# Characteristic length and time scales for granular dynamics at low density

FRANCISCO VEGA REYES

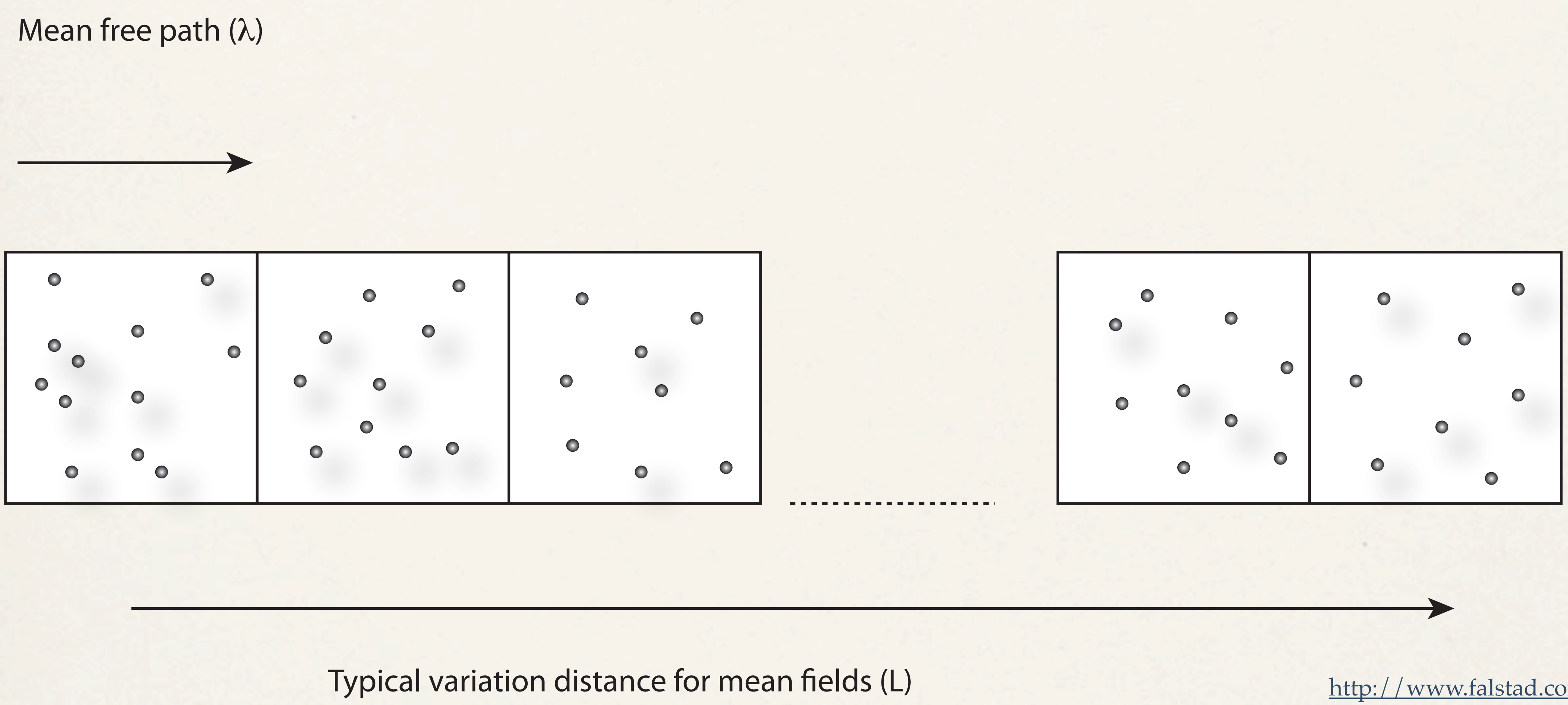
with
ANDRÉS SANTOS, VICENTE GARZÓ

and
MOISÉS G. CHAMORRO, NAGI KHALIL

*Departamento de Física, Universidad de Extremadura, Badajoz, Spain*

*"Physics Of Glassy And Granular Materials". Yukawa Institute For Theoretical Physics, Kyoto University, JAPAN, July 16th 2013.*

# The System: Inelastic hard spheres




http://www.falstad.com/gas/

Analogously, we have microscopic/macroscopic time scales: collision time ($\tau$), and characteristic time variation of mean fields (T).

# The Problem

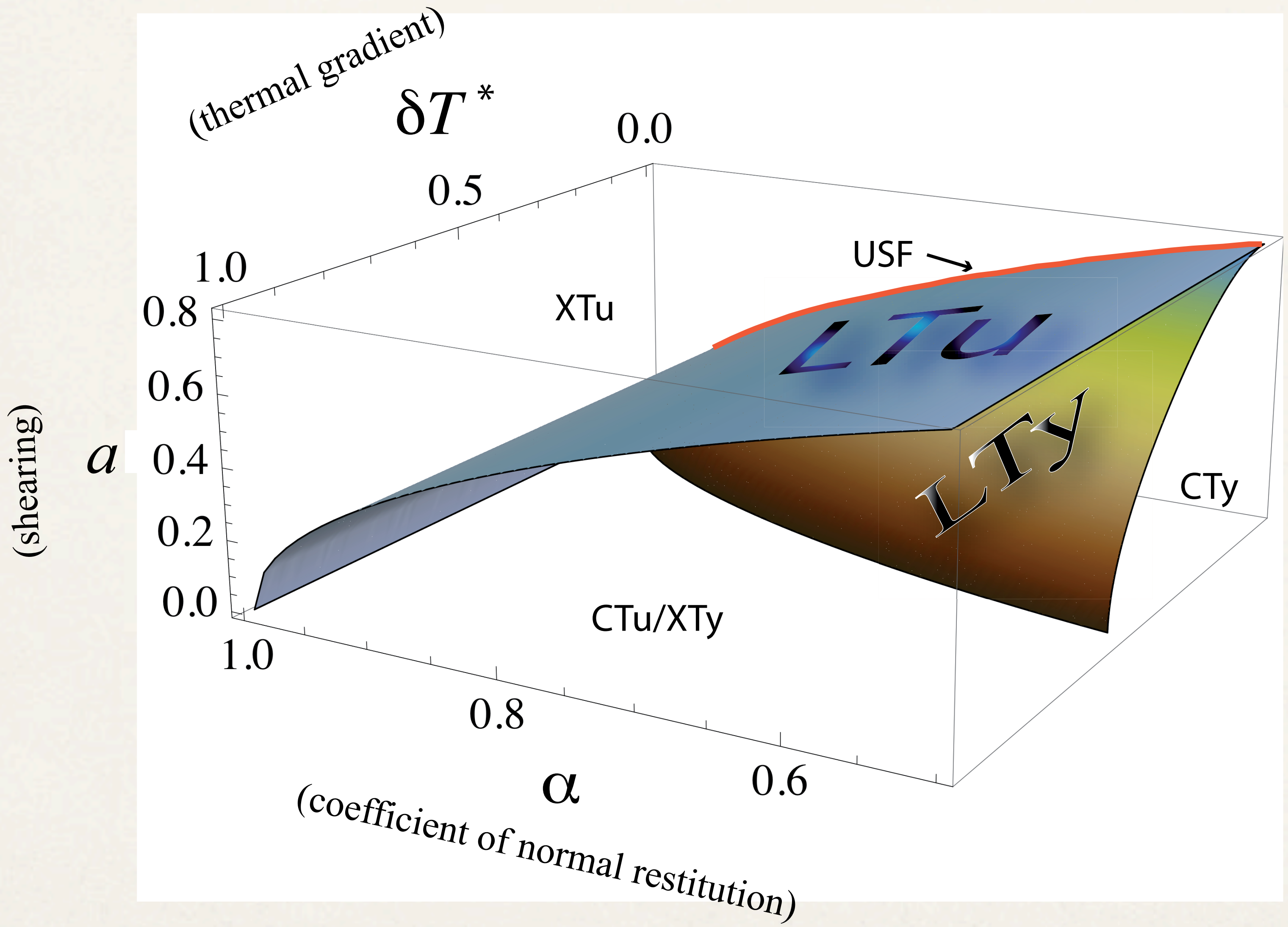
Does inelasticity alone prevents the existence of a hydrodynamic solution?

$$\mathrm{Kn} = \frac{\lambda}{L} = \frac{\tau}{T} \ll 1$$

$$\frac{\partial T}{\partial t} = -\zeta T$$

For a homogeneous system, energy balance tells us that granular temperature decreases in time according to inelasticity.

# Classes of hydrodynamic steady laminar flows




A stable laminar steady flow is always possible!

# Classes of hydrodynamic steady laminar flows

| Gas type | $-\gamma(\alpha, a)$ | $\Phi(\alpha, a)$ | Flow class |
|---|---|---|---|
| elastic, granular | $< 0$ | $< 0$ | XTu (classic Couette) |
| granular | $= 0$ | $< 0$ | LTu |
| granular | $= 0$ | $= 0$ | USF (LTu) |
| granular | $> 0$ | $< 0$ | CTu/XTy |
| granular | $> 0$ | $= 0$ | LTy |
| granular | $> 0$ | $> 0$ | CTy |

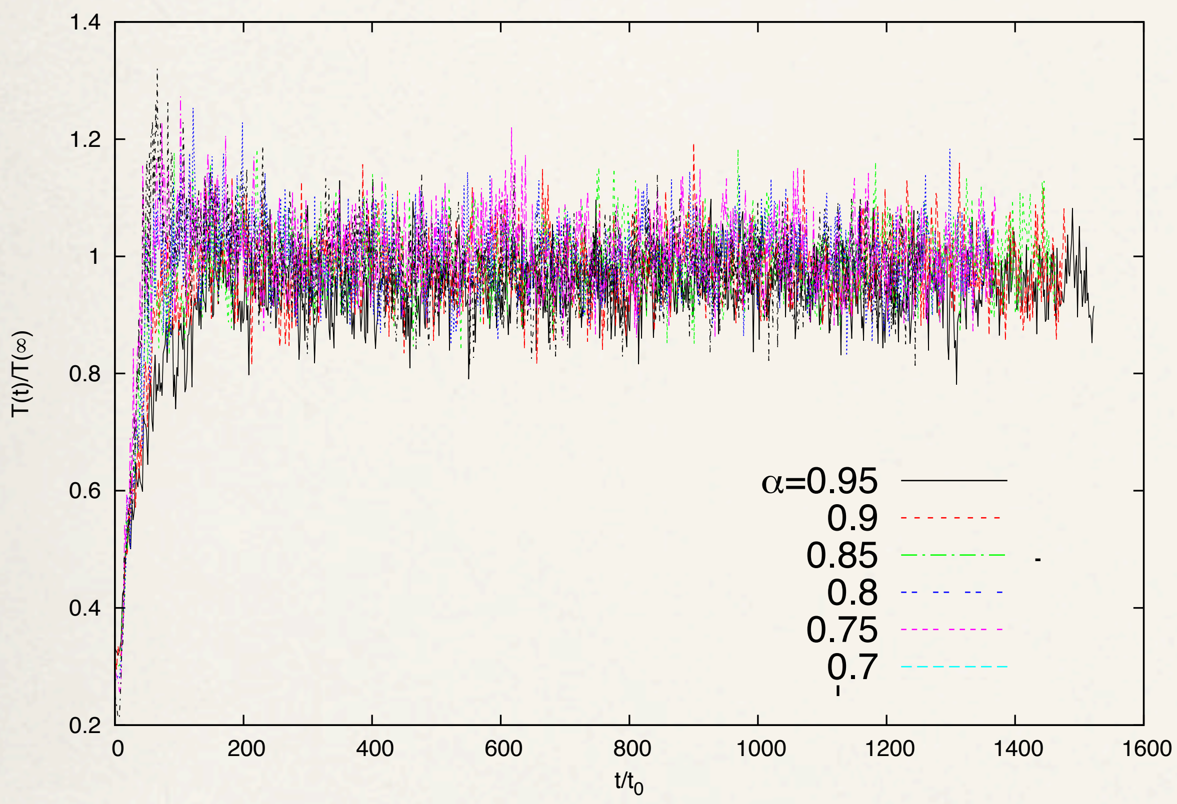
← viscous heating predominates

viscous heating
=
inelastic cooling

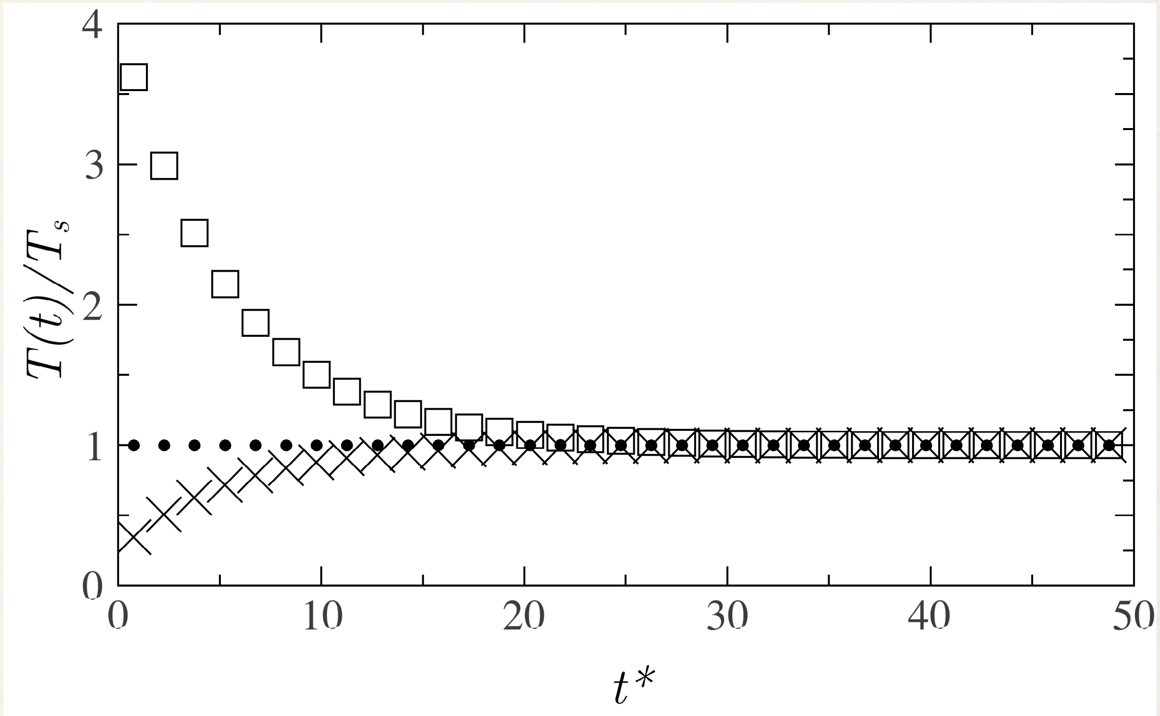
inelastic cooling predominates

# Relaxation times to hydrodynamic steady state

Couette flow




Stochastic volume forces




There is always relaxation to a hydrodynamic steady state in forced granular gases

# The kinetic equation for homogeneous state

## Boltzmann equation for inelastic rough hard spheres

$$\partial_t f(\mathbf{v}, \mathbf{w}; t) = J[\mathbf{v}, \mathbf{w}|f]$$

$J\left[\mathbf{v}|f, f\right]$, collisional operator, depends on coefficients of restitution $\alpha,\ \beta$

## Collisional rules

$$\mathbf{c}_1' = \mathbf{c}_1 - \Delta_{12}^*, \quad \mathbf{c}_2' = \mathbf{c}_2 + \Delta_{12}^*,$$

$$\mathbf{w}_1' = \mathbf{w}_1 - \frac{1}{\sqrt{\kappa\theta}}\widehat{\sigma} \times \Delta_{12}^*, \quad \mathbf{w}_2' = \mathbf{w}_2 - \frac{1}{\sqrt{\kappa\theta}}\widehat{\sigma} \times \Delta_{12}^*$$

$$\Delta_{12}^* = \widetilde{\alpha}\left(\mathbf{c}_{12} \cdot \widehat{\sigma}\right)\widehat{\sigma} + \widetilde{\beta}\left[\mathbf{c}_{12} - \left(\mathbf{c}_{12} \cdot \widehat{\sigma}\right)\widehat{\sigma} - \sqrt{\frac{\theta}{\kappa}}\widehat{\sigma} \times \left(\mathbf{w}_1 + \mathbf{w}_2\right)\right].$$

with

$$\mathbf{c} \equiv \frac{\mathbf{v} - \mathbf{u}}{\sqrt{2T_t/m}}, \quad \mathbf{w} \equiv \frac{\omega}{\sqrt{2T_r/I}},$$

$$\widetilde{\alpha} = \frac{1+\alpha}{2} \qquad \widetilde{\beta} = \frac{\kappa}{1+\kappa}\frac{1+\alpha}{2}$$

# The kinetic equation for homogeneous state

## Reduced distribution function

$$\phi(\mathbf{c},\mathbf{w}) \equiv \frac{1}{n}\left(\frac{4T_tT_r}{mI}\right)^{3/2} f(\mathbf{v},\omega).$$

## Kinetic equation for the reduced distribution functions

$$\partial_s\phi + \frac{\mu_{20}}{3}\frac{\partial}{\partial\mathbf{c}}\cdot(\mathbf{c}\phi) + \frac{\mu_{02}}{3}\frac{\partial}{\partial\mathbf{w}}\cdot(\mathbf{w}\phi) = J^*[\mathbf{c},\mathbf{w}|\phi],$$

con $\quad \mu_{pq} = -\int d\mathbf{c}\int d\mathbf{w}\, c^p w^q J^*[\mathbf{c},\mathbf{w}|\phi]. \qquad \partial_s \equiv (n\sigma^2\sqrt{2T_t/m})^{-1}\partial_t$

# The kinetic equation for homogeneous state

## Reduced distribution function

$$\phi(\mathbf{c}, \mathbf{w}) \equiv \frac{1}{n}\left(\frac{4T_t T_r}{mI}\right)^{3/2} f(\mathbf{v}, \omega).$$

## Kinetic equation for the reduced distribution functions

$$\partial_s \phi + \frac{\mu_{20}}{3}\frac{\partial}{\partial \mathbf{c}} \cdot (\mathbf{c}\phi) + \frac{\mu_{02}}{3}\frac{\partial}{\partial \mathbf{w}} \cdot (\mathbf{w}\phi) = J^*[\mathbf{c}, \mathbf{w}|\phi],$$

collision frequency

con $\mu_{pq} = -\int d\mathbf{c} \int d\mathbf{w}\, c^p w^q J^*[\mathbf{c}, \mathbf{w}|\phi].$ $\partial_s \equiv \left(n\sigma^2\sqrt{2T_t/m}\right)^{-1}\partial_t$

# Numerical solutions to the kinetic equation

1) Solution to the temporal differential equations for an expansion around the Maxwellian

$$\phi(\mathbf{c},\mathbf{w}) = \phi_M(\mathbf{c},\mathbf{w}) \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty} k_{nm\ell}\Psi_{nm\ell}(c^2,w^2,u^2),$$

Using this series expansion in the kinetic equation $\longrightarrow$

$$\boxed{\begin{aligned} -\partial_s\langle c^p w^q\rangle + \frac{1}{3}(p\mu_{20}+q\mu_{02})\langle c^p w^q\rangle &= \mu_{pq}, \\ -\partial_s\langle(\mathbf{c}\cdot\mathbf{w})^2\rangle + \frac{2}{3}(\mu_{20}+\mu_{02})\langle(\mathbf{c}\cdot\mathbf{w})^2\rangle &= \mu_b. \end{aligned}}$$

2) By means of Direct Simulation Monte Carlo (DSMC) method.

# Magnitudes definitions

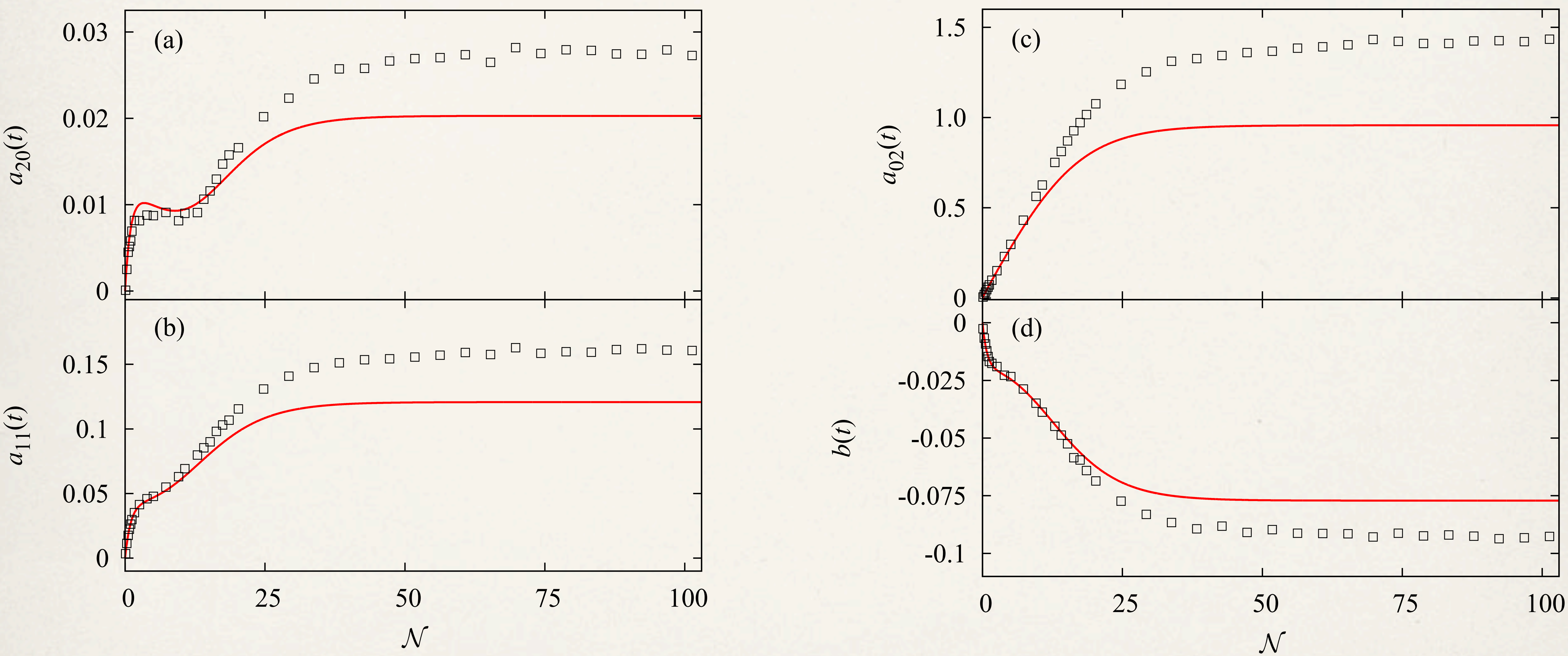
$$\phi(\mathbf{c},\mathbf{w}) \approx \phi_M(\mathbf{c},\mathbf{w})\left(1 + a_{20}L_2^{(\frac{1}{2})}(c^2) + a_{02}L_2^{(\frac{1}{2})}(w^2) + a_{11}\times L_1^{(\frac{1}{2})}(c^2)L_1^{(\frac{1}{2})}(w^2) + b\left[(\mathbf{c}\cdot\mathbf{w})^2 - \frac{c^2w^2}{3}\right]\right)$$

We neglect terms beyond second order in the expansion

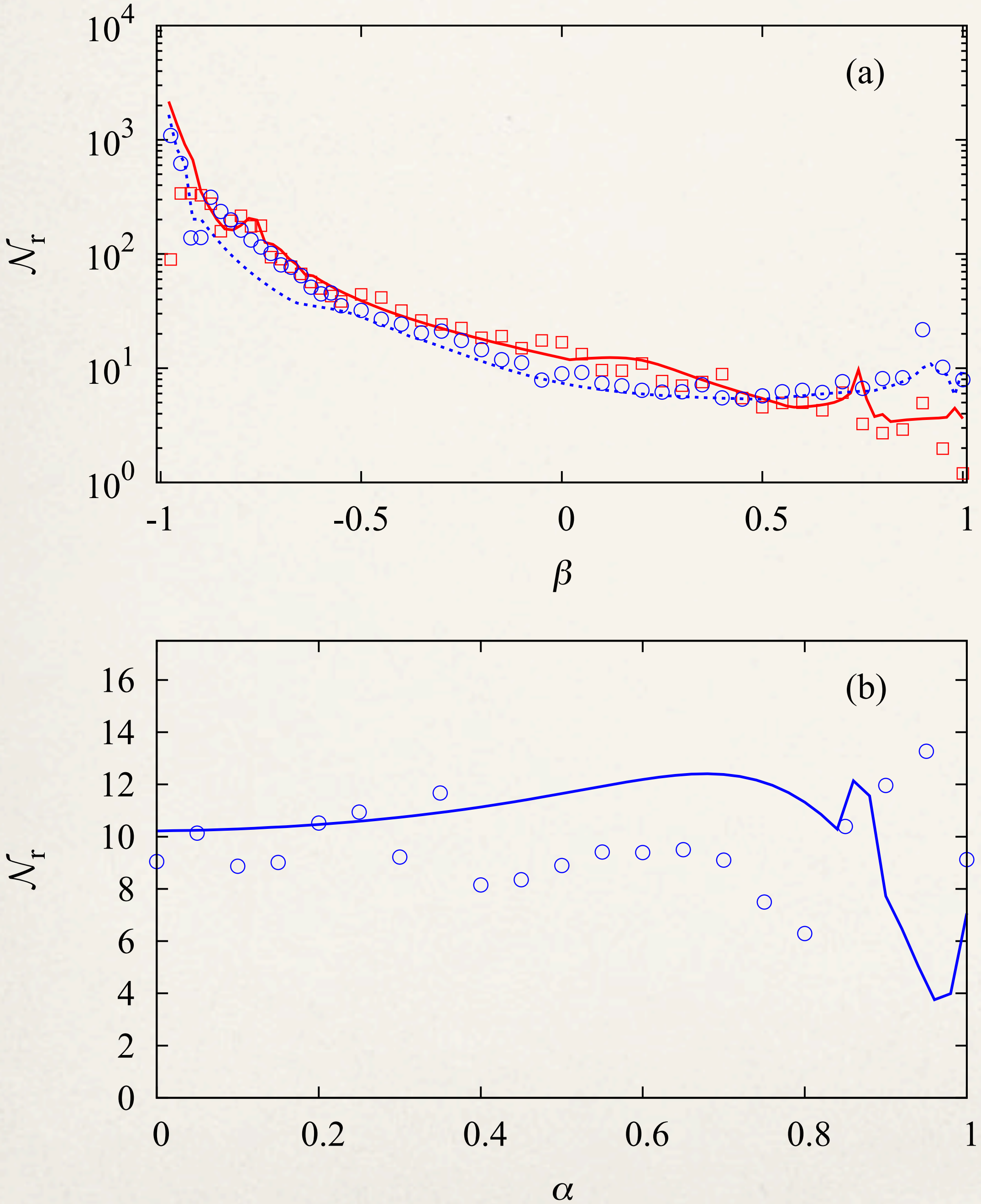
$$a_{20} = \frac{4}{15}\langle c^4\rangle - 1 \qquad a_{02} = \frac{4}{15}\langle w^4\rangle - 1$$

$$a_{11} = \frac{4}{9}\langle c^2w^2\rangle - 1 \qquad b = \frac{4}{5}\left[\langle(\mathbf{c}\cdot\mathbf{w})^2\rangle - \frac{1}{3}\langle c^2w^2\rangle\right]$$

# Comparison with Monte Carlo simulations
# Temporal Evolution.



# Comparison with Monte Carlo simulations. Relaxation to hydrodynamic state.

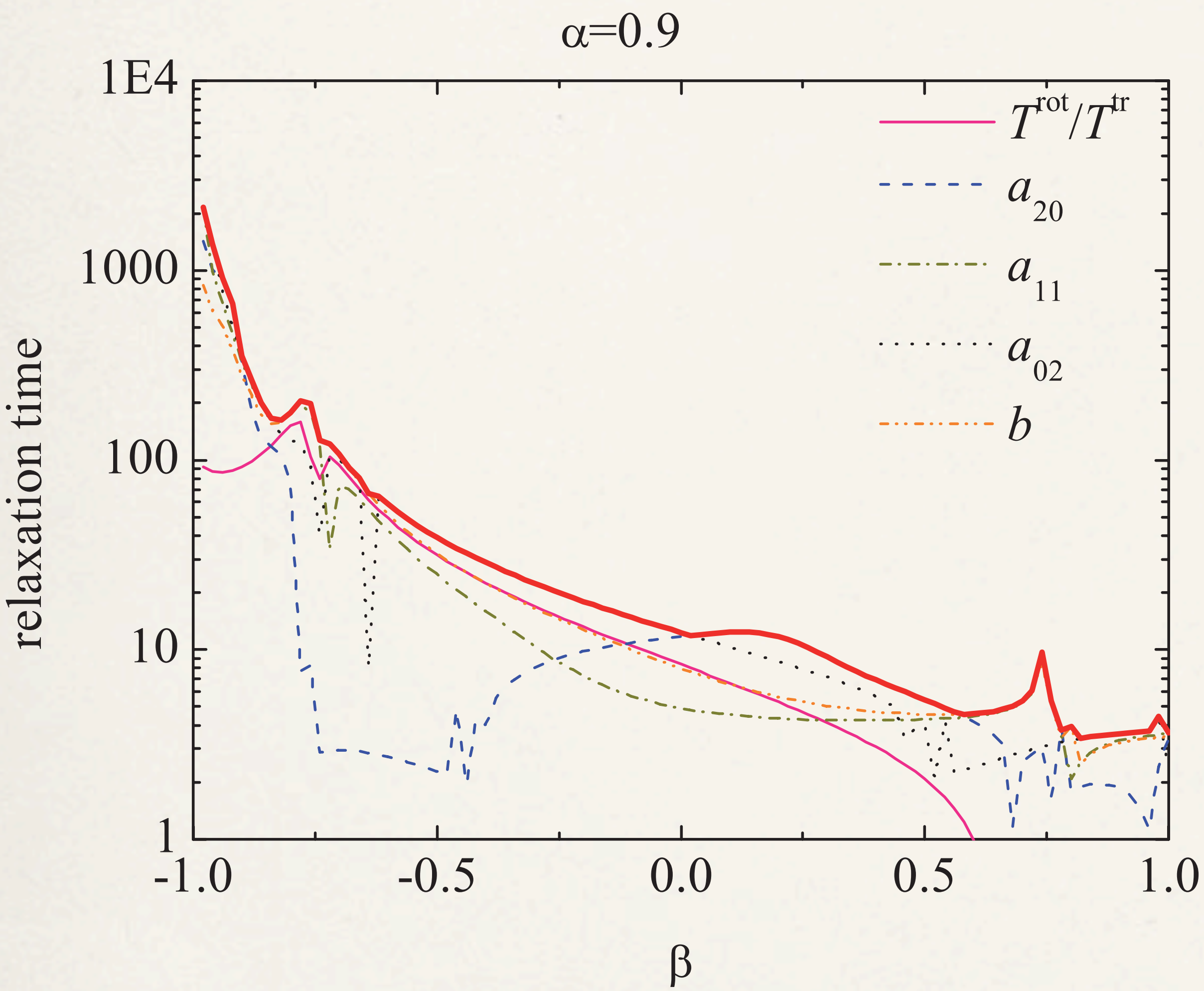


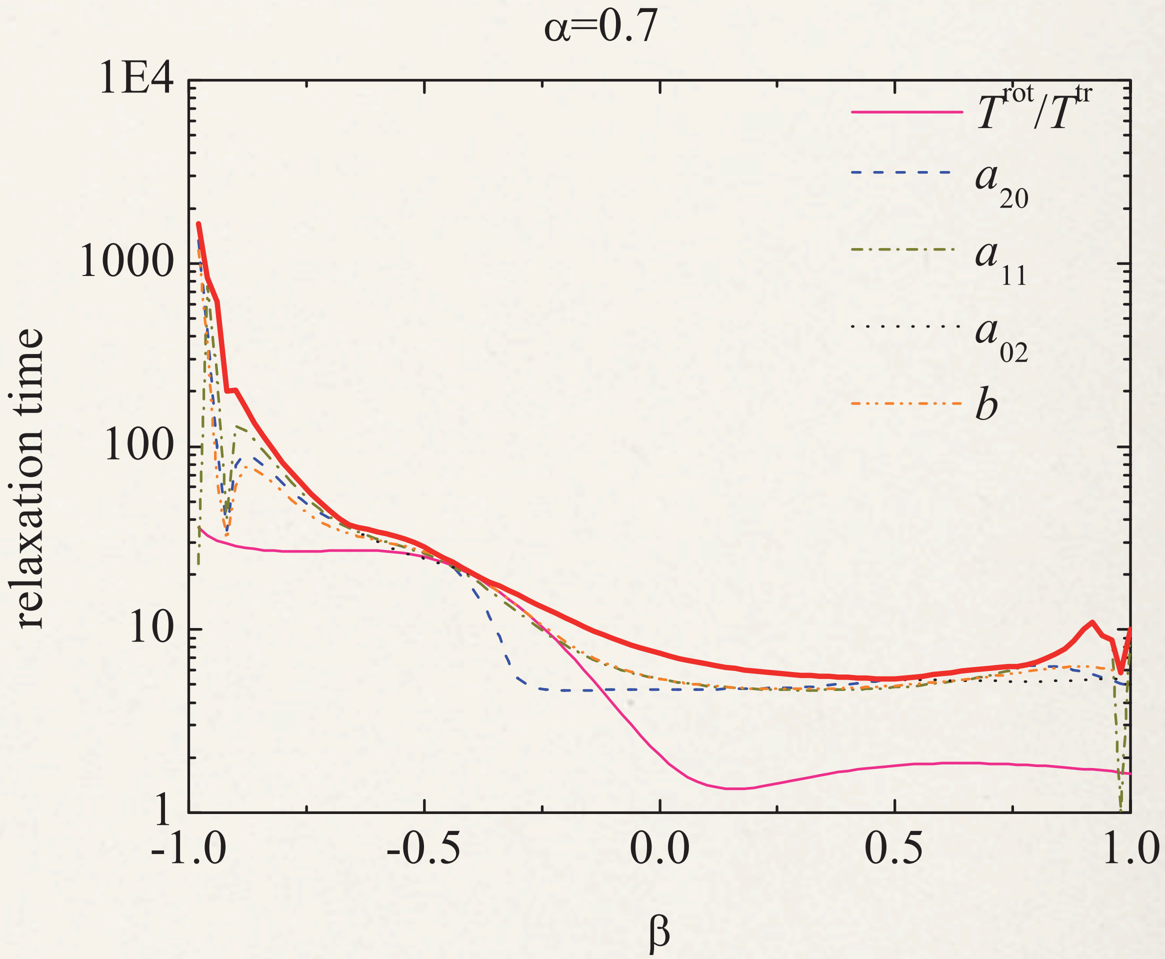

$\alpha = 0.7$ (blue)

$\alpha = 0.9$ (red)

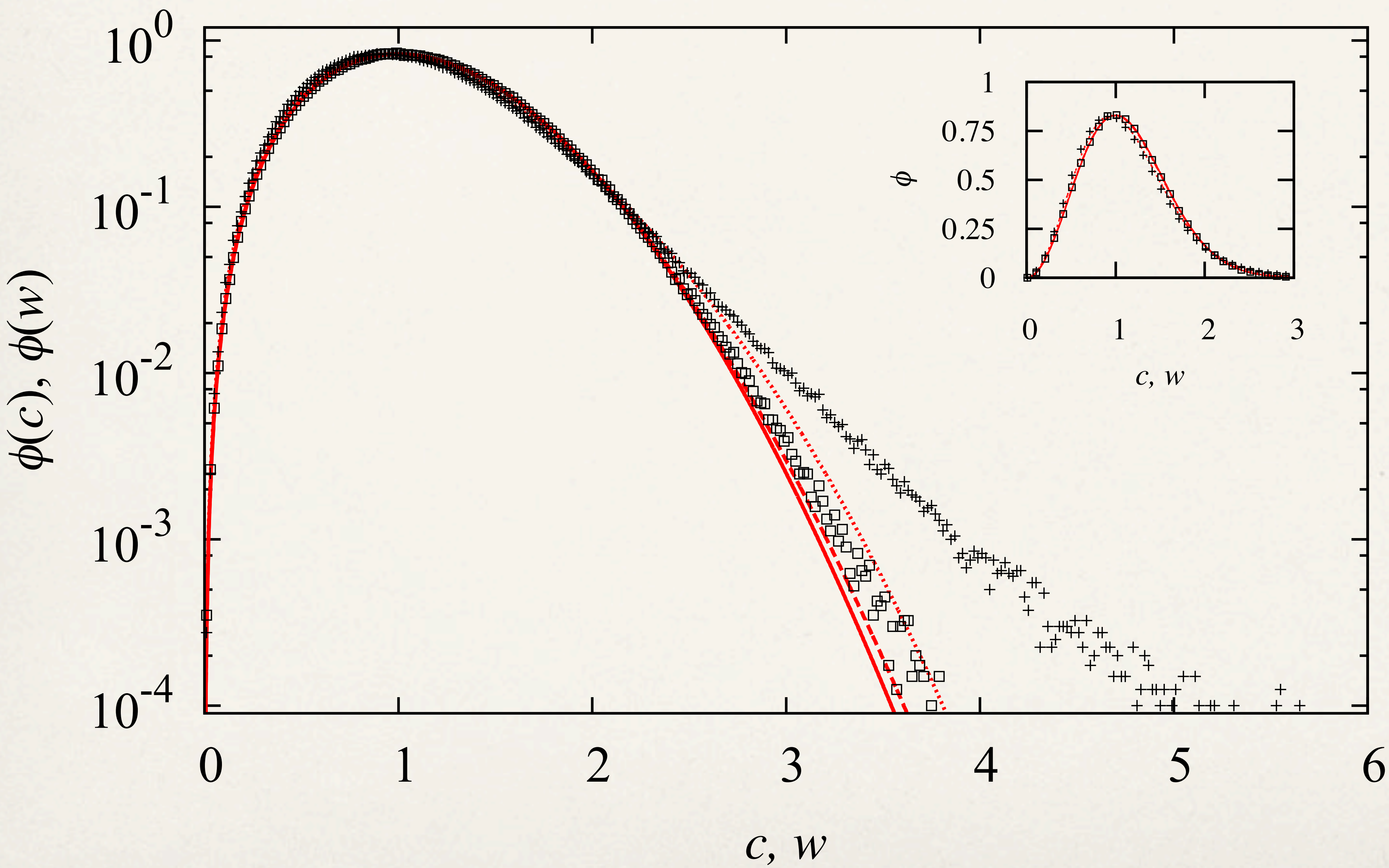
$\beta = 0$

# Comparison with Monte Carlo simulations. Relaxation to hydrodynamic state.



# Comparison with Monte Carlo simulations. Hydrodynamic distribution function.



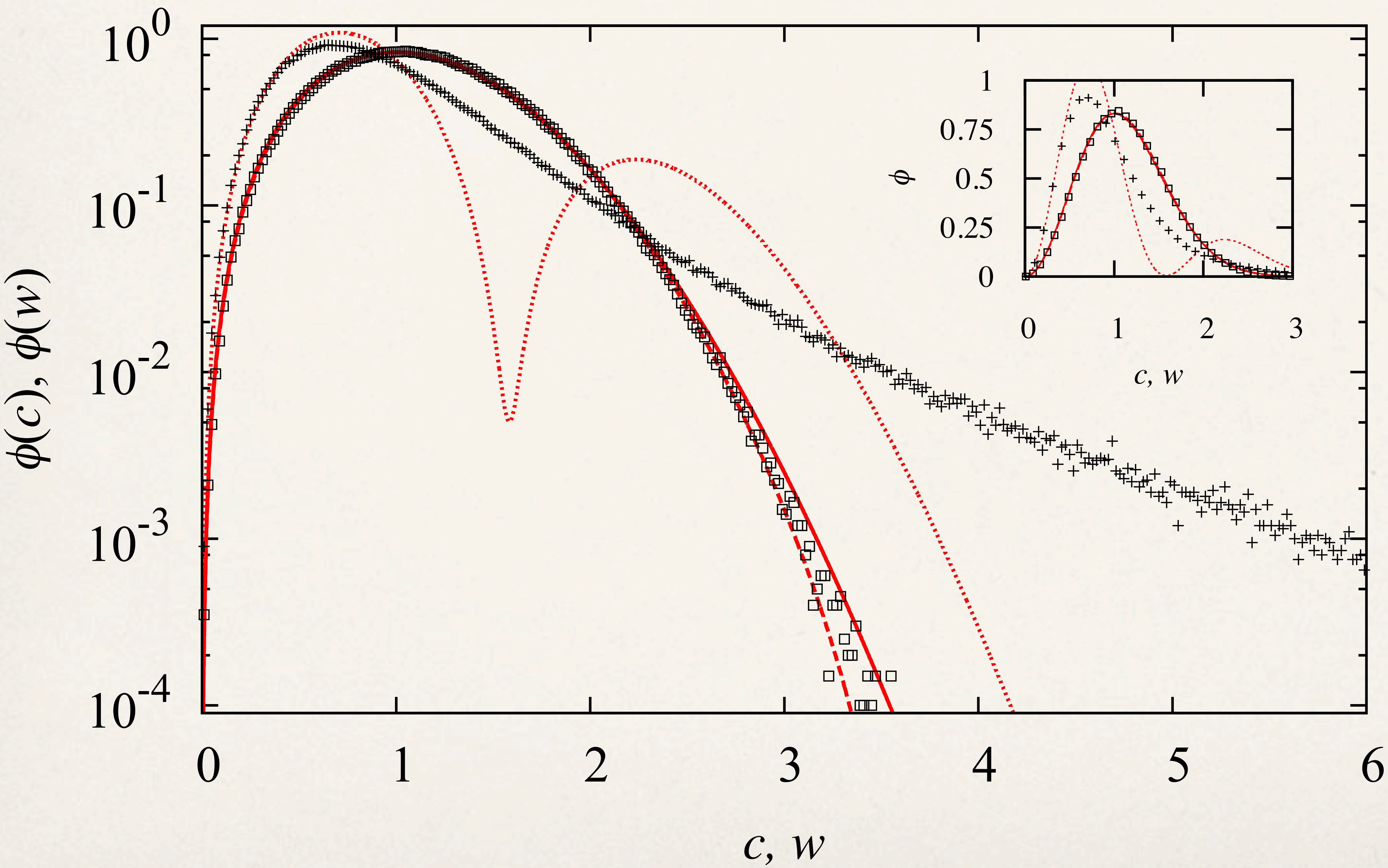

$$\alpha = 0.7$$

$$\beta = 0.5$$

$$a_{20} = 0.00971$$

$$a_{02} = 0.07204$$

# Comparison with Monte Carlo simulations. Hydrodynamic distribution function.




$\alpha = 0.9$

$\beta = -0.75$

$a_{20} = -0.02026$

$a_{02} = 0.79127$

# An experiment.
# Hydrodynamic distribution function.

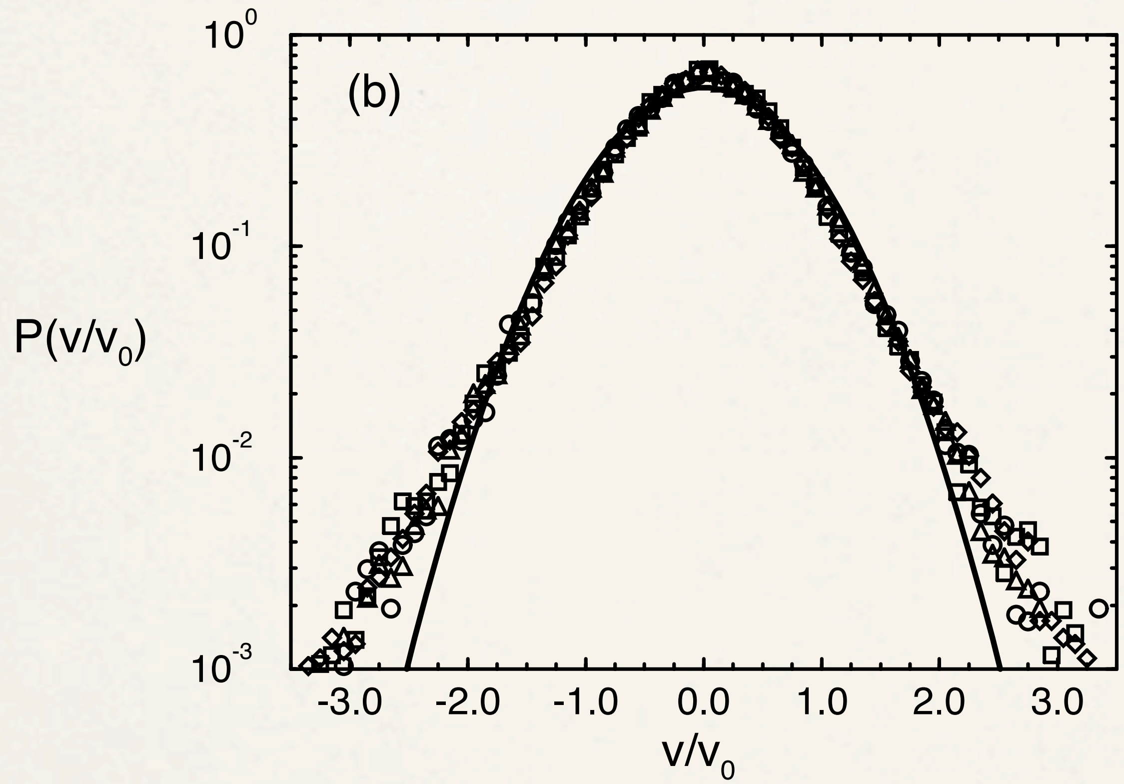



FIG. 4. Probability distribution function for a single component of the horizontal velocity on (a) linear and (b) log scales. The solid line is a Gaussian distribution. The data is (∘) $\Gamma = 1.01$, (□) $\Gamma = 0.80$, (◇) $\Gamma = 0.76$ for $N = 8000$ and $\nu = 75$ Hz; (△) $\Gamma = 1.00$ for $N = 14\,500$ and $\nu = 90$ Hz. The large population of low-speed particles is evident in (a), while (b) shows that the tails are approximately exponential. The data is scaled by $v_0 = (2v_{2m}^2)^{1/2}$.

Olafsen & Urbach,
*Phys. Rev. Lett* **81**, 4369 (1998)

# Conclusions.

- There are always hydrodynamic solutions for steady laminar flows. This is not limited by the degree of inelasticity.
- There is always a hydrodynamic solution for the homogenous cooling state, whether the spheres are smooth or rough. This is not limited by the degree of inelasticity.
- Furthermore, there is no direct relation between inelasticity and the ease with which the system reaches the hydrodynamic solution.

# Acknowledgements

Jeff Urbach and Gilberto Kremer have contributed to several of the original results presented in this talk.

# References.

1. F. Vega Reyes, J. S. Urbach, *J. Fluid Mech.* **636**, p. 279 (2009).
2. F. Vega Reyes, A. Santos and V. Garzó, *Phys. Rev. Lett.* **104**, p. 028001 (2010).
3. F. Vega Reyes, A. Santos and V. Garzó, *J. Fluid Mech.* **719**, p. 431 (2013).
4. F. Vega Reyes, A. Santos and G. M. Kremer, *"Hydrodynamic homogeneous base state for inelastic rough spheres", in preparation* (2013).
5. F. Vega Reyes, V. Garzó and N. Khalil, *"Hydrodynamic granular segregation induced by boundary heating and shear", in preparation* (2013).

# THANK YOU!!

# Expressions of polynomials in the distribution function expansion.

$$\Psi_{nm\ell}(c^2, w^2, u^2) = L_n^{(2\ell+\frac{1}{2})}(c^2) L_m^{(2\ell+\frac{1}{2})}(w^2) \times \left(c^2 w^2\right)^\ell P_{2\ell}(u)$$

$$u^2 \equiv (\mathbf{c} \cdot \mathbf{w})^2 / c^2 w^2$$

$$L_1^{(\alpha)}(x) = \alpha + 1 - x$$

$$L_2^{(\alpha)}(x) = \frac{(\alpha+1)(\alpha+2)}{2} - (\alpha+2)x + \frac{1}{2}x^2$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

# Expressions of the collisional moments.

$$\mu_{20} = 4\sqrt{2\pi}\left[\left(\widetilde{\alpha}(1-\widetilde{\alpha})+\widetilde{\beta}(1-\widetilde{\beta})\right)\left(1+\frac{3a_{20}}{16}\right)-\theta\frac{\widetilde{\beta}^2}{\kappa}\left(1-\frac{a_{20}}{16}+\frac{3a_{11}-b}{12}\right)\right]$$

$$\mu_{02} = 4\sqrt{2\pi}\frac{\widetilde{\beta}}{\kappa}\left[\left(1-\frac{\widetilde{\beta}}{\kappa}\right)\left(1-\frac{a_{20}}{16}+\frac{3a_{11}-b}{12}\right)-\frac{\widetilde{\beta}}{\theta}\left(1+\frac{3a_{20}}{16}\right)\right]$$