Histogram method in finite density lattice QCD with phase quenched simulations

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for

WHOT-QCD collaboration:

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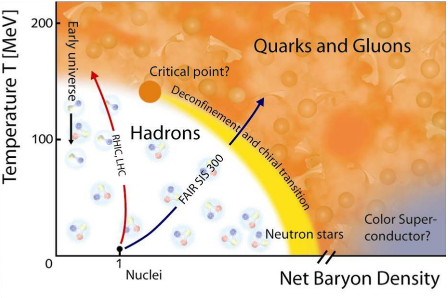
— Finite density lattice QCD —



- Various approaches
  - ✓ Reweighting method
  - ✓ Canonical approach
  - ✓ Histogram method
  - ✓ Taylor expansion
  - ✓ Imaginary chemical potential
  - ✓ Langevin approach
- Sign problem
  - ✓ Finite density QCD
  - ✓ Lattice study of θ vacuum
  - ✓ Exceptional cases
    - two-color QCD
    - Isospin chemical potential
    - Imaginary chemical potential

— Histogram method —

- ✓ All thermodynamic quantities can be derived from the free energy
- ✓ μ-dependent part of the pressure

$$\frac{p(\mu)}{T^4} - \frac{p(0)}{T^4} = \frac{1}{VT^3} \ln \left(\frac{\mathcal{Z}(\beta,\mu)}{\mathcal{Z}(\beta,0)} \right) = \frac{1}{VT^3} \ln \left\langle \frac{\det M(\mu)}{\det M(0)} \right\rangle_{\mu=0}$$

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If we are interested in observables, which can be expressed in terms of the plaquette and/or the quark determinant, we can calculate the expectation values by

1. labeling the configurations by the plaquette and the quark determinant
2. calculating the observables with the probability

— Histogram method contd. —

✓ Label gauge configurations by P (plaquette) and $F(\mu) = N_\mathrm{f} \ln \left| \frac{\det M(\mu)}{\det M(0)} \right|$

$$\frac{\mathcal{Z}(\beta,\mu)}{\mathcal{Z}(\beta,0)} = \frac{1}{\mathcal{Z}(\beta,0)} \int \mathscr{D}U e^{i\theta(\mu)} \left|\det M(\mu)\right|^{N_\mathrm{f}} e^{6\beta N_\mathrm{site} P}$$

$$= \int dP dF \langle e^{i\theta(\mu)} \rangle_{(P,F)} w_0(P,F,\beta,\mu) = \int dP dF w(P,F,\beta,\mu)$$

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✓ Probability distribution function

$$w_0(P',F',\beta,\mu) = \frac{1}{\mathcal{Z}(\beta,0)} \int \mathcal{D}U \delta(P-P')\delta(F-F') \underbrace{\left|\det M(\mu)\right|^{N_f} e^{6\beta N_{\rm site} P}}_{\text{phase quenched measure}}$$

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✓ Complex phase of the quark determinant

$$\langle e^{i\theta(\mu)} \rangle_{(P',F')} = \frac{\langle\langle e^{i\theta(\mu)} \delta(P-P')\delta(F-F') \rangle\rangle_{(\beta,\mu)}}{\langle\langle \delta(P-P')\delta(F-F') \rangle\rangle_{(\beta,\mu)}}$$

($\langle\langle \cdots \rangle\rangle_{(\beta,\mu)}$: the expectation value with the phase quenched simulations)

— Overlap problem —

$$\frac{\mathcal{Z}(\beta,\mu)}{\mathcal{Z}(\beta,0)} = \int dPdF \langle e^{i\theta(\mu)}\rangle_{(P,F)} w_0(P,F,\beta,\mu)$$

$$= \int dPdF e^{-\left[V_0(P,F,\beta,\mu) - \ln\langle e^{i\theta(\mu)}\rangle_{(P,F)}\right]} \qquad (V_0 = -\ln w_0)$$

$$= \int dPdF e^{-V(P,F,\beta,\mu)}$$

✓ The minimum of

$$V = V_0 - \ln\langle e^{i\theta(\mu)}\rangle$$

dominates the integral

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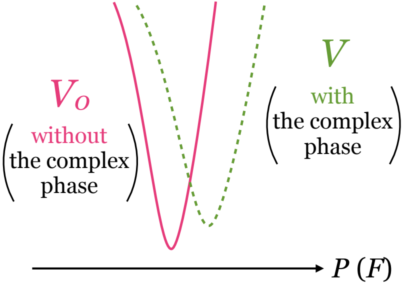
✓ If the P and/or F dependence of $\langle e^{i\theta(\mu)}\rangle_{(P,F)}$ is large, the minimum of V differs from that of V_0
$\Rightarrow$ overlap problem



— Reweighting factor —

$$w_0(P,F,\beta,\mu) = R(P,F,\beta,\beta_0,\mu,\mu_0)w_0(P,F,\beta_0,\mu_0)$$

✓ Reweighting factor

$$R = e^{6(\beta-\beta_0)N_{\rm site}P}\frac{\left\langle\left\langle \delta(P'-P)\delta(F'-F)\left|\frac{\det M(\mu)}{\det M(\mu_0)}\right|^{N_{\rm f}}\right\rangle\right\rangle_{(\beta_0,\mu_0)}}{\langle\langle\delta(P'-P)\delta(F'-F)\rangle\rangle_{(\beta_0,\mu_0)}}$$

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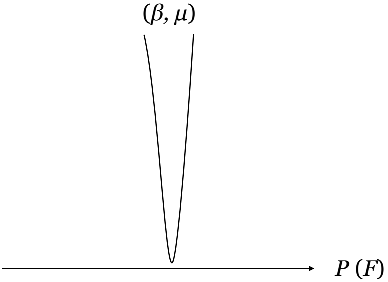

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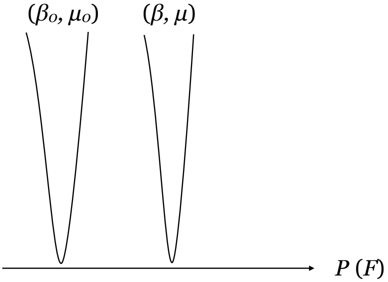

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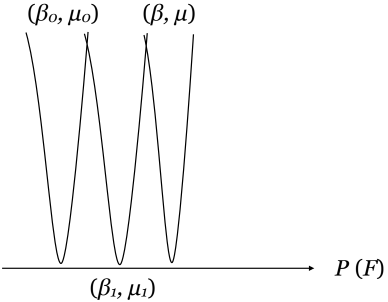

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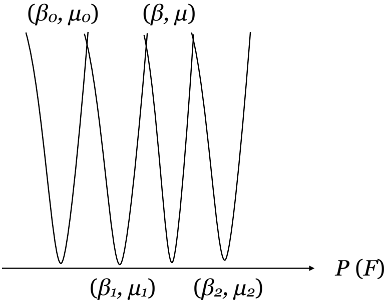

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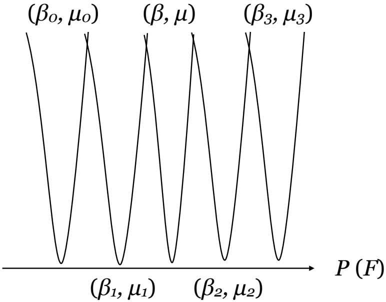

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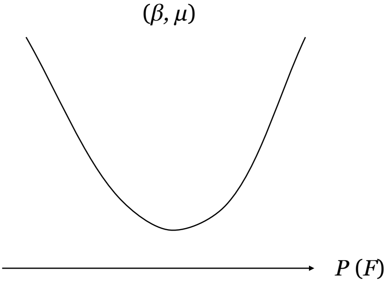

✓ We can cover wide range in P-F plane

— Sign problem —

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- ✓ Cumulant expansion of the complex phase

$$\left\langle e^{i\theta(\mu)} \right\rangle_{(P,F,\mu)} = \exp\left[i \left\langle \theta \right\rangle_c - \frac{1}{2} \left\langle \theta^2 \right\rangle_c - \frac{i}{3!} \left\langle \theta^3 \right\rangle_c + \frac{1}{4!} \left\langle \theta^4 \right\rangle_c + \cdots \right]$$

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✓ **No sign problem if the cumulant expansion converges**

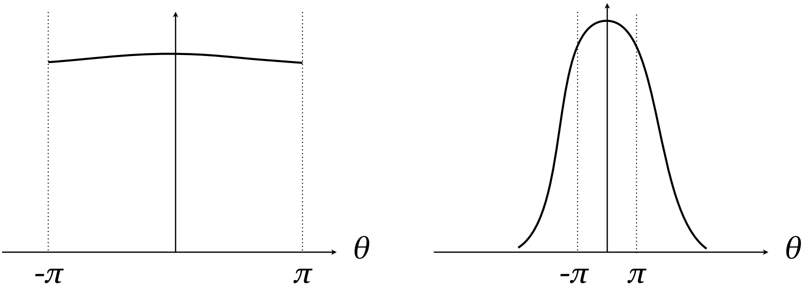
✓ Small contribution to $\mathcal{Z}(\beta,\mu)/\mathcal{Z}(\beta,0)$ for large $\left\langle\theta^2\right\rangle$

— Convergence property of the cumulant expansion —

✓ No higher-order terms if θ has a Gaussian distribution

$$\langle e^{i\theta(\mu)} \rangle_{(P,F,\mu)} = \exp\left[-\frac{1}{2} \langle \theta^2 \rangle \right]$$

✓ Bad convergence if θ is limited in the range $[-\pi, \pi]$



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✓ Need the definition of θ, giving nearly a Gaussian distribution

✓ We calculate θ as follows:

$$\theta(\mu) = N_{\rm f}\,\mathfrak{Im}\left[\ln\det M(\mu)\right] = N_{\rm f}\int_0^{\mu/T}\mathfrak{Im}\left[\frac{\partial(\ln\det M(\mu))}{\partial(\mu/T)}\right]_{\bar\mu} d\left(\frac{\bar\mu}{T}\right)$$

✓ Not a Taylor expansion, $\theta(\mu)$ not limited in the range $[-\pi,\pi]$

— Aim of this study —

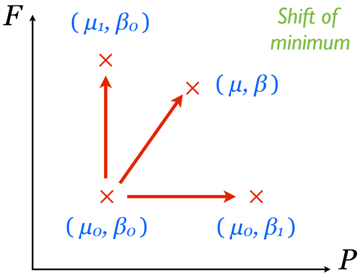
✓ Propose a new approach to finite density QCD based on the histogram method with phase quenched simulations

✓ Investigate the convergence property of the cumulant expansion of $\langle e^{i\theta} \rangle$

✓ Find parameter region (β_0, μ_0) giving large contribution to

$$\frac{\mathcal{Z}(\beta, \mu)}{\mathcal{Z}(\beta, 0)}$$

✓ EoS, quark number density, ...



— Lattice setup —

- ✓ Clover-improved Wilson quark action with $c_{SW} = (1 - 0.8412\beta^{-1})^{-3/4}$
- ✓ RG-improved Iwasaki action
- ✓ On $8^3 \times 4$ lattice, $N_{\rm f} = 2$, $m_{\rm PS}/m_{\rm V} = 0.8$
- ✓ Measurement every 10 trajectories
- ✓ Random noise method with 50 noises
- ✓ Statistics is 2000-5000
- ✓ Jackknife method with a bin size of 50 trajectories

β	c_{SW}	K	T/T_{pc}
1.50	1.853546	0.143480	0.76(4)
1.70	1.668851	0.142871	0.84(4)

— $F(\mu)$ and $\theta(\mu)$ by reweighting method —

✓ Phase quenched update (isospin chemical potential)

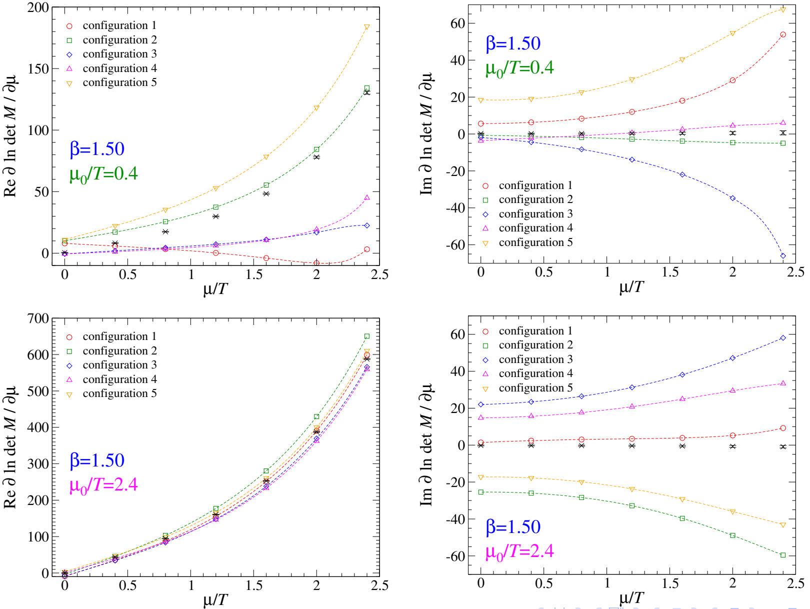
✓ Measure the μ derivative of $\ln\det M(\mu)$

✓ The quark determinant, the complex phase, and the reweighting factor,

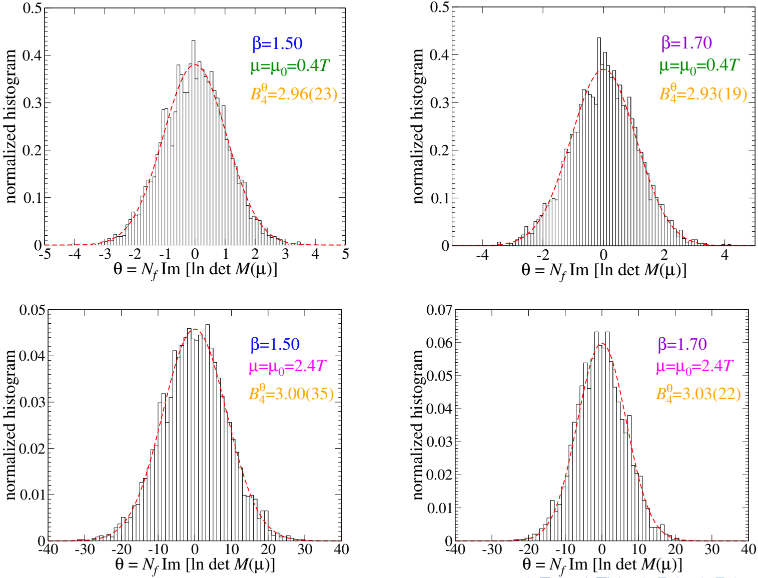
$$\begin{aligned}
F(\mu) &= N_{\rm f} \ln \left| \frac{\det M(\mu)}{\det M(0)} \right| = N_{\rm f} \int_0^{\mu/T} \mathfrak{Re} \left[\frac{\partial(\ln\det M(\mu))}{\partial(\mu/T)} \right]_{\bar\mu} d\left(\frac{\bar\mu}{T}\right), \\
\theta(\mu) &= N_{\rm f}\, \mathfrak{Im} \left[\ln\det M(\mu)\right] = N_{\rm f} \int_0^{\mu/T} \mathfrak{Im} \left[\frac{\partial(\ln\det M(\mu))}{\partial(\mu/T)} \right]_{\bar\mu} d\left(\frac{\bar\mu}{T}\right), \\
C(\mu) &= N_{\rm f} \ln \left| \frac{\det M(\mu)}{\det M(\mu_0)} \right| = N_{\rm f} \int_{\mu_0/T}^{\mu/T} \mathfrak{Re} \left[\frac{\partial(\ln\det M(\mu))}{\partial(\mu/T)} \right]_{\bar\mu} d\left(\frac{\bar\mu}{T}\right),
\end{aligned}$$

can be obtained as continuous functions of μ.

— $F(\mu)$ and $\theta(\mu)$ by μ-integration —



— $\theta(\mu)$ by μ-integration —



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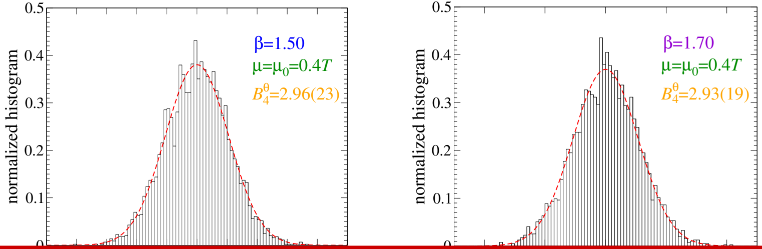
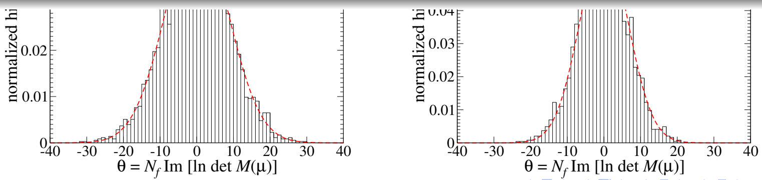
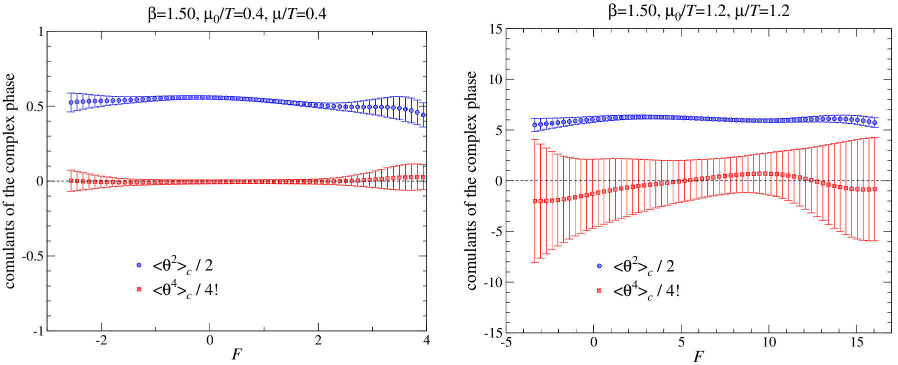


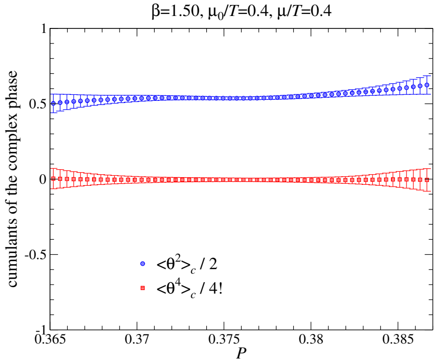
Distribution gets broader as the chemical potential increases,
and as the temperature decreases at large μ.
Well approximated by a Gaussian function.

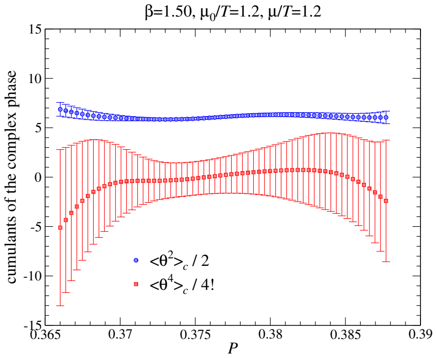


Cumulants as a function of F



Cumulants as a function of P





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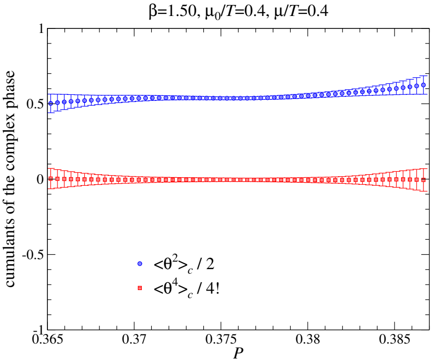
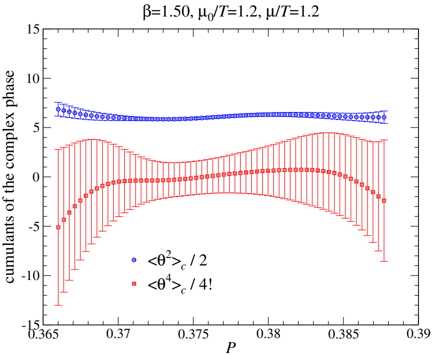
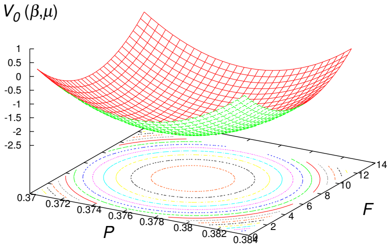
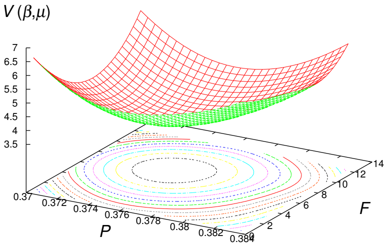


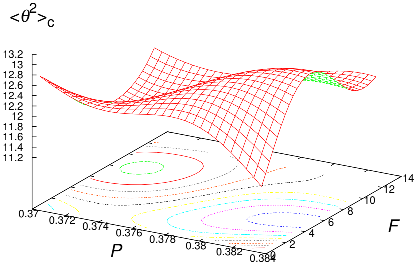


- ✓ $\langle\theta^2\rangle_c$ increases with μ
- ✓ $\langle\theta^4\rangle_c$ is zero within the statistical errors, although the errors increase with μ

Effects of the complex phase ($\beta = 1.5, \mu_0/T = \mu/T = 1.2$)







Summary and outlook

- Complex phase θ by μ-integration
  - ✓ Distribution gets broader as the chemical potential increases, and as the temperature decreases at large μ.
  - ✓ Distribution is well approximated by a Gaussian function.
- Cumulant expansion and convergence
  - ✓ $\langle \theta^4 \rangle_c$ is consistent with zero within statistical errors
- In progress
  - ✓ Simulations at several β_0 and μ_0 to cover wide range in P-F plane
  - ✓ EoS, quark number density, ...

— Cumulants —

- Culumants

$$
\begin{aligned}
\langle\theta\rangle_c &= \langle\theta\rangle_{(P,F,\mu)} \\
\langle\theta^2\rangle_c &= \left\langle\left(\theta-\langle\theta\rangle_{(P,F,\mu)}\right)^2\right\rangle_{(P,F,\mu)} \\
\langle\theta^3\rangle_c &= \left\langle\left(\theta-\langle\theta\rangle_{(P,F,\mu)}\right)^3\right\rangle_{(P,F,\mu)} \\
\langle\theta^4\rangle_c &= \left\langle\left(\theta-\langle\theta\rangle_{(P,F,\mu)}\right)^4\right\rangle_{(P,F,\mu)} - 3\left\langle\left(\theta-\langle\theta\rangle_{(P,F,\mu)}\right)^2\right\rangle_{(P,F,\mu)}^2 .
\end{aligned}
$$

- n-the power of the phase of the quark determinant

$$
\langle\theta^n(\mu)\rangle_{(P,F,\mu)} = \frac{\left\langle \theta^n(\mu)\delta(P'-P)\delta(F'-F(\mu))\left|\frac{\det M(\mu)}{\det M(\mu_0)}\right|^{N_{\mathrm{f}}}\right\rangle_{\mu_0}}{\left\langle \delta(P'-P)\delta(F'-F(\mu))\left|\frac{\det M(\mu)}{\det M(\mu_0)}\right|^{N_{\mathrm{f}}}\right\rangle_{\mu_0}} .
$$