A review on functional renormalization group (FRG)
----- phase transition -----

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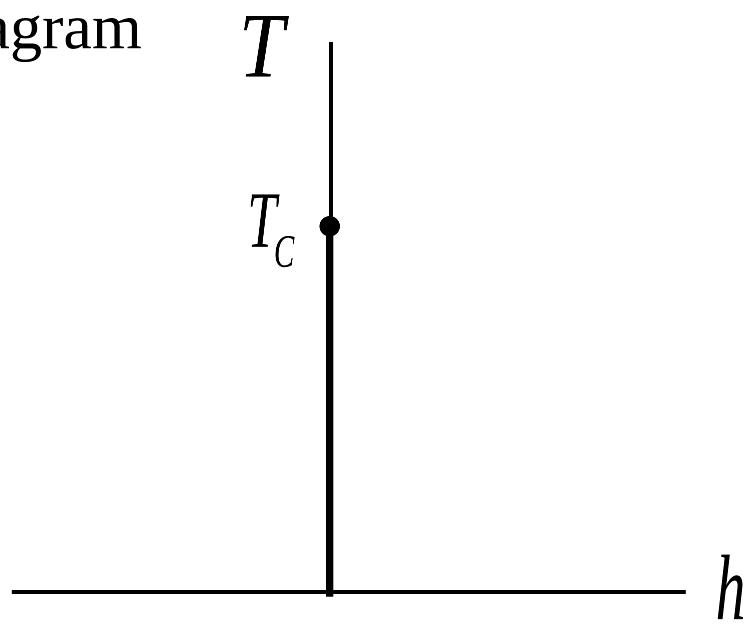
Content:

netsuba2011

1) Critical phenomena (static) --- Ising model ---

Phase diagram



Critical exponents: $t = (T - T_C)/T_C$

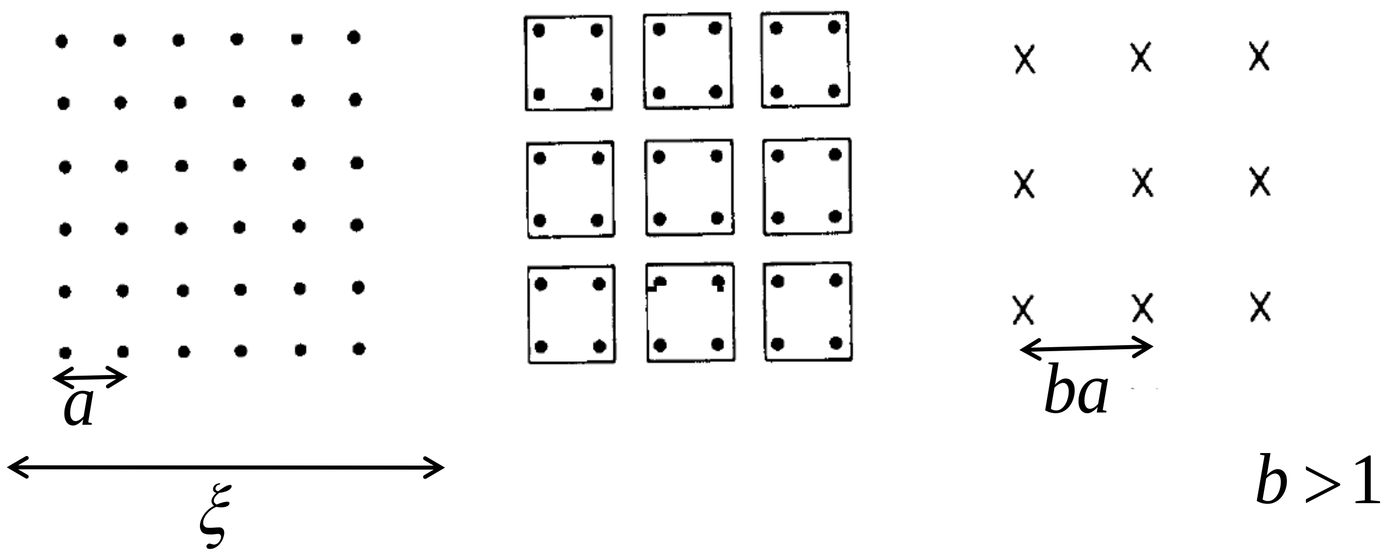
$$\xi \sim t^{-\nu}, C_h \sim t^{-\alpha}, \chi \sim t^{-\gamma}, \phi \sim |t|^{\beta} \sim h^{1/\delta},$$

Scaling relations:

$$\delta = 1 + \gamma/\beta, \alpha + 2\beta + \gamma = 2, \gamma = (2-\eta)\nu, \alpha = 2 - d\nu$$

Scaling part of thermodynamic potential

Kadanoff's block spin argument



Original spin free energy : $F_s(J, h)$

Block spin free energy : $F_s(Jb^{\Delta_t}, hb^{\Delta_h})$

zoom out by factor b $\Rightarrow F_s(J, h) = b^{-d} F_s(Jb^{\Delta_t}, hb^{\Delta_h})$

Correlation length $\xi \to \infty$ at phase transition point

Wilson's renormalization group (Z_2 scalar field theory)

Low energy effective theory with cutoff Λ

$$
\begin{aligned}
Z &= \int D\phi_{k<\Lambda} e^{-H_\Lambda[\phi]} \\
&= \int D\phi_{k<\Lambda-d\Lambda} \left[\int D\phi_{\Lambda-d\Lambda<k<\Lambda} e^{-H_\Lambda[\phi]} \right] \\
&\equiv \int D\phi_{k<\Lambda-d\Lambda} e^{-H_{\Lambda-d\Lambda}[\phi]}
\end{aligned}
$$

$$\Lambda \sim \frac{1}{a}$$

$$\Downarrow$$

$$\Lambda - d\Lambda \sim \frac{1}{ab}$$

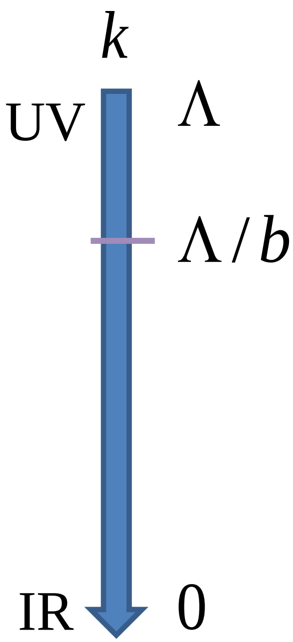
A fundamental theory at scale Λ

$$H_\Lambda = \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2} r\phi^2 + \frac{1}{4} u\phi^4$$

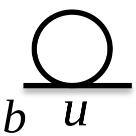
An effective theory at lower scale $\Lambda - d\Lambda$

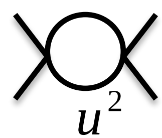
$$H_{\Lambda-d\Lambda} = \frac{1}{2} a(\nabla\phi)^2 + \cdots + \frac{1}{2} r'\phi^2 + \frac{1}{4} u'\phi^4 + \cdots$$

RG transformation = 2 steps with a parameter $b > 1$



1) Integrate the momentum shell in loop corrections $\Lambda < |k| < \Lambda / b$

$$r' = r + \int_{\Lambda/b}^{\Lambda}$$  $$+ \cdots$$

$$u' = u + \int_{\Lambda/b}^{\Lambda}$$  $$+ \cdots$$

2) Change the length scale for all variables (zoom out)

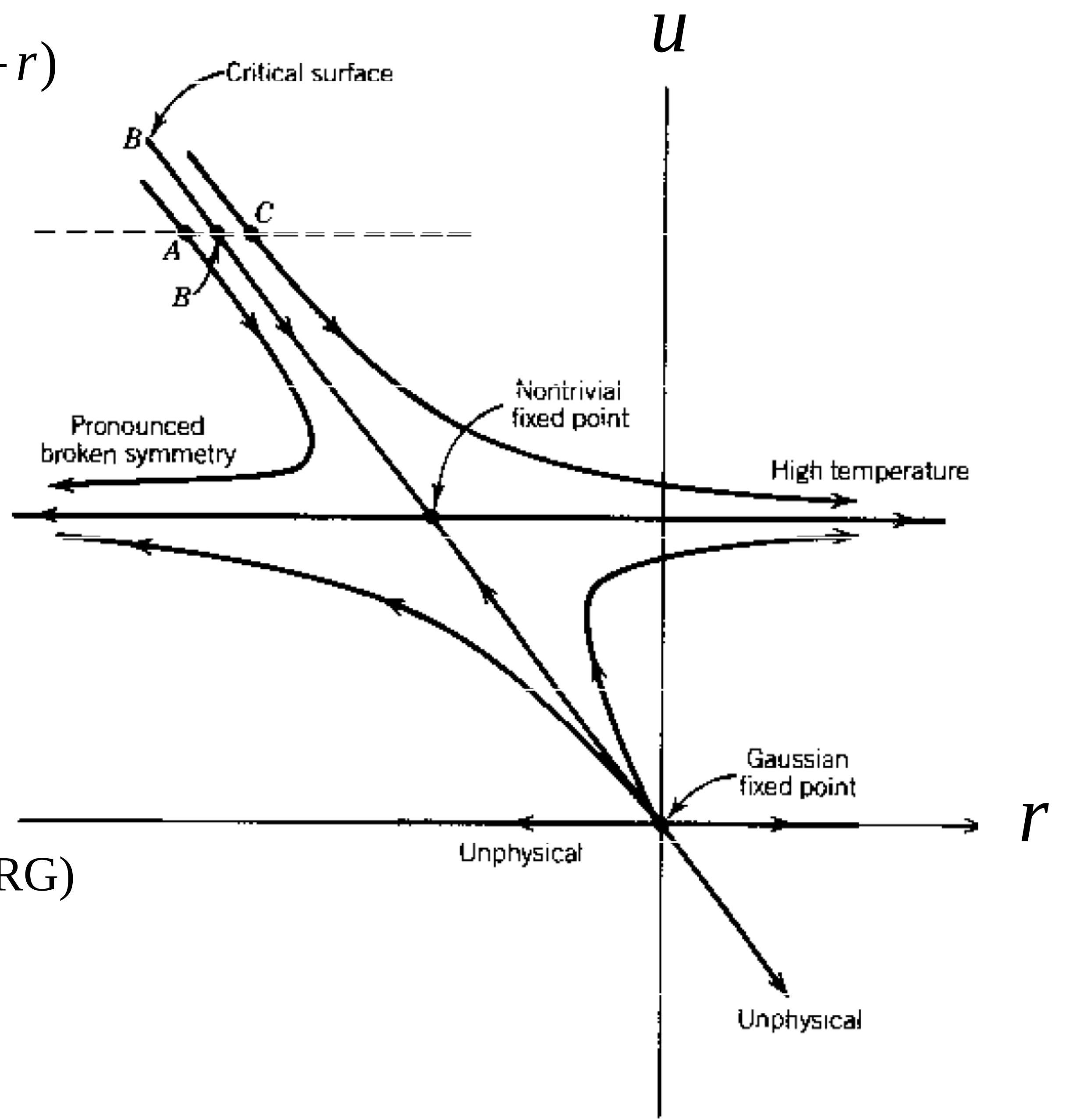
$\lambda \sim 1/k$ $\Longrightarrow$ $1/bk$

Repeat 1) and 2) = RG transformation → flows in r, u

Flow equations below critical dimension $d = 4 - \varepsilon$

$$\dot{r} = (2-\eta)r + 12\Omega_4 u(\Lambda^2 - r)$$

$$\dot{u} = (\varepsilon - 2\eta)u - 36\Omega_4 u^2$$



Non-trivial fixed poin

(a critical point):

$$r^* = -\varepsilon \frac{1}{6}\Lambda^2 + O(\varepsilon^2)$$

$$u^* = \frac{\varepsilon}{36\Omega_4} + O(\varepsilon^2)$$

Scaling dimensions

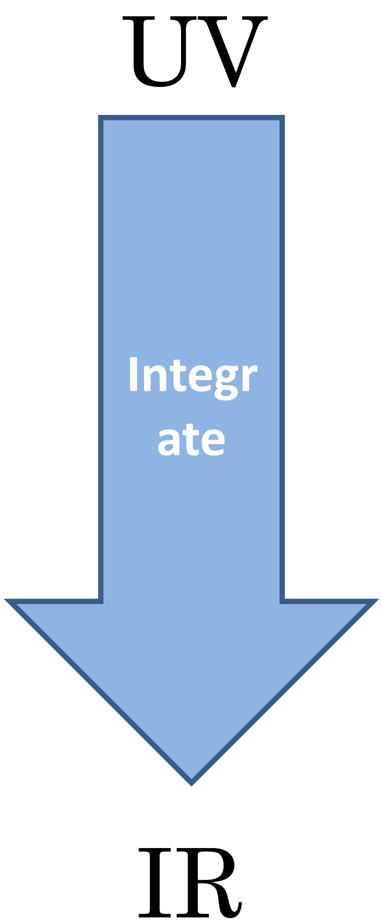
(Eigen value of Linearized RG)

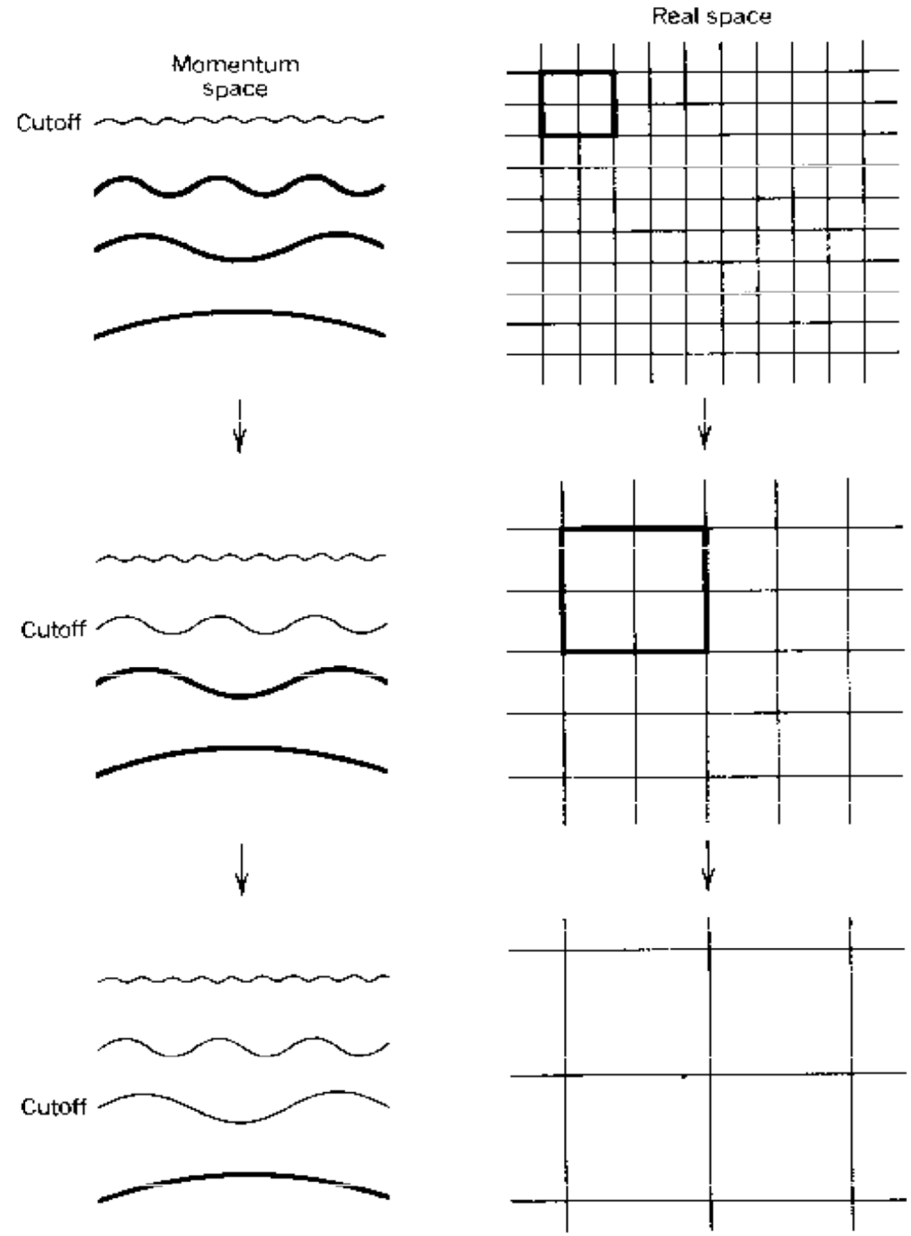
$$\Delta_t \sim 2 - \varepsilon/3 + O(\varepsilon^2)$$

$$F_s(t,0) = b^{-d} F_s(tb^{\Delta_t}, 0)$$

Functional (Exact , Non-perturbative) RG frameworks

Basic idea = Kadanof's block spin





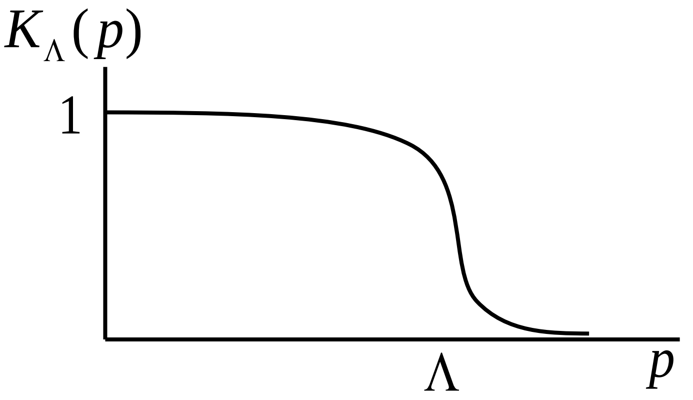
Exactly/Non-perturbatively!

FRG frameworks

1) Wilson/Wegner-Hougton/Polchinski

e.g., UV cutoff function by Polchinski
for an implementation of Wilson's RG

$$\text{Propagator} \sim \frac{K_\Lambda(p)}{p^2}$$



2) Legendre effective action (Nicoll-Chang, Wetterich, etc)

Effective average action by Wetterich with UV cutoff + IR cutoff function

Wilson /Polchinski Exact RG

introduce an inverse lattice space ~ Λ (UV cutoff)

$$S_\Lambda = \int_p \frac{1}{2} \frac{p^2}{K_\Lambda(p)} \phi_p \phi_{-p} + S_{\text{int}}[\phi]$$

$$Z[J, \Lambda, S_\Lambda] = \int D\phi \exp\left[S_\Lambda - \int_\Lambda J\phi\right]$$

and keep the generating functional invariant

$$Z[J, \Lambda, S_\Lambda] = Z[J, \Lambda - \delta\Lambda, S_{\Lambda - \delta\Lambda}]$$

$\Rightarrow$ flow equation for the action S

Again, RG equation procedure consists of 2 steps:

1) Integration of field fluctuations in shell $e^{-t}\Lambda < p < \Lambda$

$$\frac{dS_{\rm int}[\phi]}{dt} = -\int_p \frac{dK}{dp^2}\left[\frac{\delta S_{\rm int}}{\delta\phi_{-p}}\frac{\delta S_{\rm int}}{\delta\phi_p} - \frac{\delta^2 S_{\rm int}}{\delta\phi_{-p}\delta\phi_p}\right]$$

which keeps the generating functional invariant up to const.

2) Rescaling $p \longrightarrow e^t p$

$$\phi(p) = A^{d-d_\phi}\phi(Ap), \quad d_\phi = \frac{1}{2}(d-2+\eta)$$

Getting above 2 together, flow equation for the effective action

$$\frac{dS[\phi]}{dt} = -\int_p \frac{dK}{dp^2}\left[\frac{\delta S}{\delta\phi_{-p}}\frac{\delta S}{\delta\phi_p} - \frac{\delta^2 S}{\delta\phi_{-p}\delta\phi_p} + \frac{2p^2}{K}\phi_p\frac{\delta S}{\delta\phi_p}\right] - \int_p\left(\phi_p\, p\cdot\partial_p\frac{\delta S}{\delta\phi_p} + d_\phi\phi_p\frac{\delta S}{\delta\phi_p}\right)$$

Many other variants of RG equations with sharp/smooth cutoff.

2) Structure of RG flow (critical manifold, continuum limit, renormalizability)

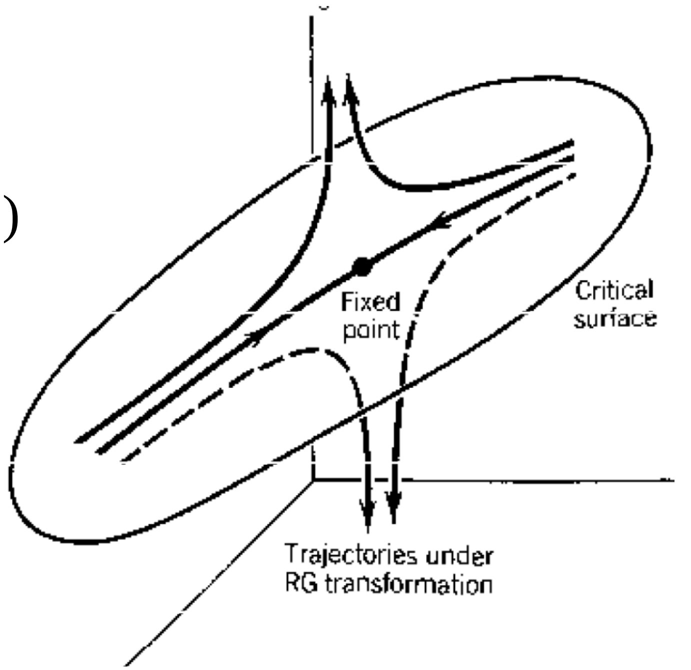
Flow in all coefficient (operator) space (Z2 symmetric theory space)

Classification of operators by mass dimension:

$$-S = \int dx^d \left[\frac{1}{2}(\nabla\phi)^2 + \lambda_2\phi^2 + \lambda_4\phi^4 + \cdots \right]$$

$$[S]=0, \quad [\phi]=(d-2)/2, \quad [\lambda_2]=2, \quad [\lambda_4]=4-d, \quad \ldots,$$

Theory space:



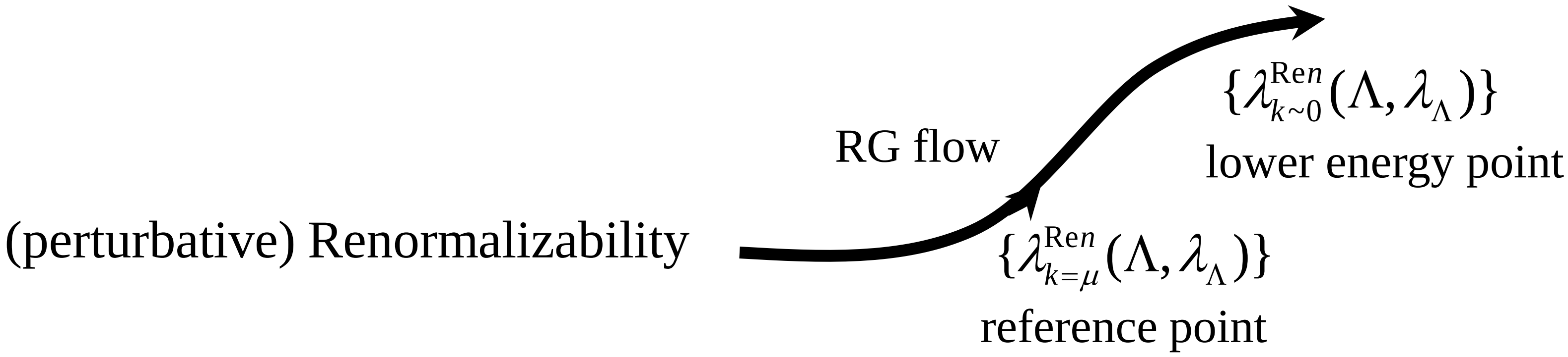
- >0 Relevant (s-renormalizable)
- <0 Irrelevant (non-renormalizable)
- =0 Marginal (renormalizable in most case?)

Tuning relevant couplings to critical surface.

Flow to IR direction,
fixed points on Critical surface
(e.g., co-dimension 2 at d=3),

Flow at lower energy scales k,
Non-renormalizable couplings (irrelevant couplings)
are guided only by renormalizable ones
(Polchinski order by order):

$$\text{for } k << \Lambda, \quad \left(\lambda_k^{Non-ren}\right)_i \cong M_{ij}\left(\lambda_k^{\mathrm{Re}n}\right)_j$$



This can be well illustrated by FRG
in case d=3 Z_2 symmetric scalar theory at critical.

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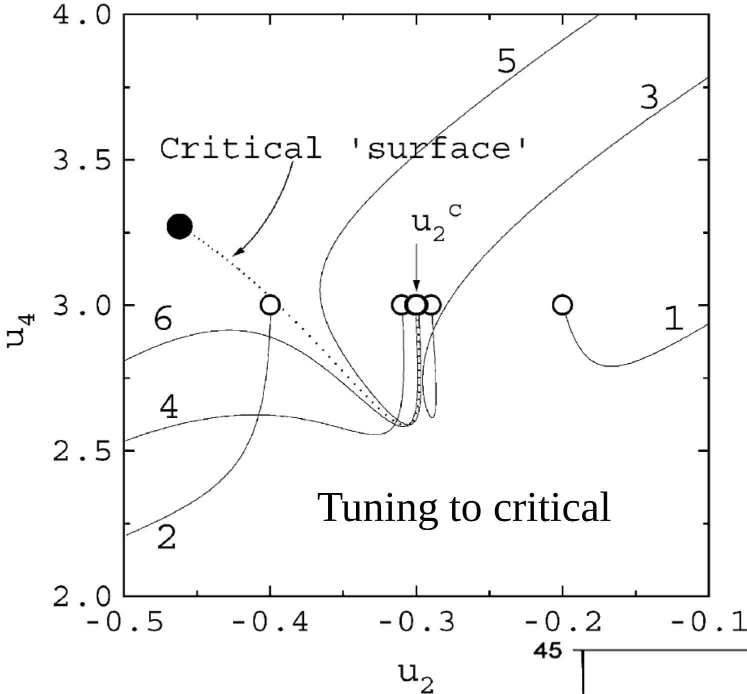
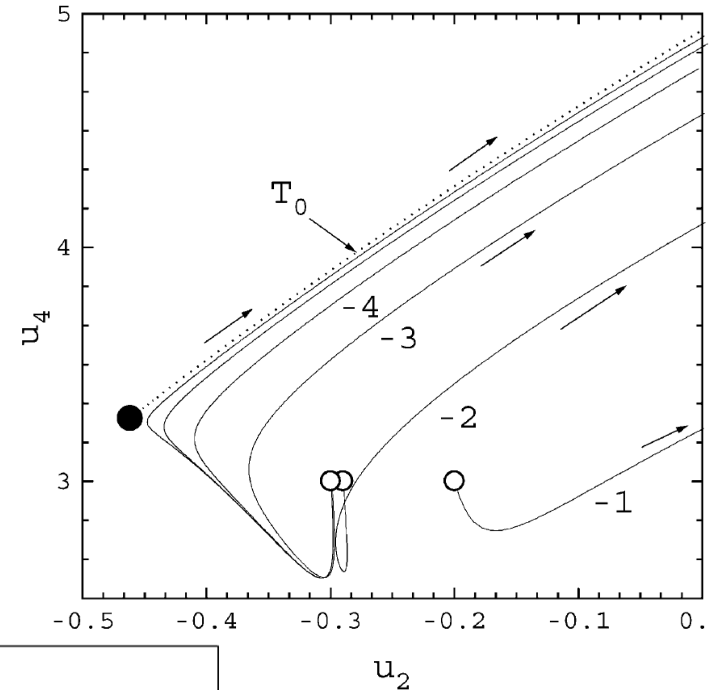
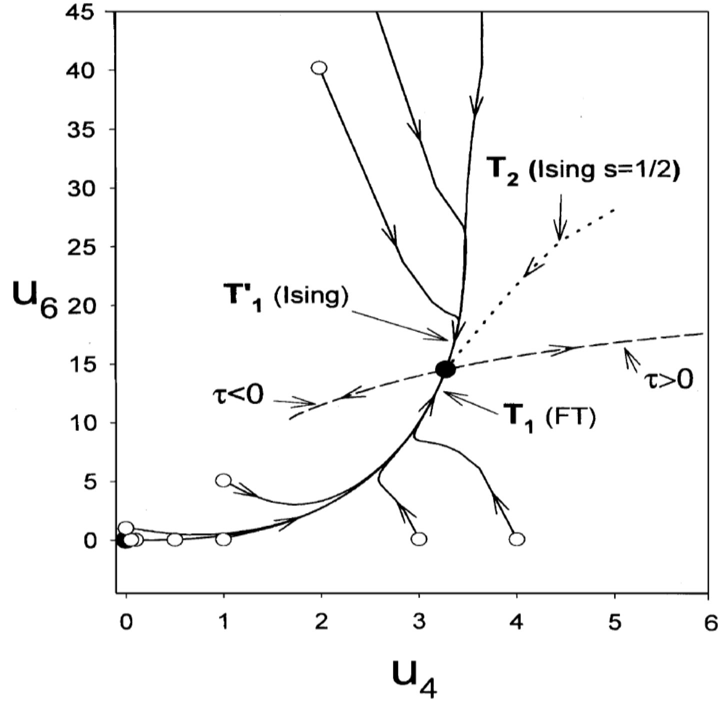






WF as UV stable
⇒ massive theory

One dimensional flow
on critical surface

G⇒WF , G as UV stable
⇒ massless theory

if starting from G
then veer away from WF
⇒UV asymptotic free

3) Functional Renormalization Group

Effective Average Action (Wetterich)

Functional Z in Euclidean space:

$$Z[J] = \int D\varphi \, e^{-S[\varphi]+\int \varphi J}$$

Generating functional for connected Green functions:

$$W[J] = \ln Z[J]$$

$$\frac{\delta W[J]}{\delta J(x)} = \langle \varphi(x) \rangle = \phi(x)$$

Effective action (Legendre transform):

$$\Gamma[\phi] = -W[J] + \int \phi \, J \quad at \quad \frac{\delta \Gamma[\phi]}{\delta \phi(x)} = J(x)$$

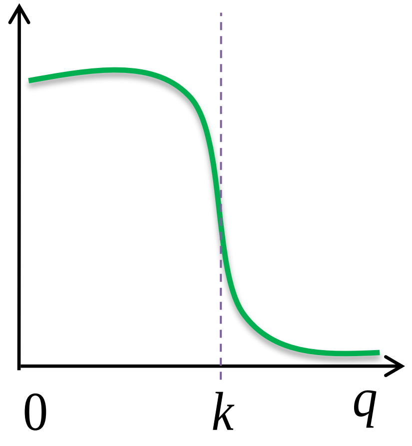
Scale dependent functional:

$$Z_\kappa[J] = \int D\varphi \exp\left[-S[\varphi] + \int \varphi J - \Delta S_\kappa[\varphi]\right]$$

$$\Delta S_\kappa[\varphi] = \frac{1}{2}\int_q R_\kappa(q)\varphi(p)\varphi(-p)$$

$R_k(q)$ IR cutoff fuction

$$\lim_{\kappa \to 0} R_\kappa(q) = 0$$



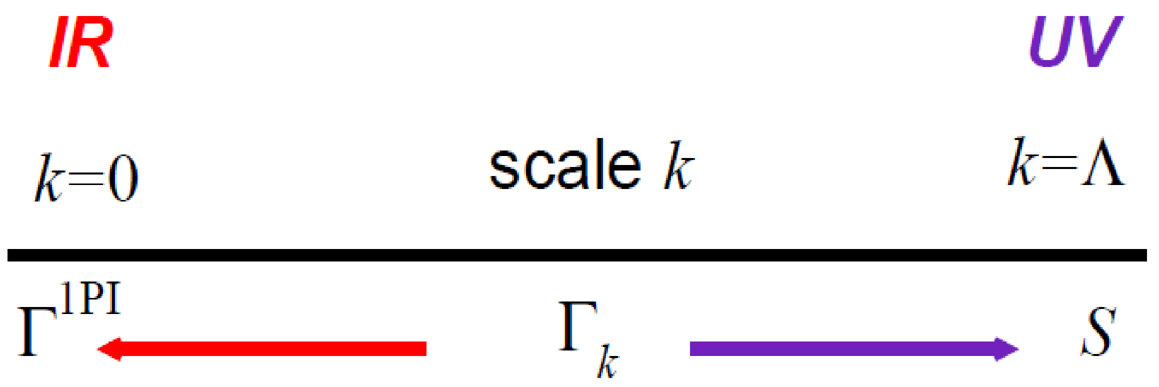
Scale dependent effective action:

$$\Gamma_\kappa[\phi] = -W_\kappa[J] + \int \phi J - \Delta S_\kappa[\phi]$$

Flow equation:

$$\partial_k \Gamma_k[\phi] = -\partial_k W_k[J] - \int \frac{\delta W_k}{\delta J(x)} \partial_k J(x) + \int \phi(x) \partial_k J(x) - \partial_k \Delta S_k[\phi]$$

$$= \frac{1}{2} \int_{x,y} G(x,y) \partial_k R_k = \frac{1}{2} \int_x \left[\left(\Gamma_k^{(2)}[\phi] + R_k \right)^{-1} \partial_k R_k \right]$$



$\Gamma_{k=0} = \Gamma$

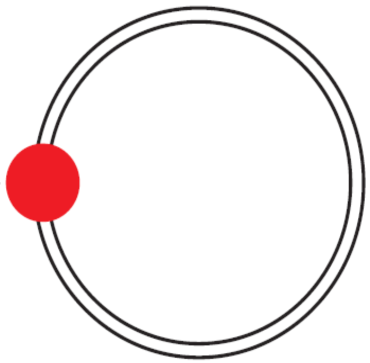
Full effective action

$\Gamma_{k=\Lambda} \equiv S$

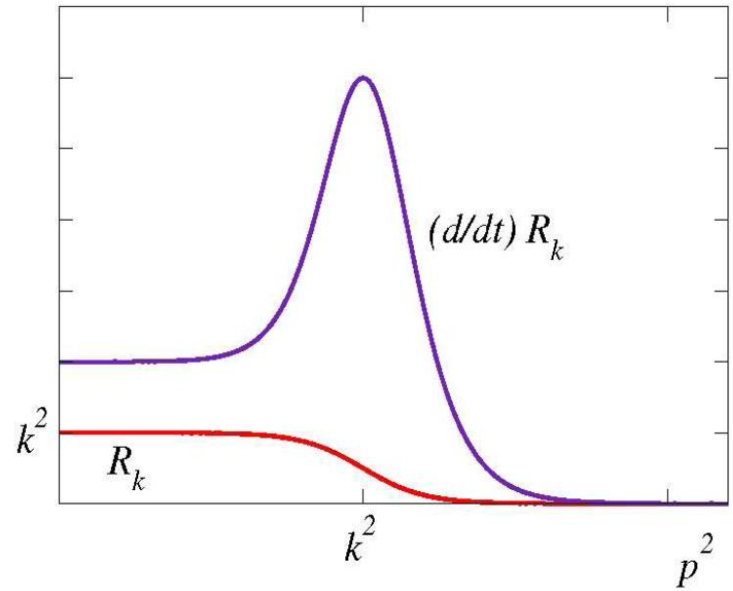
Classical /fundamental

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \int_x \left[\left(\Gamma_k^{(2)}[\phi] + R_k \right)^{-1} \partial_k R_k \right]$$

Flow equation looks like a one-loop form with full 2-point function.

$\partial_k \Gamma_k[\phi] = \frac{1}{2}$ 

Typical form of Cutoff function



Peak at *k* contributes to flow evolution !

Flow equation for n-point function:

$$\partial_k \Gamma_k^{(n)}(\phi) = \frac{\delta^n \Gamma_k(\phi)}{\delta\phi \cdots \delta\phi}$$

$$\partial_k \Gamma_k^{(2)}(\phi; p) = -\frac{1}{2}\int_q \partial_k R_k(q)\, G(q)^2 \Gamma_k^{(4)}(\phi; p, -p, q, -q);$$

$$\partial_k \Gamma_k^{(4)}(\phi; p_1, p_2, p_3, p_4) = \int_q \partial_k R_k(q^2)\, G(q)^2 \Gamma_k^{(4)}(\phi; p_1, p_2, q, -p_1 - p_2 - q) G(q + p_1 + p_2)$$
$$- \int_q \partial_k R_k(q^2)\, G(q)^2 \Gamma_k^{(6)}(\phi; p_1, p_2, p_3, p_4, q, -q);$$

$$\vdots$$

$$\vdots$$

where full propagator $G(q) = \left[\Gamma_k^{(2)}(\phi; q) + R_k(q^2)\right]_{q,-q}^{-1}$

Infinite hierarchy

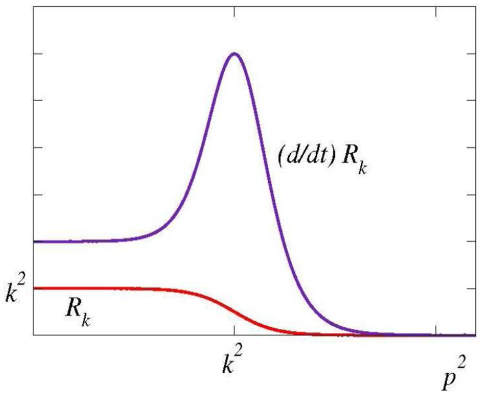
We need a truncation to make flow equation be of a closed form.
There are some truncation schemes, e.g., `` *derivative expansion* ”:

- From the structure of flow equation with cutoff function, Loop momentum is limited as

$$q \approx (<)\, k$$

- and for studies on e.g. uniform systems, external momenta are small

$$p_i < k$$



Implying that n-point functions at zero momenta are decoupled from other sectors in flow equation,

$$\Gamma_k^{(n)}(\phi; p_1, \ldots, p_n)\,|_{p_i \sim 0}$$

These functions are approximately closed in flow equation, and corrections are given by derivative expansion.

Expand the effective action by derivatives, at leading order

$$\Gamma_k(\phi) = \int U_k(\phi) + \frac{1}{2} Z_k(\phi)(\nabla\phi)^2 + O(\nabla^4)$$

$$U_k(\phi) = \sum_{n=0} \Gamma_\kappa^{(2n)}(0;0,...,0)\phi^{2n}$$

the effective potential with vertices at zero momentum,

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \int_x \left[\left(\Gamma_k^{(2)}[\phi] + R_k \right)^{-1} \partial_k R_k \right]$$

Local potential approximation (LPA): $Z_k \Rightarrow 1$

$$\partial_k U_k(\phi) = \frac{1}{2} \int_q \frac{\partial_k R_k(q^2)}{q^2 + U_k^{(2)}(\phi) + R_k(q^2)}$$

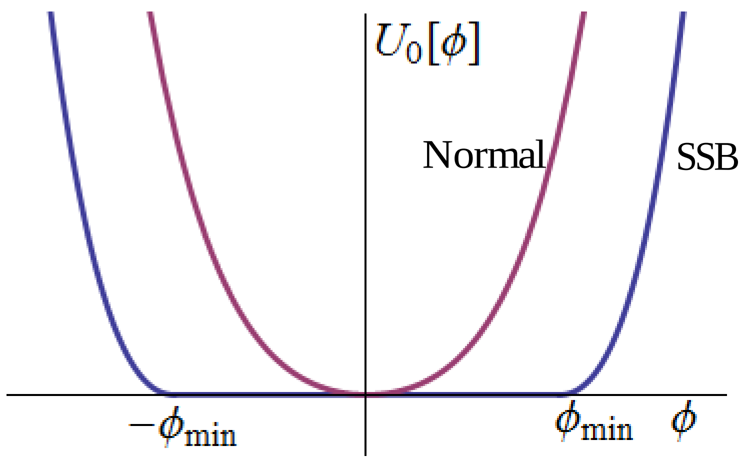
Solving flow equation in leading derivative expansion :

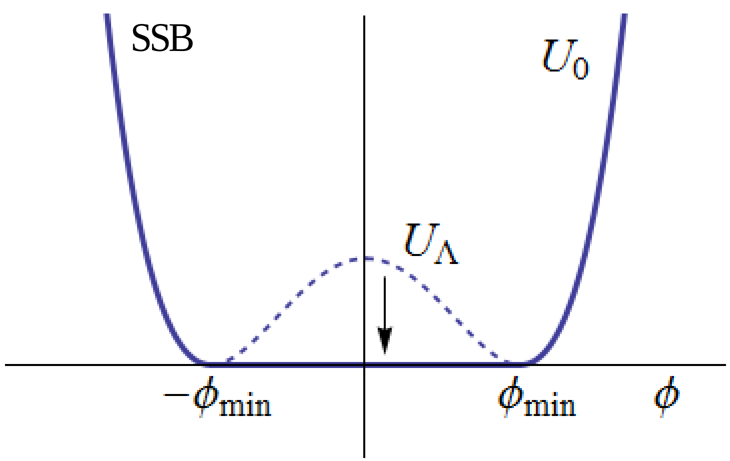
- Suitable for critical phenomena where long-range physics does matter.

- Projection of functional space onto leading order of derivative expansion without spoiling non-perturbative nature of FRG.

Start with $U_\Lambda[\phi] = \lambda_2 \phi^2 + \lambda_4 \phi^4$ $\Rightarrow$ Flow to $k=0$

without rescaling $\Rightarrow$ direct calculation of the scale dependent potential $U_k(\phi)$





$U_k(\phi) \sim -k^2\phi^2$ $\Rightarrow$ convex

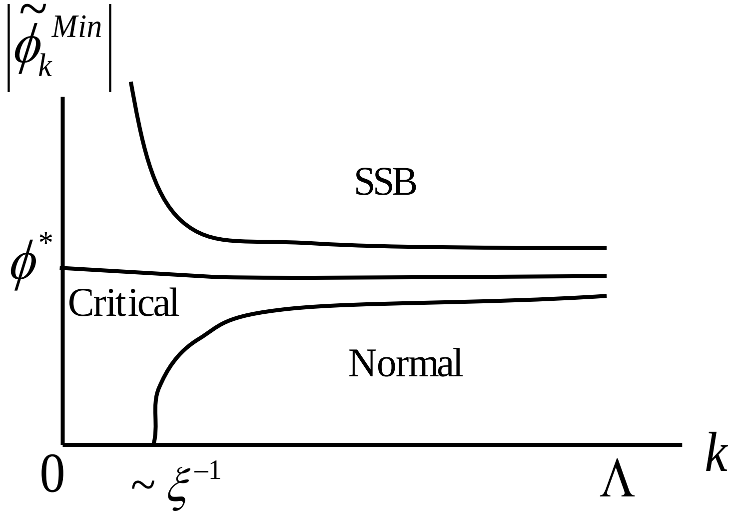
Scaling form of Flow equation

- rescaling *a la* Wilson → Rescaling (by inverse ``lattice space'' k)
- search for fixed points
- critical properties

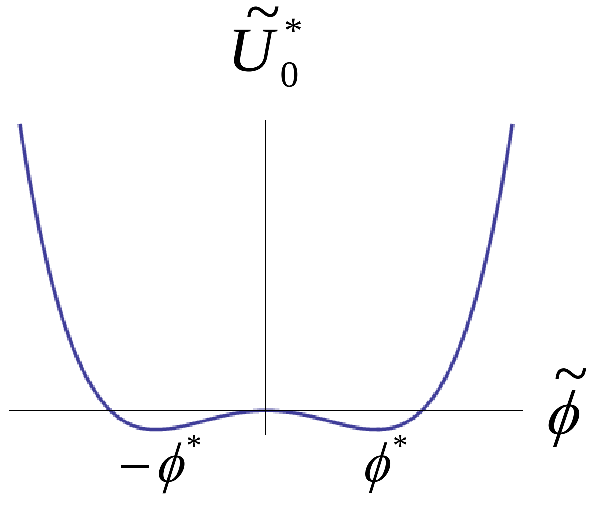
$$\tilde{\phi} = Z_k^{1/2} k^{(2-d)/2} \phi,$$
$$\tilde{U}_k(\tilde{\phi}) = k^{-d} U_k(\phi),$$
$$\vdots$$

Rescaled running minimum

$$\tilde{\phi}_k^{Min} \sim \frac{\phi^{Min}}{k^{\frac{d-2+\eta}{2}}}$$



Rescaled potential at WF fixed point



Critical exponents:

Observe how quantities approach/escape from WF fixed point (a critical point) by adjusting parameters during RG evolution.

Starting just off the critical surface and approach it by shooting some times.

$$\tilde{U}_\Lambda(\tilde{\phi}) = \lambda_{2,\Lambda}\left(\tilde{\phi}^2 - \mu_\Lambda\right) + \lambda_{4,\Lambda}\left(\tilde{\phi}^2 - \mu_\Lambda\right)^2 + \cdots$$

k has a inverse length scale.

$$\tilde{\phi}_k^{Min} \sim \frac{\phi^{Min}}{k^{\frac{d-2+\eta}{2}}} \quad \Rightarrow \quad \phi^{Min} \sim \left(\xi^{-1}\right)^{\frac{d-2+\eta}{2}}$$

$$\xi^{-2} = m_R^2 = k^2 \left.\frac{\delta^2 \tilde{U}}{\delta \tilde{\phi}_k^2}\right|_{\tilde{\phi}_k = 0} \propto \left(\mu_\Lambda - \mu_{CR}\right)^{2\nu}$$

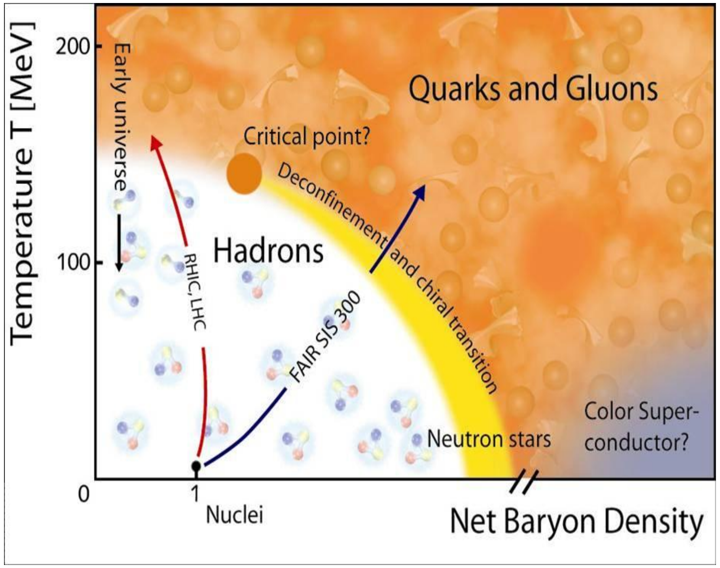
$$\mu_\Lambda - \mu_{CR} \sim T - T_C \quad \Rightarrow \quad \xi \sim \left(T - T_C\right)^{-\nu}$$

Other critical exponent can be deduced from scaling relations.

4, Chiral effective theory at finite T and μ (an exercise)

$$L_{qm} = \frac{1}{2}(\partial\sigma)^2 + \frac{1}{2}(\partial\vec{\pi})^2 + \bar{q}i\partial\!\!\!/ q - g\bar{q}(\sigma + i\vec{\tau}\cdot\vec{\pi}\gamma_5)q - U(\sigma,\vec{\pi})$$

$$U(\sigma,\vec{\pi}) = \frac{1}{2}m(\sigma^2+\vec{\pi}^2) + \frac{\lambda}{4}(\sigma^2+\vec{\pi}^2)^4 - c\sigma$$



The same universality class as QCD CEP (3d Ising model)

Derivative expansion for QM model:

$$\Gamma_k = \int d^4x \left[\frac{1}{2} Z_{\phi,k} \left(\partial_\mu \phi\right)^2 + Z_{\psi,k} \bar{\psi} \partial\!\!\!/ \psi + g \bar{\psi} \left(\sigma + i \vec{\tau} \cdot \vec{\pi} \gamma_5\right) \psi + U_k(\rho)\right]$$

Temperature:

$$\int \frac{d^d q}{(2\pi)^d} \rightarrow T \sum_{n \in \mathbb{Z}} \int \frac{d^{d-1} q}{(2\pi)^{d-1}}$$

Chemical potential:

$$\partial_0 \rightarrow \partial_0 + i\mu$$

Contributions from Bosonic and Fermionic parts.

$$\partial_k U_k(\rho) = \partial_k U_{k,B}(\rho) + \partial_k U_{k,F}(\rho)$$

Optimized cut-off functions (Litim)

$$R^{\text{opt}}_{B,k}(q^2) = (k^2 - q^2)\theta(k^2 - q^2),$$

$$R^{\text{opt}}_{F,k}(q) = q\!\!\!/ \left(\sqrt{\frac{k^2}{q^2}} - 1\right) \theta(k^2 - q^2)$$

Bosonic part: Matsubara sum, analytic thanks to optimized cut-off function

$$\partial_k \Omega_{k,B}(T,\mu;\rho) = \frac{1}{2}\int \frac{d^3\mathbf{q}}{(2\pi)^3} T \sum_{n=-\infty}^{+\infty} \frac{\partial}{\partial k}\left[\log(q_0^2 + \mathbf{q}^2 + M_\sigma^2 + k^2 - \mathbf{q}^2)\right.$$
$$\left. + \; 3\log(q_0^2 + \mathbf{q}^2 + M_\pi^2 + k^2 - \mathbf{q}^2)\right].$$

$$\partial_k \Omega_{k,B}(T,\mu;\rho) = \frac{k^4}{6\pi^2} T \sum_{n=-\infty}^{+\infty} \left(\frac{1}{q_0^2 + E_\sigma^2} + \frac{3}{q_0^2 + E_\pi^2} \right)$$

$$E_\sigma = \sqrt{k^2 + M_\sigma^2}, \qquad E_\pi = \sqrt{k^2 + M_\pi^2}$$

$$M_\sigma^2 = U_k' + 2\rho U_k'', \qquad M_\pi^2 = U_k'$$

Fermionic part: Matsubara sum, analytic thanks to optimized cut-off function

$$\partial_k U_{k,F}(\rho) = -\nu_q \int \frac{d^4q}{(2\pi)^4} \frac{\partial}{\partial k} \log\left((q_0 + i\mu)^2 + k^2 + M_q^2 \right),$$

$$M_q^2 = 2\rho g^2$$

$$\partial_k \Omega_{k,F}(T,\mu;\rho) = -2\nu_q \frac{k^4}{6\pi^2} T \sum_{n=-\infty}^{n=+\infty} \frac{1}{(q_0 + i\mu)^2 + E_q^2},$$

$$E_q = \sqrt{k^2 + 2\rho g^2}$$

$$\sum_n \frac{1}{(q_0 + i\mu)^2 + E_q^2} = \frac{1}{2E_q} \sum_n \left(\frac{E_q - \mu}{q_0^2 + (E_q - \mu)^2} + \frac{E_q + \mu}{q_0^2 + (E_q + \mu)^2} \right)$$

Flow equation for thermodynamic potential of QM model at finite T and μ :

$$\partial_k \Omega_k(T,\mu;\rho) = \frac{k^4}{12\pi^2}\left[\frac{3}{E_\pi}\left(1+2n_B(E_\pi)\right)+\frac{1}{E_\sigma}\left(1+2n_B(E_\sigma)\right) - \frac{2\nu_q}{E_q}\left(1-n_F(E_q)-\bar{n}_F(E_q)\right)\right].$$

where Bose and Fermi distribution functions:

$$n_B(E_{\pi,\sigma}) = \frac{1}{e^{E_{\pi,\sigma}/T}-1}$$

$$n_F(E_q) = \frac{1}{e^{(E_q-\mu)/T}+1}, \quad \bar{n}_F(E_q) = \frac{1}{e^{(E_q+\mu)/T}+1}$$

Grid method:

Boundary condition at cut-off scale:

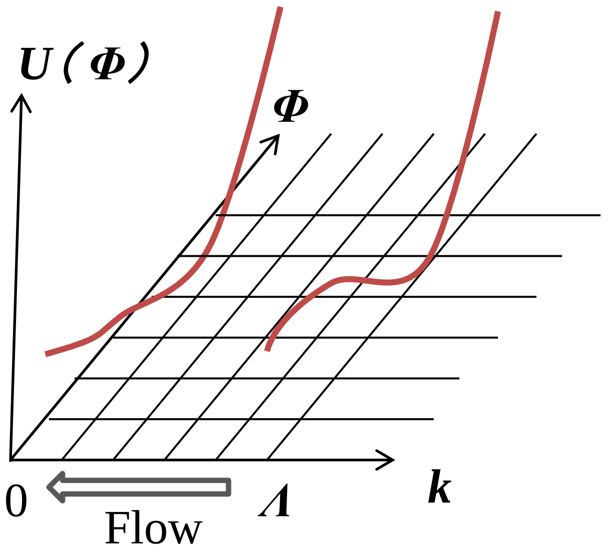
$$U_{k=\Lambda}(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{1}{4} \lambda \phi^4 - c\sigma$$

Scale evolution by flow equation

⇩

$$U_{k=0}(\phi)$$

At a fixed T



Parameters are fixed to provide low energy physics:
Pion decay constant and meson masses, etc.

Taylor expansion around minimum :

$$\Omega_k(T,\mu) = \sum_m \frac{a_{m,k}(T,\mu)}{m!}(\rho_k - \rho_0)^m - c\sigma_k$$

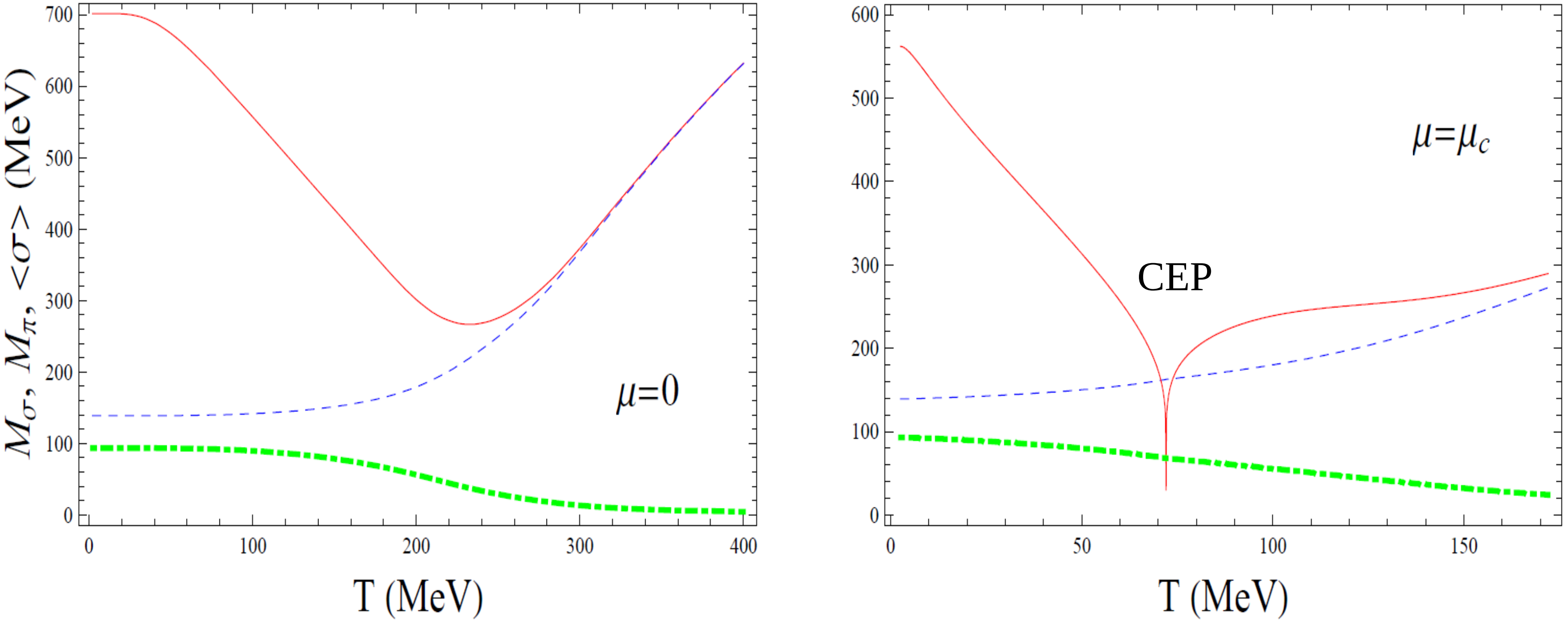
$$\sigma_0 = \sqrt{2\rho_0}$$

Flow equations with beta functions.:

$$\begin{aligned}
\frac{da_0}{dk} &= \frac{c}{\sqrt{2\rho_k}}\frac{d\rho}{dk} + \partial_k\Omega_k, \\
\frac{d\rho}{dk} &= -\frac{1}{(c/(2\rho_k)^{3/2} + a_2)}\,\partial_k\Omega_k', \\
\frac{da_2}{dk} &= a_3\frac{d\rho}{dk} + \partial_k\Omega_k'', \\
\frac{da_3}{dk} &= \partial_k\Omega_k'''\,.
\end{aligned}$$

$$m^2_{\pi,k} = \frac{c}{\sqrt{2\rho_k}}, \quad m^2_{\sigma,k} = \frac{c}{\sqrt{2\rho_k}} + 2\rho_k a_{2,k}$$

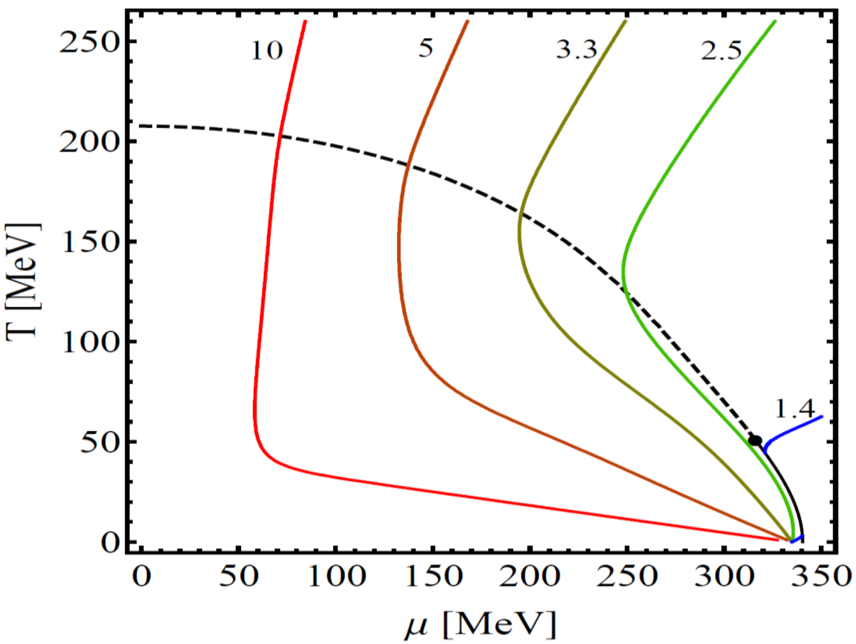
From Functional RG approach to a Chiral model

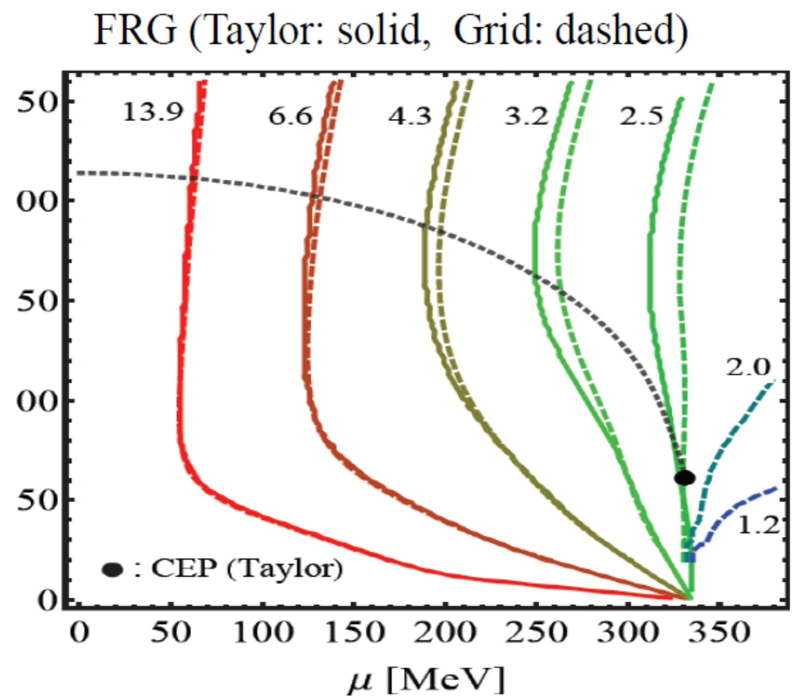


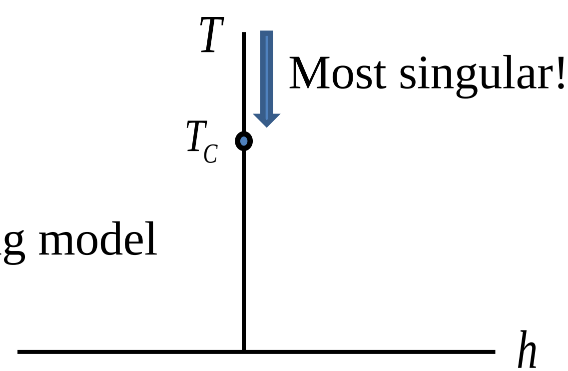
Inverse of Sigma mass → ∞

Critical region is small ?? though it is not universal quantity. What if QCD if exist?

Isentropes: $\frac{s}{n} = const.$







3d Ising model

EN et.al in PLB,
grid method: thanks to B-J Schafer

Summary

- Legendre effective action in FRG

  1) UV scale (renormalizable/non-renormalizable theories)
  and IR cutoff function

  2) non-perturbative framework at arbitrary dimension

- Derivative expansion and LPA

- Flow equation with and without rescaling

  1) direct evaluation of effective action (not only universal properties)

  2) universal scaling properties in critical regime

  3) at finite T and μ, with Fermions and ….

Ref: Delamotte, Berges-Tetradis-Wetterich, Bagnuls-Bervillier, etc

d	2	3	4
α	0	0.110(1)	0
β	1/8	0.3265(3)	1/2
γ	7/4	1.2372(5)	1
δ	15	4.789(2)	3
η	1/4	0.0364(5)	
ν	1	0.6301(4)	1/2
ω	2	0.84(4)	