Renormalization Group Approach to Casimir Effect
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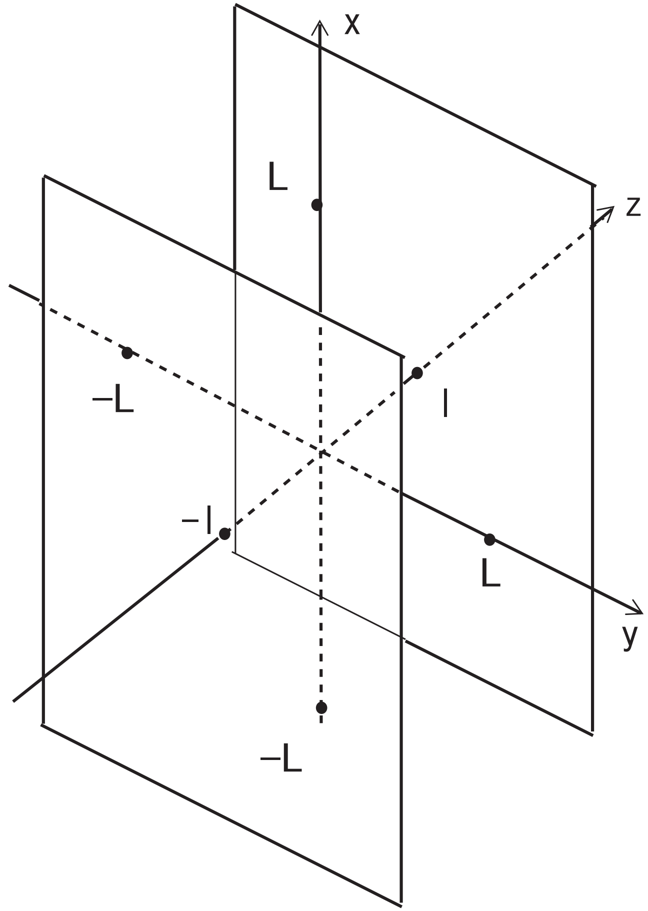
1. Introduction

Casimir energy Vacuum energy for the free part of the system

- independent of the coupling(s)
- depends on the boundary parameters and the topology
- quantum effect (zero-point oscillation)
- highly-delicate regularization is required

AdS/CFT : Geometrical interpretation of the renormalization group flow

Figure 1: Configuration of Casimir energy measurement.



2. Ordinary Regularization for Casimir Energy

1+3 D electromagnetism (free field theory) in Minkwski space:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \quad . \tag{1}$$

2 perfectly-conducting plates parallel with the separation $2l$ in the x-direction. As for y- and z-directions,the periodicity $2L$ for the IR regularization.

$$\text{Periodicity} \quad : \quad x \rightarrow x + 2l \quad , \quad y \rightarrow y + 2L \quad , \quad z \rightarrow z + 2L \quad , \\ L \gg l \quad , \tag{2}$$

the eigen frequencies and Casimir energy are

$$\omega_{n,m_y,m_z} = \sqrt{(n\frac{\pi}{l})^2 + (m_y\frac{\pi}{L})^2 + (m_z\frac{\pi}{L})^2} \quad ,$$

$$E_{Cas} = 2 \cdot \sum_{n,m_y,m_z \in \mathbf{Z}} \frac{1}{2}\omega_{n,m_y,m_z} \quad \geq 0 \quad , \tag{3}$$

($\mathbf{Z}$: all integers) $\frac{1}{2}\omega_{n,m_y,m_z}$ is the zero-point oscillation energy. Introducing the cut-off function $g(x)$ ($= 1$ for $0 < x < 1$, 0 for otherwise),

$$E^{\Lambda}_{Cas} = \sum_{n,m_y,m_z \in \mathbf{Z}} \omega_{n,m_y,m_z} g\left(\frac{\omega_{n,m_y,m_z}}{\Lambda}\right) \quad \geq 0 \quad , \quad \Lambda \ : \ \text{UV-CutOff} \ . \tag{4}$$

take the continuum limit $L \to \infty,\ L \ll l \to \infty$.

$$E^{\Lambda 0}_{Cas} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dk_y dk_z}{(\frac{\pi}{L})^2}\int_{-\infty}^{\infty}\frac{dk_x}{\frac{\pi}{l}}\sqrt{k_x^2+k_y^2+k_z^2}\ g(\frac{k}{\Lambda})$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty\,|k|\le\Lambda}^{\infty}\frac{dk_x dk_y dk_z}{(\frac{\pi}{L})^2\frac{\pi}{l}}\sqrt{k_x^2+k_y^2+k_z^2}\quad \ge 0 \quad . \tag{5}$$

Note that E_{Cas}, E^{Λ}_{Cas} and $E^{\Lambda 0}_{Cas}$ are all positive-definite. In a familiar way, regarding $E^{\Lambda 0}_{Cas}$ as the origin of the energy scale, we consider the quantity $u = (E^{\Lambda}_{Cas} - E^{\Lambda 0}_{Cas})/(2L)^2$ as the physical Casimir energy and evaluate it with the help of the Euler-MacLaurin formula as $u = (\pi^2/(2l)^3)\ (B_4/4!) = -(\pi^2/720)(1/(2l)^3) < 0$. The final result is negative. In the present analysis we take a new regularization which keeps positive-definiteness.

3. New Regularization for Casimir Energy

First we re-express $E_{Cas}^{\Lambda 0}$ using a simple identity : $l = \int_0^l dw$ (w: a regularization axis).

$$
\begin{aligned}
E_{Cas}^{\Lambda 0}/(2L)^2 &= \frac{1}{2^2\pi^3}\int_0^l dw \int_{k\leq\Lambda} P(k) 2\pi k^2 dk \\
&= \frac{1}{2^2\pi^3}\int_0^l dw(-1)\int_{r\geq\Lambda^{-1}} P(1/r)(-1)2\pi r^{-4} dr \quad . \\
& P(k)\equiv k \quad , \quad r \equiv \frac{1}{k} \quad ,
\end{aligned}
\tag{6}
$$

where the integration variable changes from the momentum (k) to the coordinate

($r = \sqrt{x^2 + y^2 + z^2}$). The integration region in (R, w)-space is the infinite rectangular shown in Fig.2.

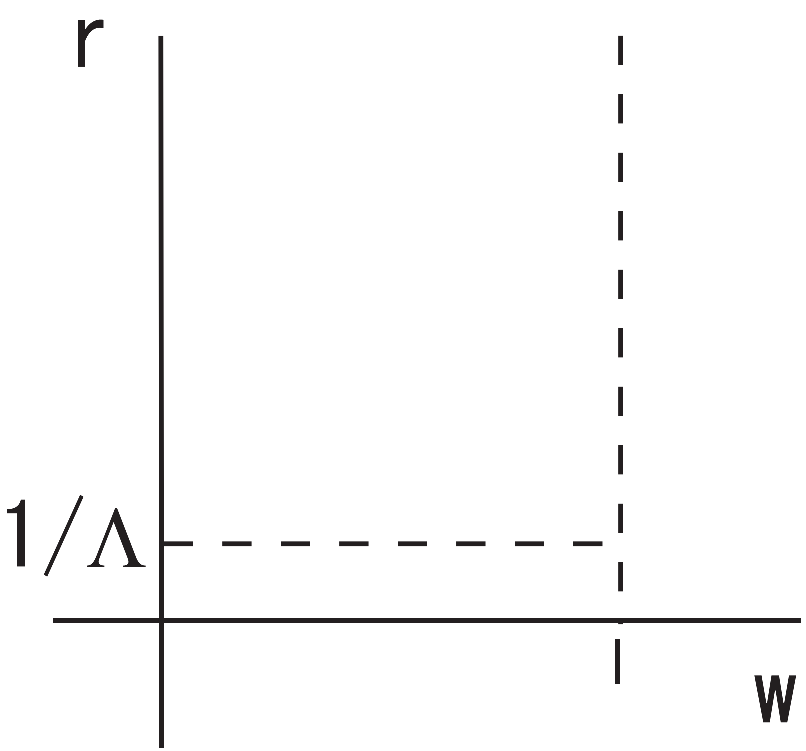


Figure 2: The integral region of (6).

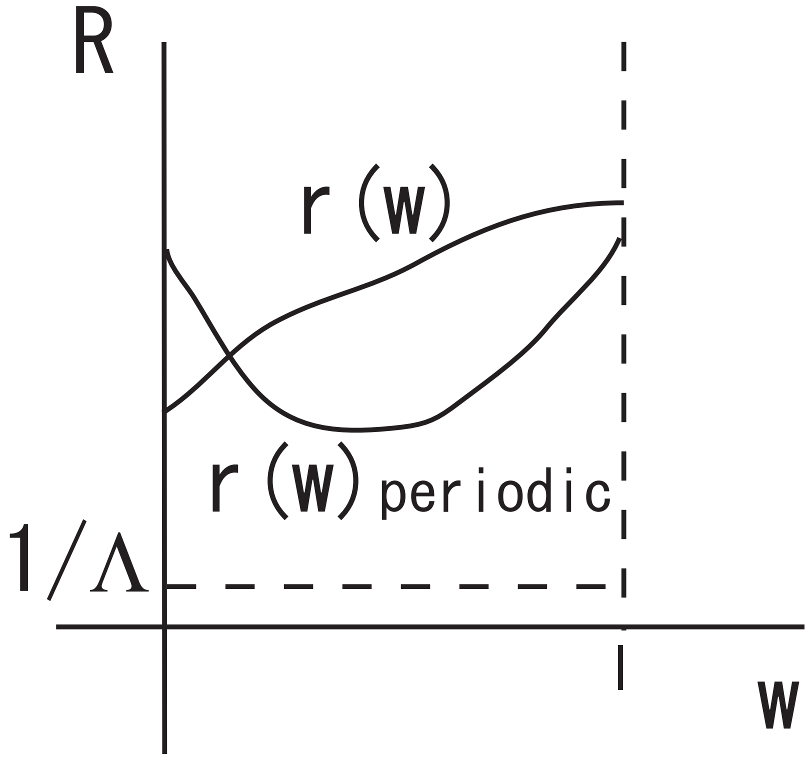


Figure 3: A general path $r(w)$ of (7) and a periodic path $r(w)$ of (8).

We regularize the above expression using the path-integral as

$$E_{Cas}^{\mathcal{W}}{}'/(2L)^2 = \frac{1}{2^2\pi^3}(2\pi)\int_{\text{all paths } r(w)}$$

$$\prod_w \mathcal{D}r(w)\left[\int dw' P(\frac{1}{r(w')})r(w')^{-4}\right]\exp\{-\mathcal{W}[r(w)]\} \quad , \tag{7}$$

where the integral is over all paths $r(w)$ which are defined between $0 \leq w \leq l$ and whose value is above Λ^{-1}, as shown in Fig.3. $\mathcal{W}[r(w)]$ is some damping functional. $\mathcal{W}[r(w)] = 0$ corresponds to (6). The slightly-more-restrictve regularization is

$$E_{Cas}^{\mathcal{W}}/(2L)^2 = \frac{1}{2^2\pi^3}(2\pi)\int_{\Lambda^{-1}}^{\infty} d\rho \int_{r(0)=r(l)=\rho}$$

$$\prod_w \mathcal{D}r(w) \left[\int dw' P(\frac{1}{r(w')}) r(w')^{-4} \right] \exp\{-\mathcal{W}[r(w)]\} \quad \geq 0 \quad , \tag{8}$$

where the integral is over all periodic paths. Note that the above regularization keep the positive-definite property. Hence the present regularization mainly defined by the choice of $\mathcal{W}[r(w)]$. In order to specify it, we introduce the following metric in (R, w)-space.

$$\text{Dirac Type} \ : \quad ds^2 = dR^2 + V(R)dw^2 \quad , V(R) = \Omega^2 R^2 \quad , \tag{9}$$

or

$$\text{Standard Type} \ : \quad ds^2 = \frac{1}{dw^2}(dR^2 + V(R)dw^2)^2 \quad , V(R) = \Omega^2 R^2 \quad . \tag{10}$$

Ω : regularization constant. (When $V(R) = 1$, w is the familiar Euclidean time.

) On a path $R = r(w)$, the induced metric and the length L is given as follows. As the damping functional $\mathcal{W}[r(w)]$, we take the length L.

$$ds^2 = dw^2(r'^2 + \Omega^2 r^2)^2 \quad , \quad r' \equiv \frac{dr}{dw} \quad ,$$

$$L = \int ds = \int (r'^2 + \Omega^2 r^2) dw \quad , \quad \mathcal{W}[r(w)] \equiv \frac{1}{2\alpha} L = \frac{1}{2\alpha} \int (r'^2 + \Omega^2 r^2) dw \quad . \tag{11}$$

α , Ω : regularization parameters. The limit $\alpha \to \infty$ corresponds to (6).

Numerical calculation can evaluate $E^{\mathcal{W}}_{Cas}$ (8), and we expect the following form[PTP121(2009)727].

$$\frac{E^{\mathcal{W}}_{Cas}}{(2L)^2} = \frac{a}{l^3}(1 - 3c \ln\,(l\Lambda)) \quad , \tag{12}$$

where a and c are some constants. a should be positive because of the positive-definiteness of (8). The present regularization result has, like the ordinary renormalizable ones such as the coupling in QED, the log-divergence. The divergence can be renormalized into the boundary parameter l. This means l flows according to the renormalization group.

$$l' = l(1 - 3c\ln(l\Lambda))^{-\frac{1}{3}} \quad , \quad \beta \equiv \frac{d\ln(l'/l)}{d\ln\Lambda} = c \quad , \quad |c| \ll 1 \quad , \tag{13}$$

where β is the renormalization group function, and we assume $|c| \ll 1$. The sign of c determines whether the length separation increases ($c > 0$) or decreases ($c < 0$) as the measurement resolution becomes finer (Λ increases). In terms of the usual terminology, attractive case corresponds to $c > 0$, and repulsive case to $c < 0$.

4. Conclusion

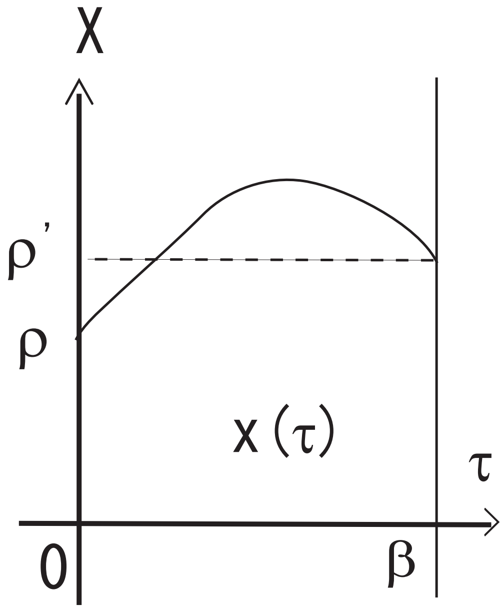
We have proposed a new regularization, in the quantum field theory, for the calculation of divergent physical quantities such as Casimir energy.

- $ds^2 = dR^2 + \Omega^2 R^2 dw^2$ (Elastic view to the space)

- Path integral using Hamiltonian (Weight functional) of length.

- Positive definite

Figure 4: A path of line in 2D Euclidean space (X,τ). The path starts at x(0)=ρ and ends at x(β)=ρ'.



2.Quantum Statistical System of Harmonic Oscillator

'Dirac' Type

$$ds^2 = dX^2 + \omega^2 X^2 d\tau^2 = G_{AB} dX^A dX^B \quad ,$$
$$(X^A) = (X^1, X^2) = (X, \tau) \quad , \quad (G_{AB}) = \mathrm{diag}(1, \omega^2 X^2) \quad ,$$
$$R = G^{AB} R_{AB} = 0 \quad , \qquad (14)$$

where $A, B = 1, 2$. Periodicity:

$$\tau \to \tau + \beta \quad , \quad \beta \ : \ \text{inverse of temperature} \quad (\beta = \frac{1}{kT}) \qquad (15)$$

The induced metric on a line

$$X = x(\tau) \quad , \quad dX = \dot{x} d\tau \quad , \quad \dot{x} \equiv \frac{dx}{d\tau} \quad , \quad 0 \leq \tau \leq \beta \quad ,$$
$$ds^2 = (\dot{x}^2 + 2V(x))d\tau^2 \quad , \quad V(x) \equiv \frac{1}{2}x^2 \quad . \tag{16}$$

Then the length L of the path $x(\tau)$

$$L = \int ds = \int_0^\beta \sqrt{\dot{x}^2 + 2V(x)} d\tau \quad . \tag{17}$$

We take the half of the length $(\frac{1}{2}L)$ as the system Hamiltonian (minimal length principle). Free energy F:

$$e^{-\beta F} = \int_{-\infty}^{\infty} d\rho \int_{\substack{x(0)=\rho \\ x(\beta)=\rho}} \prod_{\tau} \mathcal{D}x(\tau) \exp\left[-\frac{1}{2}\int_0^{\beta} \sqrt{\dot{x}^2 + 2V(x)} d\tau\right] \quad , \quad (18)$$

Normal Type

$$ds^2 = \frac{1}{d\tau^2}(dX^2)^2 + 4V(X)^2 d\tau^2 + 4V(X)dX^2 = \frac{1}{d\tau^2}(dX^2 + 2V(X)d\tau^2)^2 \quad , (19)$$

where we have the following condition.

$$d\tau^2 \sim O(\epsilon^2) \quad , \quad dX^2 \sim O(\epsilon^2) \quad , \quad \frac{1}{d\tau^2}dX^2 \sim O(1) \quad , \qquad (20)$$

Note that we do not have 2D metric in this case ('primordial' geometry). We again impose the periodicity (period: β):(15). The induced metric on the line:

$$X = x(\tau) \quad , \quad dX = \dot{x} d\tau \quad , \quad \dot{x} \equiv \frac{dx}{d\tau} \quad , \quad 0 \leq \tau \leq \beta \quad ,$$
$$ds^2 = (\dot{x}^2 + 2V(x))^2 d\tau^2 \quad . \tag{21}$$

On the path, we have this induced metric. The length L is given by

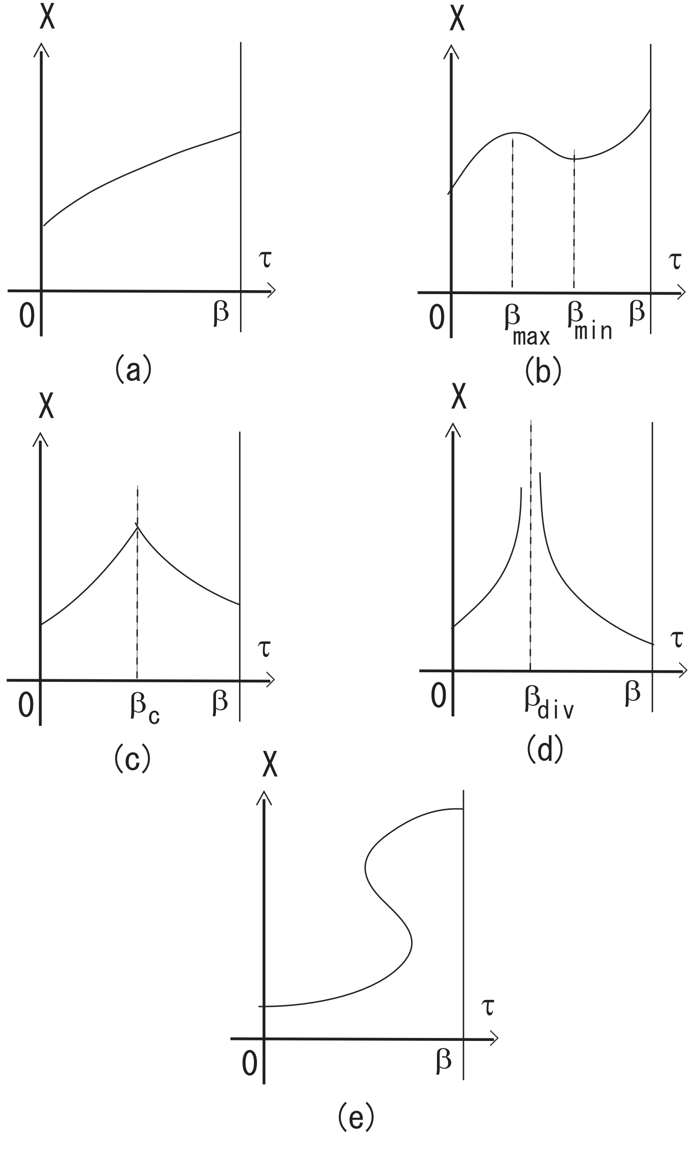
$$L[x(\tau)] = \int ds = \int_0^\beta (\dot{x}^2 + 2V(x)) d\tau \quad . \tag{22}$$

Taking $\frac{1}{2}L$ as the Hamiltonian (minimal length principle), the free energy F:

$$\mathrm{e}^{-\beta F} = \int_{-\infty}^{\infty} d\rho \int_{\substack{x(0)=\rho \\ x(\beta)=\rho}} \prod_{\tau} \mathcal{D}x(\tau) \exp\left[-\frac{1}{2}\int_0^{\beta} (\dot{x}^2 + 2V(x))d\tau\right] \quad , \quad (23)$$

This is exactly the free energy of the harmonic oscillator.

Figure 5: Singular and regular lines in 2D Euclidean space (X,τ).



5. Conclusion

We have proposed a new formalism to calculate the friction properties. The advantageous point, compared with the Langevin eq., is the use of the path-integral. It clarifies the averaging procedure.