量子情報物理学
～量子情報と量子物理の本質的融合から新たな地平を開く～
2013年12月5日（木）＠基研

エンタングルメント繰り込み群・情報幾何とAdS/CFT対応

松枝宏明（仙台高専）

HM, M. Ishihara, Y. Hashizume, Phys. Rev. D87, 066002 (2013)
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お知らせ

次回の春の物理学会
領域11・素粒子論領域・領域1合同シンポジウム
「情報・量子物理・幾何学の絡み合い」
3月27日（学会初日・午後）

松枝：趣旨説明

鈴木増雄：量子古典変換の歴史と最近の発展
奥西功一：DMRGからMERAまで
丸山勲：量子可積分系のエンタングルメント構造

押川正毅：トポロジカル量子系エンタングルメント構造と幾何学
長岡浩司：情報幾何の方法
高柳匡：MERAとAdS/CFT，今後の展開

Equivalence formulae between physically different systems

Bulk Geometry ⇔ Algebraic properties of a system on the edge

Statistical physics: Suzuki–Trotter transformation

d–dimensional "quantum" system
⇔ (d+1)–dimensional "classical" system 1975

Statistical physics: Multiscale Entanglement Renormalization Ansatz (MERA) 2007

String theory: Anti–de Sitter space / Conformal Field Theory (AdS/CFT) correspondence 1997

(d+1)–dimensional General relativity on AdS space
⇔ CFT living on the boundary of the space

Condensed matter: Edge Modes in Topological Insulators

(2+1)–dimensional Einstein–Hilbert action 1986
⇔ Chern–Simons action ⇒ Virasoro algebra

Target I: connection between MERA and AdS/CFT

Explore similarity and/or difference between MERA and AdS/CFT

MERA ⟷ **AdS/CFT**

MERA

Quantum side: CFT

Holography:

non-commutative graph

(although the tensor network seems to be discrete AdS)

AdS/CFT

Quantum side: CFT

Holography:

classical AdS space

Solution of Einstein eq.

Questions:

(a) How about finite-T properties between them ?

(b) Is the finite-T method numerically efficient ?

Anti-de Sitter (Hyperbolic) Space and CFT

$$\eta_{ij} = \begin{pmatrix} -1 & 0 & \cdots \\ 0 & 1 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Metric of AdS space   z: radial axis, z→0: boundary

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = \frac{l^2}{z^2}\left(dz^2 + \eta_{ij}\, dx^i dx^j\right)$$

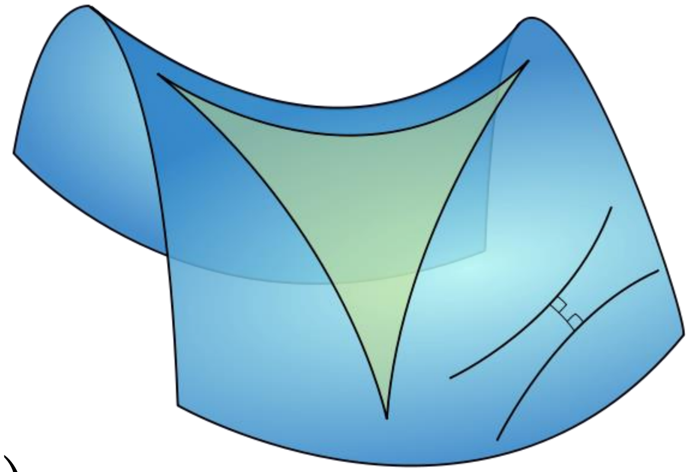
$$ds \sim \frac{l}{z} dz$$

$$s \sim l \log z$$

Infinitesimal trans.

$$\bar{x}^i = x^i + \xi^i(x)$$

$$\bar{z} = z + z\zeta(x)$$



$z \to 0$

$$d\bar{s}^2 = \bar{g}_{\mu\nu}\, d\bar{x}^\mu d\bar{x}^\nu$$

$$= ds^2 + \left(\partial_i \xi_j + \partial_j \xi_i - 2\zeta\, \eta_{ij}\right) dx^i dx^j$$

Isometry trans. ⇒ conformal Killing equation at z→0

Boundary of AdS_{d+1} ⇒ CFT_d

AdS/CFT correspondence and Holographic Entropy

Gubser-Klevanov-Polyakov(GKP)-Witten relation

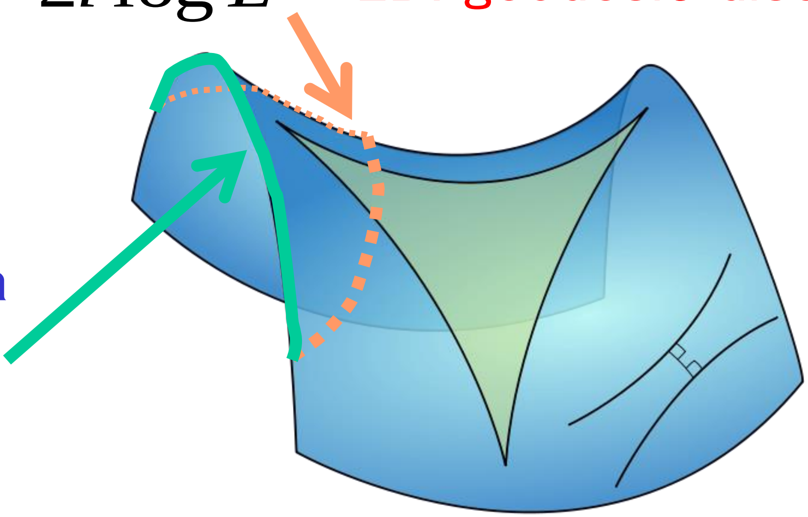
$$\langle O(x_1)\cdots O(x_n)\rangle_{CFT} = \frac{\delta}{\delta\phi(x_1)}\cdots\frac{\delta}{\delta\phi(x_n)}\exp\left(-\frac{1}{2\kappa}I(\phi(x))\right)\Big|_{\phi=0}$$

Ryu-Takayanagi Formula (extension of Beckenstein-Hawking)

2006 $$S = \frac{\gamma}{4G}$$

$$\gamma = 2l\log L$$ 2D: geodesic distance

γ : area of minimal surface



Calabrese-Cardy formula

$$S = \frac{1}{3}c\log L$$

$$c = \frac{3l}{2G}$$ Brown-Henneaux central charge

An example of tensor-network-type variational functions in quantum many-body systems

MERA = a variant of tensor-network wave functions

S=1/2 Heisenberg Antiferromagnet (2 sites)

$$H = J\,\vec{S}_1 \cdot \vec{S}_2 = \frac{J}{2}\left(S_1^+ S_2^- + S_1^- S_2^+\right) + J\,S_1^z S_2^z$$

Singlet ground state (entangled) $\quad |0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$

General variational function $\qquad$ A, B, C, D: Non-local

$$|\psi\rangle = A|\uparrow\uparrow\rangle + B|\uparrow\downarrow\rangle + C|\downarrow\uparrow\rangle + D|\downarrow\downarrow\rangle$$

→ Minimize $E = \langle\psi|H|\psi\rangle / \langle\psi|\psi\rangle$

The local approx. cannot describe the singlet.

local approximation ⇔ separable state

$a^{\uparrow}, a^{\downarrow}, c^{\uparrow}, c^{\downarrow}$: local

$$|\psi\rangle = \sum_{s_1=\uparrow,\downarrow} a^{s_1}|s_1\rangle \otimes \sum_{s_2=\uparrow,\downarrow} c^{s_2}|s_2\rangle$$

$$= \left(a^{\uparrow}|\uparrow\rangle + a^{\downarrow}|\downarrow\rangle\right) \otimes \left(c^{\uparrow}|\uparrow\rangle + c^{\downarrow}|\downarrow\rangle\right)$$

$$= a^{\uparrow}c^{\uparrow}|\uparrow\uparrow\rangle + a^{\uparrow}c^{\downarrow}|\uparrow\downarrow\rangle + a^{\downarrow}c^{\uparrow}|\downarrow\uparrow\rangle + a^{\downarrow}c^{\downarrow}|\downarrow\downarrow\rangle$$

$$|0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) \Rightarrow$$

$$a^{\uparrow}c^{\uparrow} = 0$$

$$a^{\downarrow}c^{\downarrow} = 0$$

$$a^{\uparrow}c^{\downarrow} = 1/\sqrt{2}$$

$$a^{\downarrow}c^{\uparrow} = -1/\sqrt{2}$$

Vector product state

$$|\psi\rangle = \sum_{s_1, s_2} a^{s_1} c^{s_2} |s_1 s_2\rangle \Longrightarrow \sum_{s_1, s_2} A^{s_1} C^{s_2} |s_1 s_2\rangle$$

$$A^{s_1} = \left(a_1^{s_1}, a_2^{s_1}\right)$$

$$C^{s_2} = \begin{pmatrix} c_1^{s_2} \\ c_1^{s_2} \end{pmatrix}$$

This looks local decomposition, but A and C get entangled !

Introduction of additional index that represents entanglement

$$|\psi\rangle = \sum_{\alpha=1}^{\chi=2} \left\{ \sum_{s_1=\uparrow,\downarrow} a_\alpha^{s_1} |s_1\rangle \otimes \sum_{s_2=\uparrow,\downarrow} c_\alpha^{s_2} |s_2\rangle \right\}$$

$$= \left(a_1^{\uparrow} c_1^{\uparrow} + a_2^{\uparrow} c_2^{\uparrow}\right)|\uparrow\uparrow\rangle + \left(a_1^{\uparrow} c_1^{\downarrow} + a_2^{\uparrow} c_2^{\downarrow}\right)|\uparrow\downarrow\rangle$$

$$+ \left(a_1^{\downarrow} c_1^{\uparrow} + a_2^{\downarrow} c_2^{\uparrow}\right)|\downarrow\uparrow\rangle + \left(a_1^{\downarrow} c_1^{\downarrow} + a_2^{\downarrow} c_2^{\downarrow}\right)|\downarrow\downarrow\rangle$$

$$a_1^{\uparrow} = c_2^{\uparrow} = a_2^{\downarrow} = c_1^{\downarrow} = 0$$

$$a_2^{\uparrow} c_2^{\downarrow} = 1/\sqrt{2}$$

$$a_1^{\downarrow} c_1^{\uparrow} = -1/\sqrt{2}$$

$$|\psi\rangle = |0\rangle$$

Exact for χ =2 !

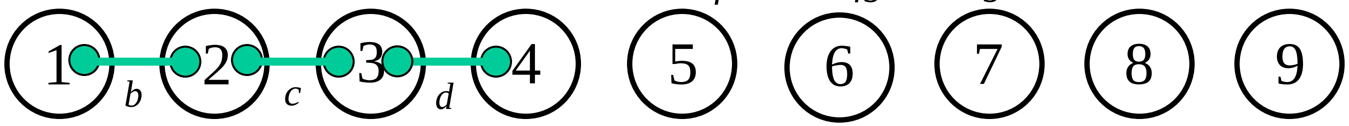
$$A^{\uparrow} = (x, y), A^{\downarrow} = (z, w), C^{\uparrow} = \begin{pmatrix} \frac{y}{xw - yz} \\ \frac{x}{yz - xw} \end{pmatrix}, C^{\downarrow} = \begin{pmatrix} \frac{w}{xw - yz} \\ \frac{z}{yz - xw} \end{pmatrix}$$

Matrix Product State (MPS): DMRG optimizes MPS

MPS for open boundary cases (vectors on two edges)

$$|\psi\rangle = \sum_{\{s_1, s_2, \cdots, s_n\}} \langle s_1 | A_2^{s_2} A_3^{s_3} \cdots A_{n-1}^{s_{n-1}} | s_n \rangle | s_1 s_2 \cdots s_n \rangle$$

$\langle s_1 | = A_b^{s_1}$ $\quad A_{bc}^{s_2}$ $\quad A_{cd}^{s_3}$ $\quad A_{de}^{s_4}$ $\quad A_{ef}^{s_5}$ $\quad A_{fg}^{s_6}$ $\quad A_{gh}^{s_7}$ $\quad A_{hi}^{s_8}$ $\quad | s_9 \rangle = A_i^{s_9}$



$A_j^{s_j}$ χ ×χ matrix, χ : unphysical, artificial degree

$s_j = \uparrow, \downarrow$: physical degree

Matrix = projection of unphysical degree on physical one

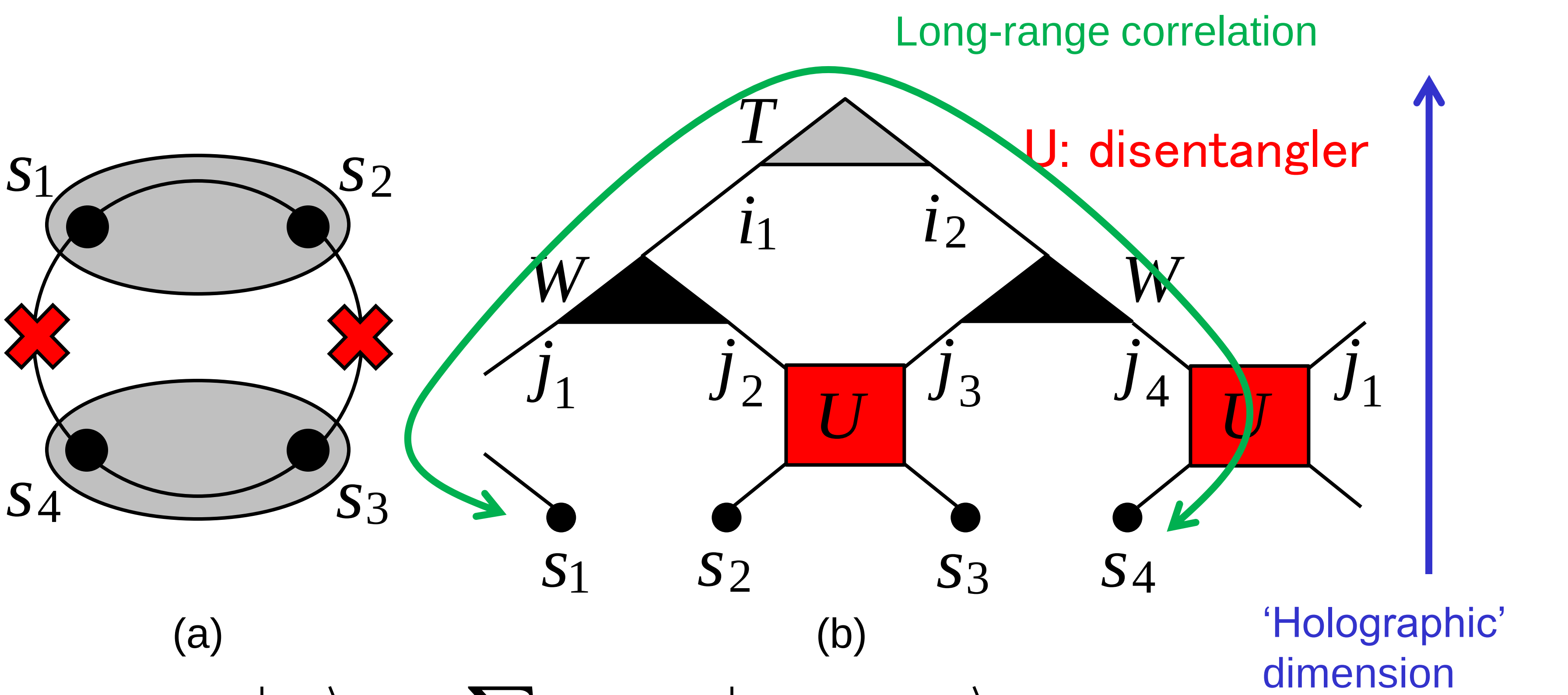
↓

What is unphysical degree of freedom ?

Ψ : product of local matrices → non-local correlation

Hierarchical Tensor Network

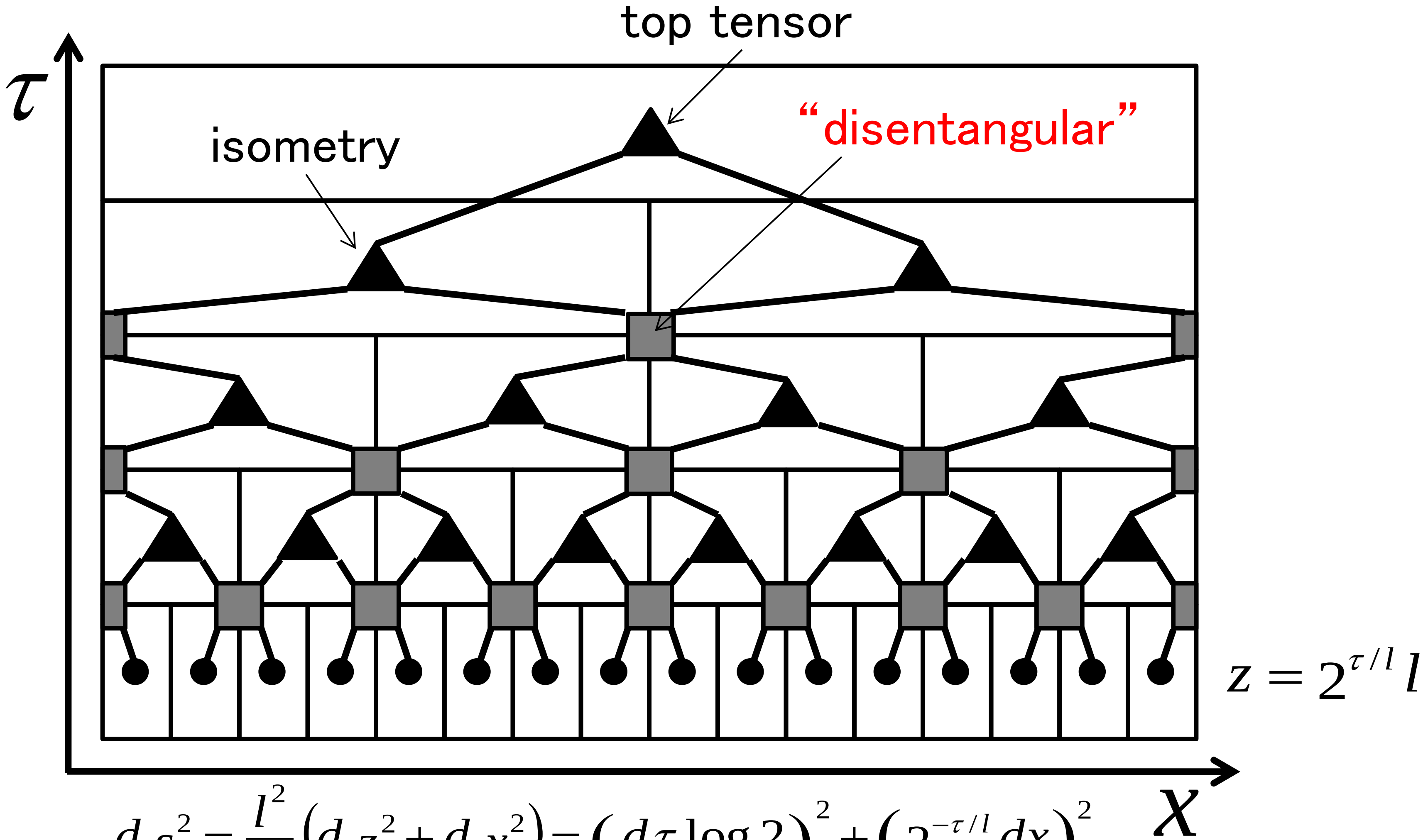
Multiscale Entanglement Renormalization Ansatz (MERA)



$$|\psi\rangle = \sum_{s_1,s_2,s_3,s_4,} T_{s_1 s_2 s_3 s_4} |s_1 s_2 s_3 s_4\rangle$$

$$|\psi\rangle = \sum_{i_1,i_2} \sum_{j_1,\ldots,j_4} \sum_{s_1,\ldots,s_4} T_{i_1 i_2} W^{i_1}_{j_1 j_2} W^{i_2}_{j_3 j_4} U^{j_2 j_3}_{s_2 s_3} U^{j_4 j_1}_{s_4 s_1} |s_1 s_2 s_3 s_4\rangle$$

(Multiscale Entanglement Renormalization Ansatz, MERA)

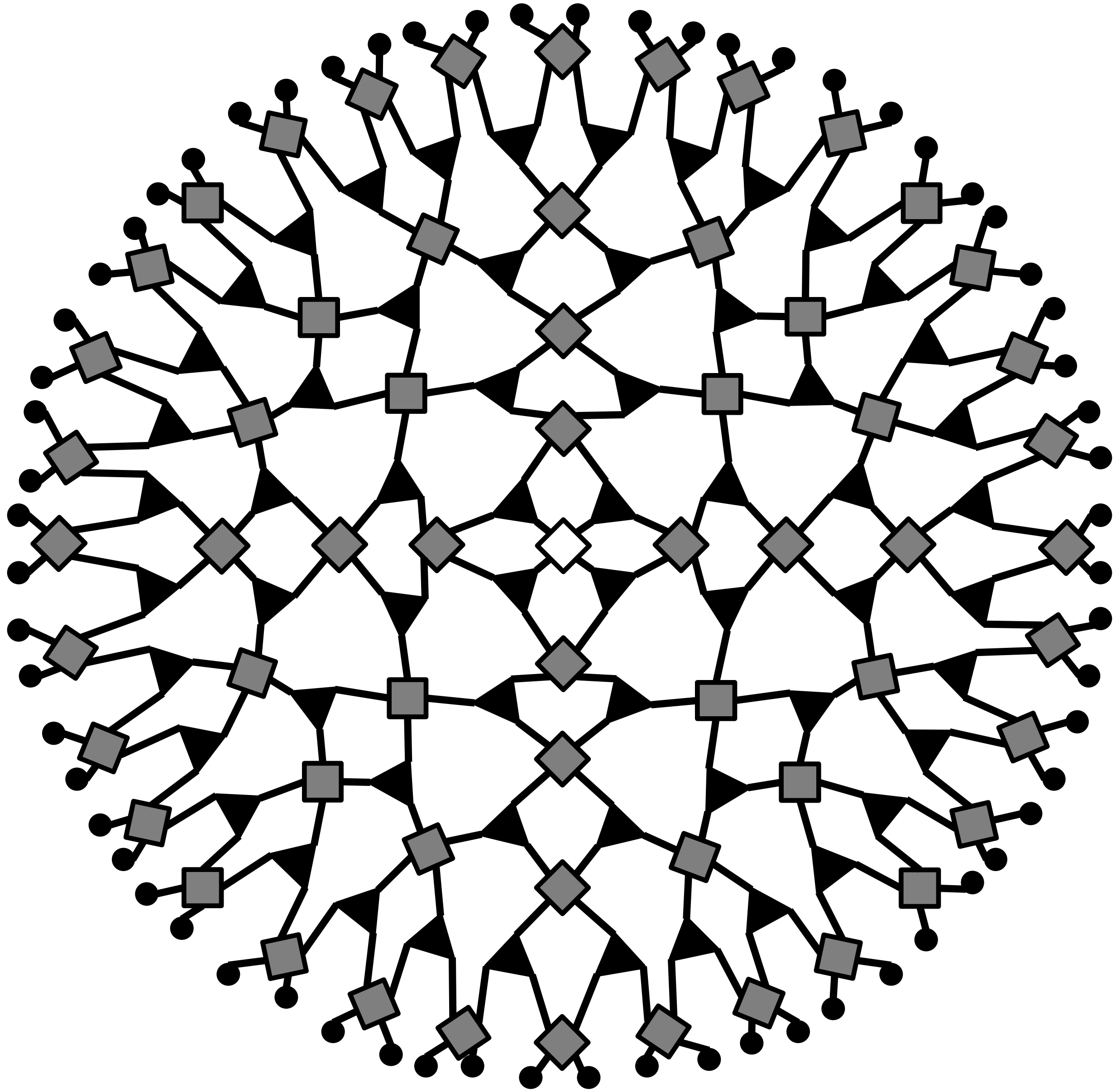


$$ds^2 = \frac{l^2}{z^2}\left(dz^2 + dx^2\right) = \left(d\tau \log 2\right)^2 + \left(2^{-\tau/l} dx\right)^2$$

MPS → decomposed into many tensors with different function

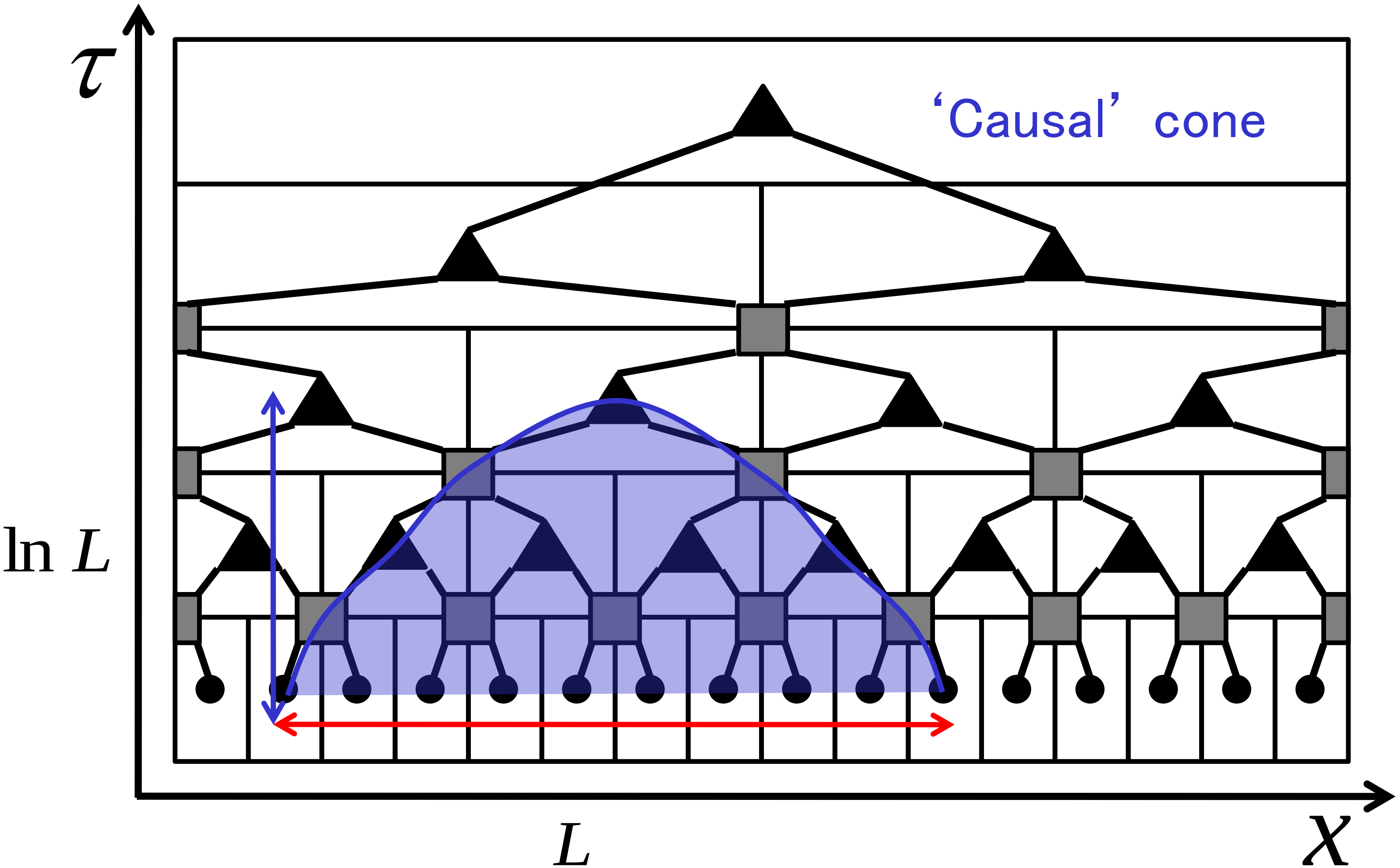
Basis change (disentangler) before renormalization

Poincare Disk Model for MERA Network



How to evaluate entanglement entropy in holographic space ?

Close connection to 'Ryu–Takayanagi formula' developed in superstring theory



S = minimal surface area in holographic space

Binary decomposition

Spatially 1D cases:

$$\underbrace{2+2+\cdots+2}_{\text{No. of boundary points: } \ln L} = 2\ln L$$

No. of boundary points: $\ln L$

Spatially 2D cases:

$$4L\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}\right)=4L\left(2-\frac{1}{2^n}\right)\rightarrow 8L$$

Thermofield dynamics (TFD) for finite-T wavefunction

Purpose: finite-T MERA and its relation with AdS/CFT

Finite-T → thermal avarage

TFD form → 'thermal vacuum'

Identity state (maximally entangled) $\left| I \right\rangle = \sum_n \left| n \right\rangle \otimes \left| \tilde{n} \right\rangle$

General representation theorem

$$\left| I \right\rangle = \sum_n \left| n \right\rangle \otimes \left| \tilde{n} \right\rangle = \sum_\alpha \left| \alpha \right\rangle \otimes \left| \tilde{\alpha} \right\rangle$$

$$\left| O(\beta) \right\rangle = \rho^{1/2} \left| I \right\rangle$$

$$\left\langle O(\beta) \right| A \left| O(\beta) \right\rangle = \sum_{m,n} \left\langle m\tilde{m} \right| \rho^{1/2} A \rho^{1/2} \left| n\tilde{n} \right\rangle$$

$$= \sum_{m,n} \left\langle m \right| \rho^{1/2} A \rho^{1/2} \left| n \right\rangle \delta_{\tilde{m}\tilde{n}} = tr(\rho A)$$

Thermal state in TFD

$$\left|\psi(\beta)\right\rangle = \sum_{\{m_j\}} \sum_{\{\tilde{n}_j\}} c^{\{m_j\}\{\tilde{n}_j\}} \left| \{m_j\}\{\tilde{n}_j\} \right\rangle$$

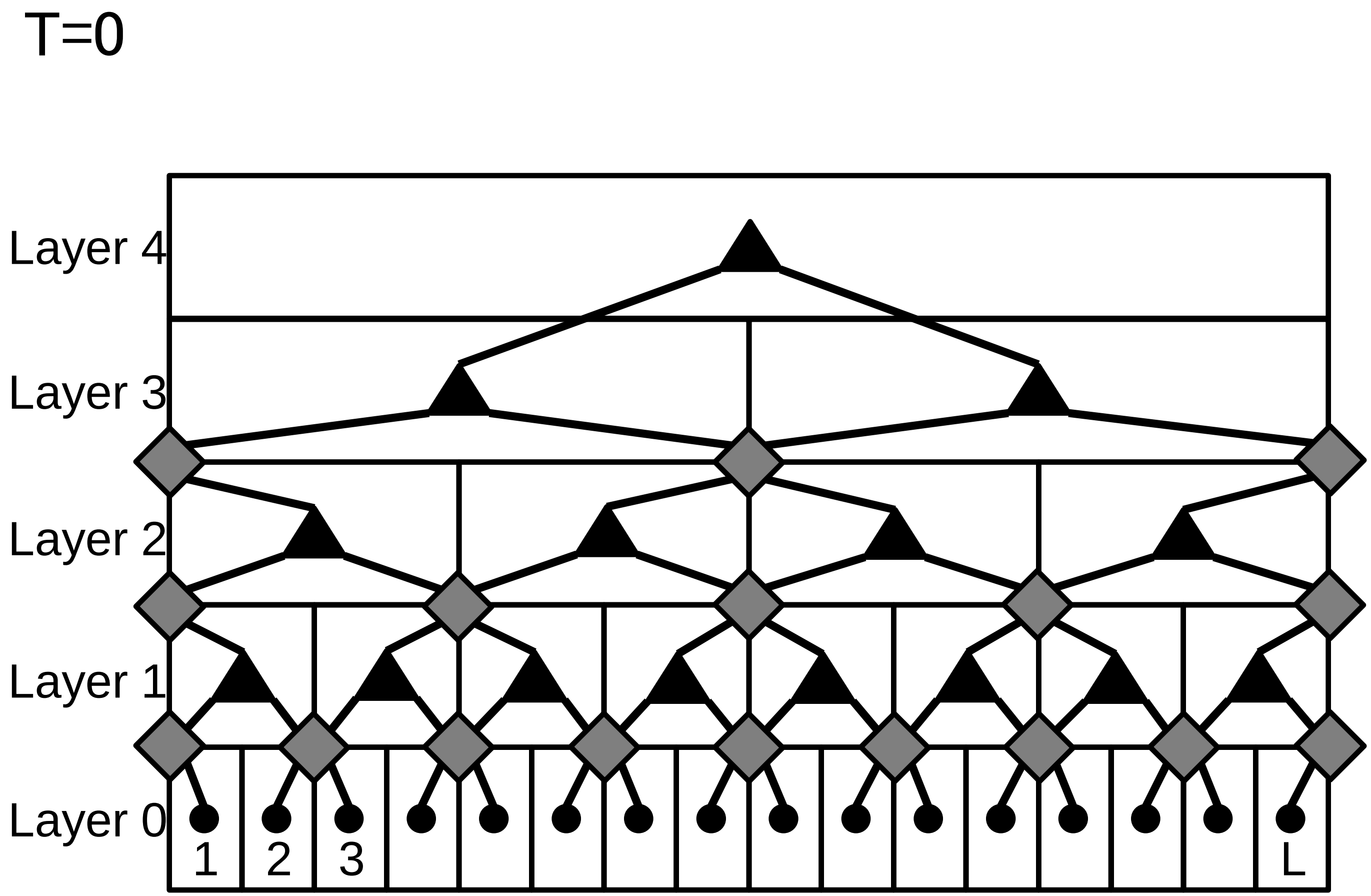
Singular value decomposition

$$c^{\{m_j\}\{\tilde{n}_j\}} = \sum_{\alpha=1}^{\chi} A_\alpha^{\{m_j\}} A_\alpha^{\{\tilde{n}_j\}}$$

α :event horizon → black hole entropy

= maximally entanglement entropy

Imagine Penrose diagrams ･･･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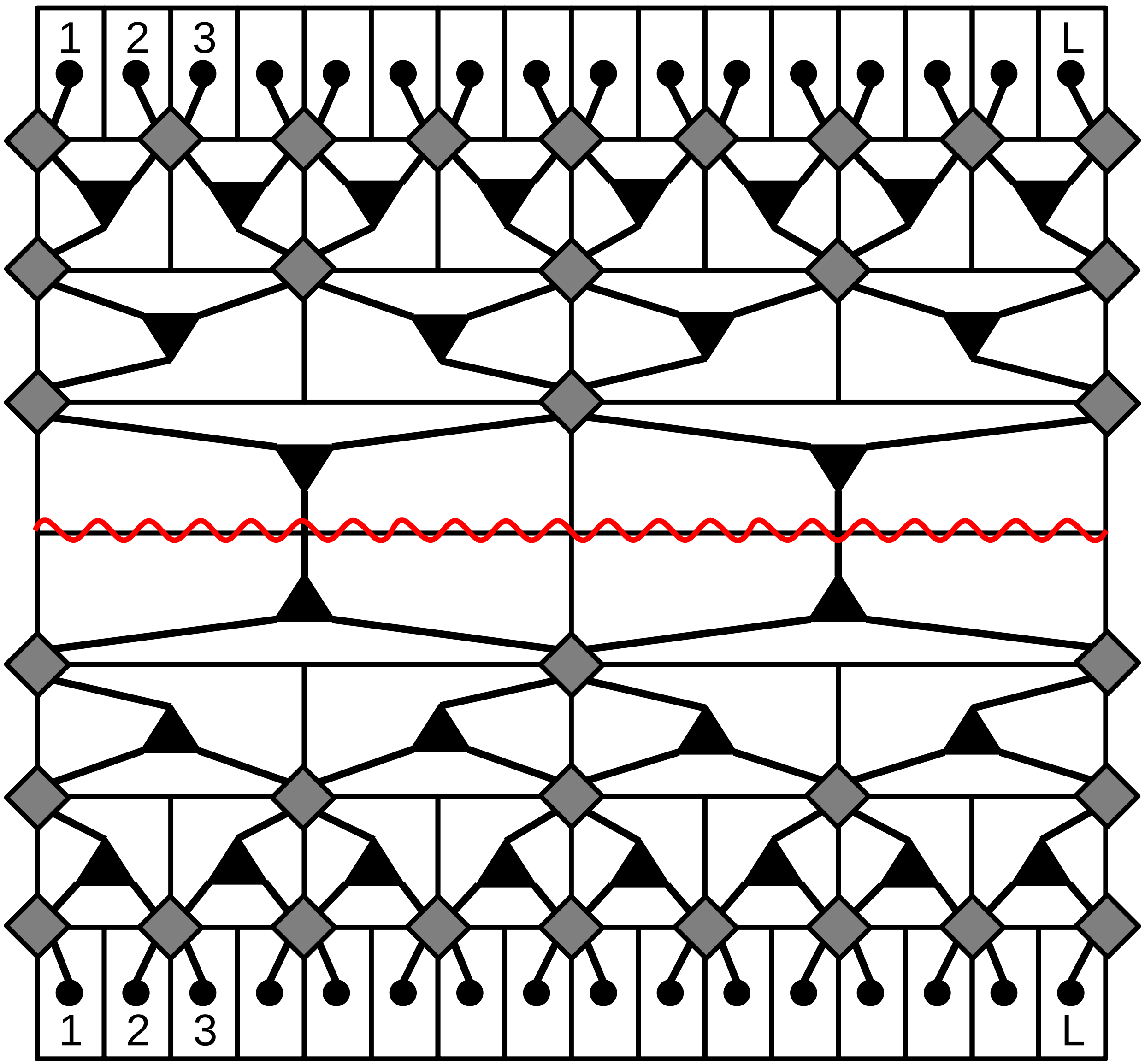

1
2
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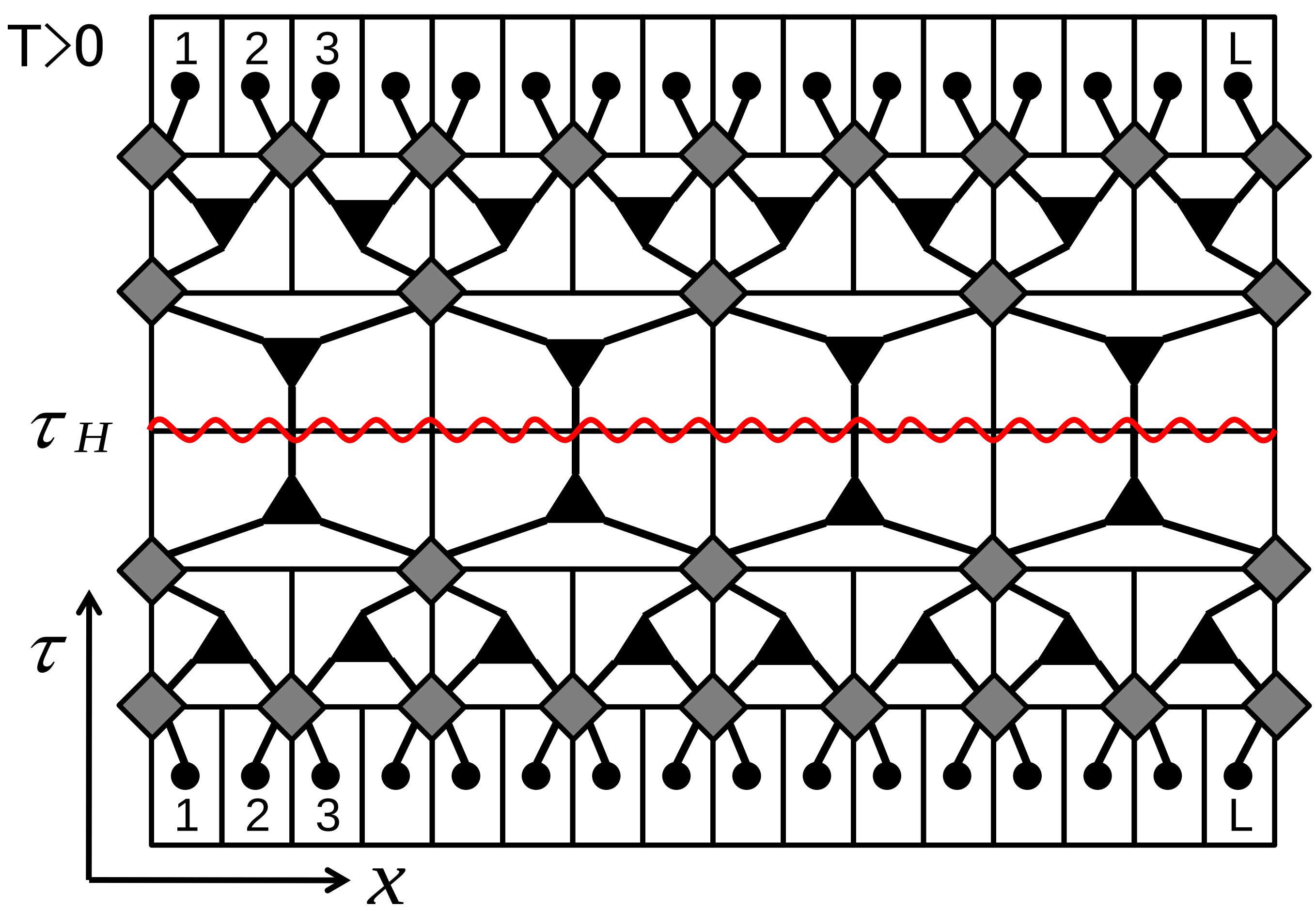
Finite-T MERA Network and AdS Black Hole



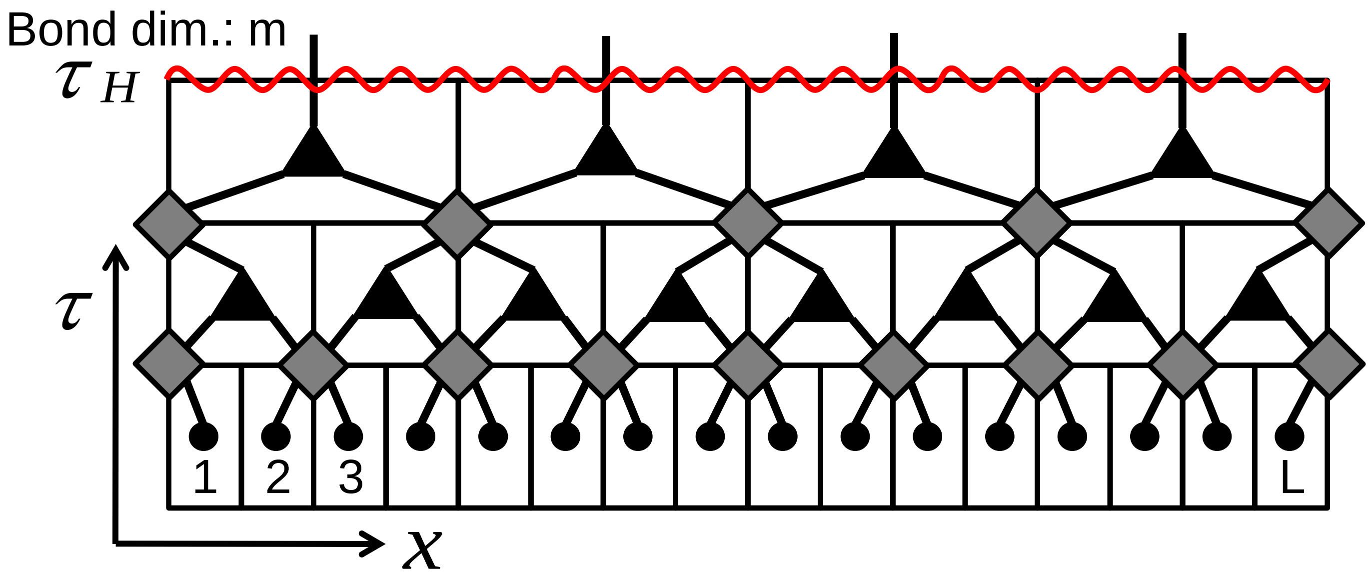
Vertical axis = energy scale, temperature scale

Wave function approach at finite-T → thermofield dynamics

→ Connection between original and tilde spaces

Temperature of MERA Network

Truncation of upper MERA layers = AdS black hole



$$z = 2^{\tau}$$

Area of interface: $\dfrac{L}{2^{\tau_H}} = A$ Total dim. at interface: $\chi = m^A$

Beckenstein–Hawking entropy & Calabrese–Cardy formula:

$$S_{BH} = A \ln m = \frac{L}{z_H} \ln m$$

$$S_{CFT} = \frac{c}{3} \ln\left(\frac{\beta}{\pi\varepsilon} \sinh\left(\frac{\pi L}{\beta} \right) \right)$$

$$k_B T = \left(\frac{3}{c\pi} \ln m \right) \frac{1}{z_H} \propto \frac{1}{z_H}$$

Banados–Teitelboim–Zanelli (BTZ) metric: black hole solution

Exact solution of Einstein eq. in (2+1)D
with negative cosmological term

＊ Schwarzschild solution
in (3+1)D flat spacetime

$$f(z) = 1 - \frac{a}{z}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = 0$$

$$ds^2 = \frac{1}{z^2}\left(-f(z)\,dt^2 + \frac{1}{f(z)} dz^2 + dx^2 \right)$$

Event horizon: $z = z_H$

$$f(z) = 1 - \left(\frac{z}{z_H} \right)^2$$

f(z)=1 → anti-de Sitter (AdS) spacetime

$$ds^2 = \frac{1}{z^2}\left(-dt^2 + dz^2 + dx^2 \right)$$

Maximally-extension of BTZ spacetime

Coordinate transformation

$$ds^2 = \frac{1}{z^2}\left(-f(z)\,dt^2 + \frac{1}{f(z)}dz^2 + dx^2 \right)$$

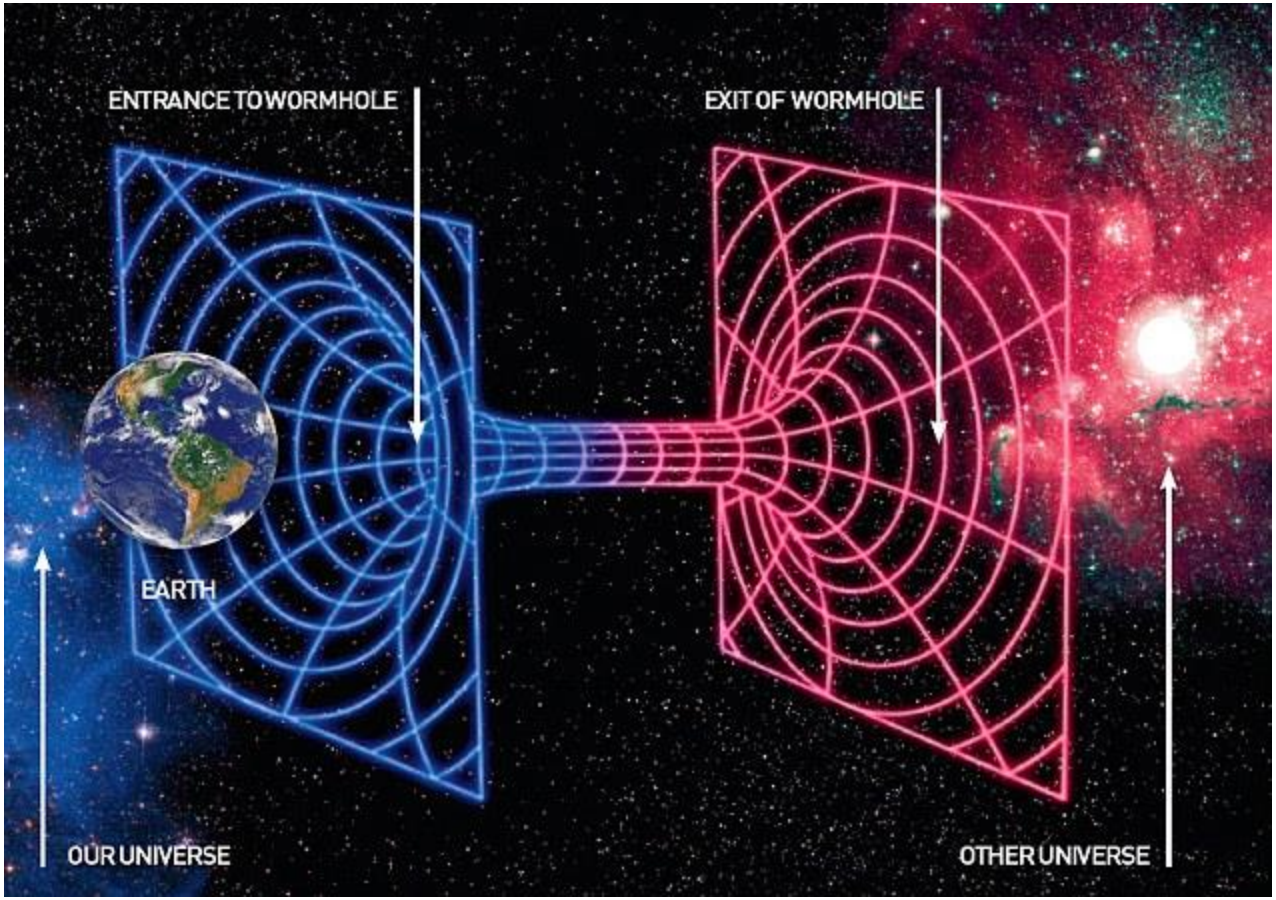
$$j = \ln\left(\frac{2z/\varepsilon}{1+\sqrt{f(z)}} \right) \qquad j_H = \ln\left(\frac{2z_H}{\varepsilon} \right)$$

ε : UV cut-off

$$ds^2 = -h(j)\,dt^2 + d(j\ln\eta)^2 + g(j)\,dx^2$$

$$\alpha = j - j_H \qquad \beta = 2e^{-j_H}\frac{x}{\varepsilon}$$

$$ds^2 = d\alpha^2 + (\cosh\alpha)^2 d\beta^2$$



ENTRANCE TO WORMHOLE
EXIT OF WORMHOLE
EARTH
OUR UNIVERSE
OTHER UNIVERSE

Summary

Finite-T MERA network

- Truncation of the network at the IR region
- T ∼ (layer number for a particular coordinate)$^{-1}$
- consistency with field-theoretical treatment
- Numerical application

Target II: information geometry of quantum states

Statistical model: $H(x) = \sum_{\mu} \theta^{\mu} F_{\mu}(x)$

Ex. transverse Ising model: $x = \{\sigma_1^x, \sigma_1^z, \sigma_2^x, \sigma_2^z, \ldots\}$

$$H = J \sum_i \sigma_i^z \sigma_{i+1}^z + h \sum_i \sigma_i^x$$

$$\theta^1 = \beta J, \theta^2 = \beta h$$

$$F_1 = \sum_i \sigma_i^z \sigma_{i+1}^z, F_2 = \sum_i \sigma_i^x$$

Microscopic (quantum) data

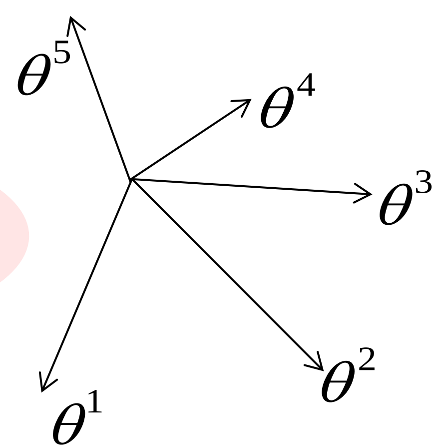
$$p(x;\theta) = \frac{1}{Z} tr\, e^{-\beta H(x)}$$

Parameter space (classical)
～"phase diagram"

Fisher metric

$$g_{\mu\nu}(\theta) = \int_X dx p(x;\theta) \frac{\partial \gamma(x;\theta)}{\partial \theta^{\mu}} \frac{\partial \gamma(x;\theta)}{\partial \theta^{\nu}}$$

$$\int_X dx p(x;\theta) = 1 \quad \gamma(x;\theta) = -\ln p(x;\theta)$$



Functionality of Fisher metric

Gaussian distribution (exponential family)

$$p(x;\theta) = \frac{1}{\left(\sqrt{2\pi\sigma}\right)^n} \exp\left\{ -\sum_i \frac{\left(x_i - \mu_i\right)^2}{2\sigma^2} \right\}$$

$$\theta = \left(\theta^1, \theta^2, \ldots, \theta^{n+1}\right) = \left(\mu_1, \mu_2, \ldots, \mu_n, \sigma\right)$$

AdS space (exact solution of vacuum Einstein equation)

$$g_{\mu\nu}(\theta) = \frac{\sum_i d\mu_i^{\,2} + 2n\, d\sigma^2}{\sigma^2}$$

μ_1 : imaginary time

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = 0$$

$\Lambda < 0$

Natural parameter representation

$$p(x;\theta)$$

$$= \frac{1}{\sqrt{2\pi\sigma}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

$$= \exp\left\{-\frac{x^2}{2\sigma^2} + \frac{x\mu}{\sigma^2} - \frac{\mu^2}{2\sigma^2} - \ln\left(\sqrt{2\pi\sigma}\right)\right\}$$

$$= \exp\left\{\left(\frac{\mu}{\sigma^2}, -\frac{1}{2\sigma^2}\right)\begin{pmatrix} x \\ x^2 \end{pmatrix} - \psi(\theta)\right\} \qquad \psi(\theta) = \ln\left(\sqrt{2\pi\sigma}\right) + \frac{\mu^2}{2\sigma^2}$$

$$\theta = \left(\theta^1, \theta^2\right) \qquad F(x) = \begin{pmatrix} F_1(x) \\ F_2(x) \end{pmatrix}$$

$$p(x;\theta) = \exp\left\{\theta^\mu F_\mu(\theta) - \psi(\theta)\right\}$$

$$\gamma(x;\theta) = \psi(\theta) - \theta^\mu F_\mu(\theta)$$

$$\partial_\mu \partial_\nu \gamma(x;\theta) = \partial_\mu \partial_\nu \psi(\theta)$$

Riemannian tensor and Hessian geometry

$$\langle O \rangle = \int_X dx p(x;\theta) O(x;\theta)$$

$$S = -\int_X dx p(x;\theta) \ln p(x;\theta) = \int_X dx p(x;\theta) \gamma(x;\theta) = \langle \gamma \rangle$$

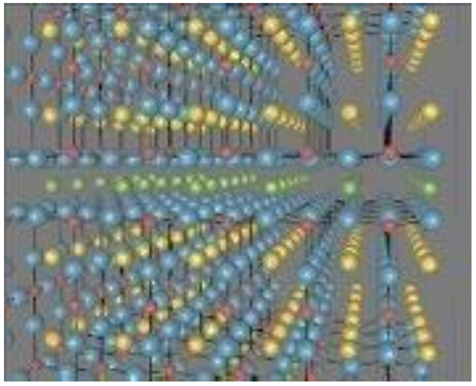
$$g_{\mu\nu} = \langle \partial_\mu \gamma \, \partial_\nu \gamma \rangle = \langle \partial_\mu \partial_\nu \gamma \rangle \qquad \langle \partial_\mu \gamma \rangle = 0$$

$$\Gamma^{\lambda}_{\mu\nu} = -\frac{1}{2} g^{\lambda\tau} T_{\mu\nu\lambda} \qquad T_{\mu\nu\lambda} = \langle \partial_\mu \gamma \, \partial_\nu \gamma \, \partial_\tau \gamma \rangle = -\partial_\mu \partial_\nu \partial_\tau \psi(\theta)$$

$$R_{\mu\nu} = \frac{1}{4} g^{\sigma\tau} g^{\rho\varsigma} \left(T_{\varsigma\mu\sigma} T_{\rho\nu\tau} - T_{\rho\sigma\tau} T_{\varsigma\mu\nu} \right)$$

$$R_{\mu\nu} \propto \partial_\mu \psi(\theta) \partial_\nu \psi(\theta) + \cdots$$

$$\gamma(x;\theta) = \psi(\theta) - \theta^\mu F_\mu(\theta)$$

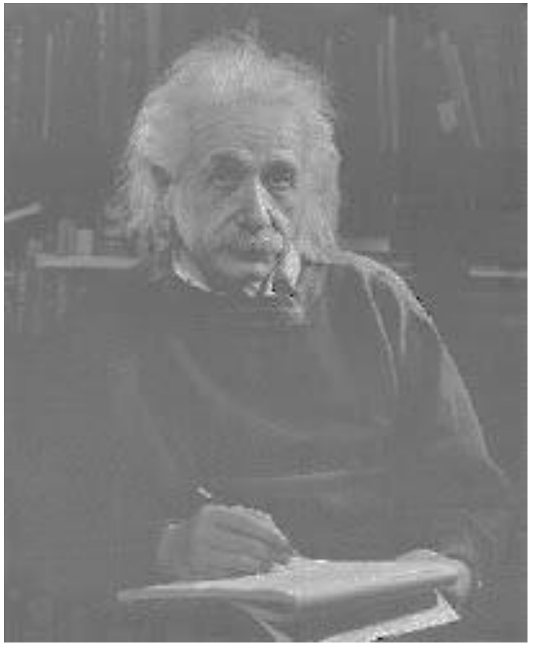


$$R_{\mu\nu} \propto \partial_\mu \psi(\theta) \partial_\nu \psi(\theta) + \cdots$$

$$L = \frac{1}{2} g^{\alpha\beta} \partial_\alpha \psi(\theta) \partial_\beta \psi(\theta)$$

$$T_{\mu\nu} = g_{\mu\sigma} \frac{\partial L}{\partial(\partial_\sigma \psi)} \partial_\nu \psi - g_{\mu\nu} L$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \propto T_{\mu\nu}$$



Information geometry for spin glass model

$$P(J_{ij}) = \frac{1}{\sqrt{2\pi}J}\exp\left\{-\frac{(J_{ij}-J_0)^2}{2J^2}\right\}$$

$$H = -\sum_{i,j} J_{ij}\sigma_i\sigma_j$$

$$p = \frac{1}{Z}\int\left\{\prod_{i,j} dJ_{ij}P(J_{ij})\right\}\exp\left(\beta\sum_{i,j}J_{ij}\sum_{\alpha=1}^{n}\sigma_i^\alpha\sigma_j^\alpha\right)$$

$$\theta = (\theta^1,\theta^2) = (J, J_0)$$

$$\frac{1}{N}\left\langle\sum_i \sigma_i^\alpha\right\rangle = m_\alpha$$

$$\frac{1}{N}\left\langle\sum_i \sigma_i^\alpha\sigma_i^\beta\right\rangle = q_{\alpha\beta}$$

$$g_{\mu\nu} = \frac{1}{J^2}(\beta J)^2\begin{pmatrix}4(\beta J)^2 v_Y & 2\beta J\, v_{XY}\\ 2\beta J\, v_{XY} & v_X\end{pmatrix}$$

$$X = \sum_\alpha m_\alpha\sum_i\sigma_i^\alpha$$

$$Y = \sum_{\alpha<\beta} q_{\alpha\beta}\sum_i\sigma_i^\alpha\sigma_i^\beta$$

(FM to SG) $\beta J = 1$

→ AdS metric after appropriate basis transformation

HM and Y. Hashizume, in preparation

Summary

Model parameter space defined by the Fisher metric

- Einstein equation ⇔ equation of states (?)
  Jacobson, Verlinde, ･･･

- Is this information-geometrical approach equivalent to AdS/CFT correspondence ?
  (dimension of classical side is not d+1 in general cases)