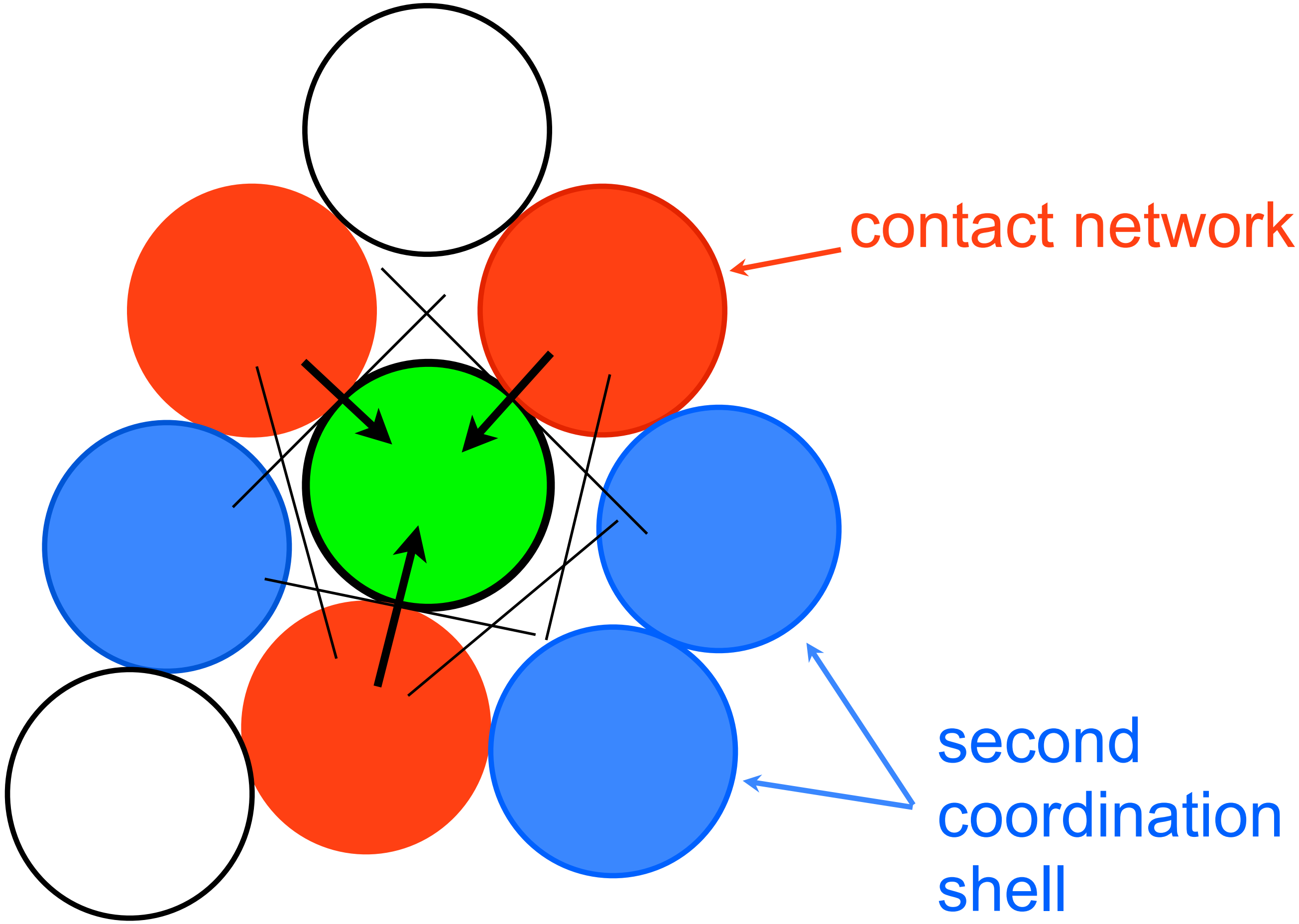
Song, Wang, Jin, Makse, Physica A (2010)

4. Cavity method for force ensemble

$$\mathcal{H} = \sum_{a=1}^{N} \left[\left(\sum_{b,(ab)\in E} f_{ab}\, \hat{n}_{ab} \right)^2 \right]$$

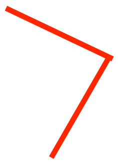
Bo, Mari, Song, Makse (2013)

The Volume function is the Voronoi volume

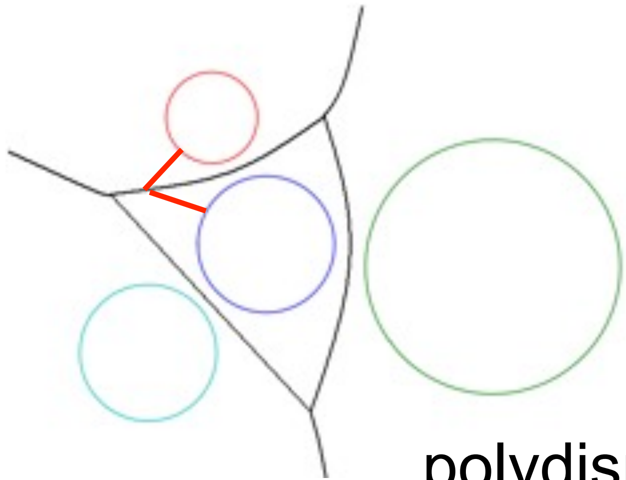


consist of all points
closer to the center of the
grain than to any other

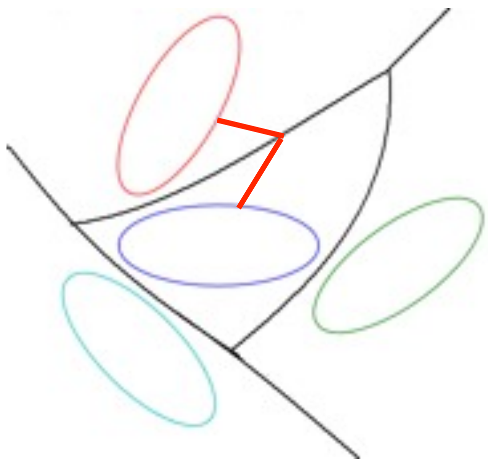
“Easily” generalizable to other systems



equal size
spheres

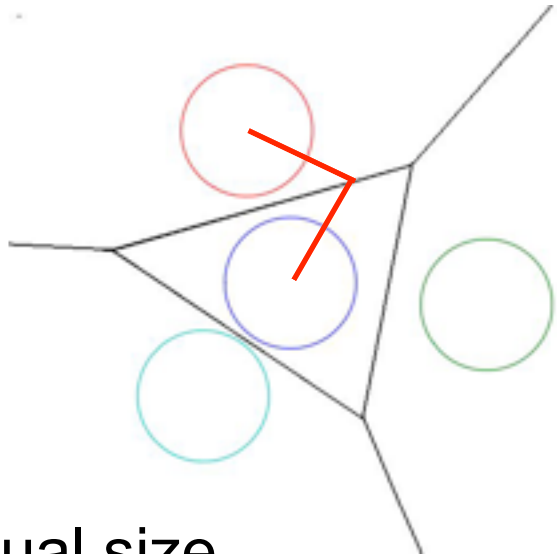


polydisperse
system



ellipsoids, spherocylinders,
non-convex particles, rods,
sphere/ellipsoids mixtures,
etc.
any dimension

“Easily” generalizable to other systems



equal size
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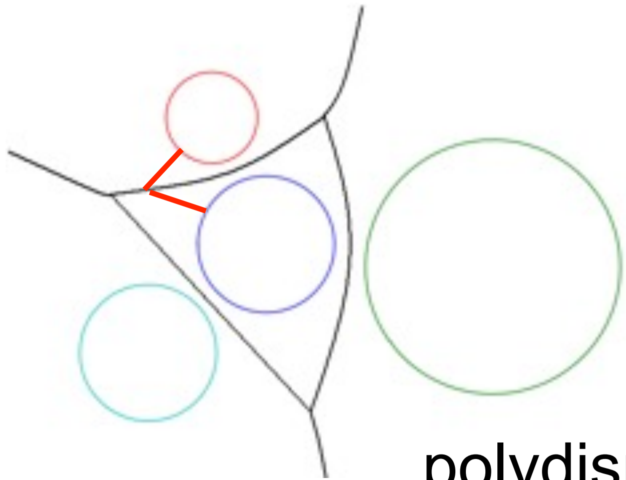

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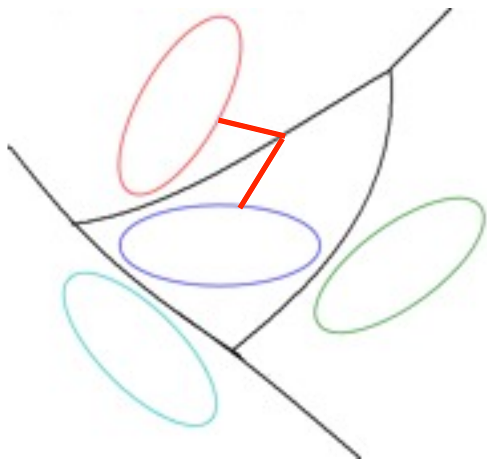

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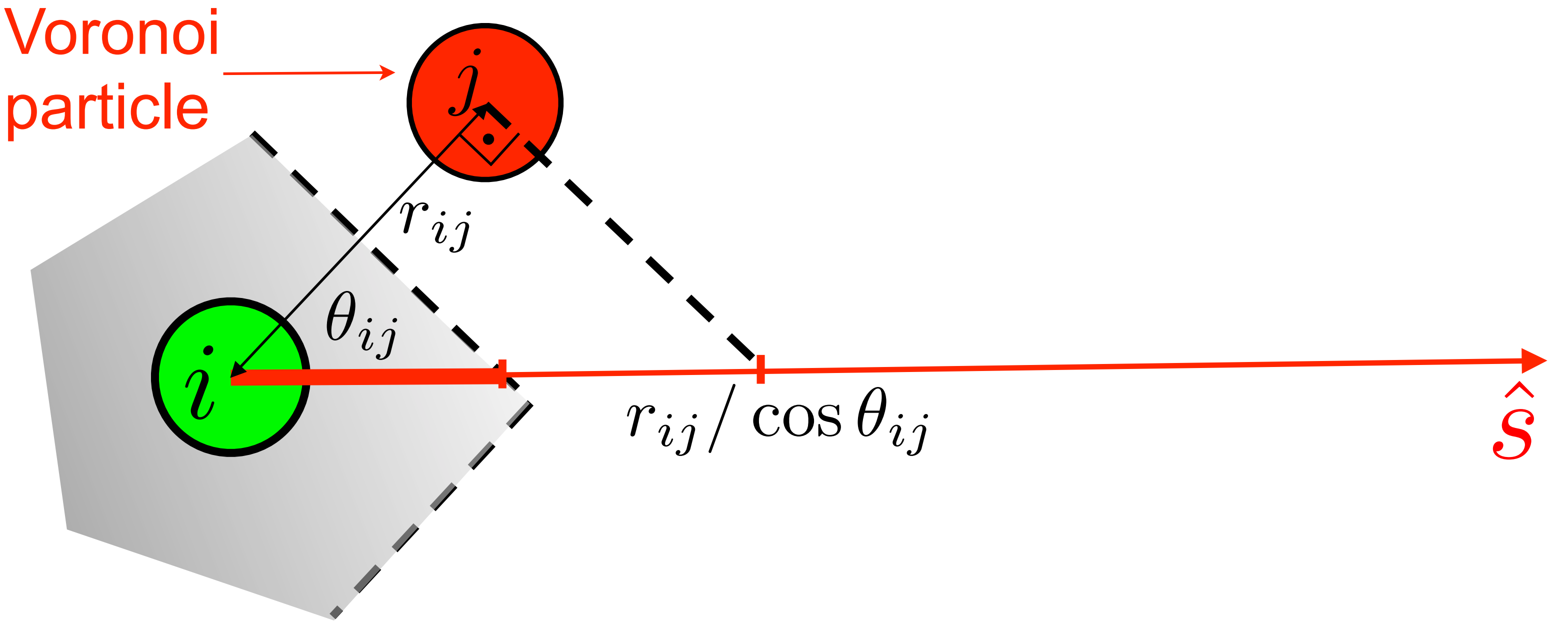
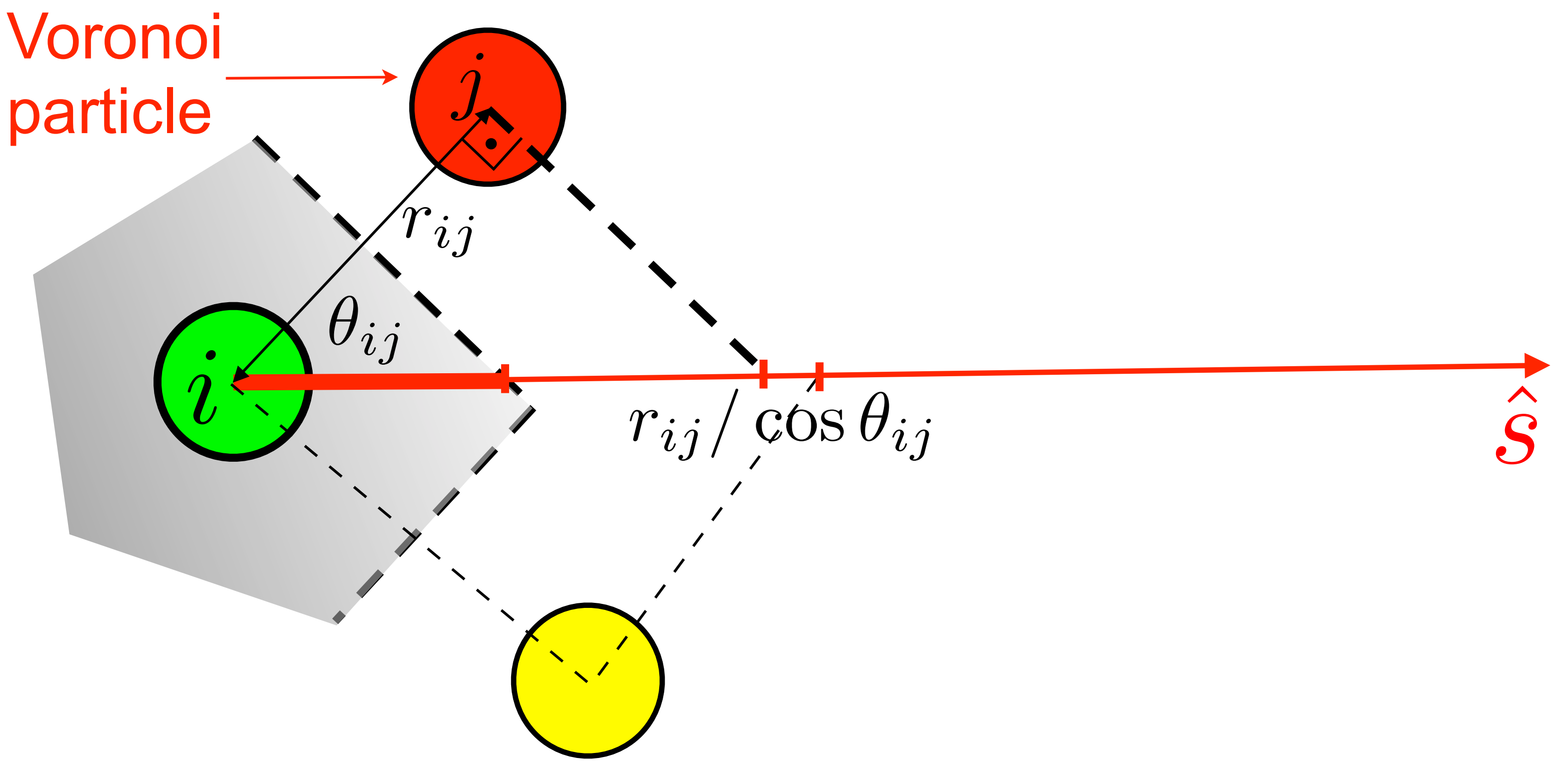
Analytical formula for Voronoi boundary



$$W_i = \frac{1}{3}\oint\left(\frac{1}{2R}\min_j \frac{r_{ij}}{\cos\theta_{ij}}\right)^3 ds$$

Important: global minimization. Reduce to one-dimension

Analytical formula for Voronoi boundary



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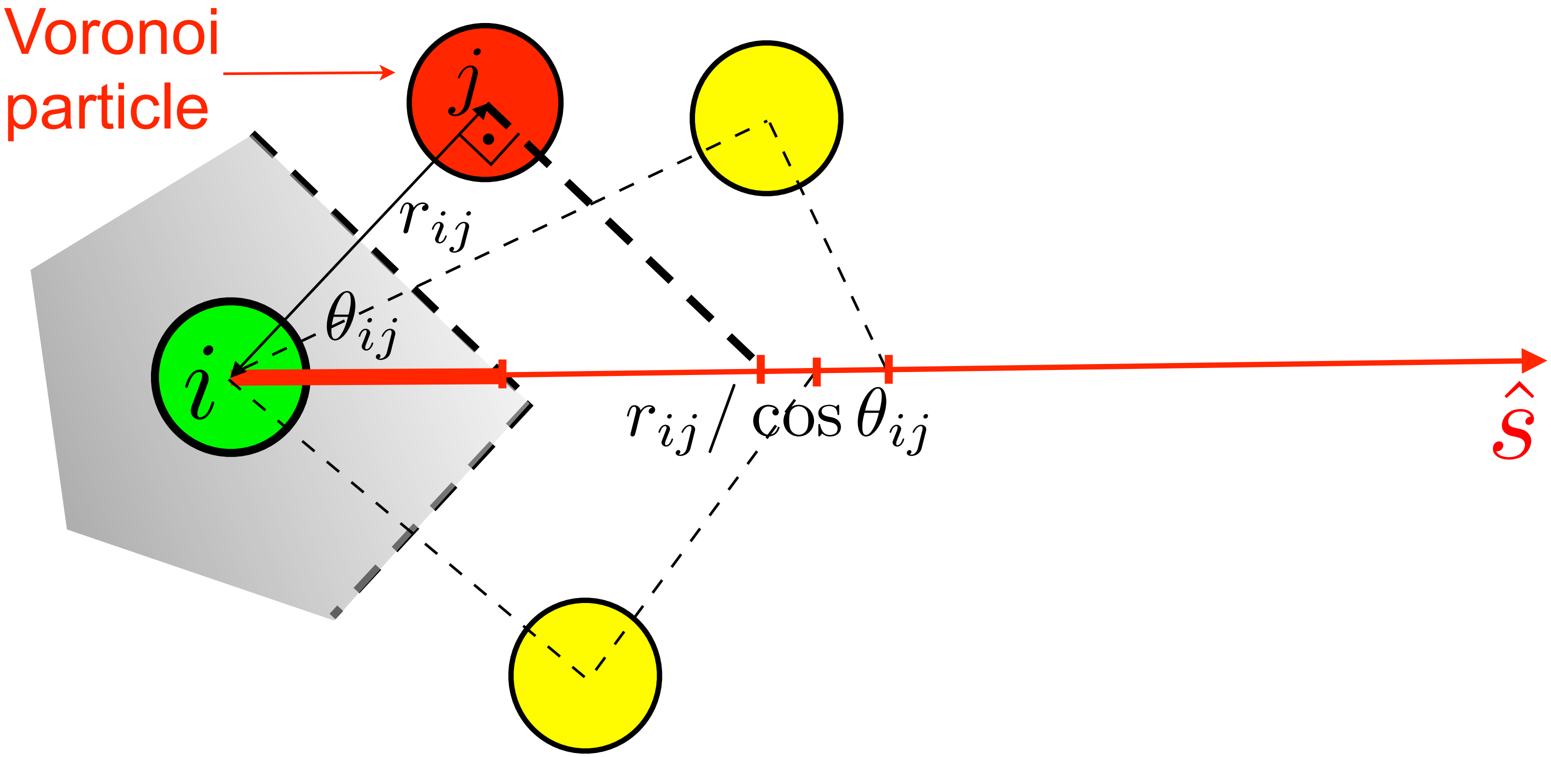

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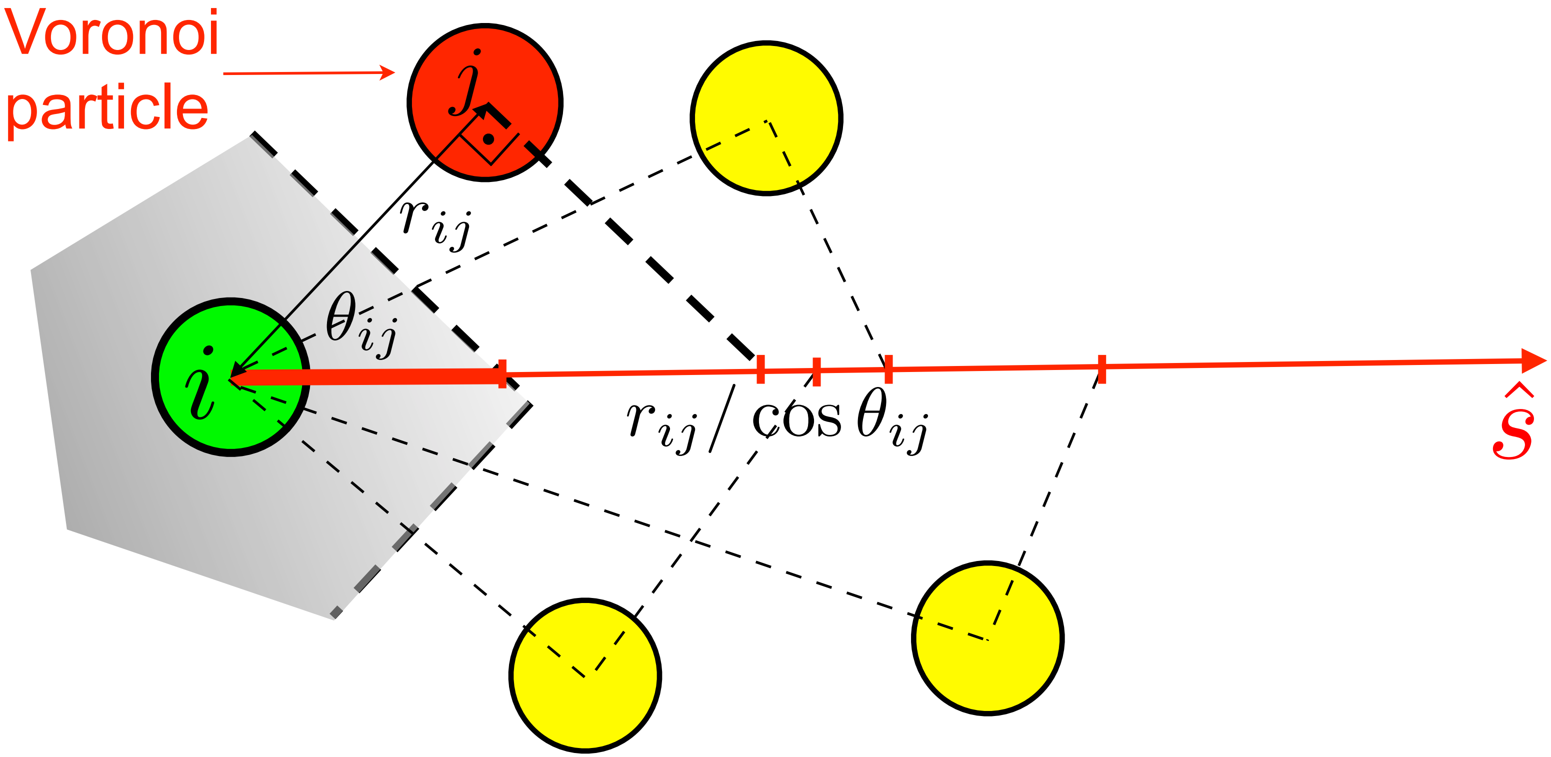

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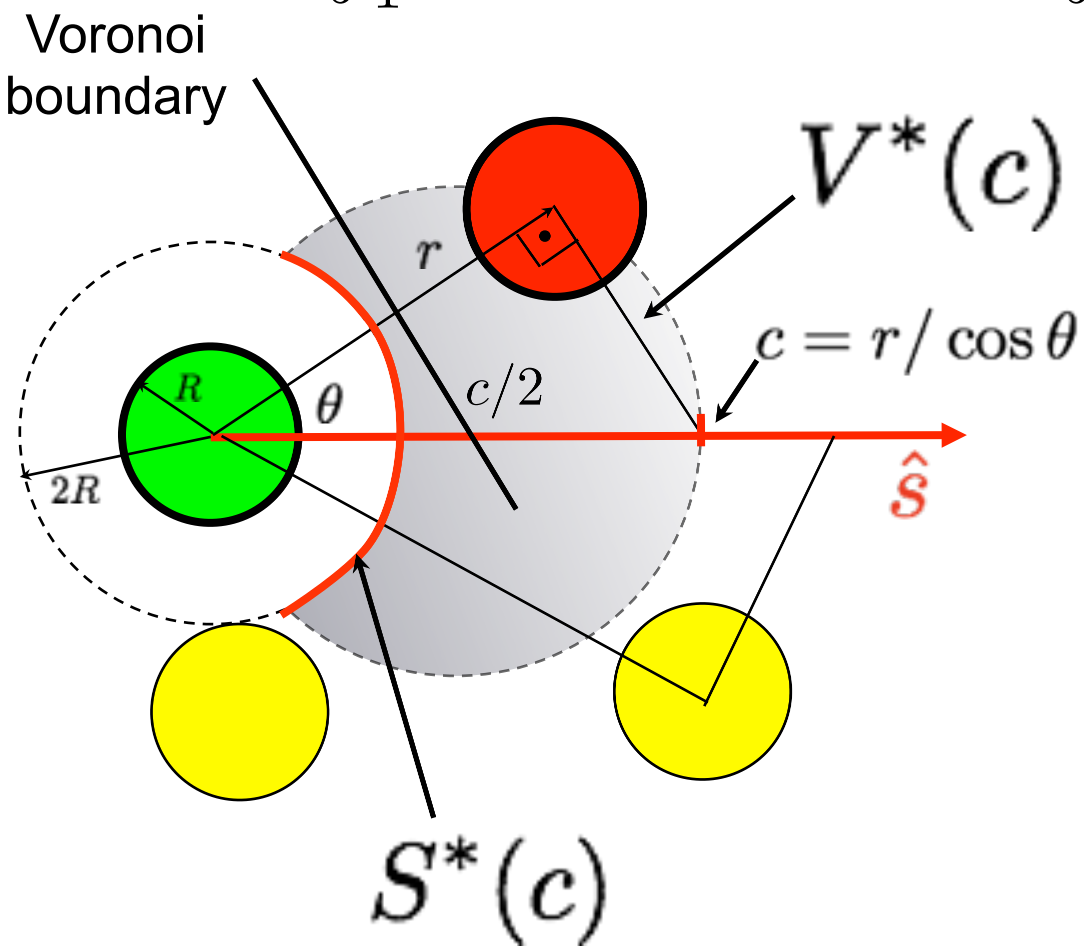
Important: global minimization. Reduce to one-dimension

Average free-volume per particle

$$w = \int_1^\infty (c^3 - 1) p(c) dc = \int_0^1 (c^3 - 1) dP_>(c)$$



Geometrical interpretation of cumulative dist:

$$P_>(c)$$

$$-\frac{\partial P_>(c)}{\partial c} = p(c)$$

Probability to find all particles outside excluded volume and surface:

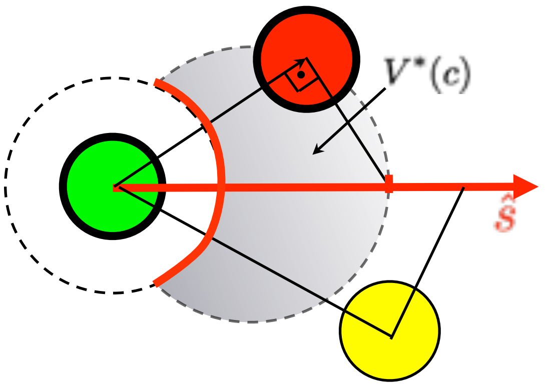
$$V^*(c) \qquad S^*(c)$$

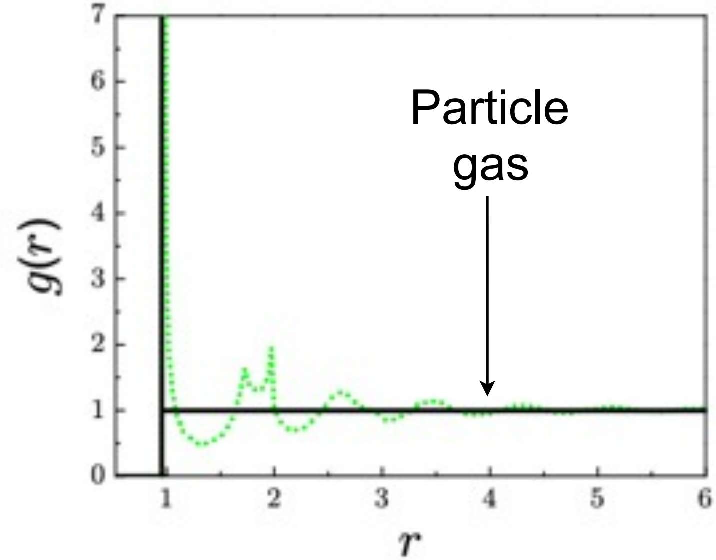
Mean-field approximation analogous to decorrelation principle

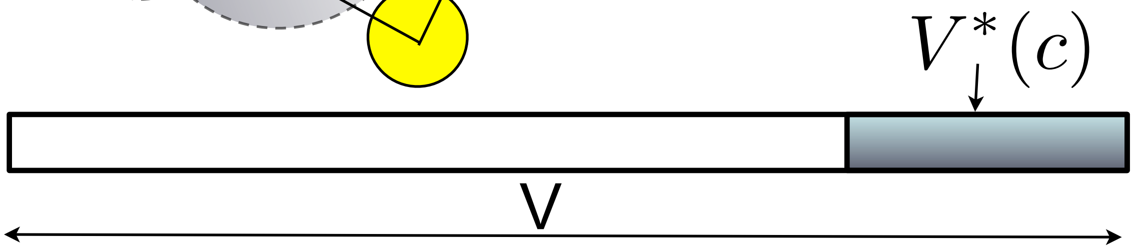
particles belong to bulk or in contact:

$$\boxed{P_{>}(c) = P_B(c) \times P_c(c)}$$

$$g_2(r) \simeq \frac{z}{\rho S_{d-1}} \delta(r-1) + \Theta(r-1)$$







Similar to car parking problem (Renyi, 1960). Probability to find a spot with $V^*(c)$ in a volume V

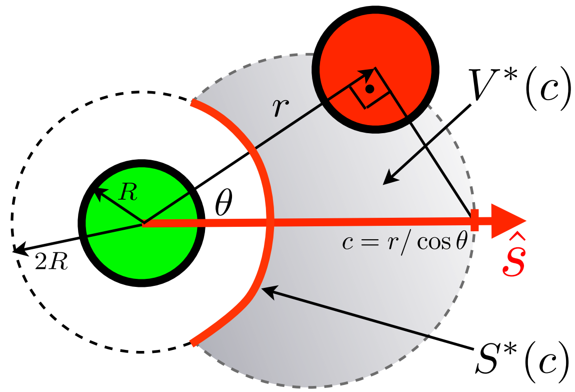
Particle gas

$$\boxed{P(V^*) = (1 - V^*/V)^N \to e^{-\rho V^*}}$$

Calculation of P>(c)

Particles are in contact and in the bulk:

$$\boxed{P_>(c) = P_B(c) \times P_c(c)}$$



Bulk term:

$$P_B(c) = e^{-\rho V^*(c)}$$

$$\rho(w) = \frac{1}{w}$$

mean free volume density

Contact term:

$$P_C(c) = e^{-\rho_s S^*(c)}$$

$$\rho_s(z) = \frac{1}{\langle S^* \rangle} = \frac{\sqrt{3}}{4\pi} z$$

mean free surface density

z = geometrical coordination number

Average Voronoi volume

$$\boxed{P_>(c) = P_B(c) \times P_c(c)}$$

$$P_>(c) = \exp\left[-\frac{1}{w}\left((c^3 - 1) - 3(1 - \frac{1}{c}) \right) - \frac{\sqrt{3}}{2} z (1 - \frac{1}{c}) \right]$$

Self-consistent equation:

$$w = \int_0^1 (c^3 - 1) \; d \exp\left[-\frac{1}{w}\left((c^3 - 1) - 3(1 - \frac{1}{c}) \right) - \frac{\sqrt{3}}{2} z (1 - \frac{1}{c}) \right]$$

equal to zero

$$\boxed{w = \frac{2\sqrt{3}}{z}}$$

represent the average free-volume of a single particle

Prediction: volume fraction vs z

free volume

$$w = \frac{2\sqrt{3}}{z}$$

volume function

$$\phi = \frac{z}{z+2\sqrt{3}}$$

Equation of state agrees well with simulations and experiments

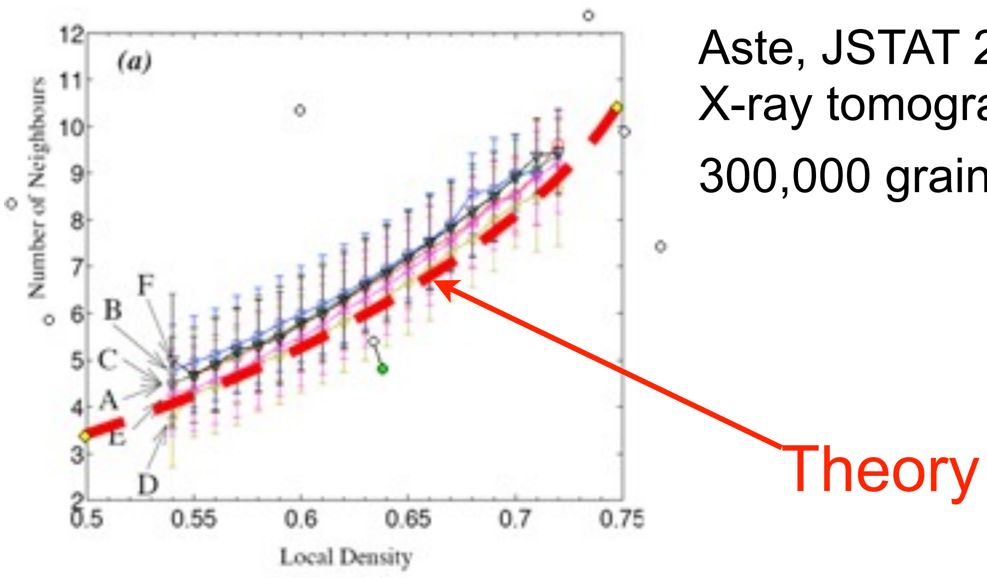
RCP

$$z = 6$$

$$w = \frac{1}{\sqrt{3}}$$

$$\phi = \frac{6}{6+2\sqrt{3}}$$

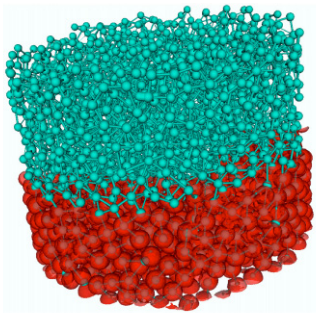
$$\phi = .634$$



Aste, JSTAT 2006

X-ray tomography

300,000 grains

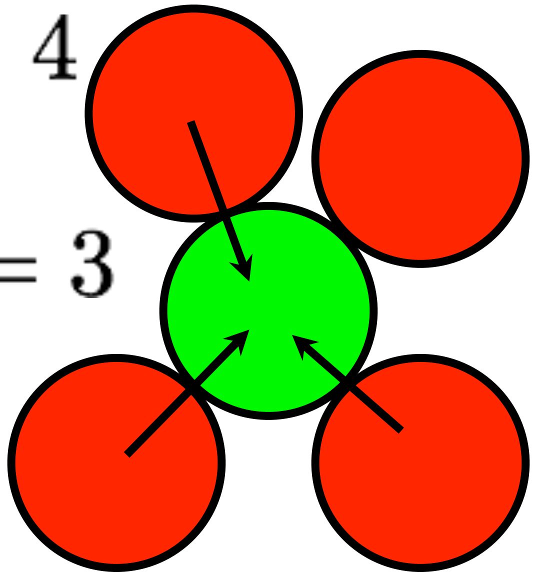


Definition of jammed state: geometric coordination z bounded by mechanical coordination Z

$$\underset{\mu=\infty}{4} = d + 1 \le Z \le 2d = \underset{\mu=0}{6}$$

$z = 4$

$Z = 3$



$z \le 2d$

$$Z \le z \le 2d = 6$$

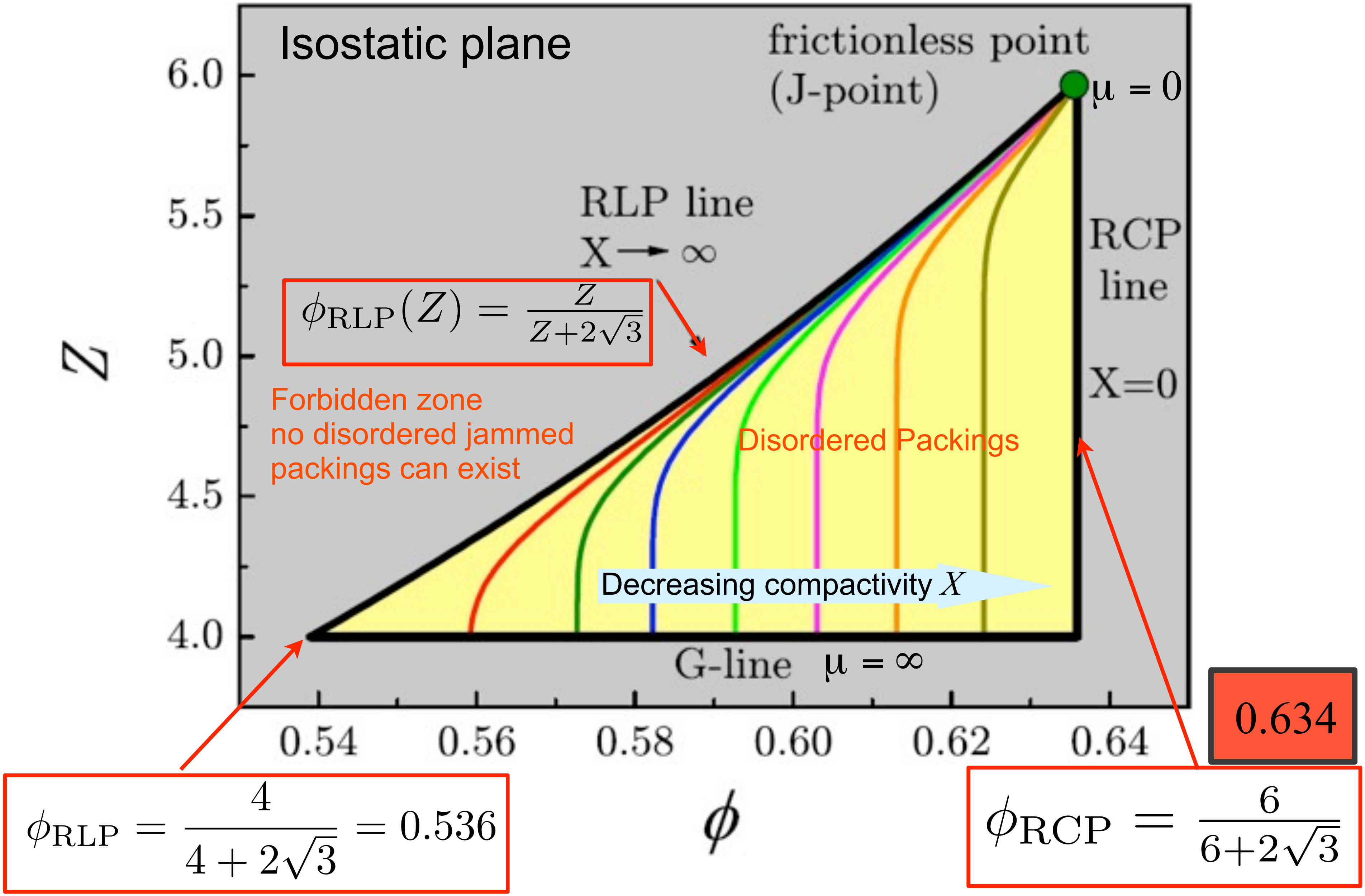
Nd positions

$zN/2$ geometrical constraints

effectively excludes the ordered states

$$|r_i - r_j| = 2R$$

Edwards phase diagram for hard spheres



Jammed packings in high dimensions

Rigorous bounds

Minkowsky lower bound: $\phi \sim 2^{-d}$

Kabatiansky-Levenshtein upper bound: $\phi \sim 2^{-0.5990\ldots d}$

Question: what's the density of RCP in high dimensions?

Conjecture: are disordered packings more optimal than ordered ones?

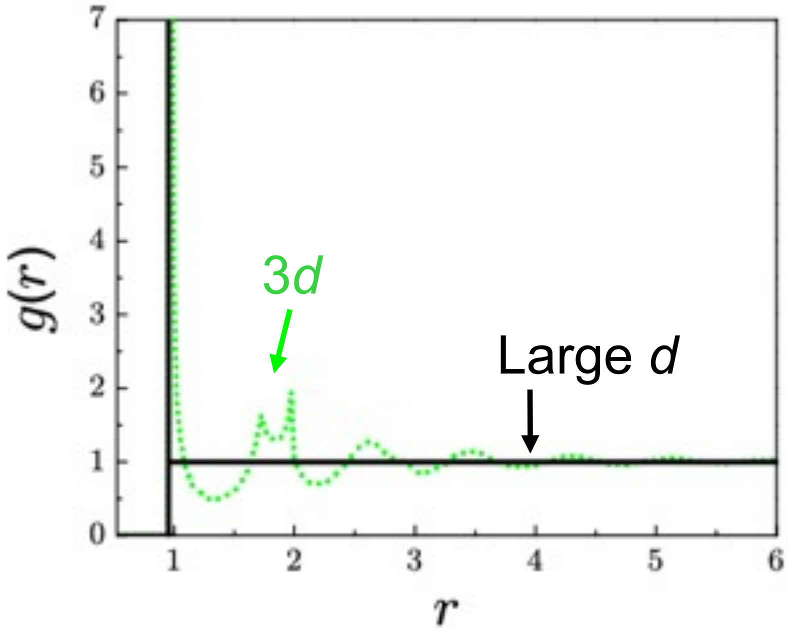
Conjecture: $P_>(c)$ becomes valid in the high-dimensional limit

(I) Theoretical conjecture of g_2 in high d

(neglect correlations)

Torquato and Stillinger, Exp. Math., 2006

$$g_2(r) \simeq \frac{z}{\rho S_{d-1}}\delta(r-1) + \Theta(r-1)$$



(II) Factorization of $P_>(c)$

$$P_>(c) = P_B(c) P_C(c)$$

Comparison with other theories

Edwards' theory

$$\phi \sim \frac{4d}{3} 2^{-d}$$

Isostatic packings ($z = 2d$) with unique volume fraction

Jin, *Charbonneau, Meyer, Song, Zamponi*, PRE (2010)

Agree with Minkowski lower bound

Glass transition RT

$$\phi \in [6.26\, d\, 2^{-d}, d\, \ln(d)\, 2^{-d}]$$

Parisi and Zamponi, Rev. Mod. Phys. (2010)

Isostatic packings ($z = 2d$) ranging volume fraction increases with dimensions

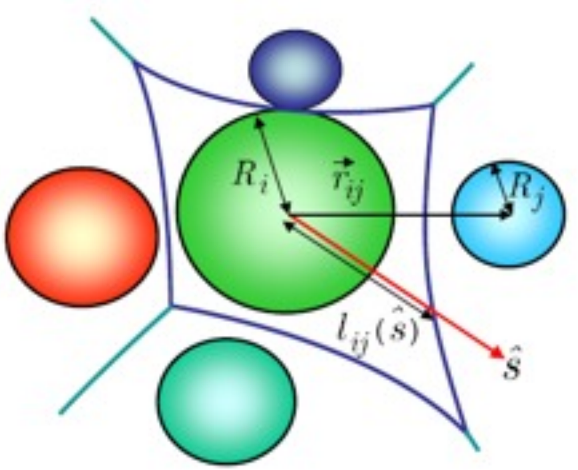
No unified conclusion at the mean-field level (infinite *d*). Neither dynamics nor jamming.

Does RCP in large *d* have higher-order correlations missed by theory?: Test of replica th.

Edwards solution seems to corresponds to ϕ_{th}. Higher entropy state.

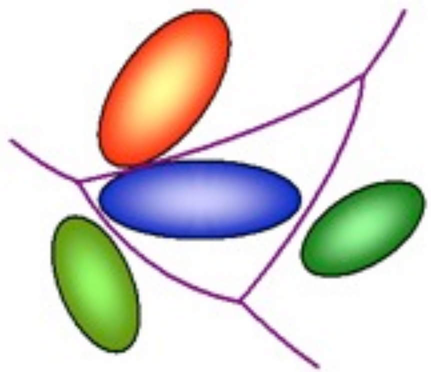
Generalizing the theory of monodisperse sphere packings

Polydisperse spheres



Non-spherical objects

(dimers, triangles, tetrahedra, spherocylinders, ellipsoids …)



Distribution of radius $P(r)$ ⟷ Distribution of angles $P(\hat{s})$

Extra degree of freedom

treated as in Onsager 1949

Optimizing random packings in the space of object shapes

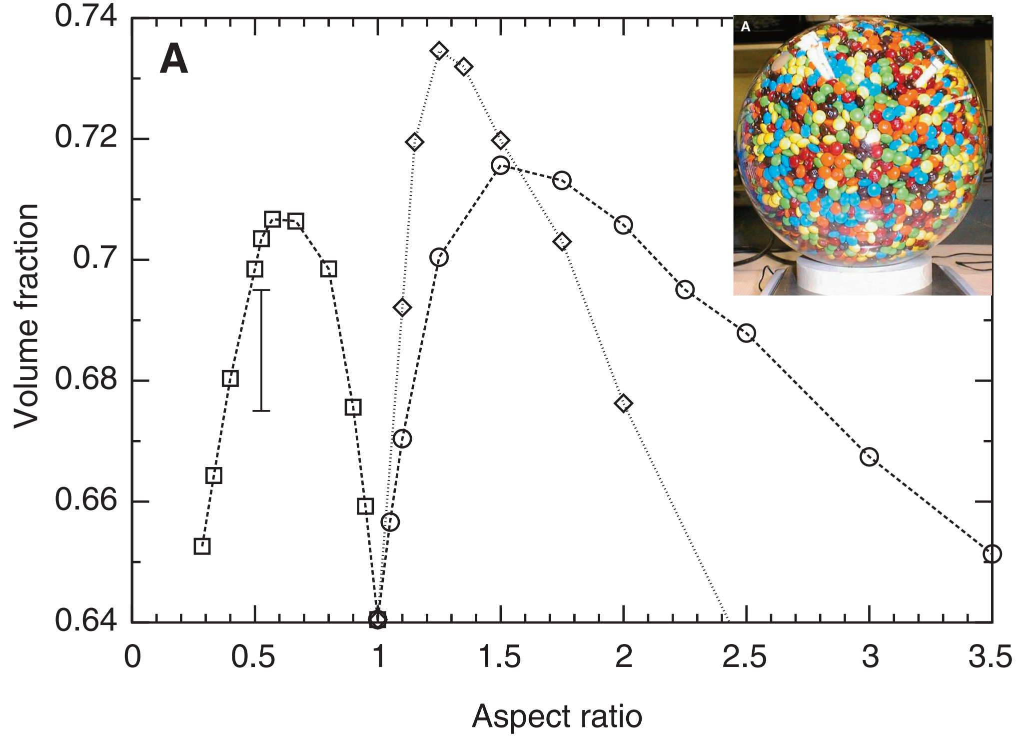
- Simulation results on packings of *ellipsoids*:

➢ Ellipsoids pack *denser* than spheres

➢ *Peak* at aspect ratio

$$\alpha \approx 1.4$$

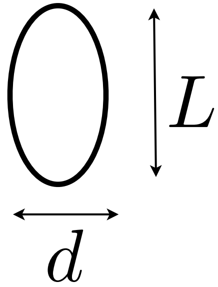
➢ Spheres appear as a *singular limit*



Donev et al, Science 2004

Edwards prediction

- Non-spherical objects:



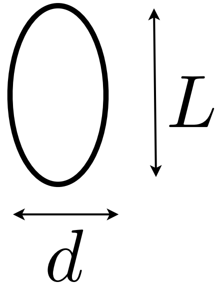
$$\alpha = \frac{L}{d}$$

$$\phi(\alpha) = \phi(Z(\alpha), \alpha)$$

$$z < 2d_f$$

Edwards prediction

- Non-spherical objects:



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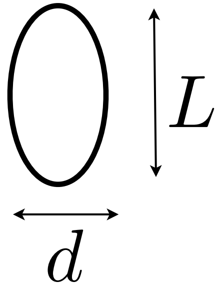
$$\phi(Z, \alpha)$$

Statistical theory of
Voronoi volume

$$z < 2d_f$$

Edwards prediction

- Non-spherical objects:



$$\alpha = \frac{L}{d}$$

$$\phi(\alpha) = \phi(Z(\alpha), \alpha)$$

$$\phi(Z, \alpha)$$

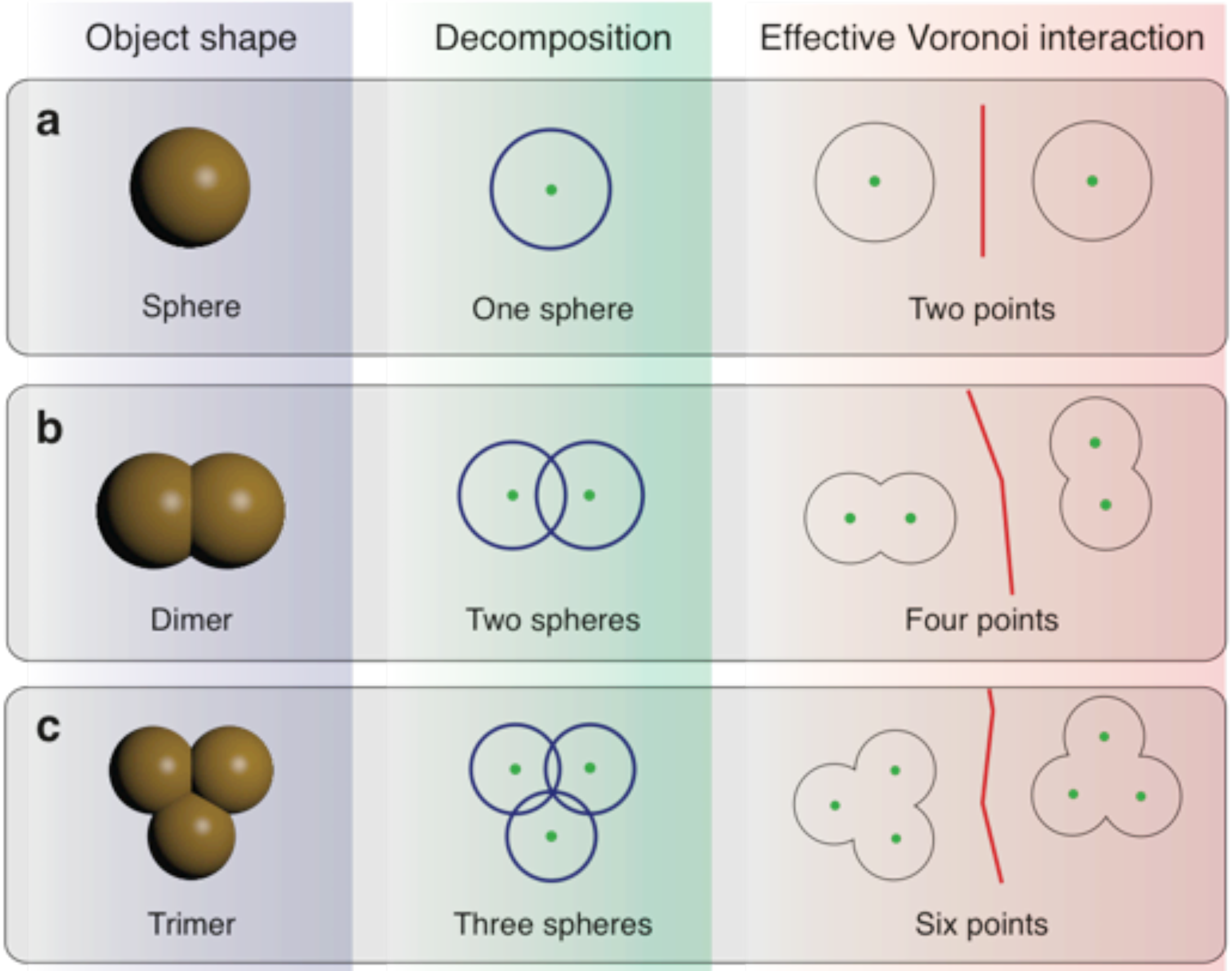
Statistical theory of Voronoi volume

$$Z(\alpha)$$

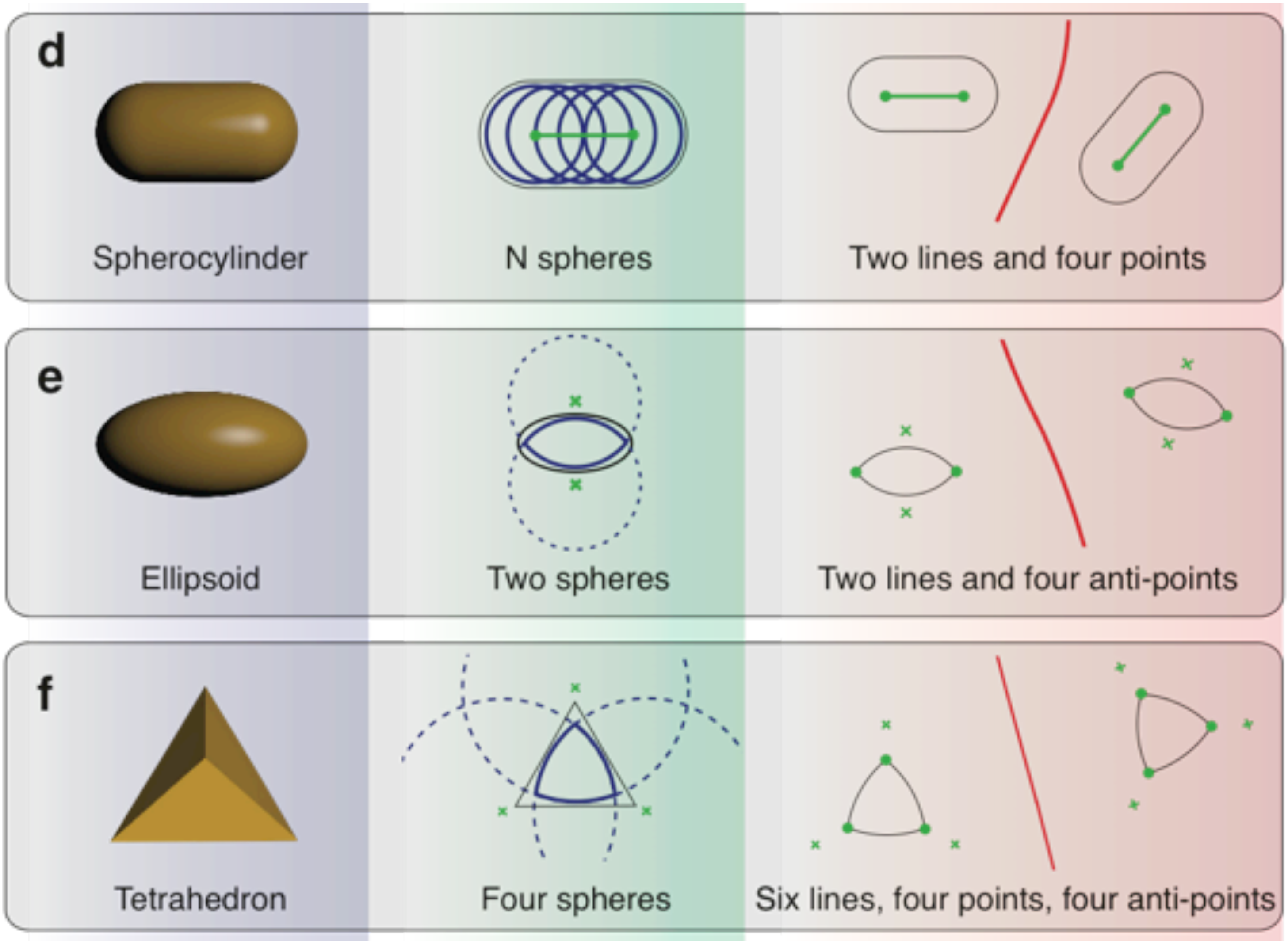
Evaluating the probability of degenerate configurations: ellipsoids are hypoconstrained

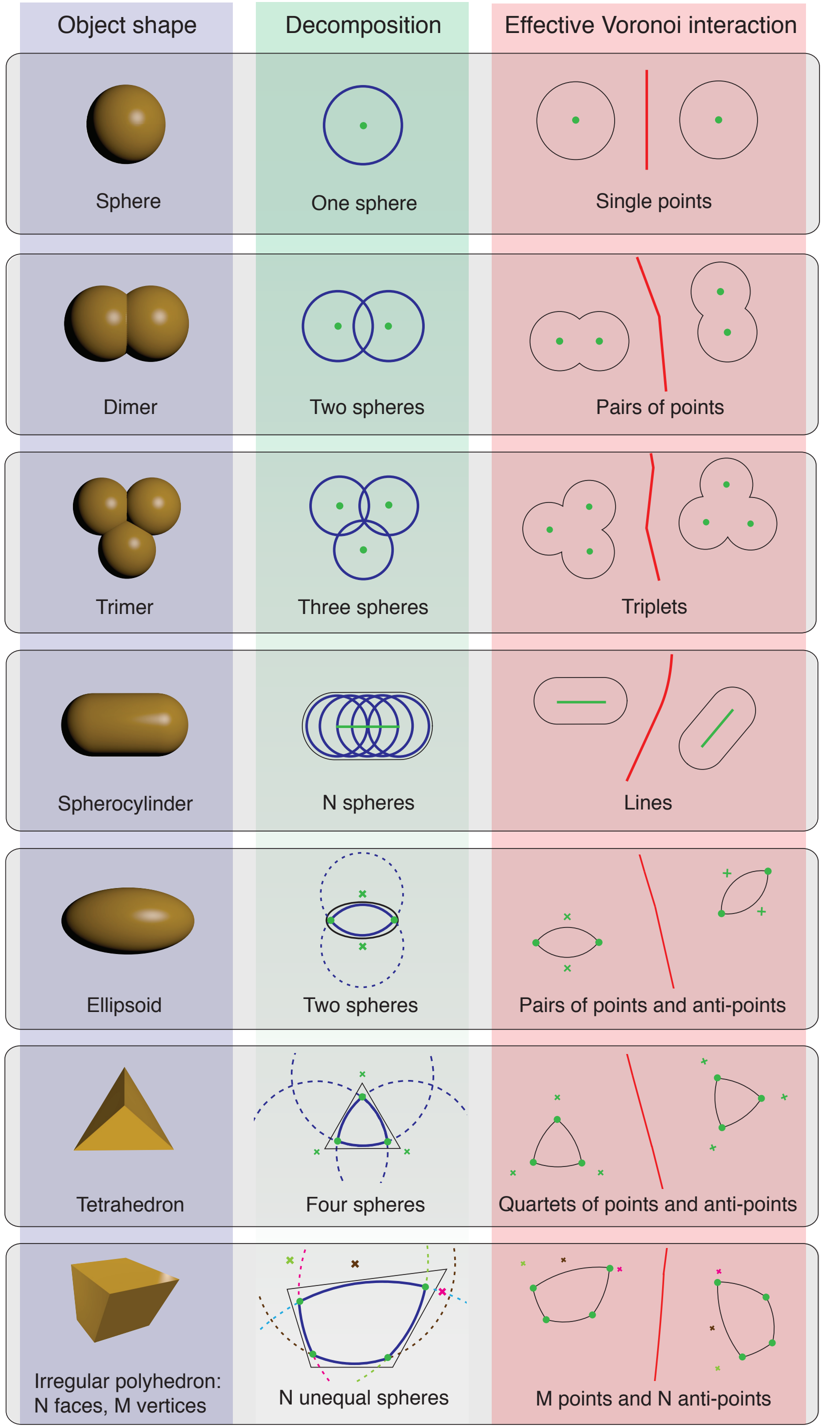
$$z < 2d_f$$

Voronoi for non-spherical shapes



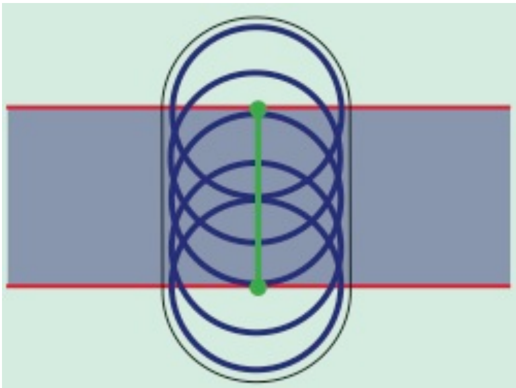
General non-spherical shapes

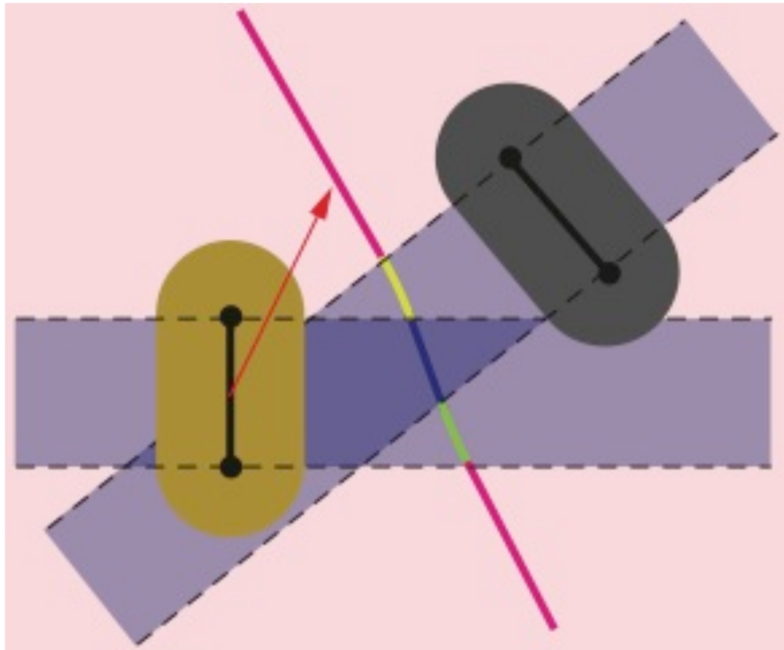



Object shape
Decomposition
Effective Voronoi interaction
Sphere
One sphere
Single points
Dimer
Two spheres
Pairs of points
Trimer
Three spheres
Triplets
Spherocylinder
N spheres
Lines
Ellipsoid
Two spheres
Pairs of points and anti-points
Tetrahedron
Four spheres
Quartets of points and anti-points
Irregular polyhedron:
N faces, M vertices
N unequal spheres
M points and N anti-points

Spherocylinders

- *Separation lines:*





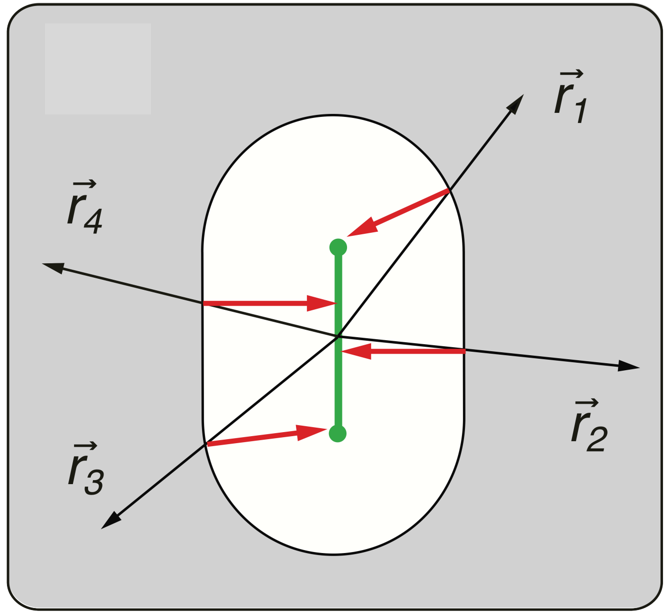
- *Four different interactions:*
  - Line – Line
  - Line – Point
  - Point – Line
  - Point – Point

Exact equation for each case $\longrightarrow$ analytic expressions for VB

Calculation of coordination number: Degenerate configurations

- Mechanical equilibrium:
  - 3 force equations
  - 2 torque equations
    (torque along symmetry axis vanishes)

Linearly independent?



$$Z_c = 2d_f = 10$$

$\longrightarrow$ Effective number of degrees of freedom can be reduced!

Degenerate configurations

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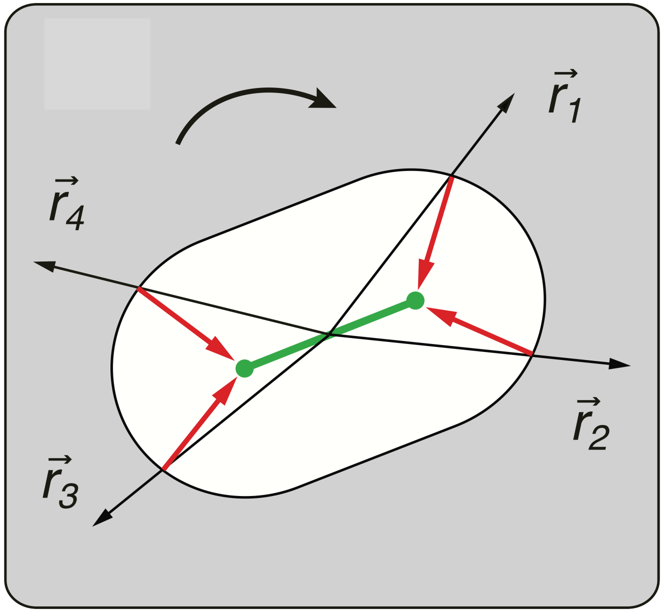
  Linearly independent?



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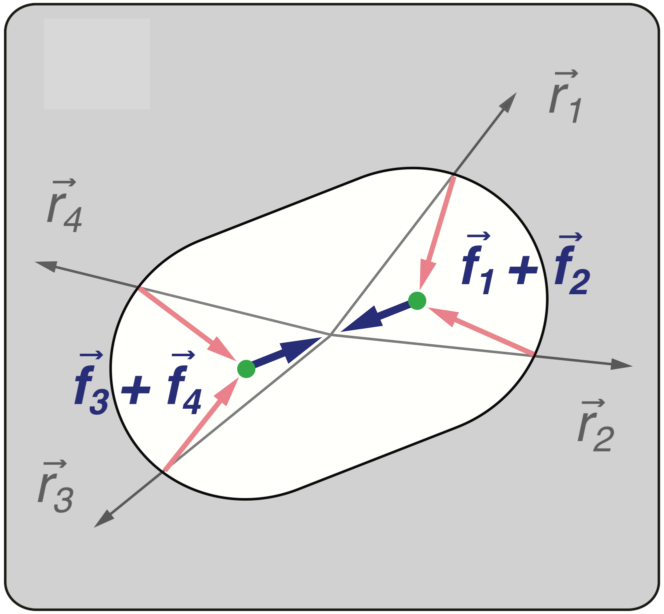
    (torque along symmetry axis vanishes)

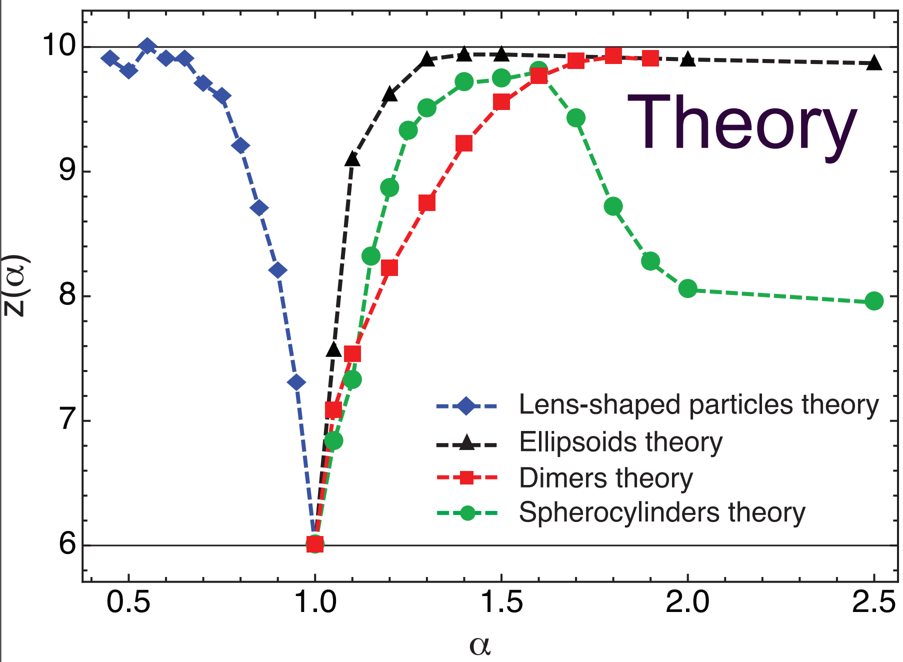
  Linearly independent?

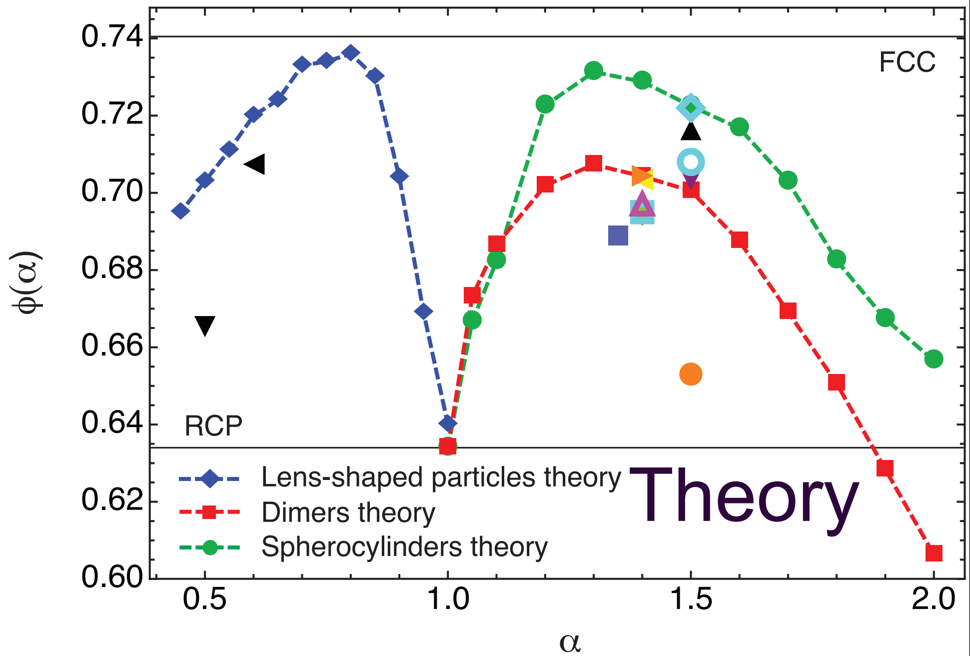


Maximal degenerate configuration: Condition of force balance automatically implies torque balance!

$$\longrightarrow \quad Z(\alpha) = 2\langle \tilde{d}_f(\alpha) \rangle$$

Theoretical predictions

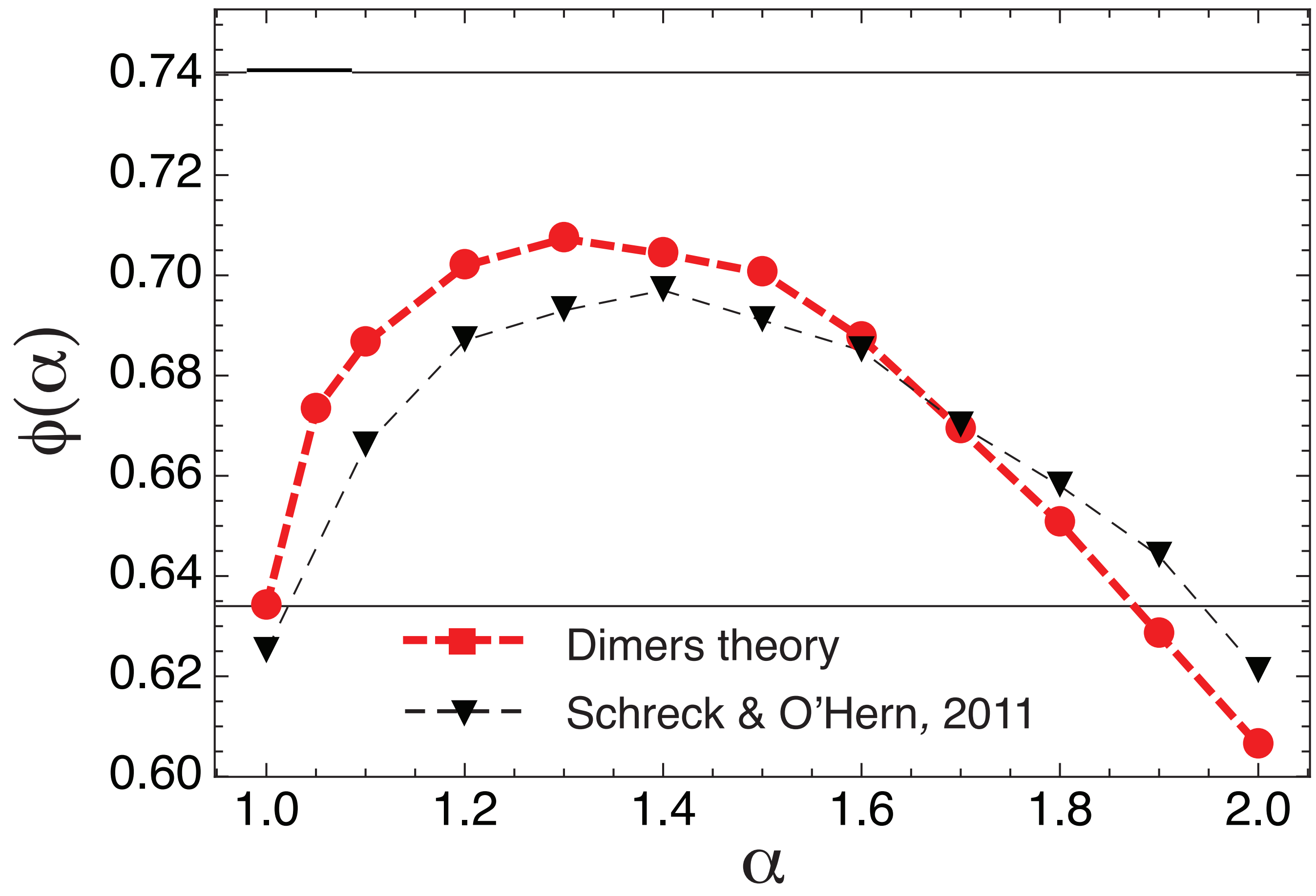




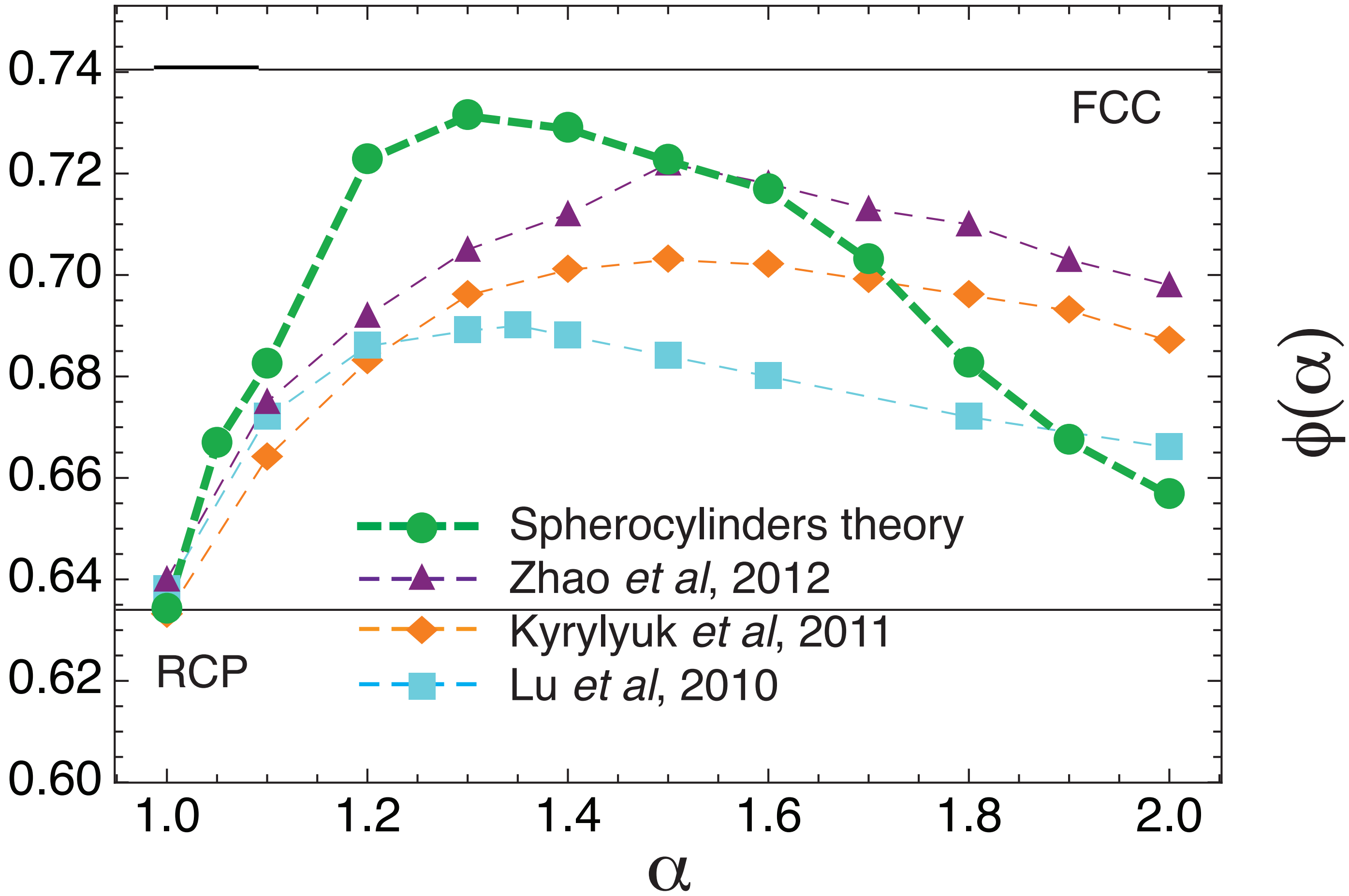
Simulations

spherocylinder	Abreu et al. 2003	●
M&M candy	Donev et al. 2004	▼
spherocylinder	Lu et al. 2010	■
spherocylinder	Jia et al. 2007	◆
spherocylinder	Williams et al. 2003	□
dimer	Schreck et al. 2011	△
dimer	Faure et al. 2009	◀
spherocylinder	Kyrylyuk et al. 2011	▼
spherocylinder	Bargiel et al. 2008	▶
oblate ellipsoid	Donev et al. 2004	◀
spherocylinder	Wouterse et al. 2009	○
prolate ellipsoid	Donev et al. 2004	▲
spherocylinder	Zhao et al. 2012	◇

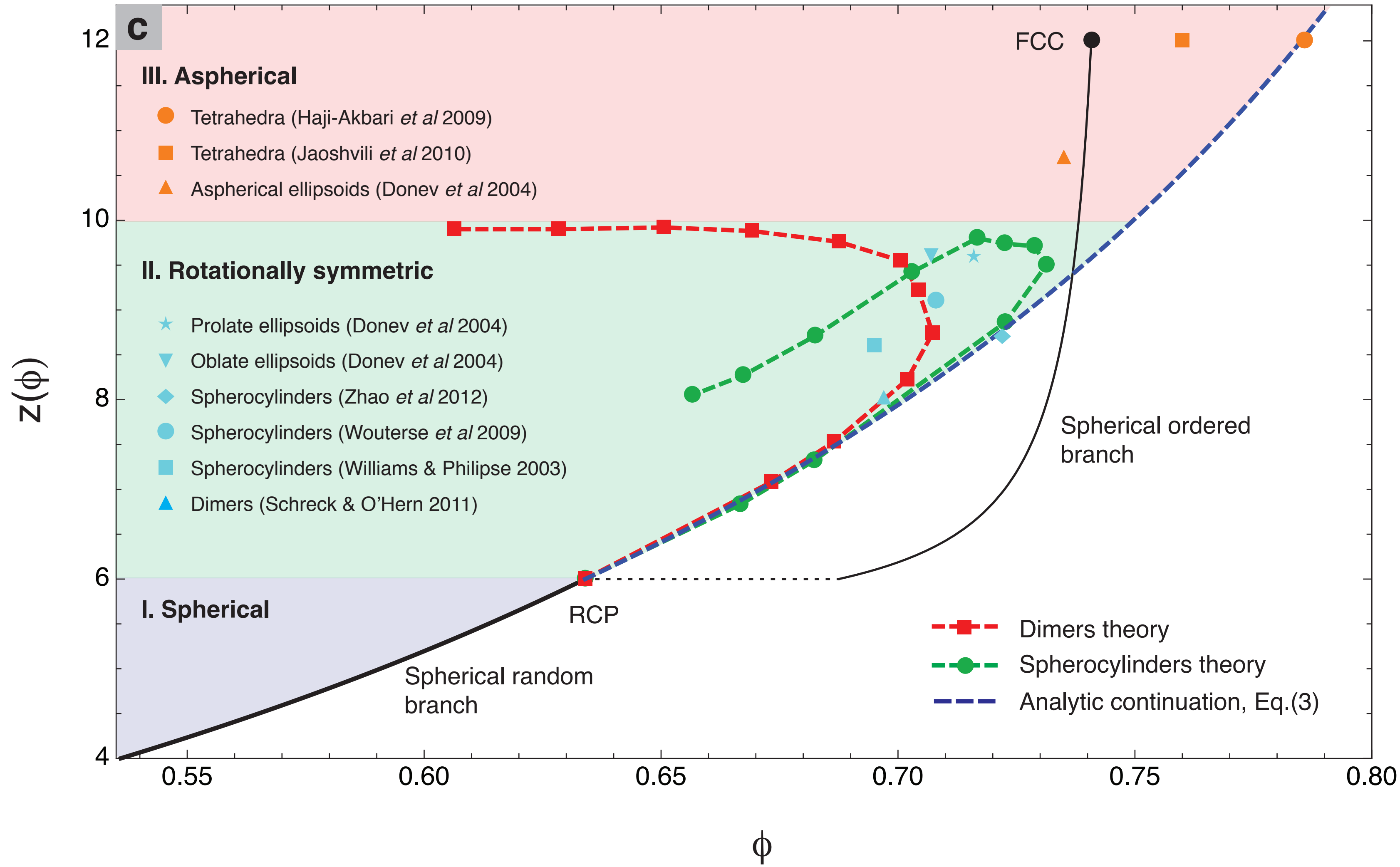
Results for packing fraction: dimers



Results for packing fraction: spherocylinders



Edwards phase diagram for many shapes



RCP is not singular: analytical continuation of spheres

Summary

Ode to Edwards!

1. Edwards ensemble to predict RCP for spheres.
2. Edwards ensemble for non-spherical particles.
3. Edwards ensemble for packings in large dimensions to compare with replica theory of hard sphere glasses.
4. Edwards replica trick or cavity method for proper average over quenched disorder for force distribution for any system: spheres, non-spheres, friction and frictionless, any dimension.
5. Extending Maxwell argument: Cavity method at RS level for solution-no solution transition to calculate Z_c from frictionless isostatic grains to frictional grains.
6. Edwards CAVEAT: 1 - 5 done at expense of drastic (yet controlled) approximations.

Cavity Method for Force Transmission

$$\mathcal{Z}(X,T) = \int \exp[-\mathcal{W}(\vec{x})/X]\mathcal{D}\vec{x} \times \int \exp[-\mathcal{H}(\vec{f})/T]\mathcal{D}\vec{f}$$

Edwards volume ensemble predicts: $\phi(Z)$

Cavity method predicts Z: $Z(\alpha)$

and Force Distribution: $P(f)$

Force transmission problem: back to Edwards (simplest model)

Edwards model = q-model = annealed disorder average

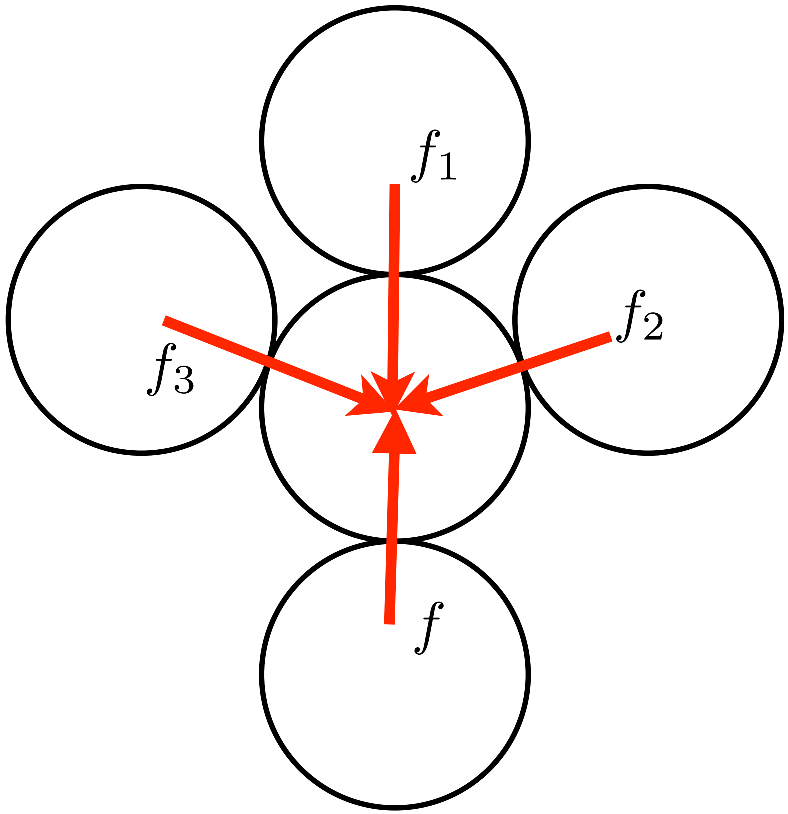
Fix $Z = 4$

Find

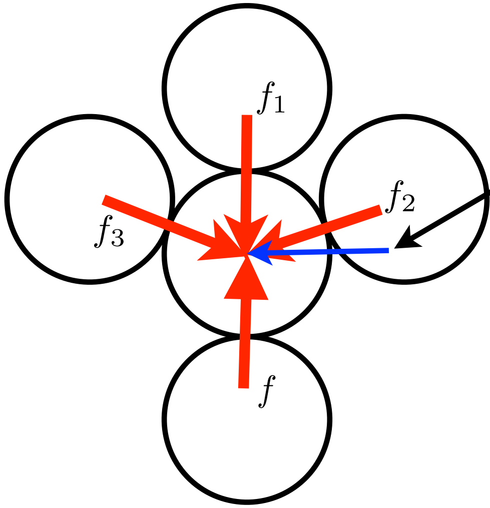
$P(f)$

with constraint

$$\vec{f} = -(\vec{f}_1 + \vec{f}_2 + \vec{f}_3)$$



Boltzmann equation for *P(f)*



$\lambda_2 f_2$: component

quenched disorder

Boltzmann equation:

assuming uncorrelated forces (MF)

$$P(f) = \int P(f_1, \lambda_1) P(f_2, \lambda_2) \boxed{\tau(\lambda_1, \lambda_2)} \delta(f - \lambda_1 f_1 - \lambda_2 f_2) d\lambda_1 d\lambda_2 df_1 df_2$$

Edwards: "Tiresomely complicated function well modelled by integrating between 0 and 1"

annealed disorder

Fourier transform:

$$\boxed{P(f) = \frac{f}{p} e^{-\frac{f}{p}}}$$

Annealed versus quenched disorder

Experimentally: first find the distribution for a fixed (quenched) packing, then average over the ensemble of packings

Average must be carried over a physical observable: free energy, not the partition function.

quenched disorder

$$F = -kT\,\overline{\ln Z}$$

↓

Replica trick(Edwards-Anderson)

$$\ln Z = \lim_{n\to 0}(Z^n - 1)/n$$

↓

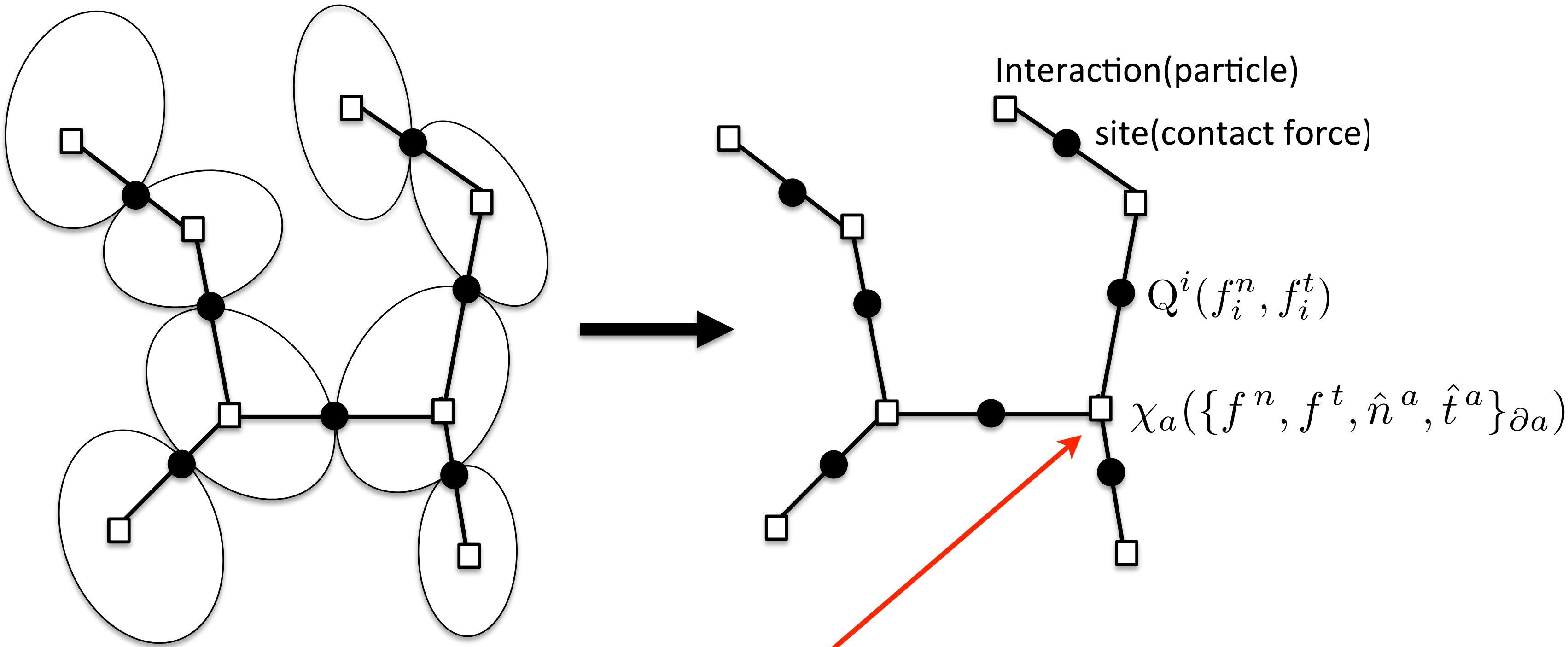
Cavity Method →

annealed disorder

$$F = -kT\,\ln \overline{Z}$$

Granular matter:
Performed average over forces
then over contact network

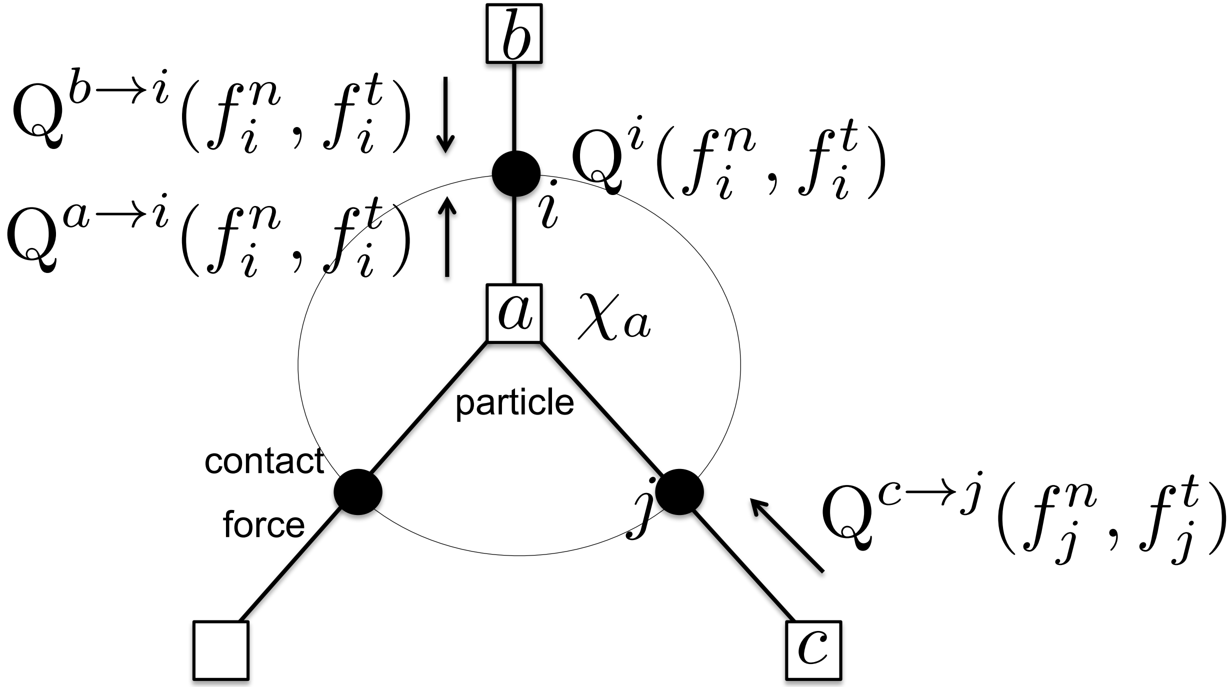
Building the factor graph of contacts from a packing



$$\chi_a(\{f^{\,n}, f^{\,t}, \hat{n}^{\,a}, \hat{t}^{\,a}\}_{\partial a}) = \delta\left(\sum_{i\in\partial a} \vec{f}_i^{\,a}\right)\delta\left(\sum_{i\in\partial a} \vec{r}_i^{\,a} \times \vec{f}_i^{\,a}\right) \times \prod_{i\in\partial a} \Theta(f_i^{\,n})\Theta(\mu f_i^{\,n} - f_i^{\,t})$$

Constraint: force balance + torque balance + repulsive + Coulomb

Compute marginal belief for a fix contact network $\mathrm{Q}^i(f_i^n, f_i^t)$



Cavity field: no average over n_j

$$\mathrm{Q}^{a\to i}(f_i^n, f_i^t) = \frac{1}{Z^{a\to i}} \int \mathrm{d}\hat{t}_i \prod_{\substack{j\in\partial a - i\\ c=\partial j - a}} \mathrm{d}f_j^n \mathrm{d}f_j^t \mathrm{d}\hat{t}_j\, \mathrm{Q}^{c\to j}(f_j^n, f_j^t)\chi_a(\{f^n, f^t, \hat{n}, \hat{t}\}_{\partial a})$$

Belief propagation

$$\mathrm{Q}^i(f_i^n, f_i^t) = \frac{1}{Z^i}\mathrm{Q}^{a\to i}(f_i^n, f_i^t)\,\mathrm{Q}^{b\to i}(f_i^n, f_i^t), \qquad \{a, b\} = \partial i$$

Force probability over an ensemble of random graphs $P(f^n)$

Degree distribution

Joint distribution of contacts positions on one particle

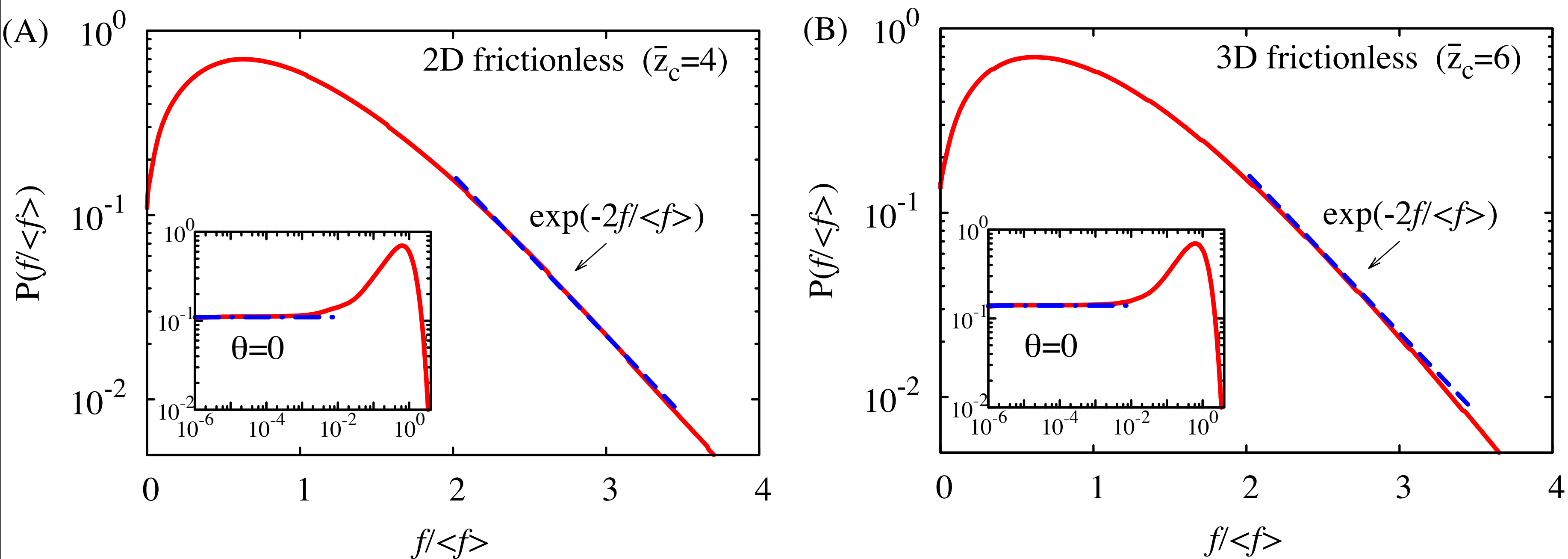
Solved with Population Dynamics

$$\mathcal{Q}(\mathrm{Q}^{\to}) = \frac{1}{\mathcal{Z}} \sum_z z\, \mathrm{R}(z) \int \Omega(\{\hat{n}_j\}) \prod_{j=1}^{z-1} \mathrm{d}\hat{n}_j \mathrm{D}\mathrm{Q}^{\to j} \mathcal{Q}(\mathrm{Q}^{\to j}) \delta\left[\mathrm{Q}^{\to} - \mathcal{F}_{\to}(\{\mathrm{Q}^{\to j}\})\right]$$

Cavity equation

$$\mathrm{P}(f^n, f^t) = \langle \mathrm{Q}^i(f^n, f^t) \rangle = \frac{1}{\mathrm{Z}} \left[\int \mathrm{D}\mathrm{Q}^{a\to i} \mathcal{Q}(\mathrm{Q}^{a\to i}) \mathrm{Q}^{a\to i}(f^n, f^t) \right]^2$$

Prediction: $P(f) \sim f^{\theta}$ $\theta = 0$ signature of jamming



The Population Dynamics Algorithm

1

- G(V, E), Initialize all cavity fields $\{\psi^{i\to a}(f_i)\}$.

2

- Draw an integer z with (edge-perspective) degree distribution P(z).
- Then pick at random z-1fields $\psi^{j\to b}(f_j)$ from the population of N fields.

3

- Generate a set of relative contact directions $n_{i(1)}, \ldots, n_{i(z-1)}$ with uniform distribution ; Particles do not overlap.

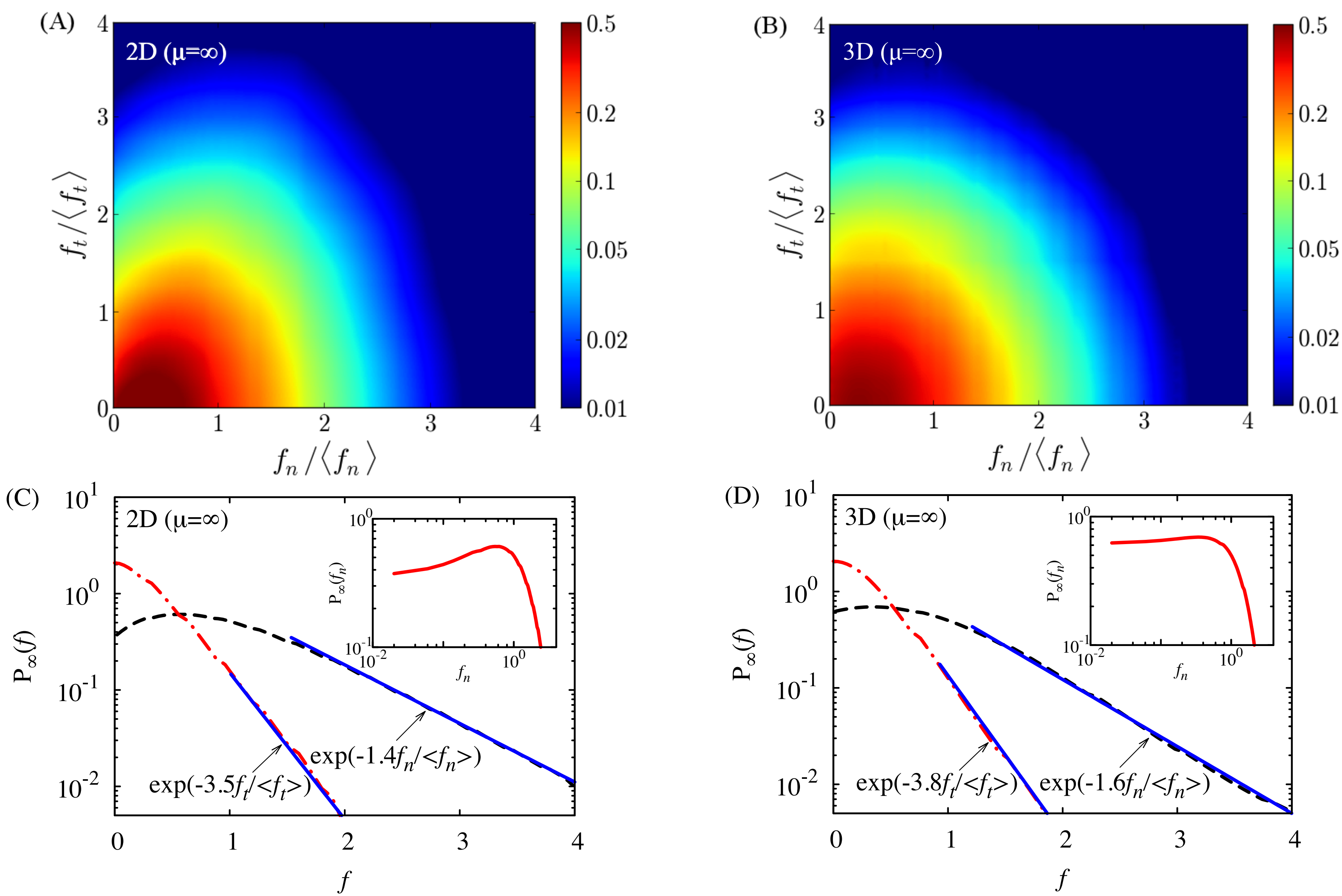
4

- update the new cavity field $\psi^{i\to a}(f_i)$ by using the incoming fields according to cavity equation.

5

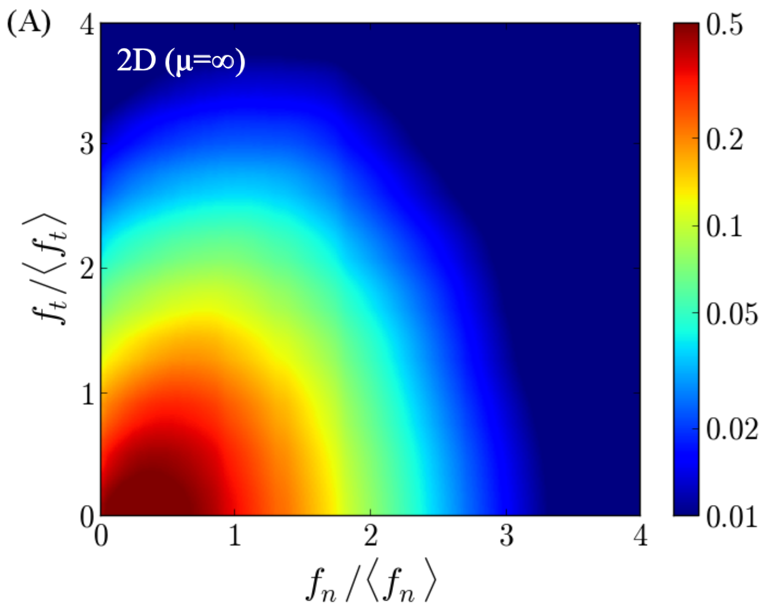
- Update all cavity fields to generate a new population. Rescale <f>=const. Run until convergence, or until the number of iterations exceeds T_{max}.

Force probability over an ensemble of random graphs $P_\mu(f^n, f^t)$

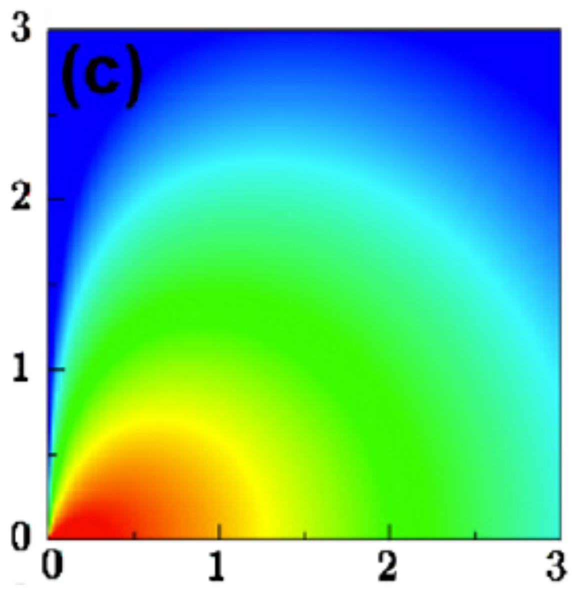


Comparison with simulations

Cavity method



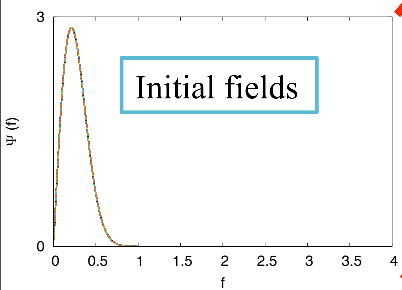
P. Wang et al. Physica A (2010)

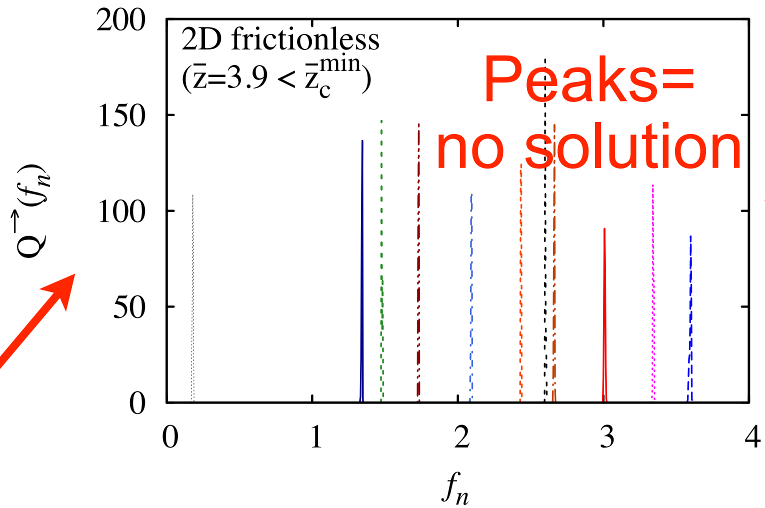


Solution-no solution transition at Zc

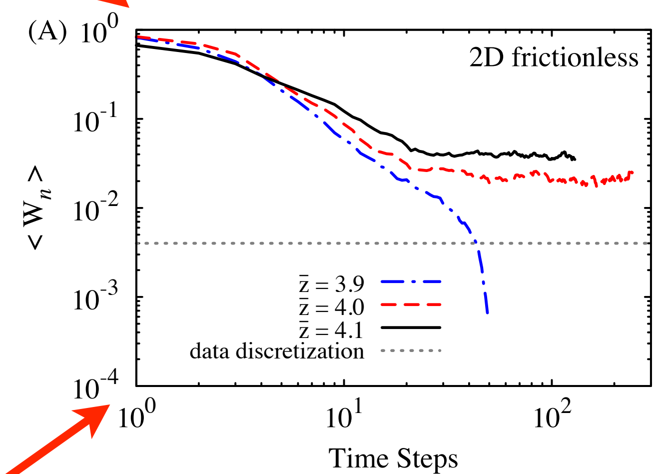
$$\mathcal{Q}(\mathrm{Q}^{\to}) = \frac{1}{\mathcal{Z}}\sum_{z} z\,\mathrm{R}(z)\int \Omega(\{\hat{n}_j\})\prod_{j=1}^{z-1} \mathrm{d}\hat{n}_j \mathrm{D}\mathrm{Q}^{\to j}\,\mathcal{Q}(\mathrm{Q}^{\to j})\,\delta\Big[\mathrm{Q}^{\to}-\mathcal{F}_{\to}(\{\mathrm{Q}^{\to j}\})\Big]$$

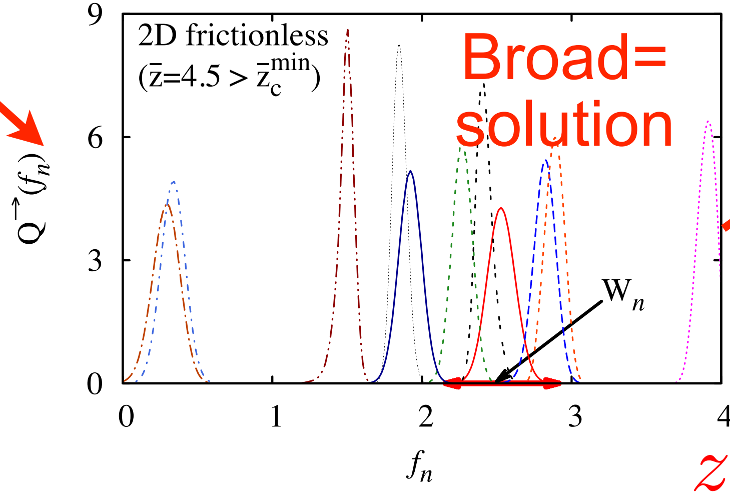
$z < z_c(\mu)$





order parameter:
WIDTH

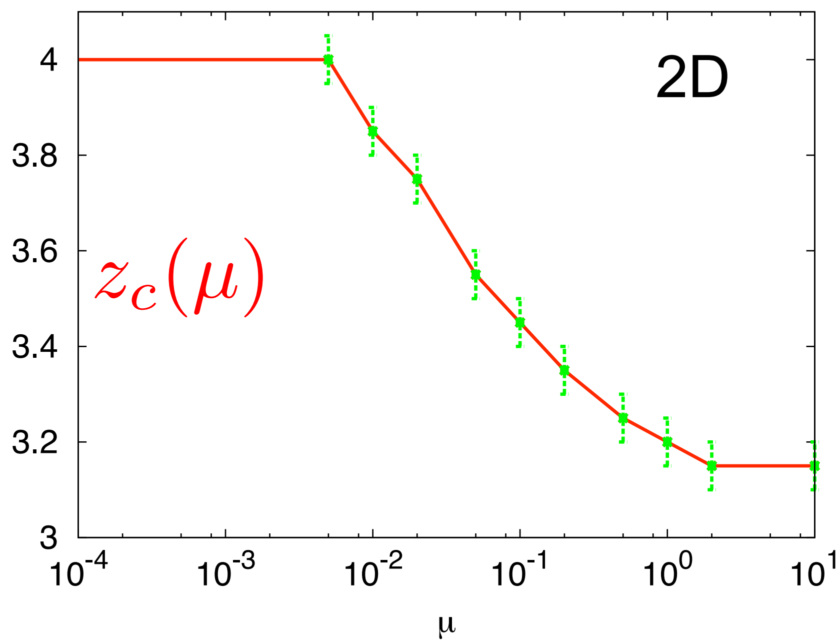




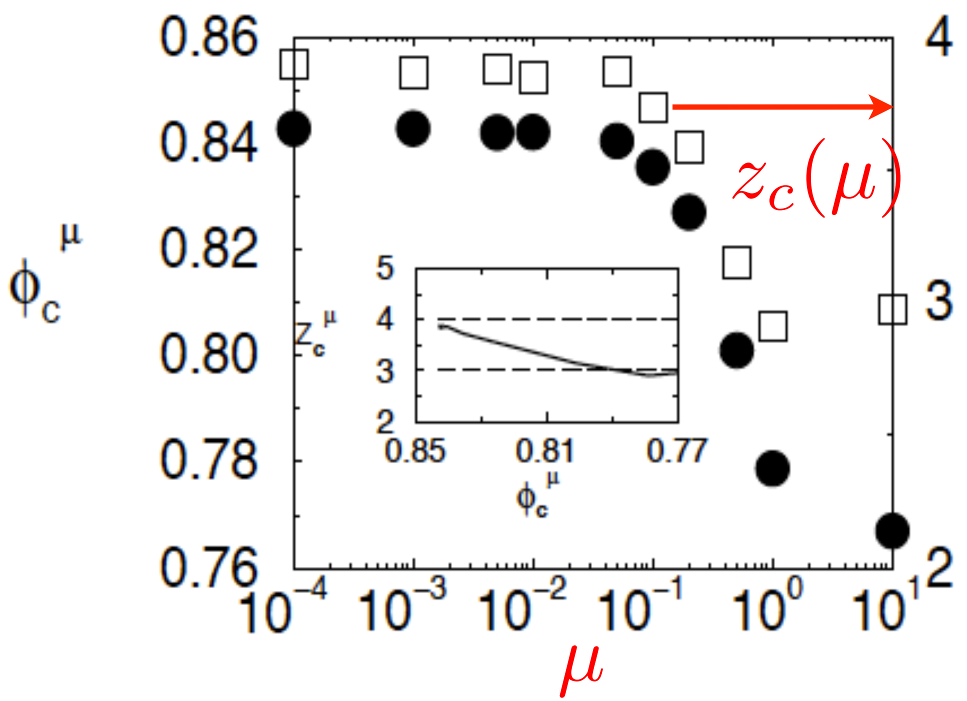
$z > z_c(\mu)$

Comparison with simulations

Cavity method



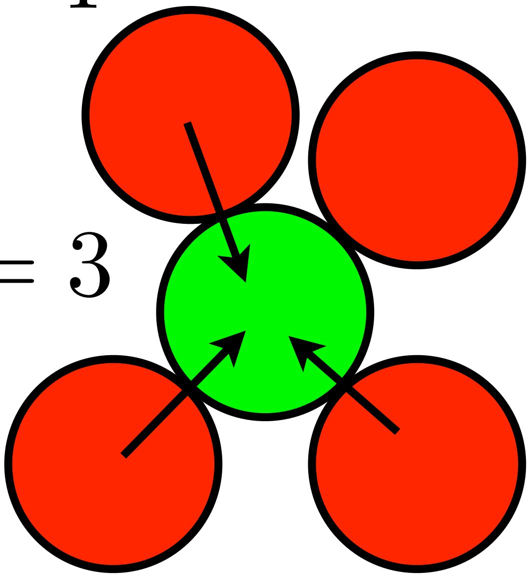
Silbert et al. PRE (2002)



Consistent with interpretation of $z_c(\mu)$ as a lower bound

Definition of jammed state: isostatic condition on Z

$z = 4$



$Z = 3$

z = geometrical coordination number. Determined by the geometry of the packing.

$$Z \leq z \leq 2d = 6$$

Z =mechanical coordination number.

Determined by force/torque balance.

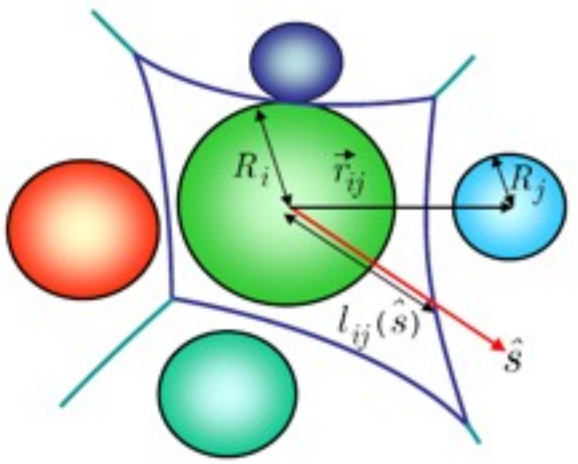
$$\underset{\mu=\infty}{4 = d+1} \leq Z \leq \underset{\mu=0}{2d = 6}$$

Generalizing the theory of monodisperse sphere packings

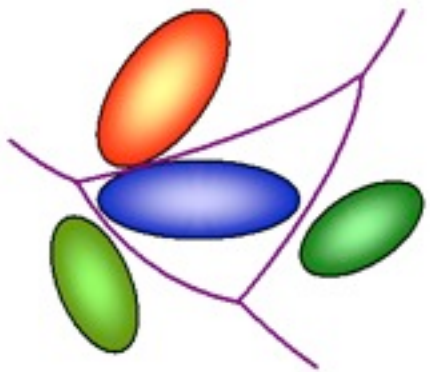
Theory of monodisperse spheres

Polydisperse (binary) spheres

Non-spherical objects



(dimers, triangles, tetrahedrons, spherocylinders, ellipses, ellipsoids …)



Distribution of radius $P(r)$ ⟷ Distribution of angles $P(\hat{s})$

Extra degree of freedom

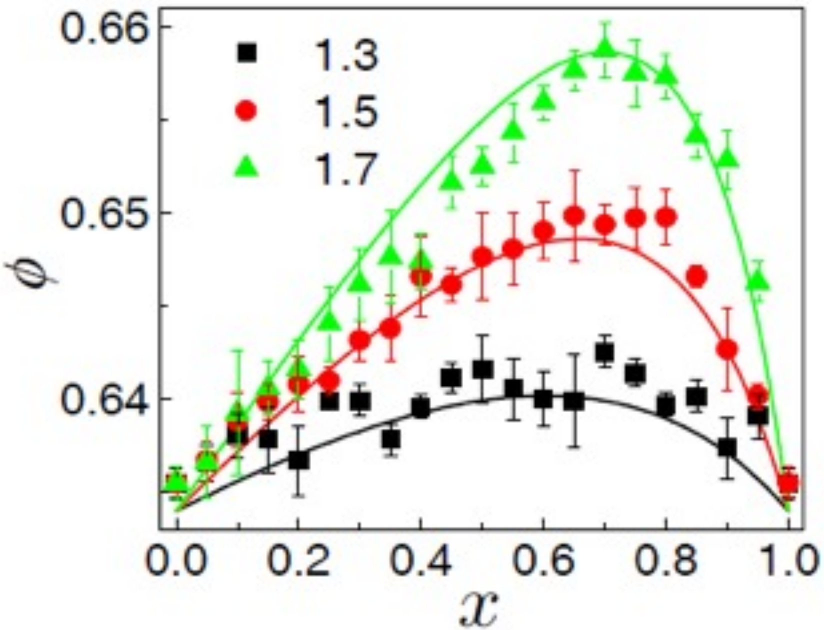
Onsager 1949

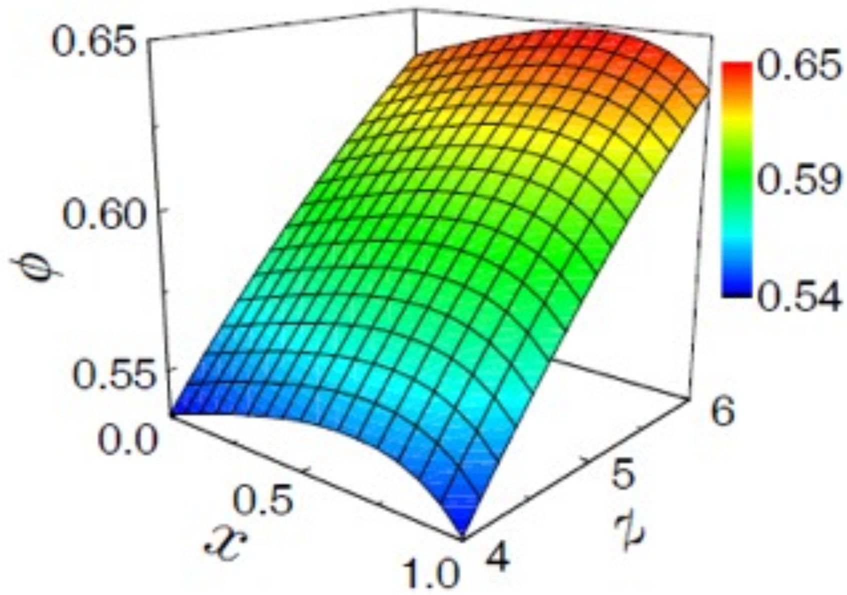
Result of binary packings

Binary packings

$$W = \frac{\pi}{2} \sum_{i=1}^{2} x_i \int_0^\infty c^2 \exp\left\{ \sum_j \left[-\rho_{ij}^s(z,x) S_{ij}^*(c) - \rho_j(W) V_{ij}^*(c) \right] \right\} dc$$

RCP (Z = 6)





Danisch, Jin, Makse, PRE (2010)

The partition function for hard spheres

Volume Ensemble + Force Ensemble

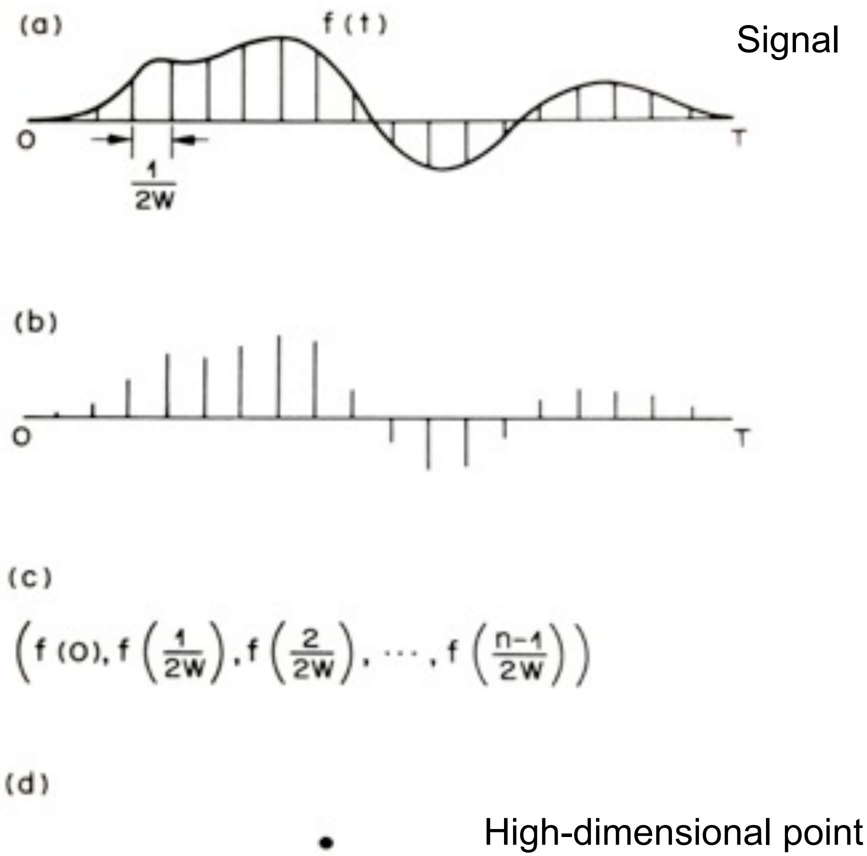
1. The Volume Function: W (geometry)

$$\boxed{\mathcal{Z}(X) = \int \exp\left[-\frac{W(\vec{x})}{X}\right] \Theta_{\mathrm{jam}}\left(\vec{x}, \vec{f}\right) \mathcal{D}\vec{x}\, \mathcal{D}\vec{f}}$$

2. Definition of jammed state: force and torque balance

Solution under different degrees of approximations

Jammed packings in infinite dimensions



Sloane

Most efficient design of signals
(Information theory)

Sampling theorem

Optimal packing
(Sphere packing problem)

Rigorous bounds

Minkowsky lower bound: $\phi \sim 2^{-d}$

Kabatiansky-Levenshtein upper bound: $\phi \sim 2^{-0.5990\ldots d}$

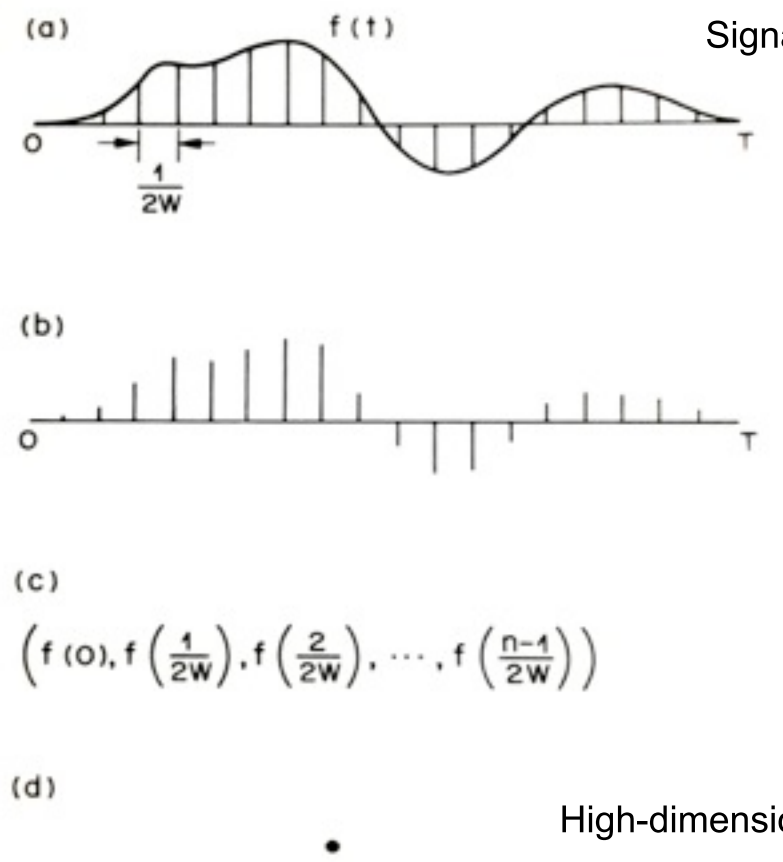
Question: what’s the density of RCP in high dimensions?

Conjecture: are disordered packings more optimal than ordered ones?

Sphere packings in high dimensions

Sloane

Signal



High-dimensional point

Most efficient design of signals
(Information theory)

Sampling theorem

Optimal packing
(Sphere packing problem)

Rigorous bounds

Minkowsky lower bound: $\phi \sim 2^{-d}$

Kabatiansky-Levenshtein upper bound: $\phi \sim 2^{-0.5990\ldots d}$

Question: what's the density of RCP in high dimensions?

Determination of a lower bound on average coordination number $\bar{z}_c^{\min}(\mu)$

$$\mathcal{Q}(\mathrm{Q}^{\rightarrow}) = \frac{1}{\mathcal{Z}} \sum_z z\,\mathrm{R}(z) \int \Omega(\{\hat{n}_j\}) \prod_{j=1}^{z-1} \mathrm{d}\hat{n}_j \mathrm{DQ}^{\rightarrow j} \mathcal{Q}(\mathrm{Q}^{\rightarrow j}) \delta\left[\mathrm{Q}^{\rightarrow} - \mathcal{F}_{\rightarrow}(\{\mathrm{Q}^{\rightarrow j}\})\right]$$