Percolation in $2d$ coarsening dynamics

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Plan

- Introduction and Motivations
- Model and simulations
- Approach to percolation
- Consequences : Correlation function and Finite temperature
- Extensions
- Conclusions

Introduction and Motivations

Introduction and Motivations

- The dynamics after a quench at low temperature of a ferromagnetic system is studied since a very long time.
- For a dynamics with a non conserved order parameter, the evolution is controlled by the growth of a characteristic length scale $R(t) \simeq t^{1/2}$ with an equilibrium reached when $R(t) \simeq L$ with L the linear size of the considered system. Natural time scale is then t/L^2.
- Finite size spin clusters can also be considered. It can be seen that these clusters will shrink and disappear due to a curvature-driven ordering processes described by the Allen-Cahn equation : the local velocity of an interface is proportional to the local curvature.
- For crossing or wrapping interfaces, the curvature is zero.

Introduction and Motivations

- This explains the existence of metastable stripe states. Krapivsky, Redner and collaborators have obtained recently results on the probability existence of such metastable strip states for the ferromagnetic $2d$IM model at $T = 0$.
- While it was observed since a very long time (2001) that the proportion is $\simeq 1/3$ for strip states and $\simeq 2/3$ for ground states, it is only recently (2009) that a link with percolation was obtained.
- For the FBC case, the proportion of ground states is $2\pi_{hv} = \frac{1}{2} + \frac{\sqrt{3}}{2\pi} \log\left(\frac{27}{16}\right) = 0.64424..$ and the probability of strip states is $\pi_h + \pi_v = \frac{1}{2} + \frac{\sqrt{3}}{2\pi} \log\left(\frac{27}{16}\right) = 0.35576...$, (J. Cardy, 1992, G. Watts, 1996).
- Similar results for PBC and also with an aspect ratio $r = L_X/L_Y$.

Introduction and Motivations

- A link with the percolation was already observed in a serie of works by Arenzon, Bray, Cugliandolo and collaborators (2007) who considered the statistics of spin clusters for the Ising model after a quench at a subcritical point and observed that after few steps, the distribution scales like for percolation.
- Question : where and how does the percolation come in this problem ?

Model and simulations

Model and simulations

- Ising model defined with a spin variable $S = \pm 1$ on each site of a lattice:

$$H = -J \sum_{\langle ij \rangle} S_i S_j, \qquad (1)$$

with $\langle ij \rangle$ the sum on nearest neighbours and $J = 1$. We will consider the square lattice, the triangular lattice, the kagome, the bowtie-a or the hexagonal lattice with $N = L \times L$ spins and either the free boundary conditions (FBC) or the periodic boundary conditions (PBC). For each of these lattice, a 2nd order phase transition at a finite T_c separates a paramagnetic phase from a ferromagnetic phase.

- The choice of boundary conditions can have some influence on the final state after a quench from a paramagnetic state to zero temperature.

Model and simulations

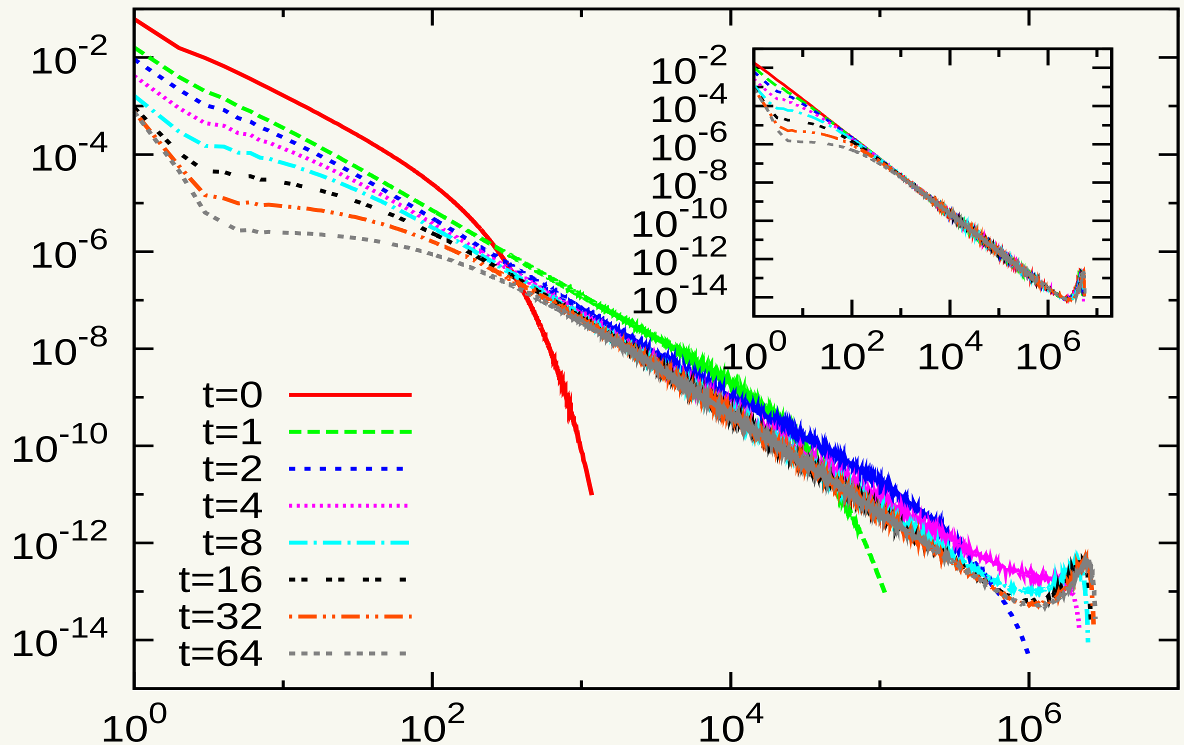
- We consider dynamics with non conserved order parameter : Glauber type. At the time $t = 0$, instantaneous change $1/T = 0$ to $T = 0$.
- At $T = 0$, the dynamics is particularly simple since the system tries to minimise its energy: To each spin S_i is associated a local field $h_i = \sum_{|i-j|=1} S_j$. We choose at random a position i. If $S_i h_i < 0$, the spin is reversed, otherwise if $S_i h_i = 0$, the spin is reversed with a probability $1/2$.
- After an equilibration time $t_{eq} \simeq L^2$, finite domains have disappeared and the configuration is either completely magnetised or in a striped state.

Approach to percolation

Approach to percolation

- Arenzon et al. have shown that after a subcritical quench starting from infinite temperature, the distribution of the spin clusters $\mathcal{N}(A,t)$, as a function of their area A, is similar to the one of percolation.
- This distribution is related to the one of the percolation after a very short time $t \simeq 10$, with a power law behaviour of the form $\mathcal{N}(A,t) \simeq A^{-\tau_A}$.
- This was established by looking at the behaviour for small A and the value of the overall constant which is known exactly in the case of the 2d percolation or for the 2d critical Ising model.

Approach to percolation



$\mathcal{N}(A,t)$ vs. A at different times t after a quench from infinite temperature to $T_c/2$ at $t=0$ and for $L=2560$. Inset : quench from T_c.

Approach to percolation

- For $t \geq 16$ we can clearly distinguish the two parts of the distribution (2). A first part in the range $1000 \leq A \leq 10^6$ with a power law behaviour. The second part is the small bump at around $A \simeq 2.10^6$.
- We measure $\tau_A = 2.020 - 2.040$ which is close to both the exponent for percolation, $\tau_A = 2 + 5/91 \simeq 2.05495$ and for the 2d critical Ising model, $\tau_A = 2 + 5/187 \simeq 2.02674$. Difficult to distinguish between these two cases ...
- A better way to distinguish between these two distributions is to look at the part of the distribution corresponding to the large (percolating) clusters.
- A more complete version of the distribution is

$$\mathcal{N}(A,t) \simeq A^{-\tau_A} + N_p(A/L^{2-\beta/\nu}, t) \ . \tag{2}$$

The second part corresponds to the percolating states.

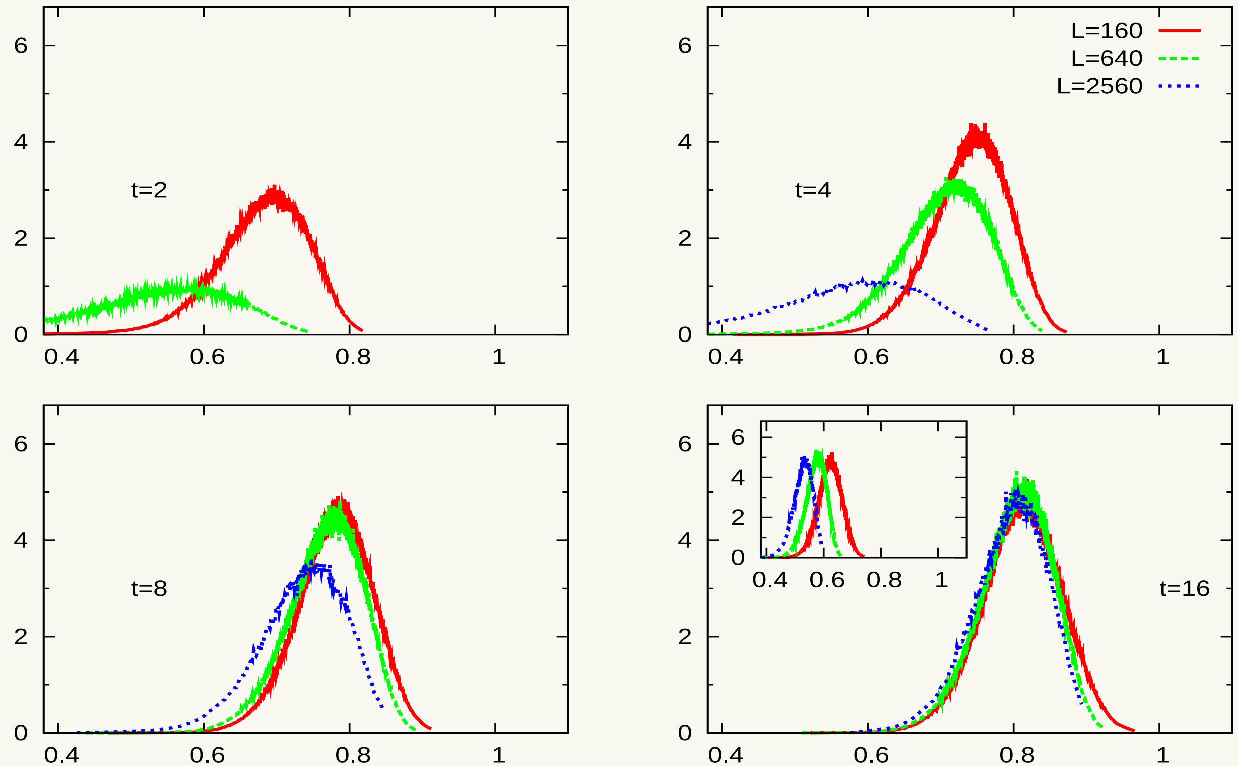
Approach to percolation

- $L^{2-\beta/\nu}$ corresponds to the average size of the percolating states, with β/ν the order parameter critical exponent.

$$\beta/\nu = d\left(\frac{\tau_A - 2}{\tau_A - 1}\right) \; . \tag{3}$$

- $A^{\tau_A}\mathcal{N}(A,t)$ vs. $A/L^{2-\beta/\nu}$ with the parameters of the percolation. For $t \simeq 2$, the distributions depend on the size, while for $t \simeq 16$, they all become similar and percolation like. Inset : similar plot but with parameters of critical Ising.

Approach to percolation

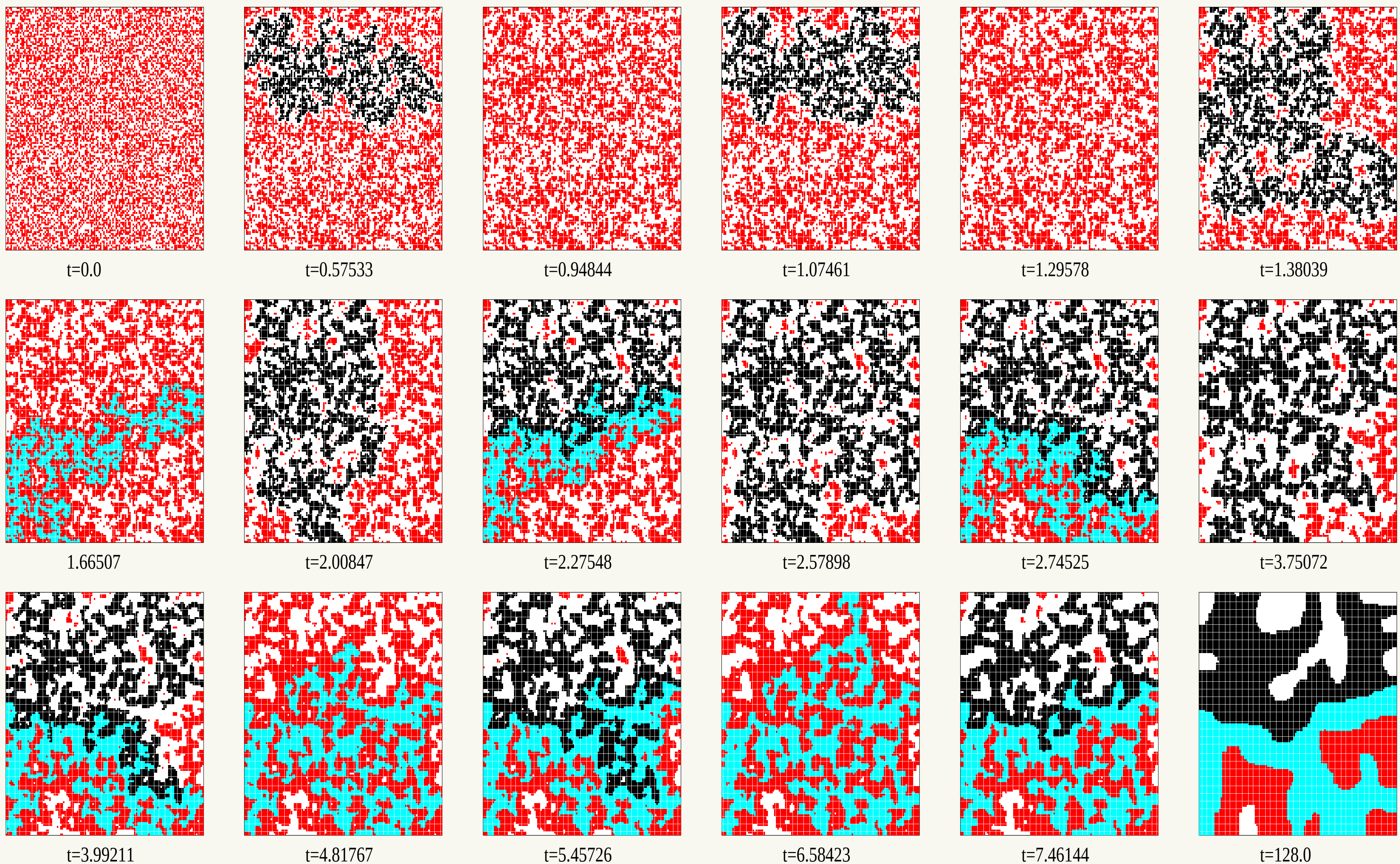


$A^{\tau_A}\mathcal{N}(A,t)$ vs. $A/L^{2-\beta/\nu}$ at different times t after a quench from infinite temperature to $T_c/2$ at $t=0$. τ_A and β/ν for percolation and for $2d$IM in the inset.

Approach to percolation

- $t = 2, L = 160, 640 \leftrightarrow t = 4, L = 640, 2560$
  $t = 4, L = 160, 640 \leftrightarrow t = 8, L = 640, 2560$.
- Saturation : $t = 4$ for $L = 160$, $t = 8$ for $L = 640$, $t = 16$ for $L = 2560$.
- This suggests a time dependance of the form $t/L^{1/2}$ up to some saturation at $t_p \simeq L^{1/2}$.
- Note that the existence of a percolating clusters is not enough to predict the faith of the configuration.
- In the next figure, we show snapshots at different times of a single configuration with 128×128 spins and FBC after a quench from infinite temperature to $T = 0$ at initial time t. Percolating clusters are shown in a different colour.

Approach to percolation



Approach to percolation

- We observe that percolating cluster already appear at a very earlier time, $t = 0.57533$ but next it can disappear, re-percolate again, etc. It is only after a much later time, $t = 7.46144$ that the configuration reach a final percolating state.
- We want a more accurate way of measuring the time $t_p(L)$ it takes to reach a percolating state : after a quench from $1/T = 0$ to $T = 0$ we let evolve the system up to $t = t_w$, then we make two identical copies of the configuration, $s_i(t_w) = \sigma_i(t_w)$. Next we let evolve each copy with a different history.
- We then compute the overlap between the two copies at the subsequent times:

$$q_{t_w}(t, L) = \frac{1}{N} \sum_i \langle s_i(t) \sigma_i(t) \rangle \ . \tag{4}$$

Approach to percolation

- It is only after having let the system evolve with the $T = 0$ dynamics for some time (the t_p of prevision section !!!) that a percolating state is reached.
- If we let the system evolve beyond t_p, and we make the two copies at $t_w \geq t_p$, the two clones should be strongly correlated for all subsequent times, with an asymptotic finite overlap.
- We observe that if $t_w(L)$ increases as $L^{1/2}$, the overlap remains finite and close to 1. This indicates that $t_p \simeq L^{1/2}$ is the time it takes for a totally disordered configuration to reach a percolating state.

Approach to percolation

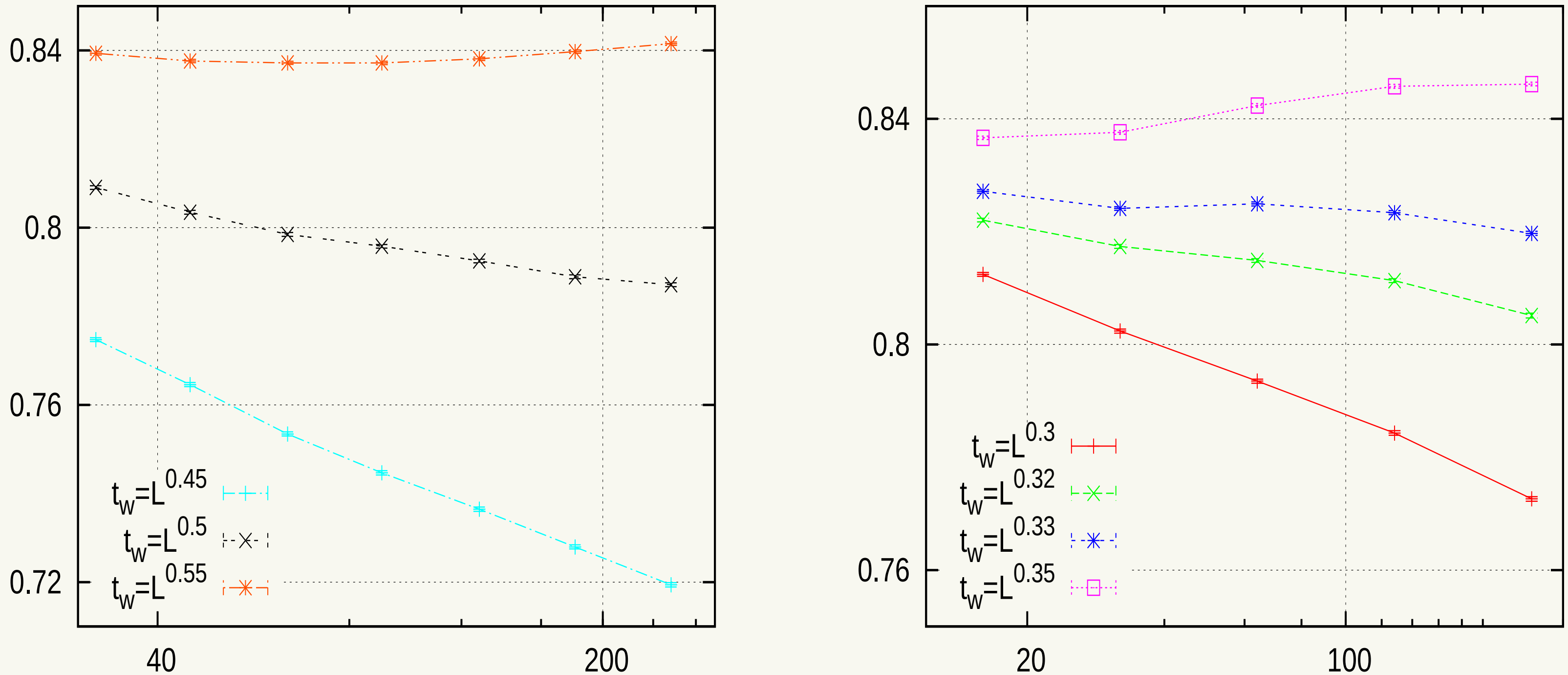
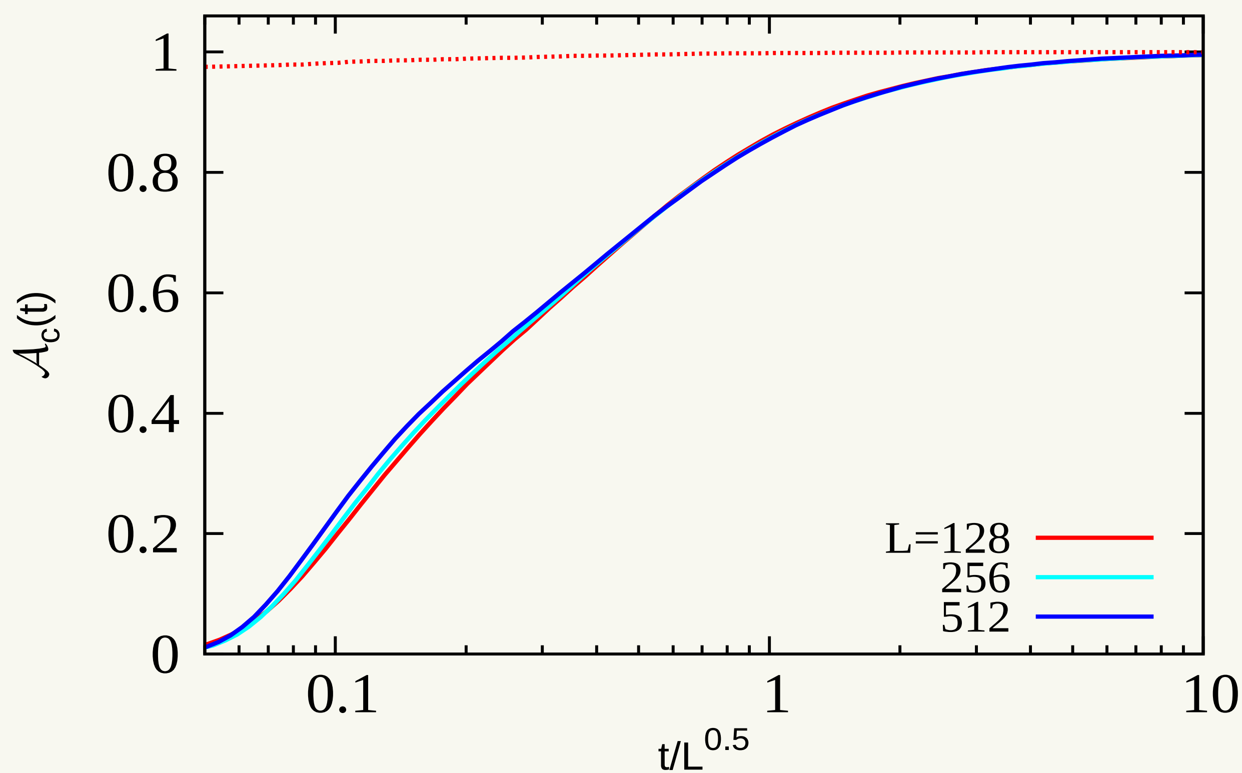


Figure 1: $q_{t_w}(t)$ between two copies vs. the size L for a quench at $t = 0$ and a common evolution up to $t_w(L)$. Left panel, FBC of the square lattice. Right panel, PBC for the triangular lattice.

Approach to percolation

Another way of investigating the approach to percolation is by computing the overlap of the number of crossings between a given time t_w and at the final time $\mathcal{A}_c(t) = \langle \delta_{n_c(t), n_c(t_{eq})} \rangle$ as a function of t/L^x



Consequence : correlation function and finite temperature

Consequence

- We want to show some of the consequences of the existence of the percolating time t_p.
  We consider a two points correlation function defined as

$$G(r,t) = \langle S_i(t) S_j(t) \rangle = f\left[\frac{r}{\xi(t)}\right] = g\left[\frac{r}{\xi(t)}, \frac{\mathcal{L}(L)}{\xi(t)}\right] \tag{5}$$

  with $r = |i-j|$, $\xi(t) = t^{1/z}$ the characteristic length and $\mathcal{L}(L) = \xi(t_p) \simeq L^{0.5/z}$. In the following figure, we show the correlation function as function of $\frac{r}{\xi(t)}$ and also in the case we impose the condition $\frac{\mathcal{L}(L)}{\xi(t)} = cst$.

Consequence

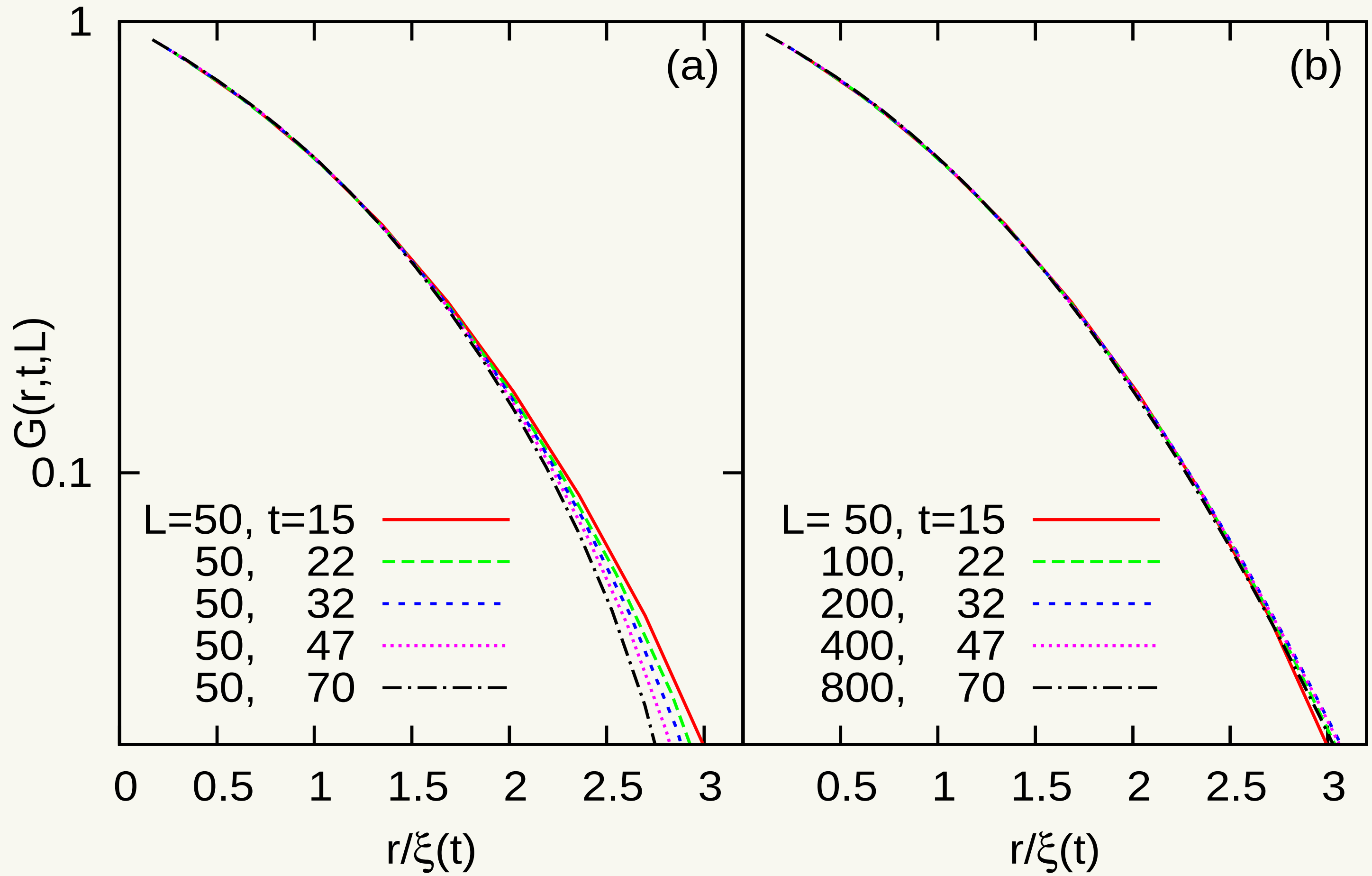


Figure 2: $G(r,t,L)$ vs. $r/\xi(t)$ for the $2d$IM after a quench from $T=0$.

Consequence

- With a final state at finite temperature we expect that the thermal fluctuations will destroy the crossing states and the system will end in a completely magnetised state.
- At $T = 0$, the magnetisation converges to a finite value $\simeq 0.733181 = 2\pi_{hv} + (\pi_h + \pi_v)1/4$, with $2\pi_{hv} = \frac{1}{2} + \frac{\sqrt{3}}{2\pi}\log\frac{27}{16}$ and $\pi_h + \pi_v = 1 - 2\pi_{hv}$
- For finite temperature, the behaviour is similar up to $t/L^2 \simeq 1$ for $T < T_c$.
- For $t/L^2 > 1$, the magnetisation will eventually go to 1 but after a time which increases with the size and the distance from T_c.

Consequence

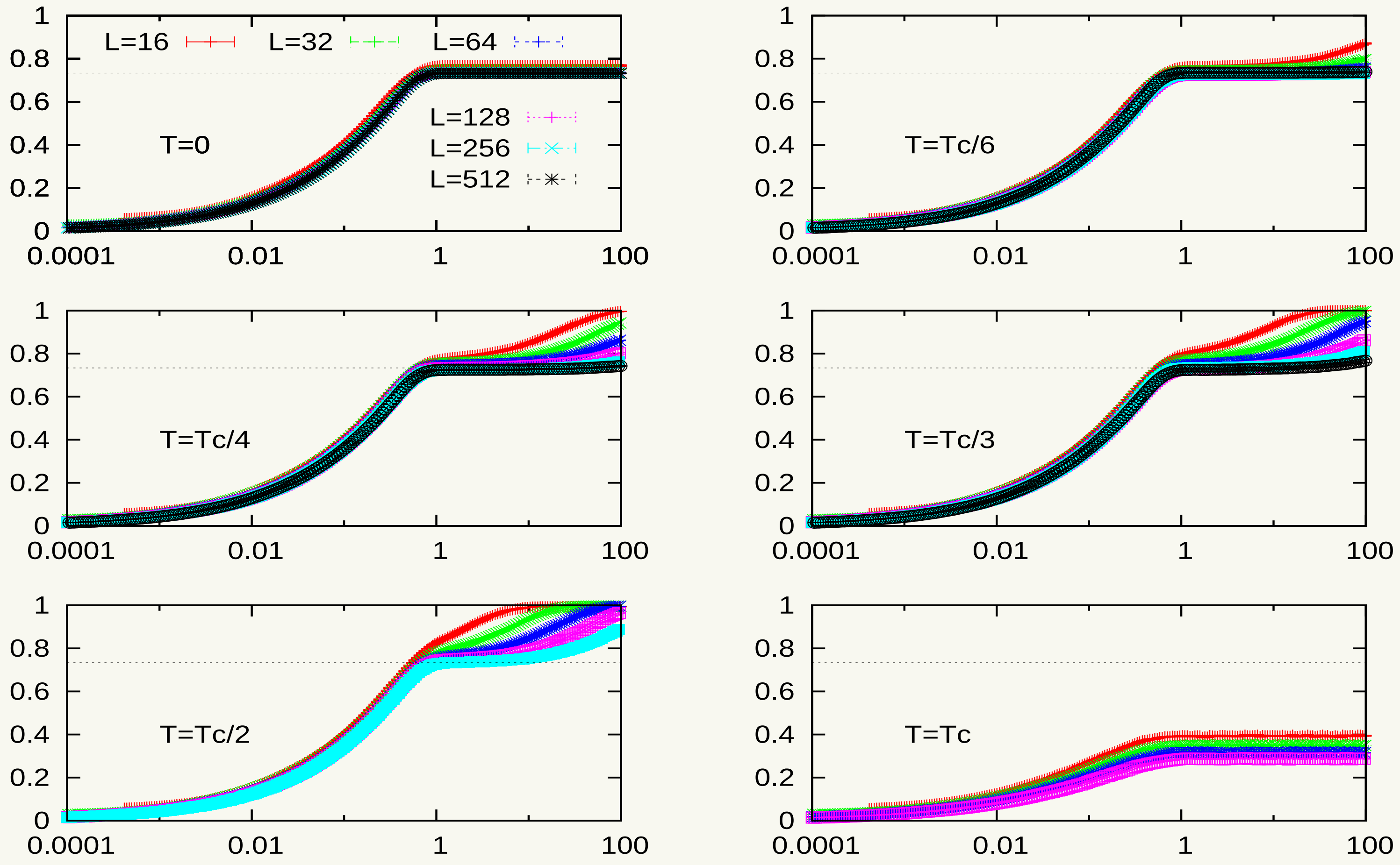


Figure 3: Mag vs. t/L^2 for different final temperatures T.

Consequence

- We can also look the restricted overlap of the number of crossings with a final state i defined as

$$\mathcal{A}_c^{(i)}(t) =< \delta_{n_c(t),i} > \quad . \tag{6}$$

Clear correspondence between $\mathcal{A}_c^{(1)}(t)$ and the evolution of the magnetisation. ((1) = crossing in both directions)

- In the following figures, we show $\mathcal{A}_c^{(1)}(t)$ as a function of t/L^2, $t/L^{0.5}$ and $t/L^{3.333}$.
- We observe that the earlier dynamics scales as a power of $t/L^{0.5}$ up to the value $\mathcal{A}_c^{(1)}(t/L^{0.5} = 1) \simeq 2\pi_{hv} = 0.64424$ corresponding to the final value at zero final temperature.
- The late dynamics is controlled by a scaling of $t/L^{3.333}$.

Consequence

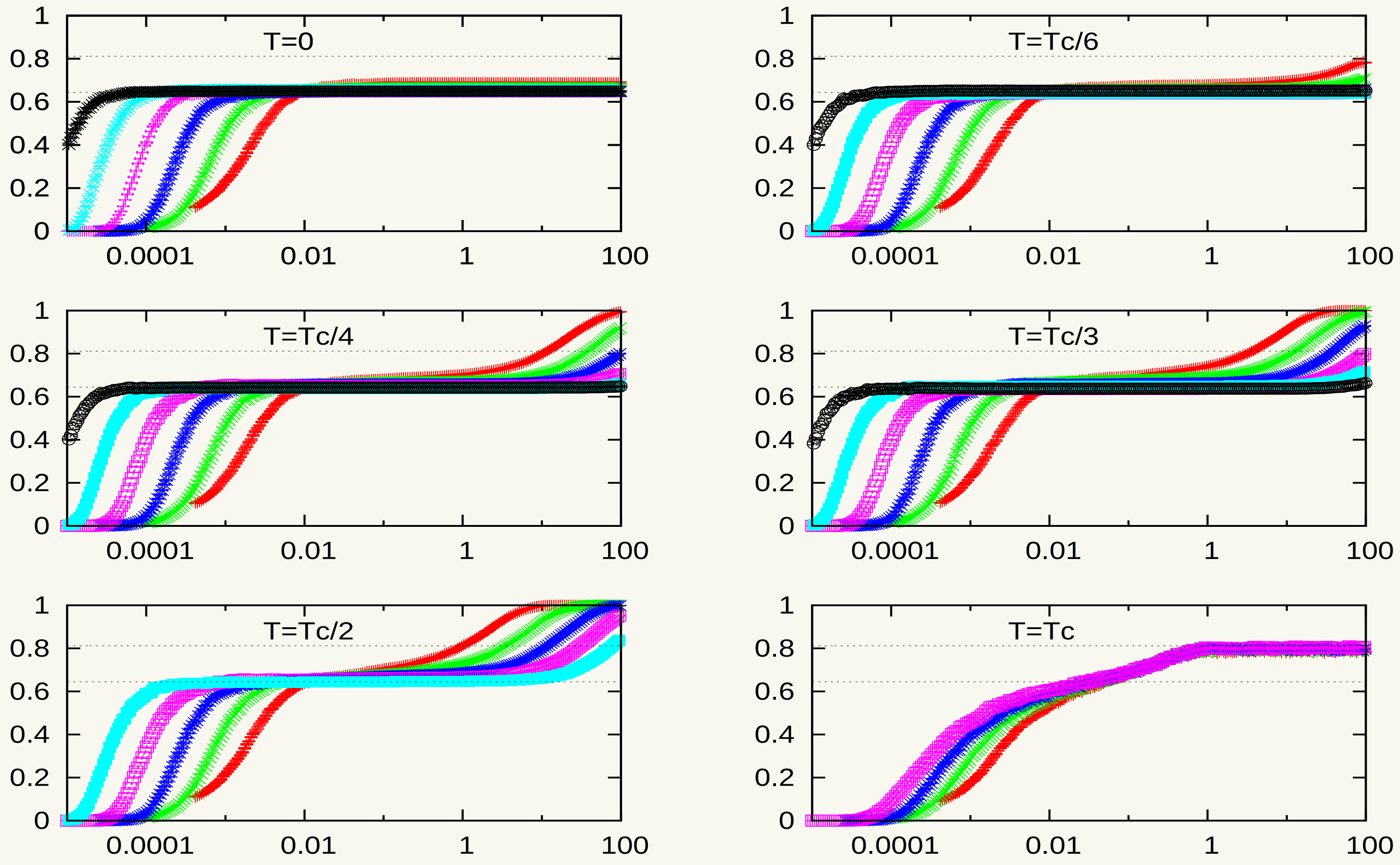


Figure 4: $\mathcal{A}_c^{(1)}(t)$ vs. t/L^2 for different final temperatures T.

Consequence

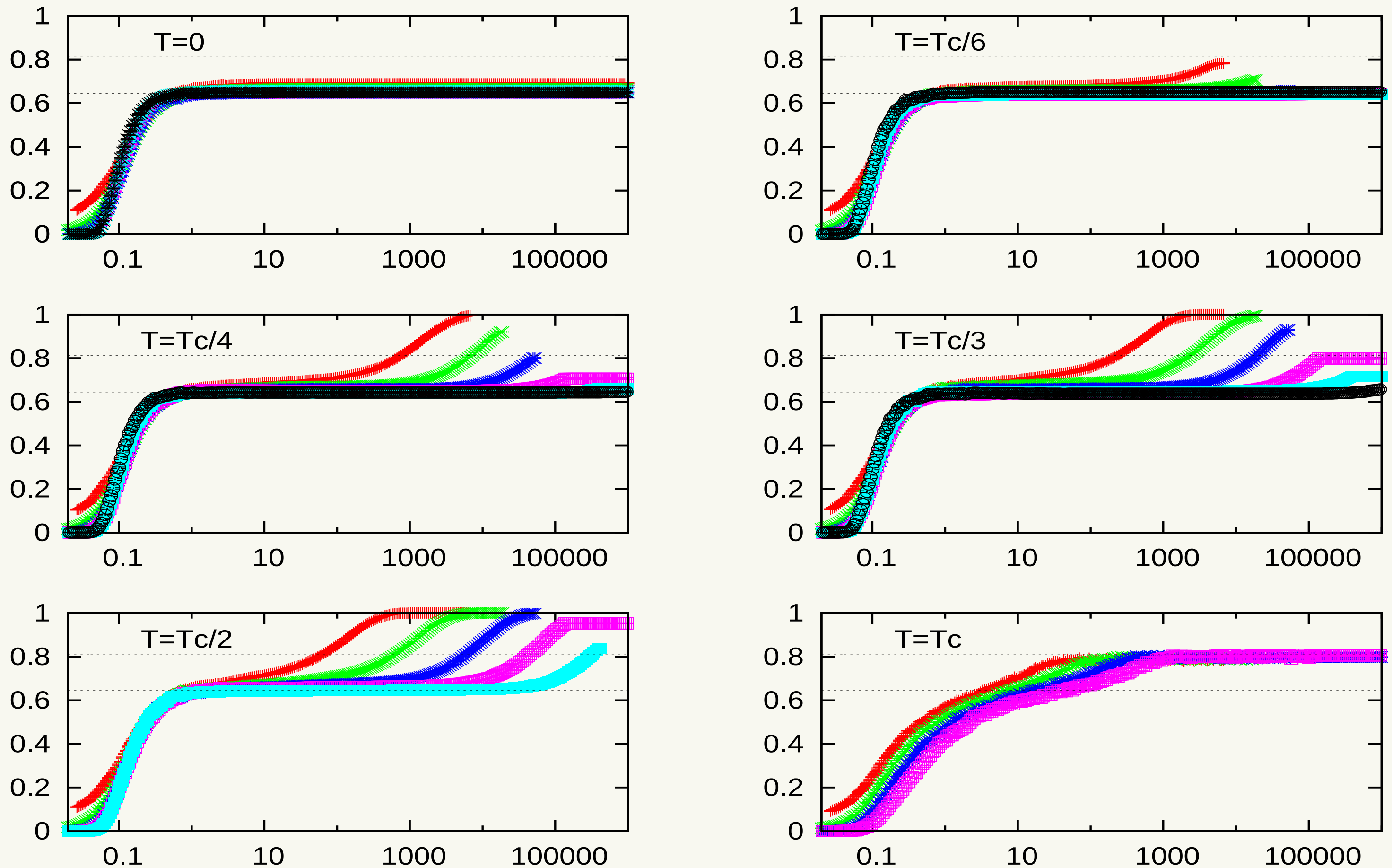


Figure 5: $\mathcal{A}_c^{(1)}(t)$ vs. $t/L^{0.5}$ for different final temperatures T.

Consequence

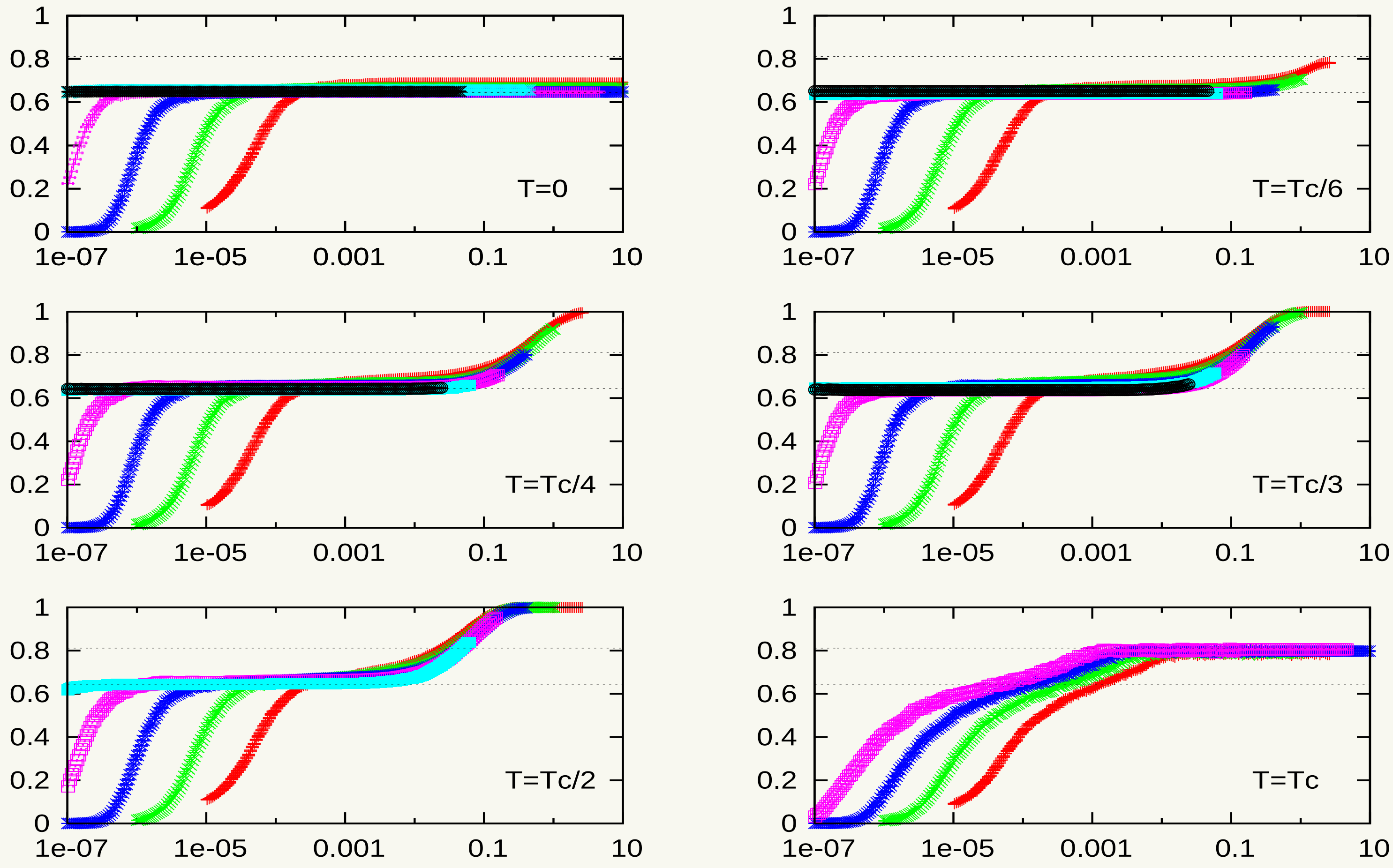


Figure 6: $\mathcal{A}_c^{(1)}(t)$ vs. $t/L^{3.333}$ for different final temperatures T.

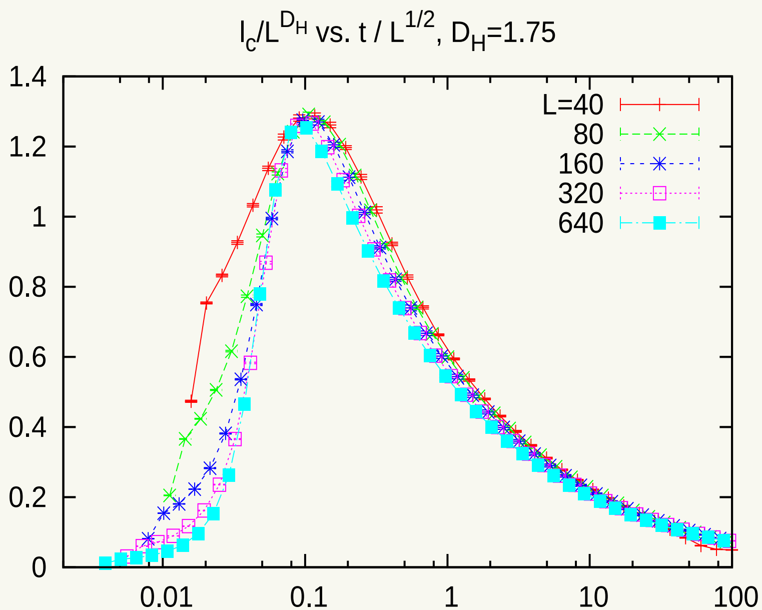
Extensions

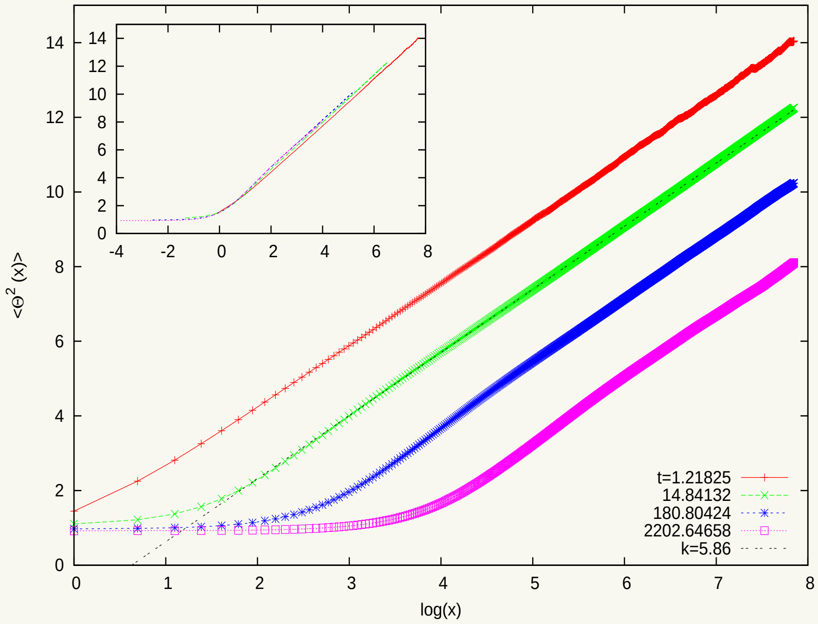
Extensions

- Other $2d$ lattices : triangular : $t_p \simeq L^{2/6}$; kagome : $t_p \simeq L^{2/4}$; bowtie-a : $t_p \simeq L^{2/5}$; hexagonal : $t_p \simeq \log(L) \to t_p = L^{z/n_c}?$
- the hexagonal (or honeycomb) lattice is particular since the state is blocked very quickly due to the existence of clusters with 6 spins which will never disappear at $T = 0$ (Takano and Miyashita, 1993). Still the percolation is present at $T = 0$.
- Other dynamics : Voter model $t_p \simeq 1.666$.
- Similar results also for the directed Ising model, Godrèche and Pleimling, 2015
- $d = 3$ dimension ? For the 3d Ising model, the percolation threshold is at $p_c \simeq 0.3$. So starting from the paramagnetic state, we already have two percolating states.

Extensions

Other quantities can also be considered like the fractal dimension $D_H = 1.75$ associated to the length interface l_c of the percolating cluster or the variance of the winding angle $< \theta^2(x) >$ which has to behave has $a + \frac{4k}{8+k} \log x$ with $k = 6$ for percolation.





Conclusion

Conclusion

- The dynamics after a quench from an high temperature ($T > T_c$) to a low temperature ($T < T_c$) is described by the coarsening of finite clusters and physical quantities are functions of t/L^2.
- The final state is controlled by the existence of percolating states. These states appear after a time $t_p \simeq L^{1/2}$ for the square lattice and $t_p \simeq L^{1/3}$ for the triangular lattice. $t_p \simeq L^{z/c}$ with c the lattice coordination number ?
- These percolation states will become stripe states which will be present with a finite probability at $T = 0$ in the large time limit.
- At finite temperature $< T_c$, the stripe states can also be observed and will disappear, due to thermal fluctuations after a time $\simeq L^{3.33}$.