A Lattice QCD Simulation using Quaternion basis

Sadataka Furui

School of Science and Engineering

Teikyo University, Utsunomiya

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Strong coupling gauge theory in LHC Era proc.(2011) p.398

INTRODUCTION AND MOTIVATION

- In QCD, there are ultraviolet fixed point and infrared fixed point.
- In perturbative QCD, Banks and Zaks(1982) showed if the number of flavors N_f is reduced to just below 11N/2, an infrared fixed point will appear. And if N_f is reduced further, chiral symmetry breaking and confinement set in.
- Appelquist et al(2008) claimed in a lattice simulation using staggered fermion and Schrödinger functional(SF) scheme, that for $12 \le N_f \le 16$ infrared behavior is governed by the fixed point.

- Staggered fermion has 4 times taste degeneracy and $N_f=12$ observed by Appelquist and Fodor et al (2011) may not correspond to effective $N_f=12$ of Wilson fermion.
- Dietrich and Sannino(2007) studied two-loop β function for a generic non-Abelian gauge theory with fermionic matter in a given representation of SU(N). In case of N=3, the conformal window appears for $12\le N_f\le 16$. Non-perturbative corrections were not discussed.
- Vicinity to the conformality is characterized by walking behavior of the running coupling.
- Nakajima and S.F. (2008) observed using the 2+1 flavors Wilson fermion gauge configuration of RBC/UKQCD , the walking behavior.

- The walking behavior is observed also in light-front holography of Brodsky et al(2010) and an experimental analysis of JLAB group(2006).
- In infrared region, where non-perturbative effects are essential, the conformal field amplitude $D(Q^2) = D_{PT}(Q^2)$ acquires a correction $D_{NP}(Q^2)$. Grunberg (2001) incorporated a scale parameter ρ such that $\beta_1 = \rho\, \beta_0$ and calculated higher order terms consistent by taking into account the cancellation of $D_{NP}(Q^2)$ and $D_{PT}(Q^2)$.

The modified Banks-Zaks fixed point predicts $4 \le N_f \le 6$ for the conformal window.

The $N_f = 3$ system might be not far from the window.

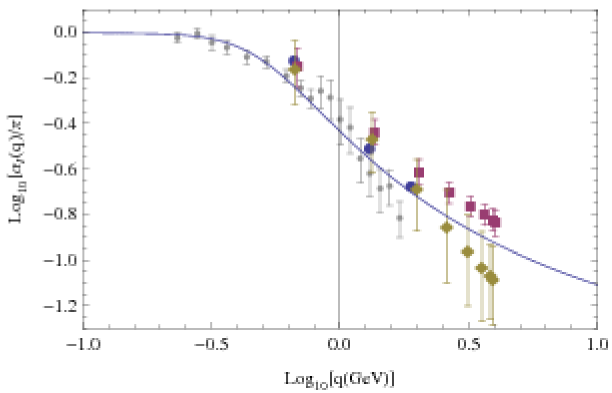

Fig.1:The running coupling of the domain wall fermion. Coulomb gauge gluon-ghost coupling of $m_u = 0.01/a$(square), $0.02/a$(diamond), and quark-gluon coupling of $m_u = 0.01/a$(large disks). Small disks are the α_{s,g_1} derived from the spin structure function of the JLab group (2008) and the solid curve is their fit.

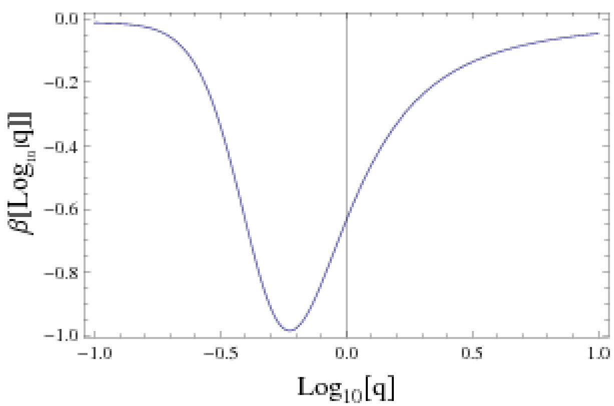

Fig.2: β function obtained from the fitted function of the JLab group. In Brodsky et al., the phenomenological $\alpha_{g1}(q)$ in $q < 1$GeV and the AdS/QCD theoretical values $\alpha_{g1}^{AdS}(q)$ for $q > 1$GeV are connected. In this figure I derived β from the $\alpha_{g1}(q)$.

- Braun and Beneke (1994) pointed out that the BLM scale fixing of Q^* via quark loop insertion has the same effect as the scale changing of Grunberg, and that the different N_f dependence could manifest itself.
- Fermions are expressed as spinors or Grassmann variables. E.Cartan(1938) showed that the Pauli spinors can be treated as a member of quaternions **H,** whose automorphism group is SO(3) that rotates i , j and k. With a new imaginary unit ℓ, **H**+**H** ℓ makes an octonion **O**. Octonion have G2 symmetry and the basis has the triality symmetry.
- I study possible roles of triality in the infrared renormalization effects in the QCD.

QUATERNION AND OCTONION

- The quark $q = \begin{pmatrix} \phi \\ \psi \end{pmatrix}$ anti − quark $q' = \begin{pmatrix} C\phi \\ C\psi \end{pmatrix}$

- With use of quaternion bases $1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, the spinors ϕ and $C\phi = \phi'$ are defined as

$$\phi = \xi_0 + \xi_{14}\boldsymbol{i} + \xi_{24}\boldsymbol{j} + \xi_{34}\boldsymbol{k} \quad (1)$$
$$C\phi = \xi_{1234} - \xi_{23}\boldsymbol{i} - \xi_{31}\boldsymbol{j} - \xi_{12}\boldsymbol{k}. \quad (2)$$

Similarly, ψ and $C\psi = \psi'$ are defined as

$$\psi = \xi_4 + \xi_1\boldsymbol{i} + \xi_2\boldsymbol{j} + \xi_3\boldsymbol{k} \quad (3)$$
$$C\psi = \xi_{123} - \xi_{234}\boldsymbol{i} - \xi_{314}\boldsymbol{j} - \xi_{124}\boldsymbol{k}. \quad (4)$$

There are two semi-spinors which have a quadratic form invariant with respect to the group of rotation

$$\Phi = {}^t\phi C\phi = \xi_0\xi_{1234} - \xi_{23}\xi_{14} - \xi_{31}\xi_{24} - \xi_{12}\xi_{34} \quad (5)$$
$$\Psi = {}^t\psi C\psi = \xi_4\xi_{123} - \xi_1\xi_{234} - \xi_2\xi_{314} - \xi_3\xi_{124} \quad (6)$$

and vectors with a quadratic form

$$F = x^1x^{1'} + x^2x^{2'} + x^3x^{3'} + x^4x^{4'} \quad (7)$$

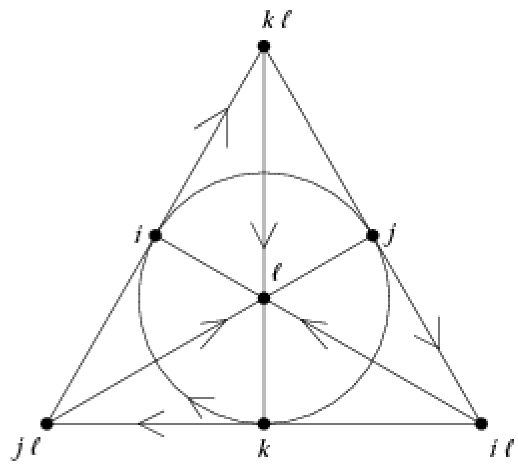

Fig.3: The multiplication table of octonion. The 7 oriented lines represent 7 quaternionic triplets.

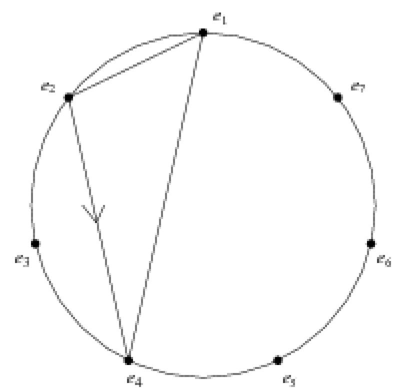

Fig.4: The multiplication rule of octonion. The relation $\mathbf{e}_i \circ \mathbf{e}_j = \pm \mathbf{e}_k$ and be found from rotating $\mathbf{e}_1 \circ \mathbf{e}_2 = \mathbf{e}_4$ by an integral multiple of $2\pi/7$. e.g. $\mathbf{e}_2 \circ \mathbf{e}_5 = -\mathbf{e}_3$.

- É. Cartan introduced in the $\mathbf{R}^8$ space, 24 dimensional bases

$$\{\xi_0, \xi_1, \xi_2, \xi_3, \xi_4\}, \quad \{\xi_{12}, \xi_{31}, \xi_{23}, \xi_{14}, \xi_{24}, \xi_{34}\}, \quad \{\xi_{123}, \xi_{124}, \xi_{314}, \xi_{234}, \xi_{1234}\}$$

$$\{x^1, x^2, x^3, x^4\}, \quad \{x^{1'}, x^{2'}, x^{3'}, x^{4'}\}$$

- The trilinear form in these bases is

$$\begin{aligned}
\mathcal{F} &= \phi^T C X \psi = x^1(\xi_{12}\xi_{314} - \xi_{31}\xi_{124} - \xi_{14}\xi_{123} + \xi_{1234}\xi_1) \\
&+ x^2(\xi_{23}\xi_{124} - \xi_{12}\xi_{234} - \xi_{24}\xi_{123} + \xi_{1234}\xi_2) \\
&+ x^3(\xi_{31}\xi_{234} - \xi_{23}\xi_{314} - \xi_{34}\xi_{123} + \xi_{1234}\xi_3) \\
&+ x^4(-\xi_{14}\xi_{234} - \xi_{24}\xi_{314} - \xi_{34}\xi_{124} + \xi_{1234}\xi_4) \\
&+ x^{1'}(-\xi_0\xi_{234} + \xi_{23}\xi_4 - \xi_{24}\xi_3 + \xi_{34}\xi_2) \\
&+ x^{2'}(-\xi_0\xi_{314} + \xi_{31}\xi_4 - \xi_{34}\xi_1 + \xi_{14}\xi_3) \\
&+ x^{3'}(-\xi_0\xi_{124} + \xi_{12}\xi_4 - \xi_{14}\xi_2 + \xi_{24}\xi_1) \\
&+ x^{4'}(\xi_0\xi_{123} - \xi_{23}\xi_1 - \xi_{31}\xi_2 - \xi_{12}\xi_3)
\end{aligned}$$

- The vector x^i and $x^{i'}$ are defined as Plücker coordinates expressed by the fermionic spinors.

THE DOMAIN WALL FERMION

DWF and the mass of a quark

- Domain wall fermion has specific boundary condition in the 5^{th} dimension, and could make qualitative difference from the boundary condition of the SF scheme.
- The DWF of Shamir(1993) and Narayanan and Neuberger(1993) are expressed by Grassmann variables.
- We solve the coupled differential equation in $L^3 \times T \times L\_s \times$ (3 color) $\times$ (4 component spinor) space, and adopt results at the middle of the L_s space to calculate the running coupling. (RBC/UKQCD config.

L=16, T=32,L_s=16) (L=24,T=64,L_s=16 is under way)

- In the DWF method, the mass of the fermion is

$$m(s) = M\, sign(s)$$

The continuum fermion propagator is constructed from

$$D_F = i\gamma_\mu \partial_\mu + i\gamma_5 \partial_s + im(s)$$

When a plane wave is assumed in the first four cordinates, the effective one dimensional problem becomes

$$\hat{D}_F = i\gamma_5 \partial_5 + im(s) + \gamma \cdot p$$

It is a linear ordinary differential equation of first order, and the solution $\psi = U(s,\lambda,p)u(\lambda,p,\alpha)$ becomes

$$U(s,\lambda,p) = \begin{cases} \cosh(s\kappa_+) + \kappa_+^{-1}\gamma_5[i(\gamma \cdot p - \lambda) - M]\sinh(s\kappa_+) & s \geq 0 \\ \cosh(s\kappa_-) + \kappa_-^{-1}\gamma_5[i(\gamma \cdot p - \lambda) + M]\sinh(s\kappa_-) s \leq 0 \end{cases}$$

where $U(s,\lambda,p)$ is a 4×4 matrix, $u(\lambda,p,\alpha)$ is an s independent spinor, $\kappa_\pm = (i\lambda \pm M)^2 + p^2$.

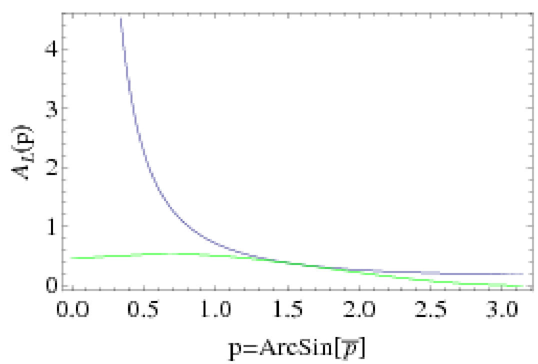

Fig.5: Left handed chiral fermion amplitude $A_L(p)$(blue) and $A_L(p)\bar{p}^2$(green). $M = 1$

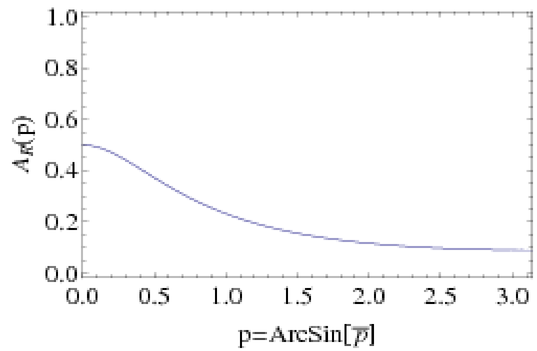

Fig.6: Right handed chiral fermion amplitude $A_R(p)$. $M = 1$

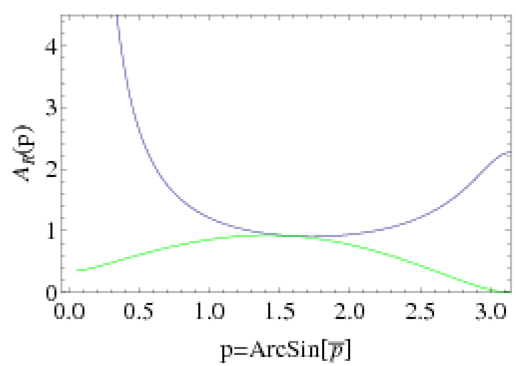

Fig.7 Negative mass chiral fermion amplitude $A_R(p)$(blue) and $A_R(p)\bar{p}^2$(green). $M = -1.8$

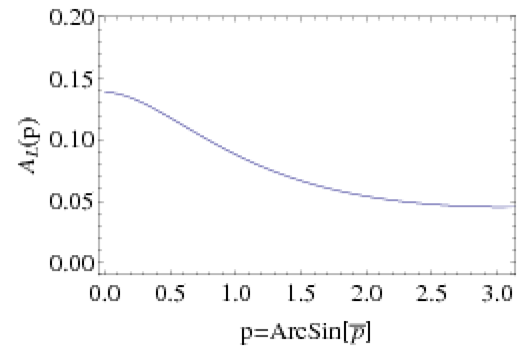

Fig.8: Negative mass chiral fermion amplitude $A_L(p).M = -1.8$

- I adjust the phase η in the 5th dimension of the wave function $\phi_{L/R}(p,s)$ such that both $\mathrm{Tr}\langle\chi(p,0)\phi_L(p,0)\rangle$ and $\mathrm{Tr}\langle\chi(p,L_s-1)\phi_R(p,L_s-1)\rangle$ are close to a real number. Namely, I define

$$e^{i\theta_L} = \frac{\mathrm{Tr}\langle\chi(p,0)\phi_L(p,0)\rangle}{|\mathrm{Tr}\langle\chi(p,0)\phi_L(p,0)\rangle|},$$

$$e^{-i\theta_R} = \frac{\mathrm{Tr}\langle\chi(p,L_s-1)\phi_R(p,L_s-1)\rangle}{|\mathrm{Tr}\langle\chi(p,L_s-1)\phi_R(p,L_s-1)\rangle|}$$

and sample-wise calculate $e^{i\eta}$ such that

$$|e^{i\theta_L}e^{i\eta}+1|^2+|e^{i\theta_R}e^{-i\eta}-1|^2$$

is the minimum.

- Effectively, it makes a correlation between $\phi_L(p,0)$ and $\phi_R(p,L_s-1)$, and keep these wave functions in one triality sector.

INFRARED PROBLEM IN THE QCD THERMODYNAMICS

- The gluon self energy diagrams of the order g^6 including two self-dual vector field exchange consist of diagrams like the followings.

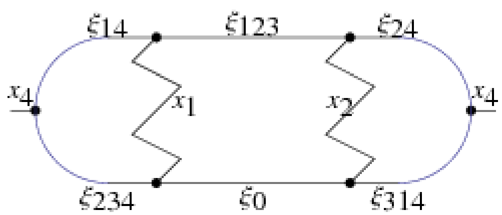

Fig.9

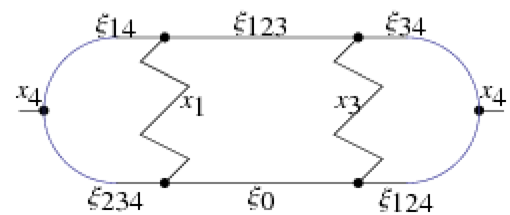

Fig.10:

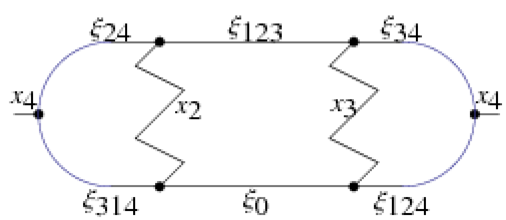

Fig.11:

- At $p_0 = 0$ and in the limit of $\mathbf{p} \to \mathbf{0}$, the 3-loop diagrams expected to yield the self energy $g^6 T^3/m(T)$, where $m(T)$ is the infrared cut off.

- These contributions are expected to play roles in cancelling the unwanted pole at $p = g^2 N_c T \frac{8+(\xi+1)^2}{64}$ from the inverse of the following eq. (Kalashnikov-Klimov(1979))

$$p^2 + \Pi_T(p_0 = 0, p) = p^2 - g^2 N_c T \frac{8 + (\xi + 1)^2}{64} p \tag{8}$$

  where ξ is the gauge parameter of covariant gauges, N_c is the number of colors.

- In a perturbative analysis of finite temperature QCD, the inverse gluon propagator goes like (Linde (1980))

$$p^2 + a_1 g^2 T p + a_2 g^4 T^2 + a_3 \frac{g^6 T^3}{p} + \cdots . \tag{9}$$

- Whether the 3-loop diagram with exchange of two self-dual gluon fields dominates in g^6 term, as expected from the conjecture of D'Adda and Di Vecchia(1978) is to be investigated.

CONCLUSION AND DISCUSSION

- The Dirac spinor is expressed by a quaternion.
- A quaternion that operates on the left-handed, and another that operates on right-handed spinor make an octonion.
- The octonion possesses the triality symmetry. Physical domainwall fermion propagator chooses one triality sector .
- In the MOM scheme, one selects the triality sector. In the SF scheme the effective number of N_f becomes 3 times larger.
- Walking behavior of the running coupling is an indication of the proximity of the system to the conformal window. It is necessary to extend the simulation to larger lattices.
- The g^6 order diagrams necessary to solve the negative pressure problem etc. may be provided by the three loop diagram with two self-dual vectors exchange.
- Do particles in a triality sector different from that of electrons in the detector behave like unparticles of Georgi(2007)?

Thank you very much for your attention.