Exact non-equilibrium dynamics in 1D integrable quantum systems

Ochanomizu University
Tetsuo Deguchi

Jun Sato [A], Eriko Kaminishi [A]

Pulak Ranjan Giri [B], Ryoko Hatakeyama [A]

[A]Univ. of Tokyo, [B] IIP Natal (Brazil)

``Thermalization in isolated quantum systems ''

New Frontiers in Non-equilibrium Physics 2015,

YITP, Kyoto Univ., Kyoto, Aug 3-7, 2015

``Ochanomizu'' means

Ocha (Tea or Green Tea) + no (for) + Mizu (Water)

Contents

- Part 0: A review:
  Equilibration of isolated quantum systems

- Part 1: Introduction to Algebraic Bethe ansatz

- Part 2: Exact Relaxation Dynamics of the 1D Bose Gas

- Part 3: Exact Relaxation Dynamics of the XXX spin chain

  Relaxation time of fidelity

  Power-law relaxation of local magnetization< $\sigma^z{}_m$>

Part 0

Thermalization (equilibration) of isolated quantum systems

- We say that a local operator of an isolated quantum many-body system thermalizes or equilibrates (Cf. M. Rigol et al., (2007, 2008)) when the expectation value at time *t* approaches a constant value as *t* goes to infinity:

$$\langle A(t) \rangle \rightarrow \langle A(\infty) \rangle \qquad (t >> 1)$$

(i) For non-integrable systems, it is conjectured that the asymptotic value follow the average of the Gibbs distribution

$$\rho_{eq} = \exp(-\beta E)/Z$$

(ii) For integrable systems, it is conjectured that the asymptotic value follow the average of the generalized Gibbs ensembles (GGE)

$$\rho_{eq} = \exp(-\sum_j \lambda_j I_j)/Z$$

Here I_j (j=1, 2, …, N) denote **quasi-local** conserved quantities.
N is the degrees of freedom, the number of sites or particles.

- We recall that the quantum state does no change at all in time.

A fundamental inequality of Typicality

A. Sugita (2007); P. Reimann, PRL (2007))

$$E\,[\,(\Delta <A>)^2\,] \leq \frac{|A|_{op}{}^2}{d+1}$$

(1) E[B]: ensemble average of *B* over states in a energy shell [E- ΔE, E]

(2) $|A|_{op}$: the largest eigenvalue of operator A (operator norm);

(3) <A> = $<\psi|\, A \,|\psi>$: expectation value of A ;

(4) d: the number of all energy levels in energy shell [E- ΔE, E];

(5) $\Delta <A> = <A> - <A>_{eq}$: deviations from the equilibrium value

Assumption in the derivation

The inequality is shown by assuming the isotropic Haar measure for the probability distribution on the unit sphere (A. Sugita (2007)):

For a given quantum state

$$| \psi > = \Sigma_j c_j |E_j \rangle , \quad \text{for } E_j \text{ in energy shell } [E- \Delta E, E],$$

we assume the probability of having $\{c_j\}$ as

$$P(\{c_j\}) = C \, \delta (1- \Sigma_j |c_j|^2) \ .$$

Part 1: A brief introduction to Algebraic Bethe ansatz

(1) 1D Bose gas;
(2) the XXZ chain

(1) The one-dimensional Bose gas:

Particles ineteract each other through repulsive delta-function potentials

- Lieb-Liniger Hamiltonian

$$H_{Lieb-Liniger} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^{\,2}} + \sum_{j,k=1}^{N} c\delta(x_j - x_k)$$

- *L*: system size, *N*: the number of bosons
- P. B. C. (Periodic Boundary Conditions): ϕ(x)= ϕ(x+L)
- Physical background: In 1dimension, the s-wave is enough to describe the scattering effect.

(2) Spin-1/2 XXZ spin chain)

The Hamiltonian of the spin-1/2 XXZ spin chain under P.B.C.

$$\mathcal{H}_{\mathrm{XXZ}} = \frac{1}{2}\sum_{j=1}^{L}\left(\sigma_j^X\sigma_{j+1}^X + \sigma_j^Y\sigma_{j+1}^Y + \Delta\sigma_j^Z\sigma_{j+1}^Z\right) .$$

Here σ_j^a $(a = X, Y, Z)$ are Pauli matrices on the jth site. We define q by

$$\Delta = (q + q^{-1})/2 \qquad (q = \exp\eta)$$

Quantum phase transitions at $\Delta = \pm 1$:

For $-1 < \Delta \leq 1$, $\mathcal{H}_{\mathrm{XXZ}}$ is gapless. $(\Delta = \cos\zeta$ by $\boldsymbol{q = e^{i\zeta}}$, $0 \leq \zeta < \pi.)$

Low excited spectrum is consistent with **CFT with** $c = 1$

For $\Delta > 1$ or $\Delta < -1$, it is gapful. $(\Delta = \pm\cosh\zeta$ by $q = e^{-\zeta}$, $0 < \zeta.)$

Scheme of the Algebraic Bethe ansatz

- R-matrix: (the Boltzmann weights of the 6-verstex model (E.H. Lieb, 1966))

the solution of the Yang-Baxter equations of the 6-vertex model:

Anti-Ferroelectric model on the square lattice

- Product of R-matrices -> Monodromy matrix

--> global operators: A(k), B(k), C(k), D(k)

whose commutation relations are given by the Yang-Baxter equations

- Transfer matrix for the XXZ spin chain (that of the 6-vertex model)
  t(k) = A(k) + D(k)

- B(k) : generators of Bethe states
  C(k) : conjugate of B(k)

Theorem

$B(k_1) \cdots B(k_N) \mid 0 >$ is an eigenstate of XXZ Hamiltonian ,

if k_j satisfy the Bethe-ansatz equations

Determinant formula of the scalar product in the Bethe ansatz (N. Slavnov, 1989)

``Scalar product": $<0|\ C(q_1) \cdots C(q_N)\ B(k_1) \cdots B(k_N)\ |0>$

is expressed in terms of the determinant if q_j or k_j satisfy the Bethe-ansatz equations.

$$<0|\ C(q_1) \cdots C(q_N)\ B(k_1) \cdots B(k_N)\ |0>$$
$$= \text{(some factors of } q_j \text{ and } k_j\text{)} \times \det H(\{q_j\}, \{k_j\}),$$

Matrix H is called Slavnov's matrix.

Practical merits of determinant expressions of form factors (and scalar products)

- <A | O | B>: the computation time becomes N^3 or less if it is expressed as a determinant.

If one calculate the inner product directly,

it will cost an exponential time: 2^N time

- Numerically stable if it is expressed as a Fredholm determinant in the large N limit.

Another technique for the XXZ chain: Quantum Inverse Scattering Problem (Kitanine, Maillet and Terras (1999)

- Any local operator is expressed in terms of the global operators of algebraic BA:

  A(k), B(k), C(k), D(k)

-> Matrix elements (form factors) of any local operator can be evaluated through Slavnov determinant

Part 2: Exact dynamics of 1D Bose gas

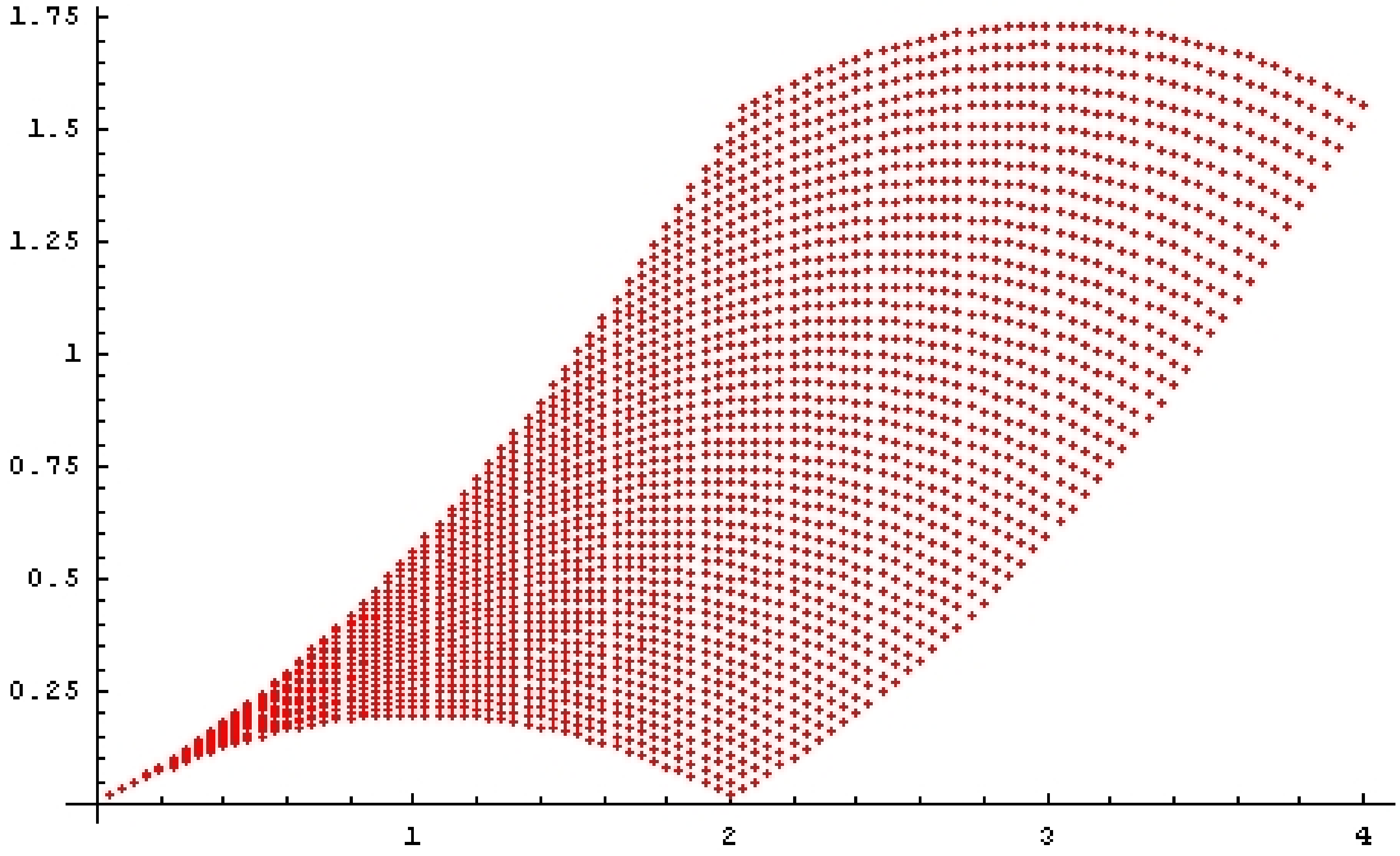
Relaxation for large systems: N=1000;

Animations produced by Jun Sato

J. Sato, R. Kanamoto, E. Kaminishi and T.D., PRL vol. 108, 110401 (2012)

Energy -Momentum Spectrum of 1D Bose Gas (c=100)

$$H_{Lieb-Liniger} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^{\,2}} + \sum_{j,k=1}^{N} c\,\delta(x_j - x_k)$$



For the 1D Bose gas (the LL model), the Bethe-ansatz eigenstates are complete.

- Bethe eigenstates of N particles $|k_1, k_2, \cdots k_N >$

$$\Psi(x_1, \cdots, x_N) = \Sigma_{p \in S(N)} A_P \exp(i \ \Sigma_{j=1}^{N} k_{Pj} \, x_j)$$

- Pseudo-momenta k_a's satisfy the Bethe-ansatz eqs:

$$\exp(L \, i \, k_a) = \Pi_{b \neq a} (k_a - k_b + ic) / (k_a - k_b - ic)$$

$$(a = 1, \cdots, N)$$

Energy eigenvalue $E = \Sigma_j k_j^2$

The Bethe eigenstates are complete.
(Cf. T.C. Dorlas, CMP(1993))

Time evolution of the density profile of 1D Bose gas

- The density operator $\rho(x) = \psi(x)^{\dagger}\psi(x)$
- The density profile at time t is calculated with the form factor expansion:

$$
\begin{aligned}
& < X(t) \,|\, \rho(x) \,|\, X(t) > \\
& = \frac{1}{N^2} \sum_{p,p'=0}^{N-1} \exp(i(P - P')x - i(E_p - E_{p'})t) \\
& \times < p \,|\, \rho(0) \,|\, p' >
\end{aligned}
$$

where $P = 2\pi p / L, P' = 2\pi p' / L$

$<p \mid \rho(0) \mid p'>$ can be calculated by a determinant.

Construction of the quantum state of a ``dark soliton'' (J. Sato et al., arXiv:1204.3960)

- We take superposition of excited states with on hole: $(q=0, 1,\cdots,N-1)$

$|X, N\rangle = \Sigma_{p=0}^{N-1} \exp(2\pi i pq) |p\rangle$

Here $|p\rangle$ denotes the Bethe eigenstate with one hole corresponding to momentum $2\pi p/L$,

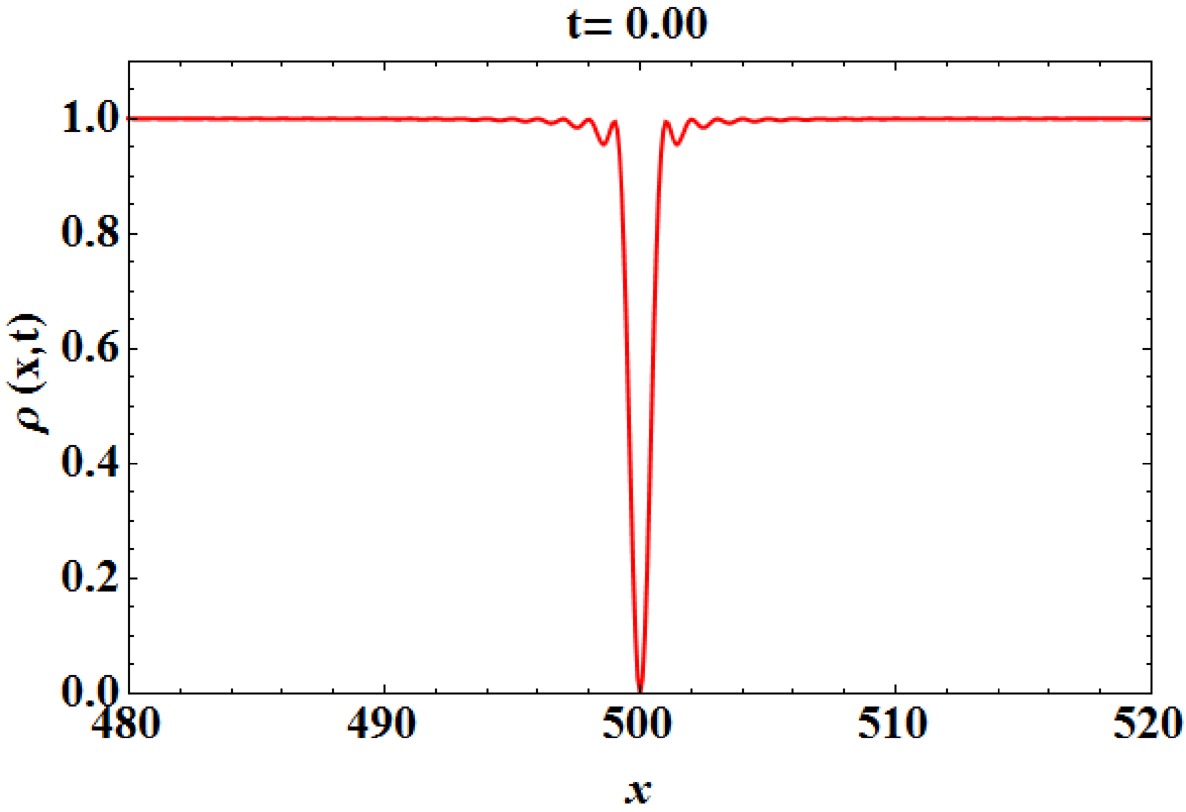
and $q=0, 1,\cdots,N-1$.

``Delta function'' becomes a ``dark soliton''

Observation of relaxation processes

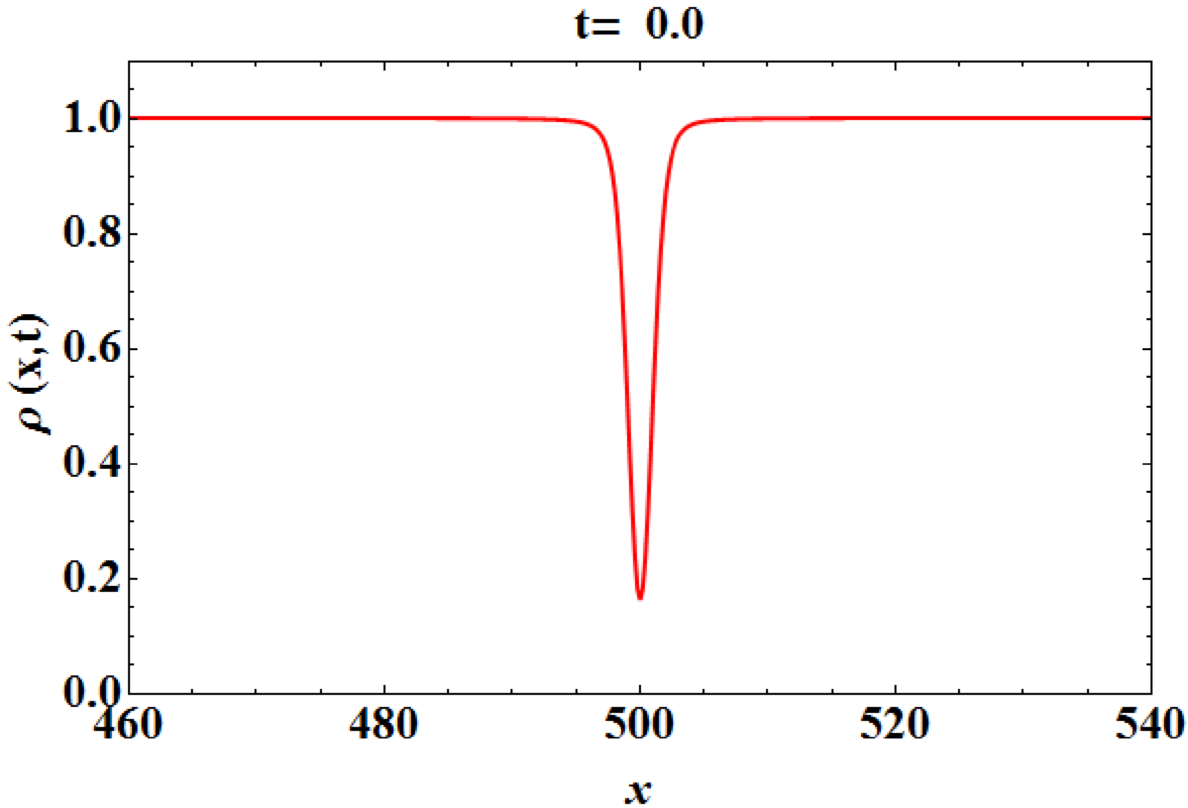
Relaxation for $N = 1000$; $\rho\,(\mathrm{x,t})$ density profile

J. Sato, R. Kanamoto, E. Kaminishi and T.D., PRL (2012)



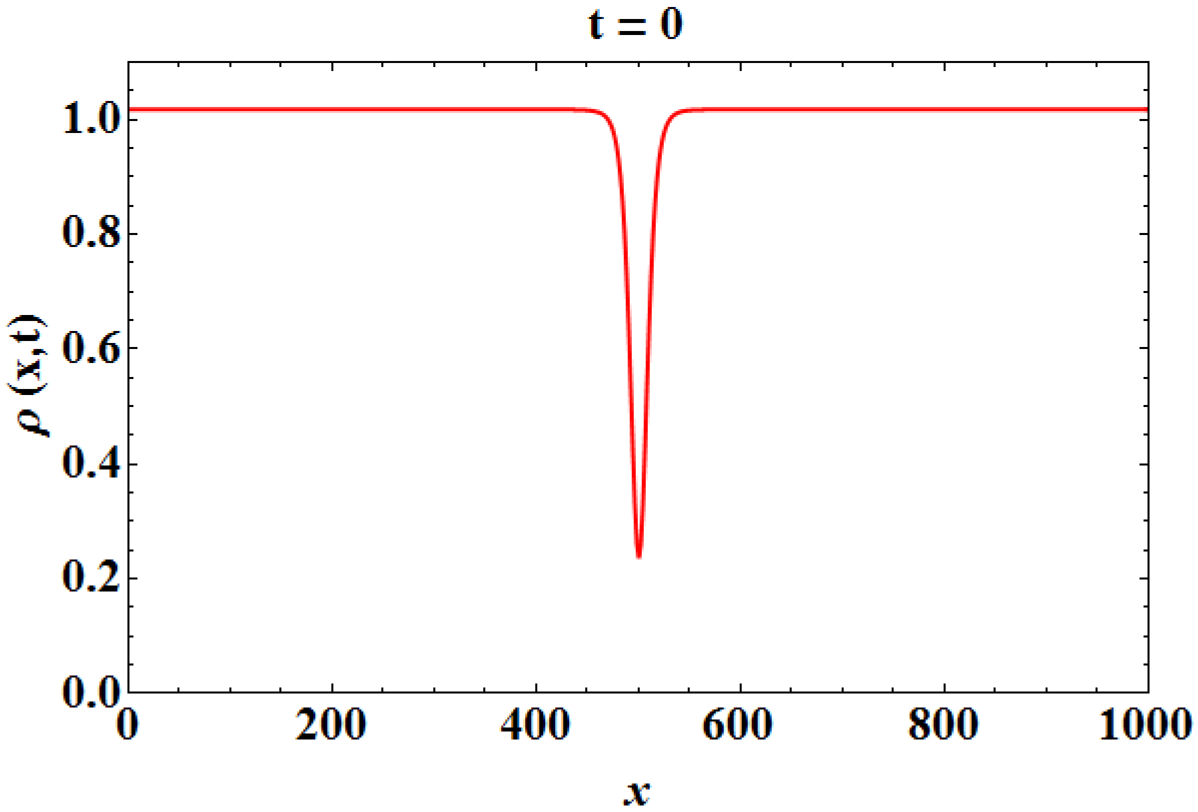
$c = 100$

$N = 1000$



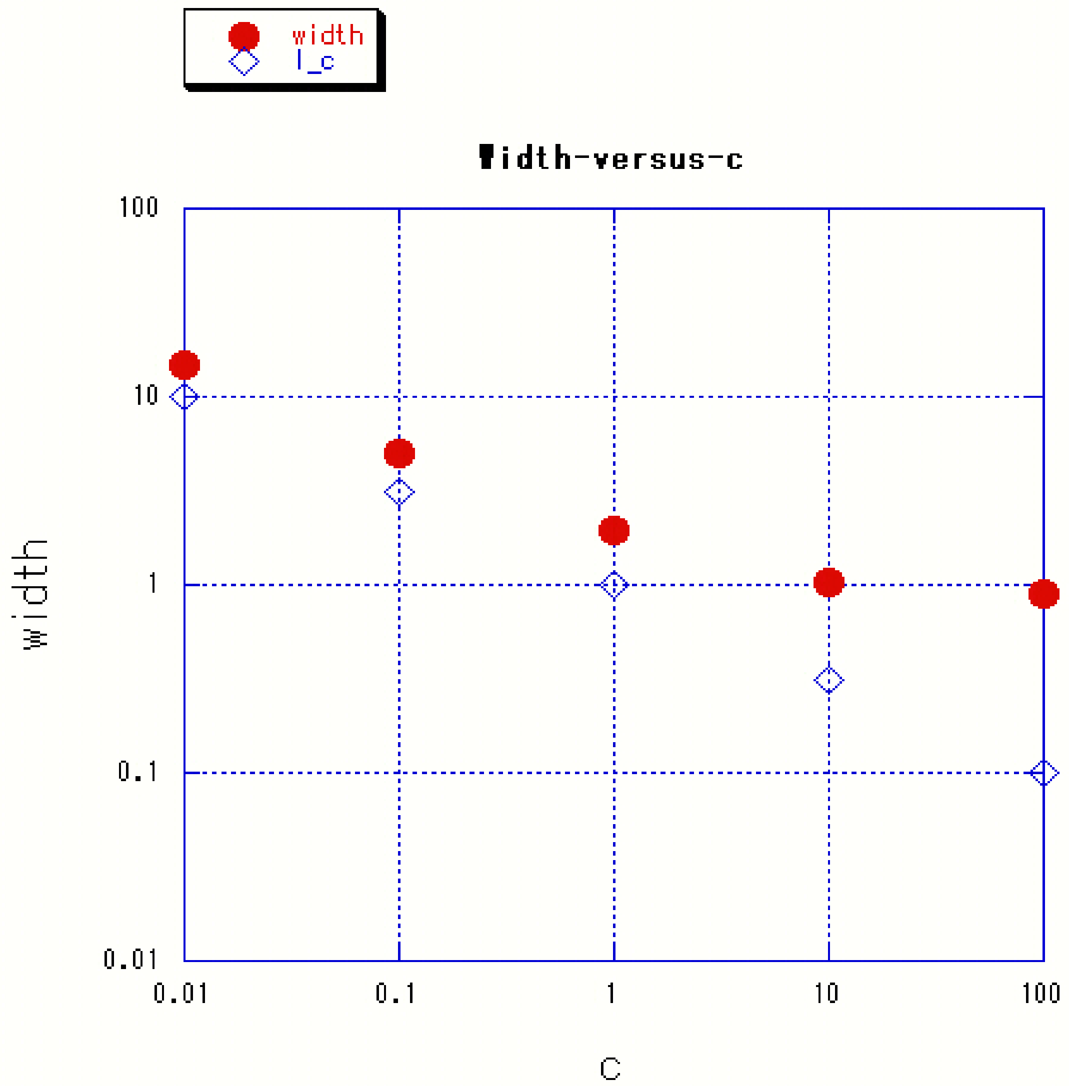
$c = 1$

$N = 1000$



$c = 0.01$

The width of dipped region at the initial profile is proportional to healing length $l_c=1/(\mathrm{cn})^{1/2}$, n=N/L.



Important observations

- Dipped dark soliton-like profile relaxes to a flat profile

- The life-time of the ``quantum dark soliton'' becomes longer as the coupling constant becomes smaller.

- However, the correlation among bosons becomes stronger as system size increases if particle density N/L and coupling constant c are kept constant.

Ref. J. Sato, E. Kaminishi, and T. Deguchi, arXiv:1303.2775
Finite-size scaling behavior of Bose-Einstein condensation in the 1D Bose Gas

Part 3: Relaxation dynamics in XXX chain

How does a local quantity equilibrate in time ?

(1) Time evolution of fidelity
(2) Time evolution of local magnetization < $\sigma^z{}_m$ >

T. Deguchi, P.R. Giri and R. Hatakeyama
arXiv: 1507.07470

A motivation: Statistical behavior in fully interacting quantum systems

- R.V. Jensen and R. Shanker, PRL **54**, 1879 (1879)

  Statistical Behavior in Deterministic Quantum Systems with Few Degrees of Freedom

- K. Satio, S. Takesue and S. Miyashita,

  J. Phys. Soc. Jpn **65**, 1243 (1996)

  System-Size Dependence of Statistical Behavior in Quantum System

Spin-1/2 XXX chain (Δ=1)

The Hamiltonian of the spin-1/2 XXZ spin chain under P.B.C.

$$\mathcal{H}_{\rm XXZ} = \frac{1}{2}\sum_{j=1}^{L}\left(\sigma_j^X\sigma_{j+1}^X + \sigma_j^Y\sigma_{j+1}^Y + \Delta\sigma_j^Z\sigma_{j+1}^Z\right) .$$

Here σ_j^a $(a = X, Y, Z)$ are Pauli matrices on the jth site. We define q by

$$\Delta = (q + q^{-1})/2 \qquad (q = \exp\eta)$$

Quantum phase transitions at $\Delta = \pm 1$:

For $-1 < \Delta \leq 1$, $\mathcal{H}_{\rm XXZ}$ is gapless. ($\Delta = \cos\zeta$ by $\boldsymbol{q = e^{i\zeta}}$, $0 \leq \zeta < \pi$.)

Low excited spectrum is consistent with **CFT with** $c = 1$

For $\Delta > 1$ or $\Delta < -1$, it is gapful. ($\Delta = \pm\cosh\zeta$ by $q = e^{-\zeta}$, $0 < \zeta$.)

The Bethe-ansatz equations for the XXX spin chain. In the M down-spin sector for M rapidities $\lambda_1, \lambda_2, \ldots, \lambda_M$ they are given as follows.

$$\left(\frac{\lambda_\alpha + i/2}{\lambda_\alpha - i/2} \right)^N = \prod_{\beta \neq \alpha} \frac{\lambda_\alpha - \lambda_\beta + i}{\lambda_\alpha - \lambda_\beta - i}, \quad \text{for } \alpha = 1, 2, \ldots, M \, . \tag{1}$$

By taking the logarithm of the both hand sides of (1) we have

$$\begin{aligned} 2\tan^{-1}(2\lambda_\alpha) &= \frac{2\pi}{N} J_\alpha + \frac{1}{N} \sum_{\beta=1}^{M} 2\tan^{-1}(\lambda_\alpha - \lambda_\beta), \\ & \qquad \text{for } \alpha = 1, 2, \ldots, M. \end{aligned} \tag{2}$$

Here we call J_α the **Bethe quantum numbers** (Hagemans and Caux (2007)). They are given by integers or half-integers by the condition

$$J_\alpha = \frac{1}{2}(N - M + 1) \mod 1 \, . \tag{3}$$

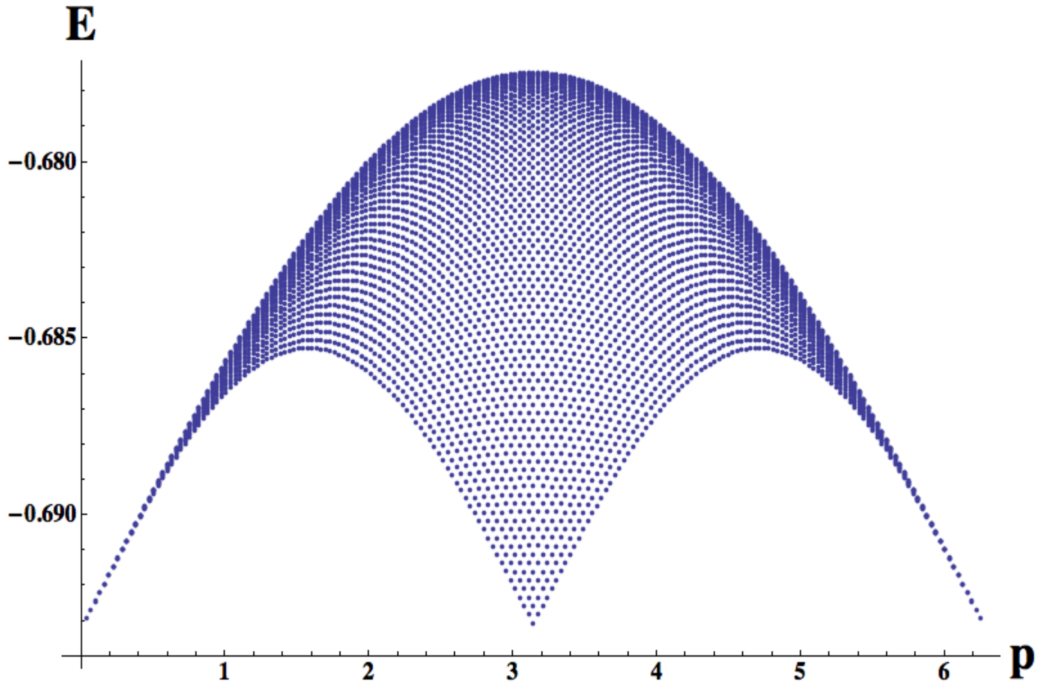
(1) **Time evolution of fidelity** for real and complex solutions of BAE for the spin-1/2 XXX chain (M=N/2-1)

M: the number of down spins

- Spinons; kinks, lowest excitations of spin-½ XXX chain (N^2 states)
- We consider quantum states with the sum of

(i) all spinons with equal weight: all-spinon state

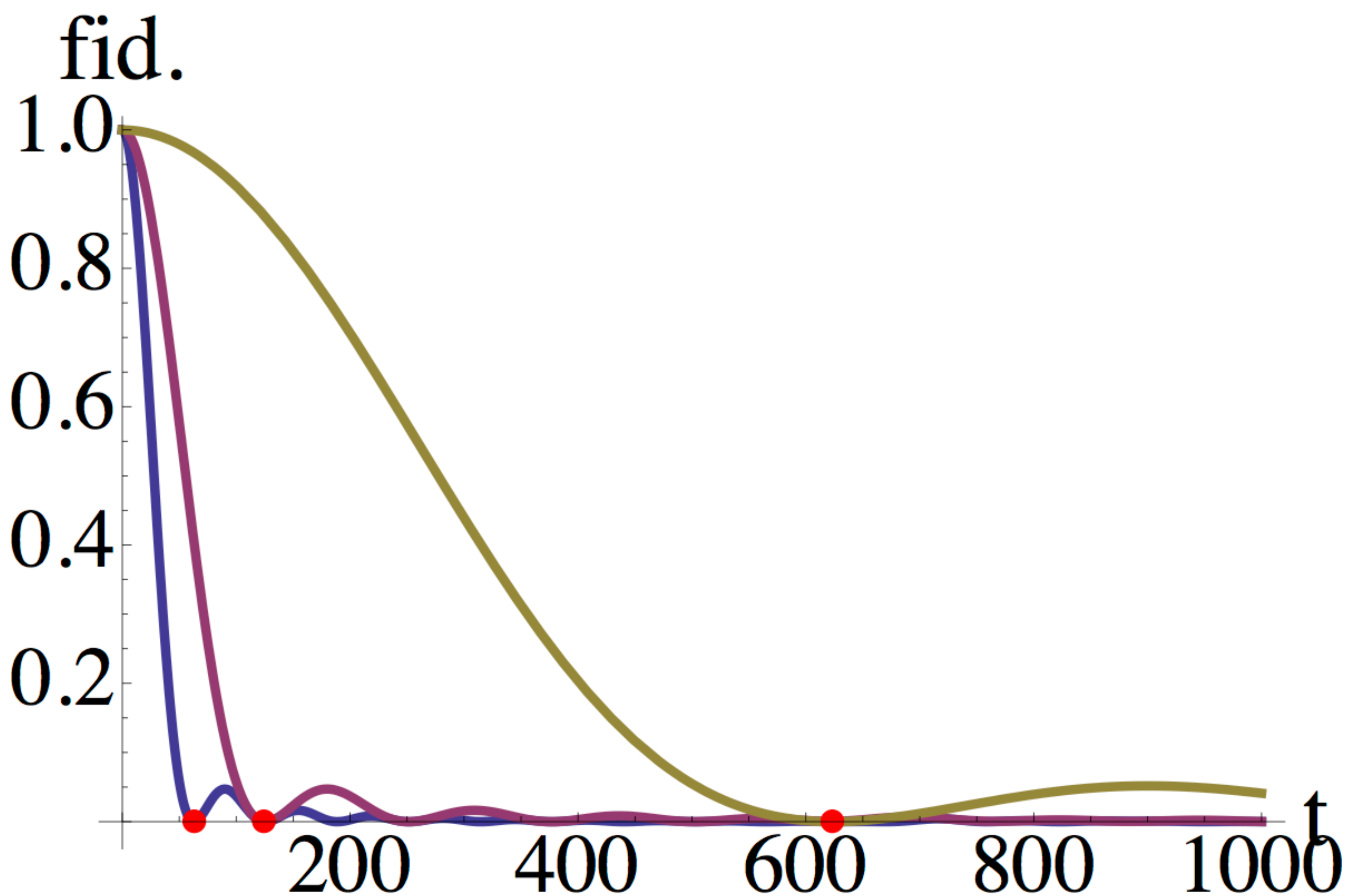
(ii) n-string solutions (bound states) (n>1)



Time evolution of the fidelity for spinons of the spin-1/2 XXX chain: N=1000 and M=N/2-1=499. ΔE＝0.01, 0.05, 0.1

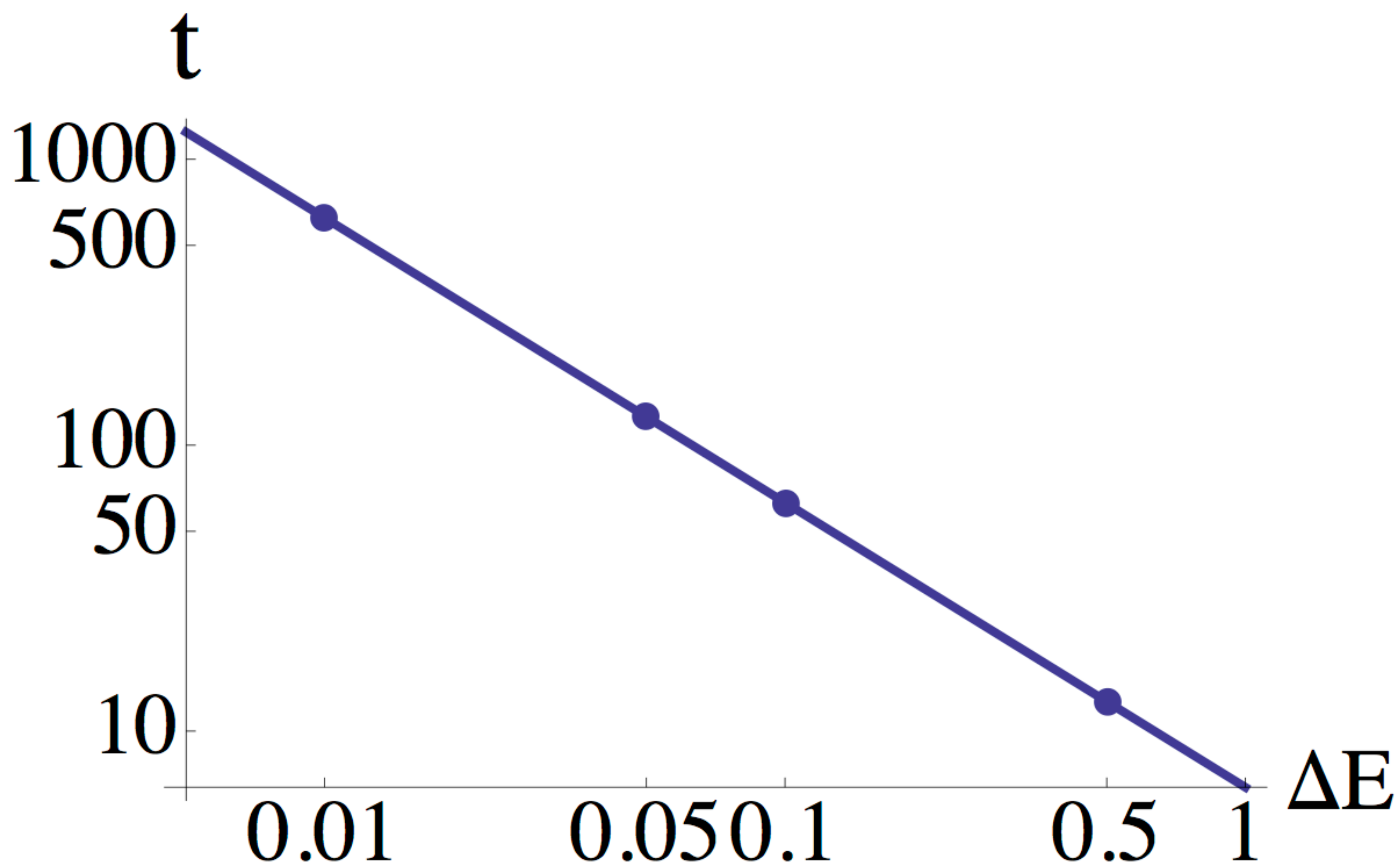
It is fitted by Monnai' s approximate formula of fidelity (T. Monnai, J. Phys. Soc. Jpn. **83**, 064001(2014) Lorentzian + oscillation

Cf. E.J. Torres-Herrera and L.F. Santos, Phys. Rev. A **90**, 033623 (2014)

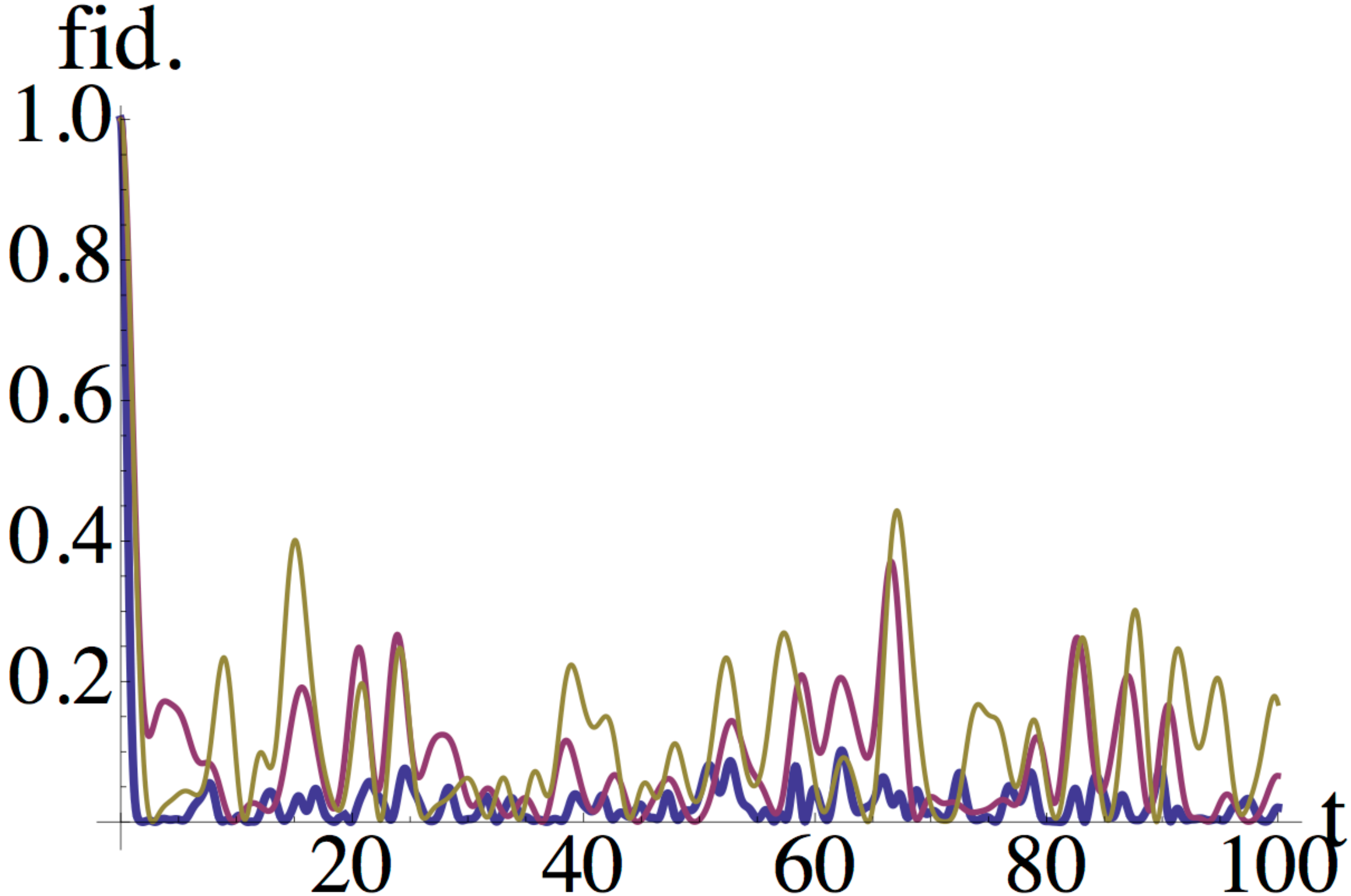


Relaxation time of fidelity versus energy width $\triangle E$ for real BAE solutions (by Ryoko Hatakeyama) :

It is given by the Boltzmann time , consistent with rigorous study for relaxation of generic systems: S. Goldstein, T. Hara, and H. Tasaki, New J. Phys. **17** (2015) 045002 : $T_R \cong h/\triangle E$



Fidelity of N=10 XXX chain: (M=N/2-1) all solutions (purple); only real solutions (yellow); all string solutions in the same range as real ones (red) (Ryoko Hatakeyama: BAE solutions confirmed by P.R. Giri; Cf. R. Hagemans and J.-S. Caux (2007))



We assume the following form of a 2-string solution to the BAEs in the two down-spin sector ($M=2$) :

$$\begin{aligned}\lambda_1 &= x+\frac{i}{2}(1+2\delta)\,,\\ \lambda_2 &= x-\frac{i}{2}(1+2\delta)\,.\end{aligned} \tag{4}$$

We assume that both string center x and string deviation δ are real

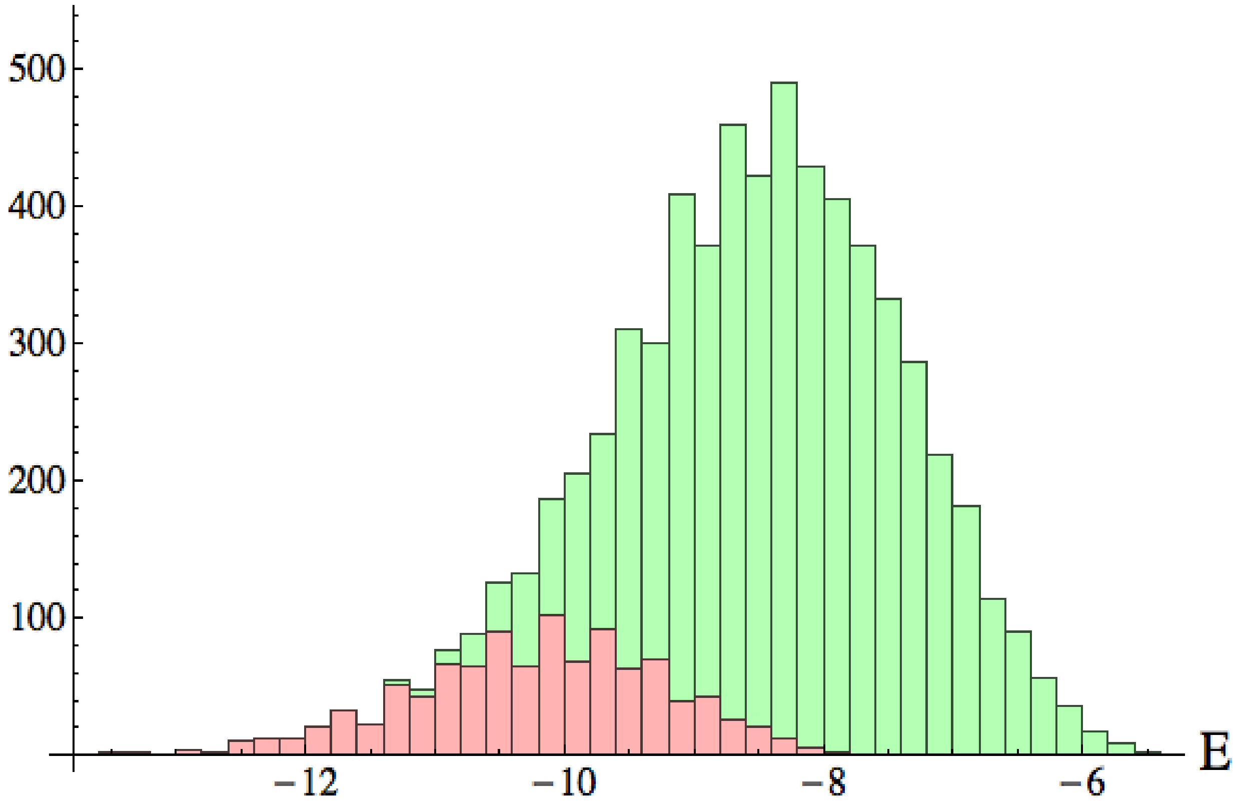
The BAEs in the logarithmic form for a 2-string with $M=2$ are given by

$$2\tan^{-1}\left(2x+i(1+2\delta)\right) = \frac{2\pi}{N}J_1+\frac{1}{N}2\tan^{-1}\left(i(1+2\delta)\right)\,, \tag{5}$$

$$2\tan^{-1}\left(2x-i(1+2\delta)\right) = \frac{2\pi}{N}J_2+\frac{1}{N}2\tan^{-1}\left(-i(1+2\delta)\right)\,. \tag{6}$$

Histogram of real and complex solutions for N=20

XXX chain: real & real + 2-string (pink); real & real + 2-string & real + 3-string & real + 2 x 2-string (green) (M=N/2-1=9) (R. Hatakeyama)



(2) Time evolution of local magnetization

$$< \sigma^z_m >$$

- We evaluate the time evolution of the expectation value of σ^z_m by the form-factor expansion
- For a given quantum state $|\varphi>$ we have

$$<\phi|\, \sigma^z_m\, (t)|\phi > = \Sigma_{n,\, n'} <\varphi|n'> <n'|\, \sigma^z_m\, |n> <n|\varphi> \\ \times \exp(- i\, (E_{n'}-E_n)t)$$

We make use of the completeness: $I = \Sigma_n\, |n> <n|$

- Recently, quasi-soliton scattering of XXZ chain is studied by R. Vlijm, M. Ganahl, D. Fioretto, M. Brockmann, M. Haque, H.G. Everz, and J.-S. Caux, arXiv:1507.08624
- Cf. Form factor expansion is also used for the 1D Bose gas: J. Sato et al, PRL **108,**110401 (2012)

Form factors and the spectral functions

[1] N. Kitanine, J.M. Maillet and V. Terras, Nucl. Phys. B **554** [FS] (1999) 647–678

Form factors: $\langle\mu|\sigma_m^z|\lambda\rangle, \quad \langle\mu|\sigma_m^{\pm}|\lambda\rangle$

Quantum Inverse Scattering $+$ Scalar product formula

[2] M. Karbach et al., (2002);
J. Sato, M. Shiroishi and M. Takahashi, (2004);
J-.S. Caux, R. Hagemans and J.-M. Maillet (2005)

A technique in Time evolution of local magnetization $< \sigma^z_m >$

We factor out the Cauchy determinant from the form factor of σ^z_m (in the XXZ spin chain)

- $< \mu | \; \sigma^z_m | \lambda >$
  $= \text{“Cauchy} \;\; \det \; (\mu - \lambda) \text{''} * \det(I + U)$

Here | μ > and | λ > are Bethe eigenstates of the XXZ spin chain.

The above formula holds for real and complex solutions in the XXX limit, or if zeta is enough smaller than π for the XXZ chain.

The det (I + U) leads to a **Fredholm determinant** in the large *N* limit .

Initial states: all-spinon state and partial sums over spinon states

- All real spinon-state with equal weight

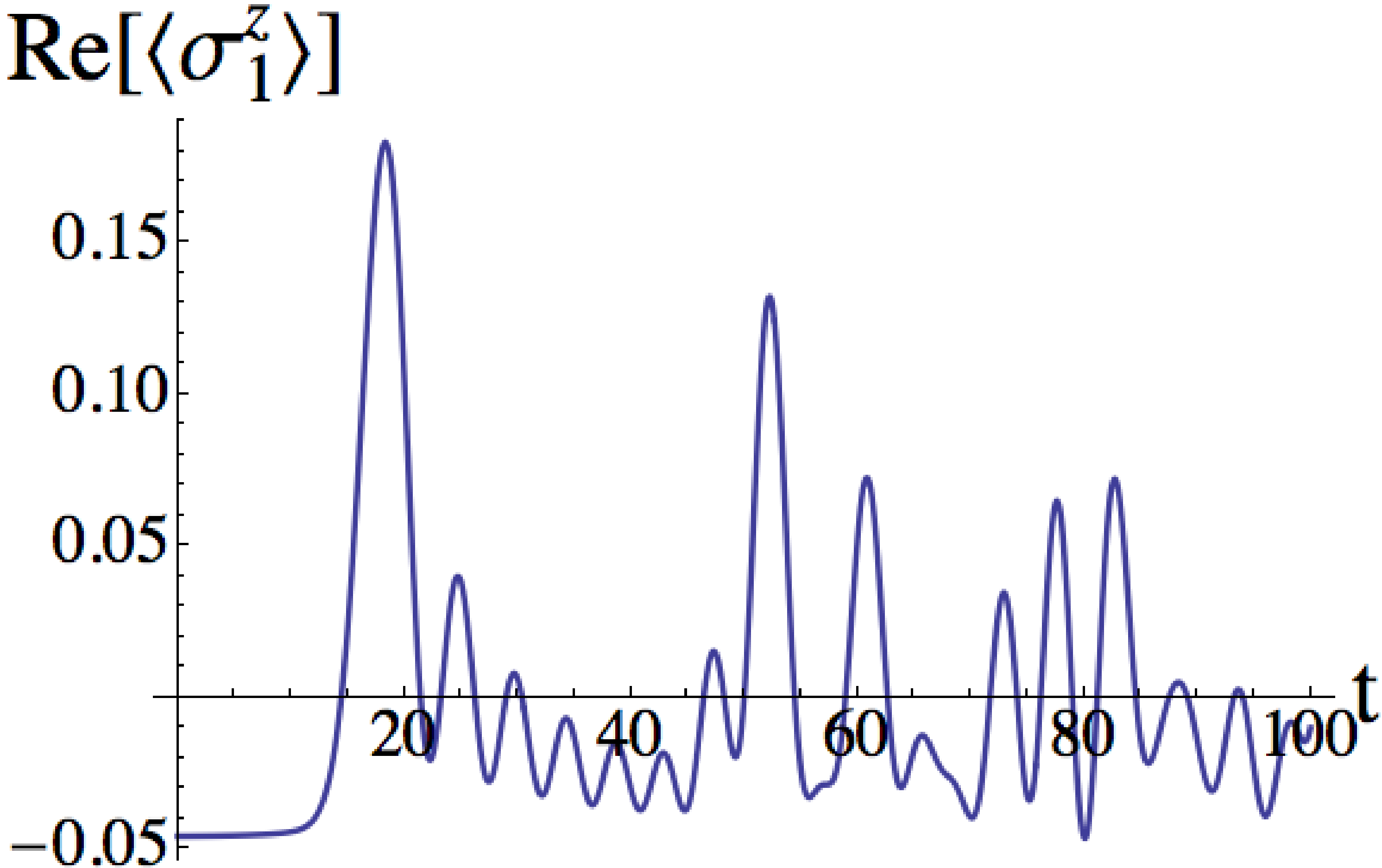
  (or random weight, in a Energy shell)

Local density is localized at one site at initial time.

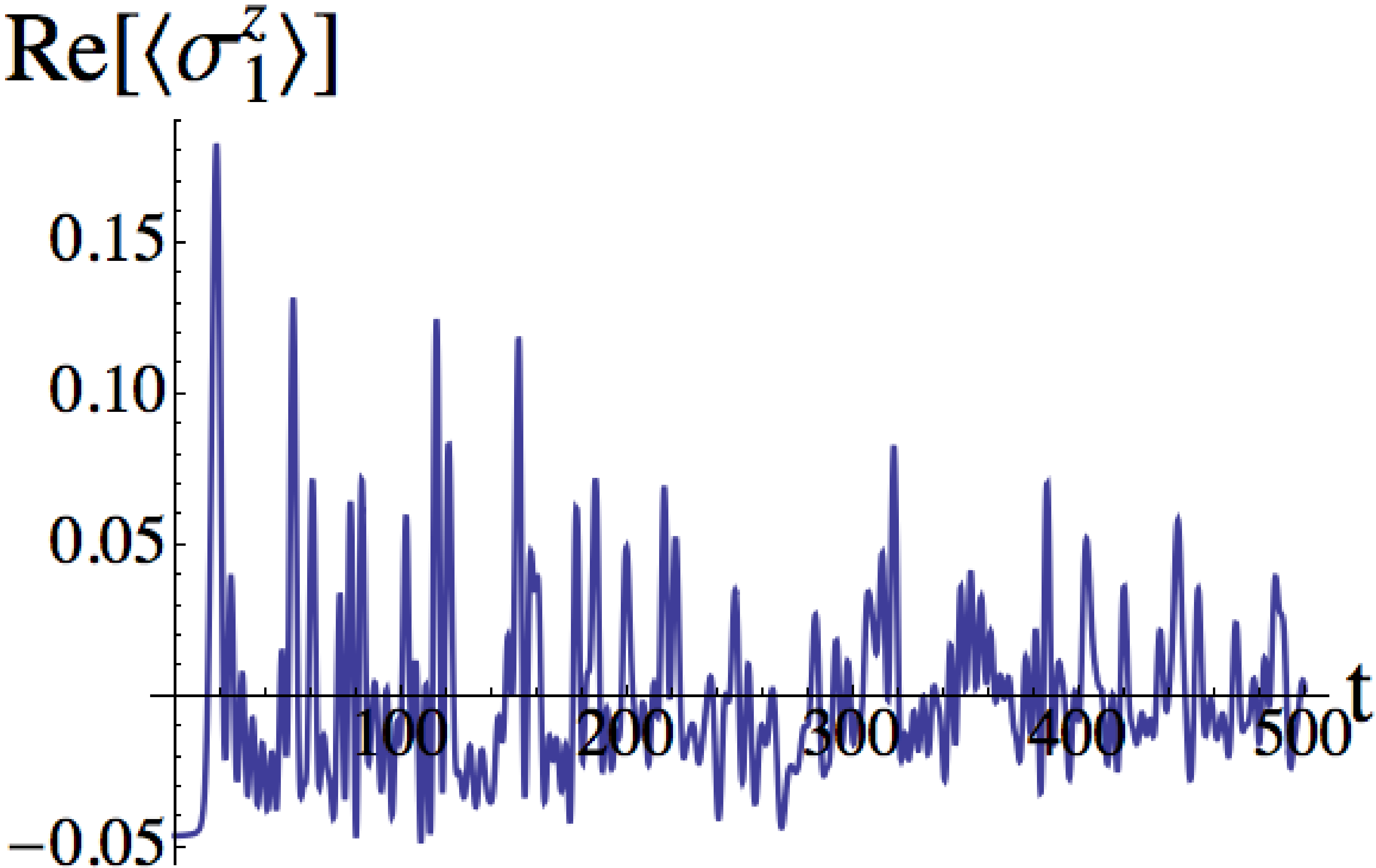
- Cf. For 1D Bose gas, `quantum soliton state' was constructed in

  J. Sato et al, PRL **108,**110401 (2012)

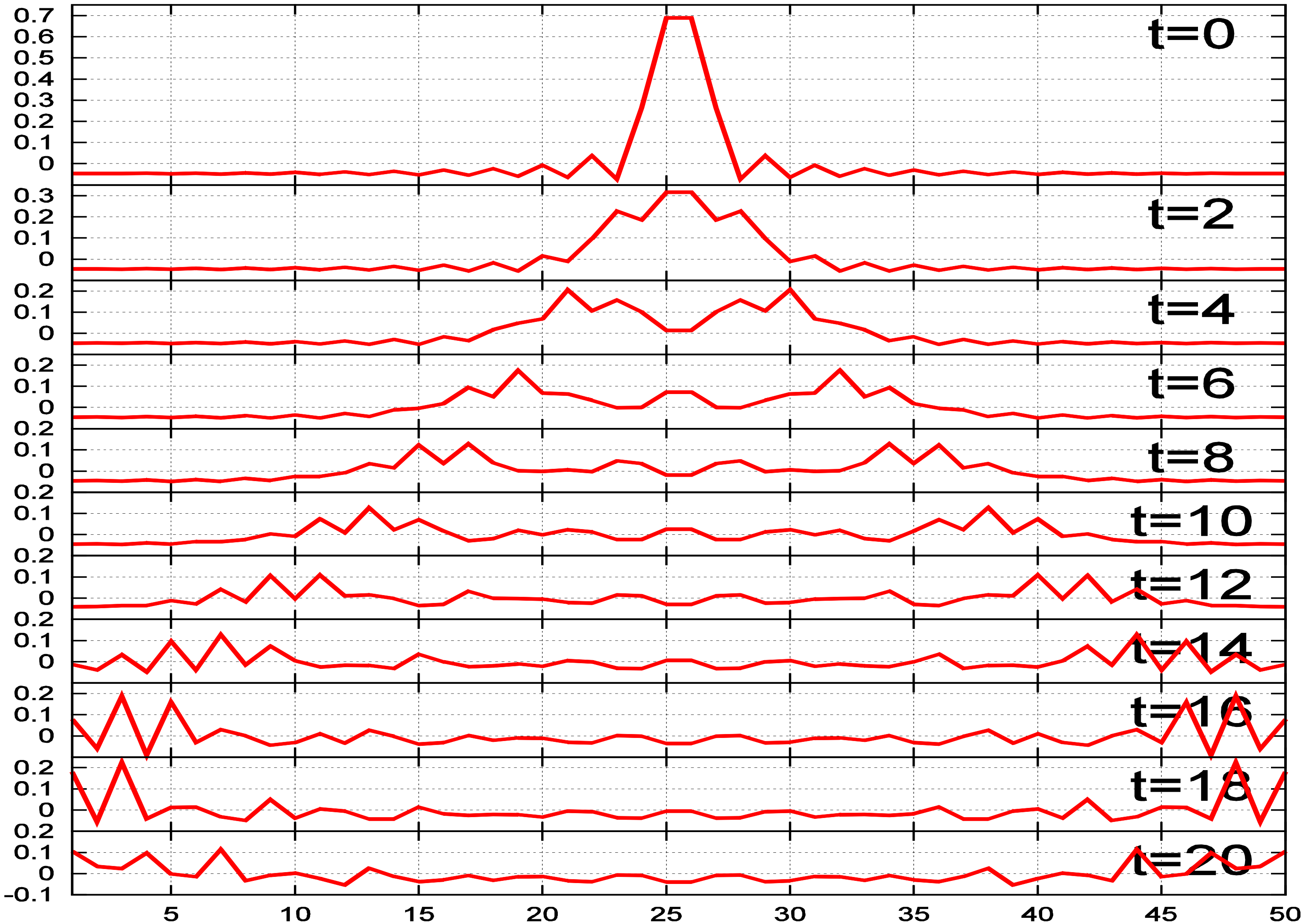
< $\sigma^{z}{}_{m}$> with m=1 (N=50) for the all-spinon state (Local magnetizaion of all spinon state)



Relaxation of local magnetization: $<\sigma^z_m>$ with m=1 (N=50) for the all-spinon state: $\Delta<\sigma^z_m>$ of the order of 1/N remains after long time (by Ryoko Hatakeyama) T=500



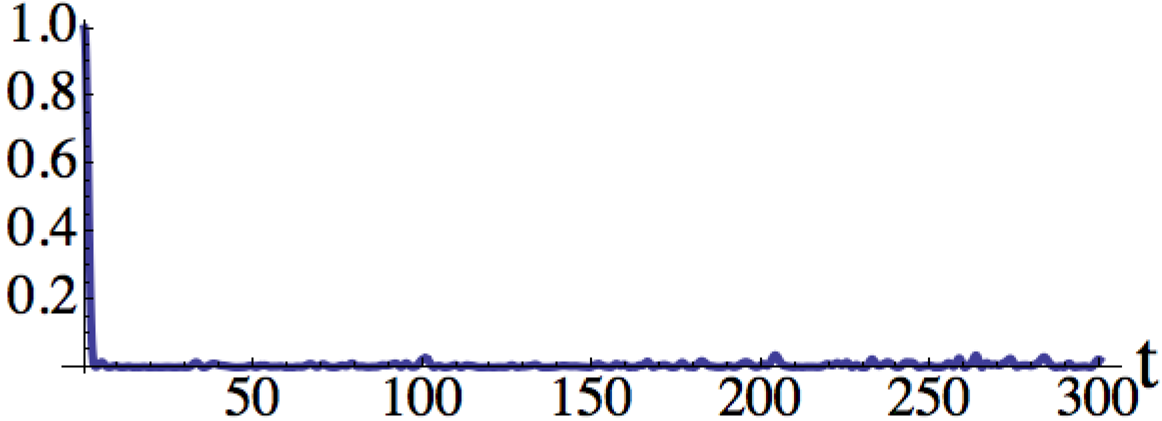
All spinon state is localized initially, propagates and collapses in time. (N=50)) (by Ryoko Hatakeyama)

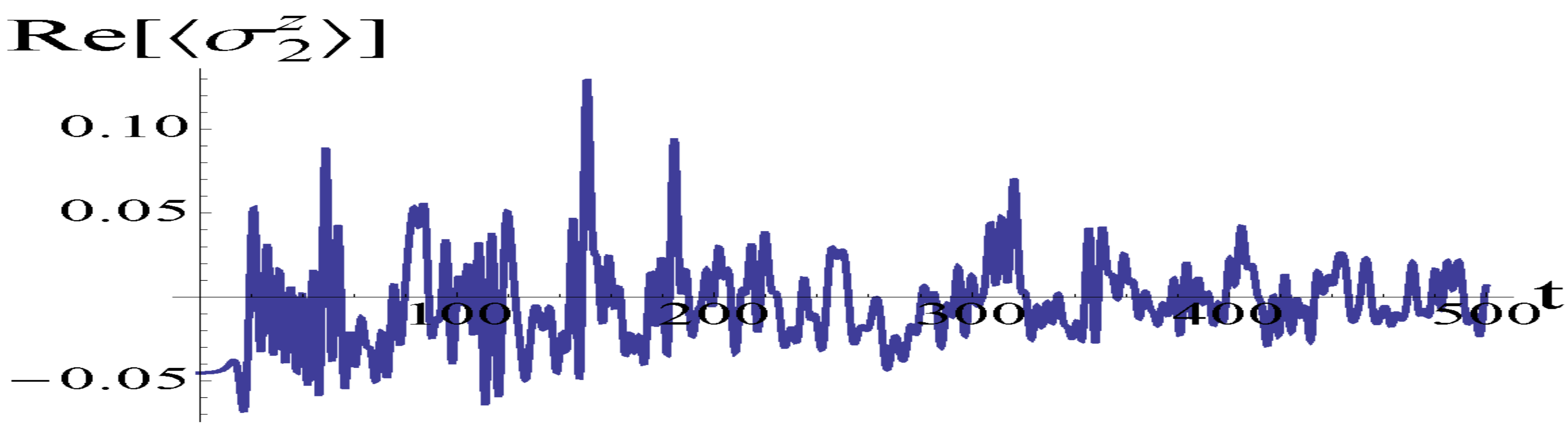


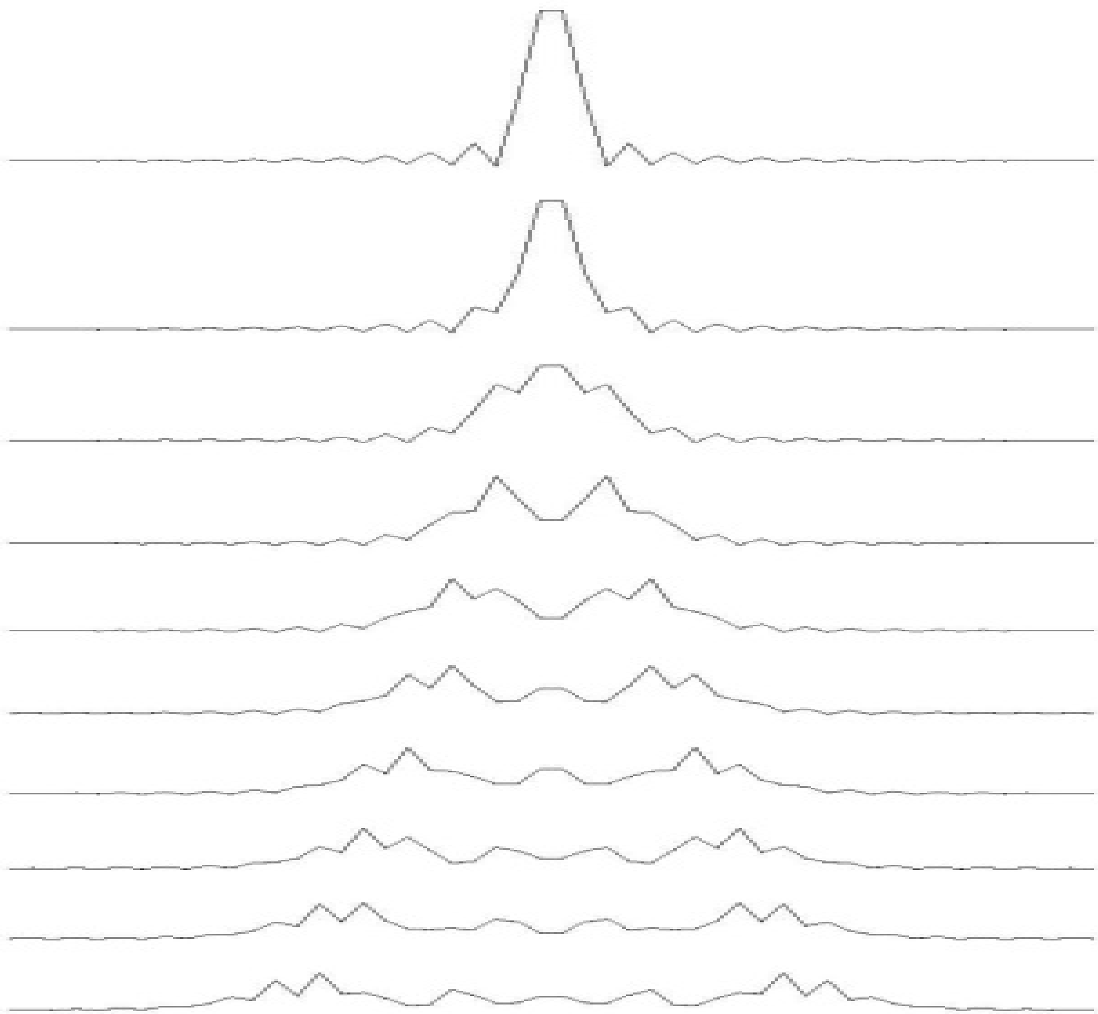
Fidelity versus local magnetization for the all-spinon state:

$<\sigma^z_m>$ for m=2 (N=50, T=500) (by Ryoko Hatakeyama)

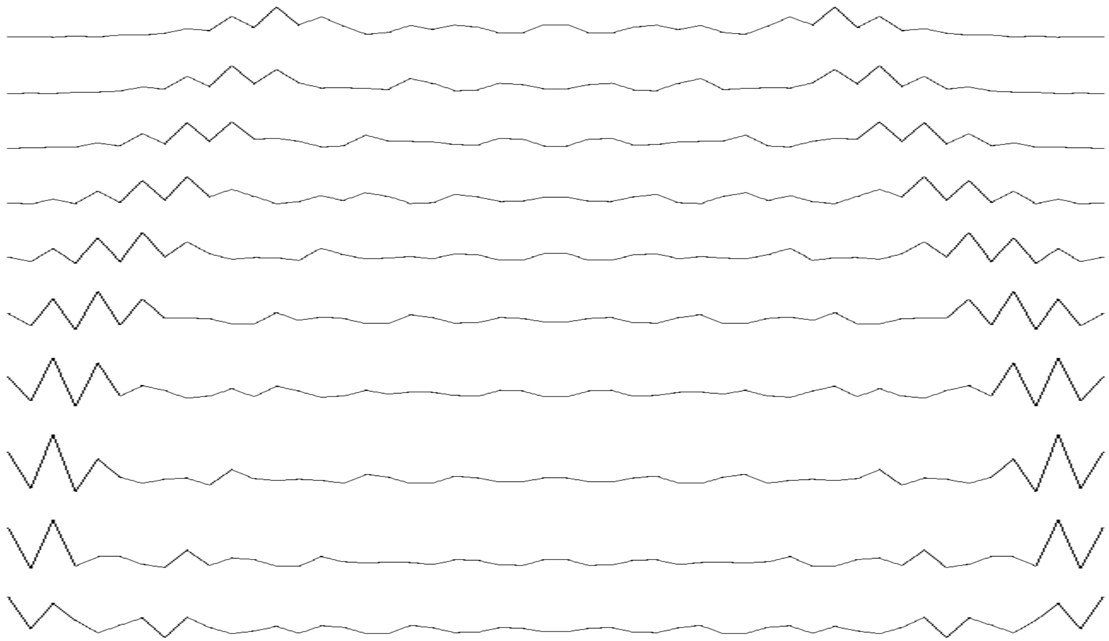
Relaxation time of $<\sigma^z_m>$ is much longer than that of fidelity.



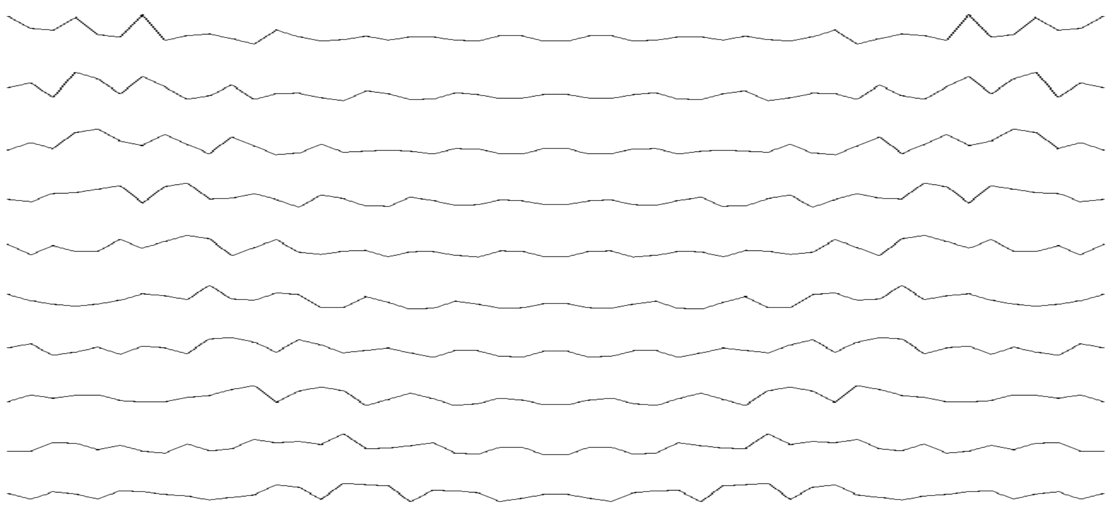




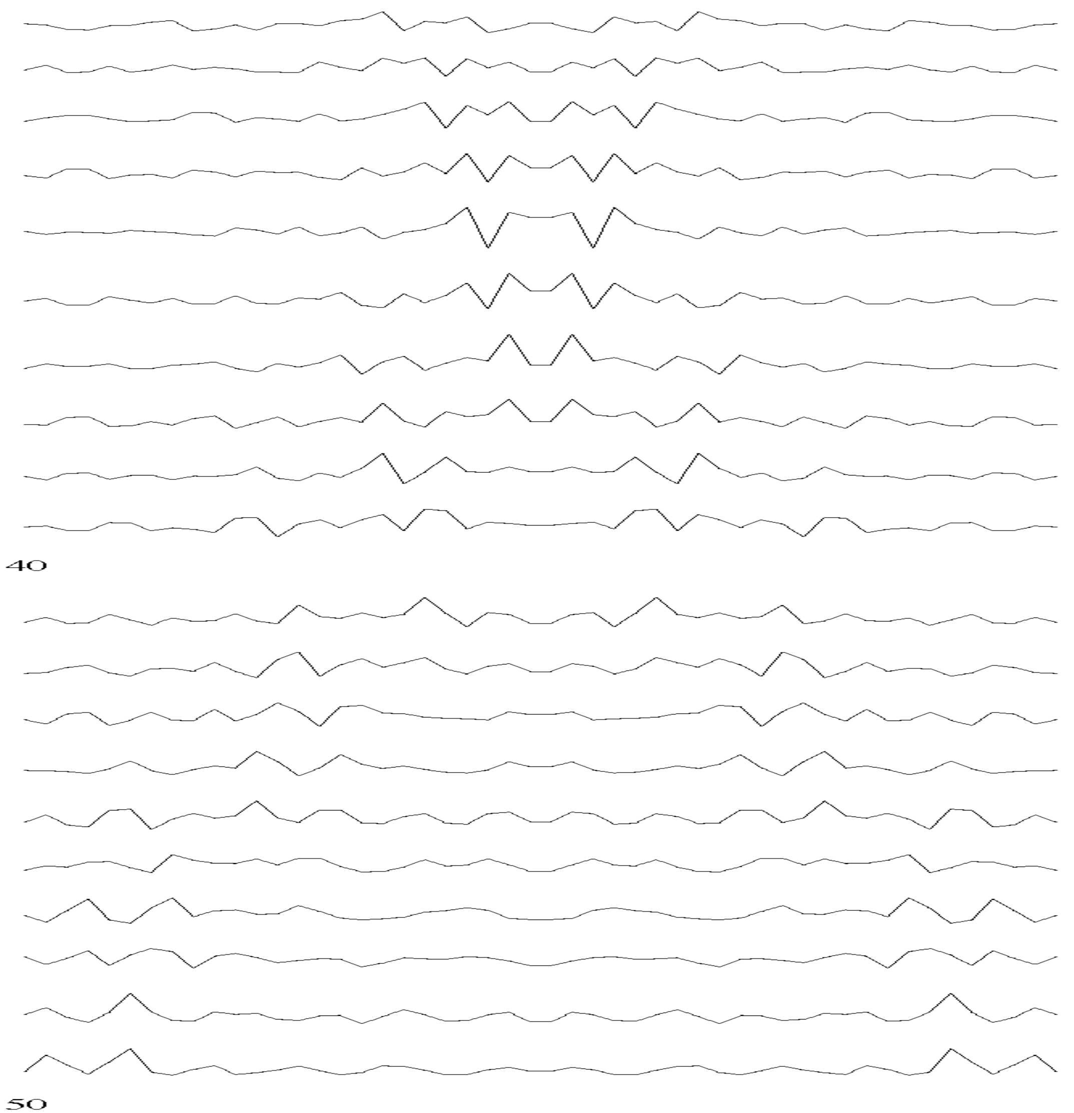
10

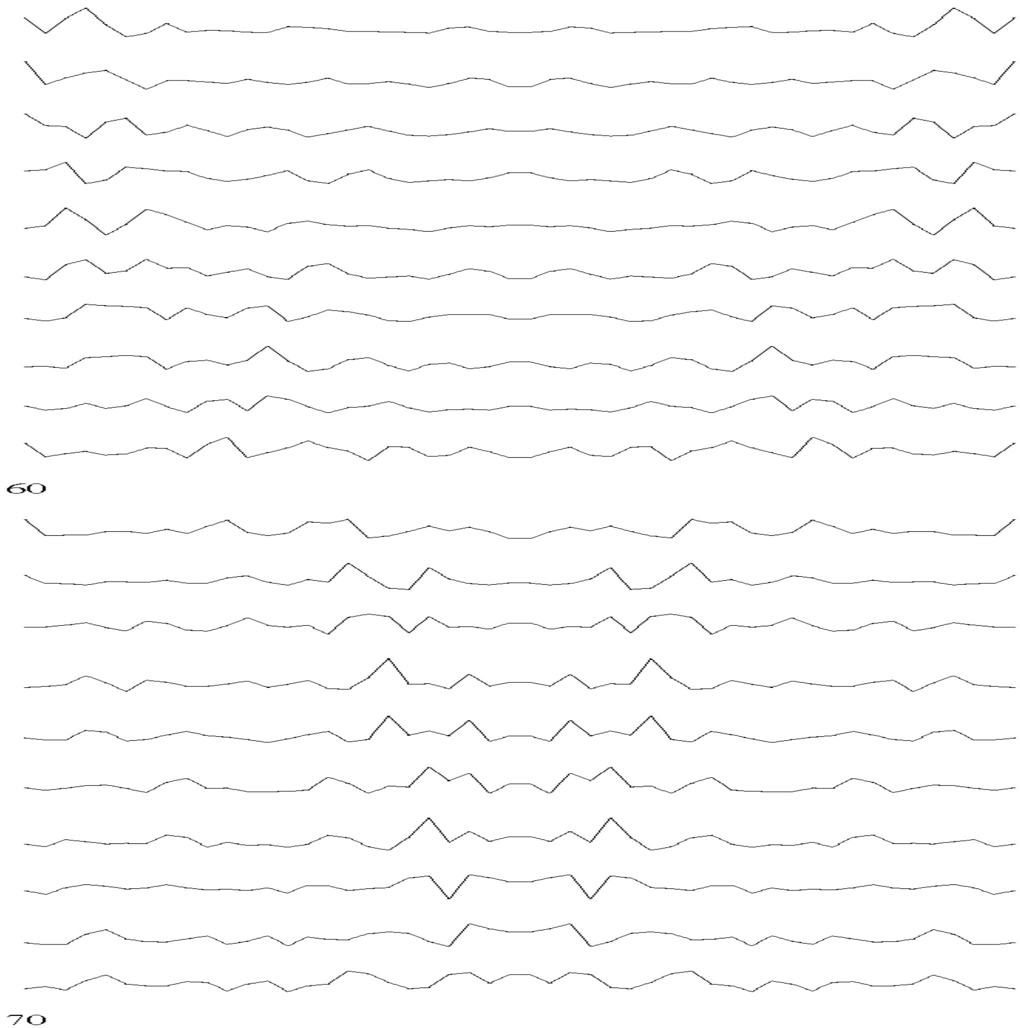


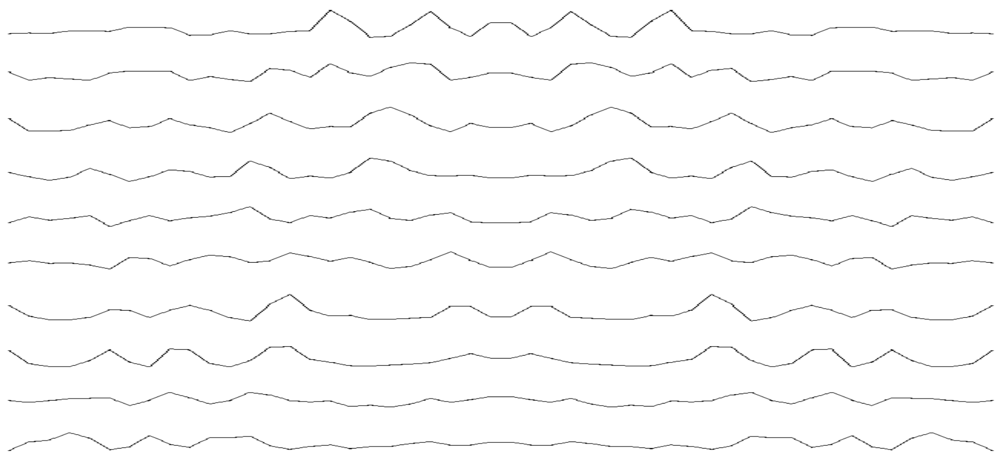
20



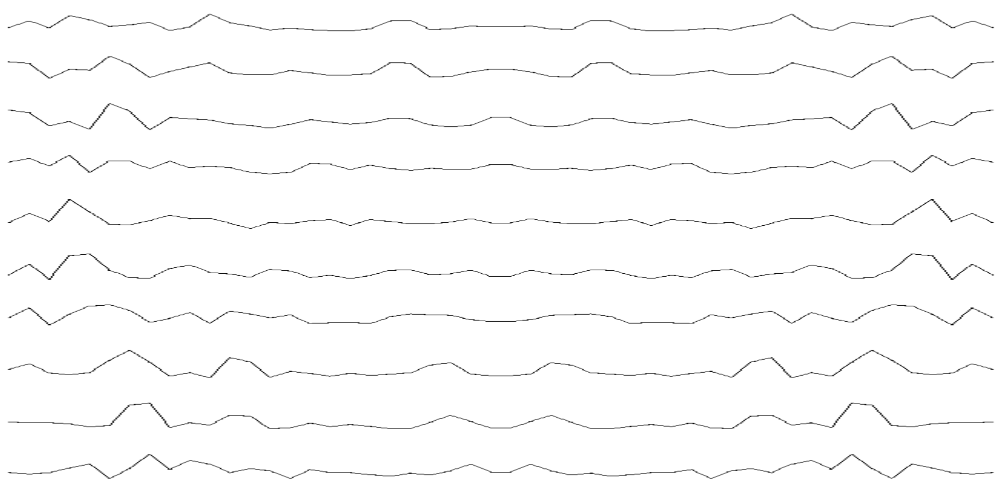
30


40
50

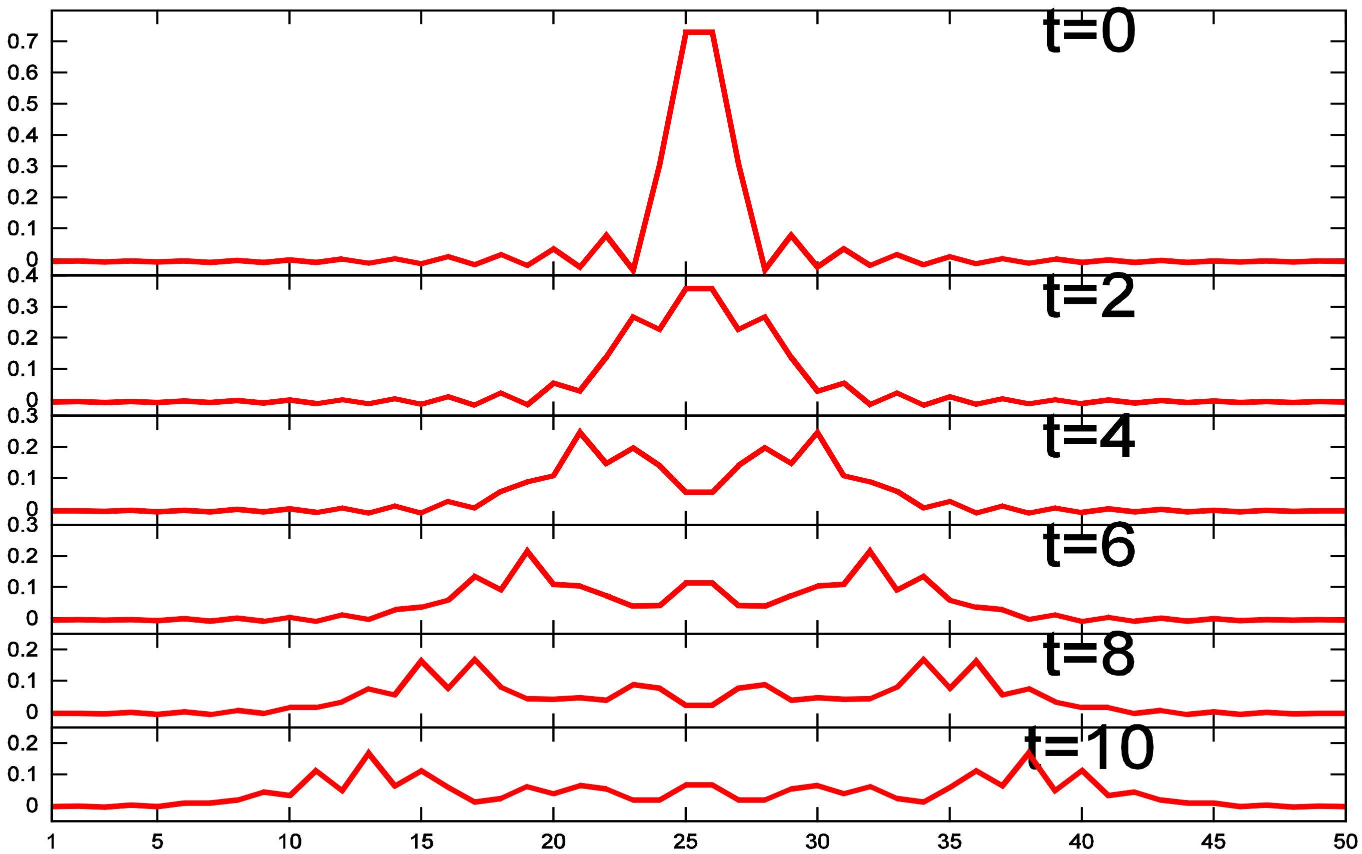

60
70



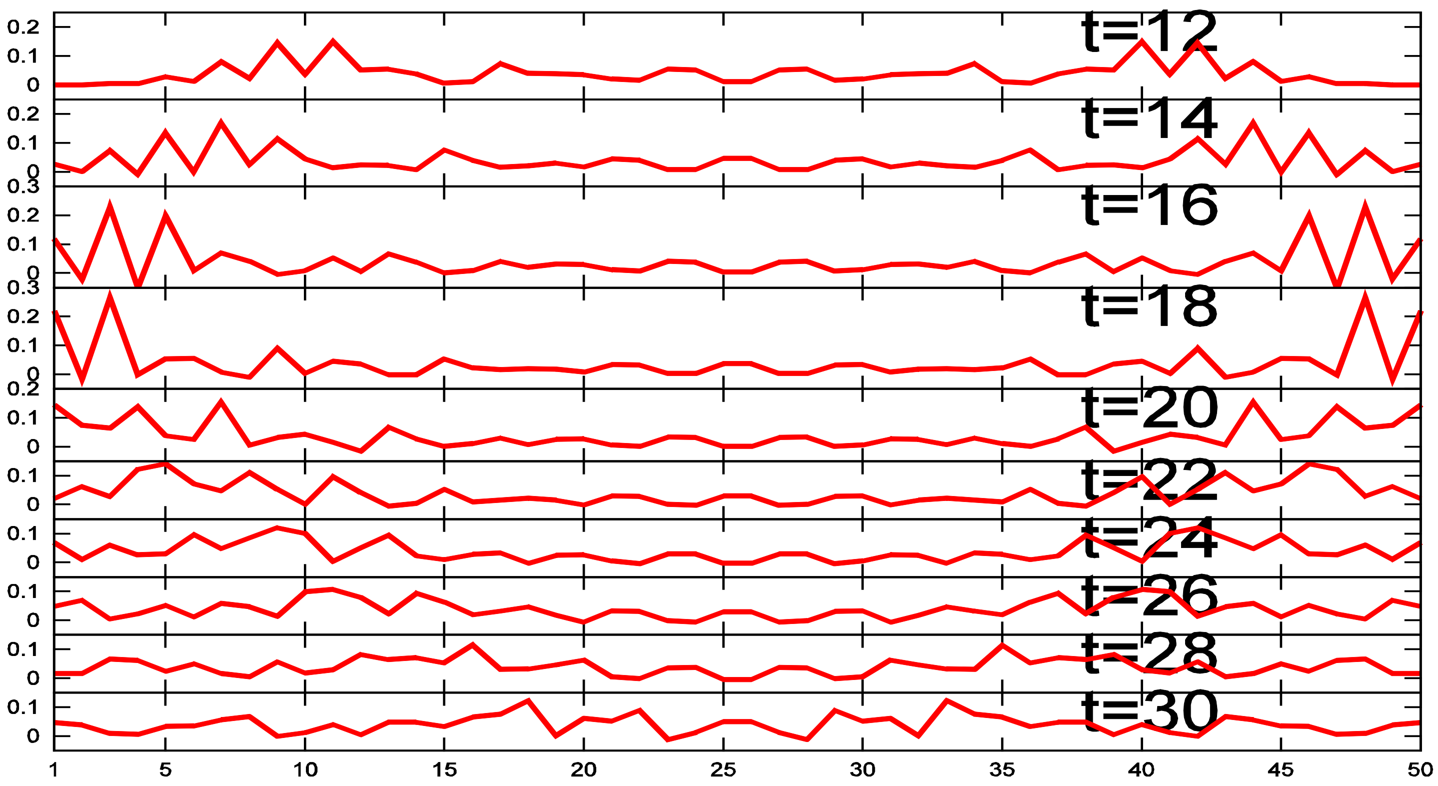
80



90



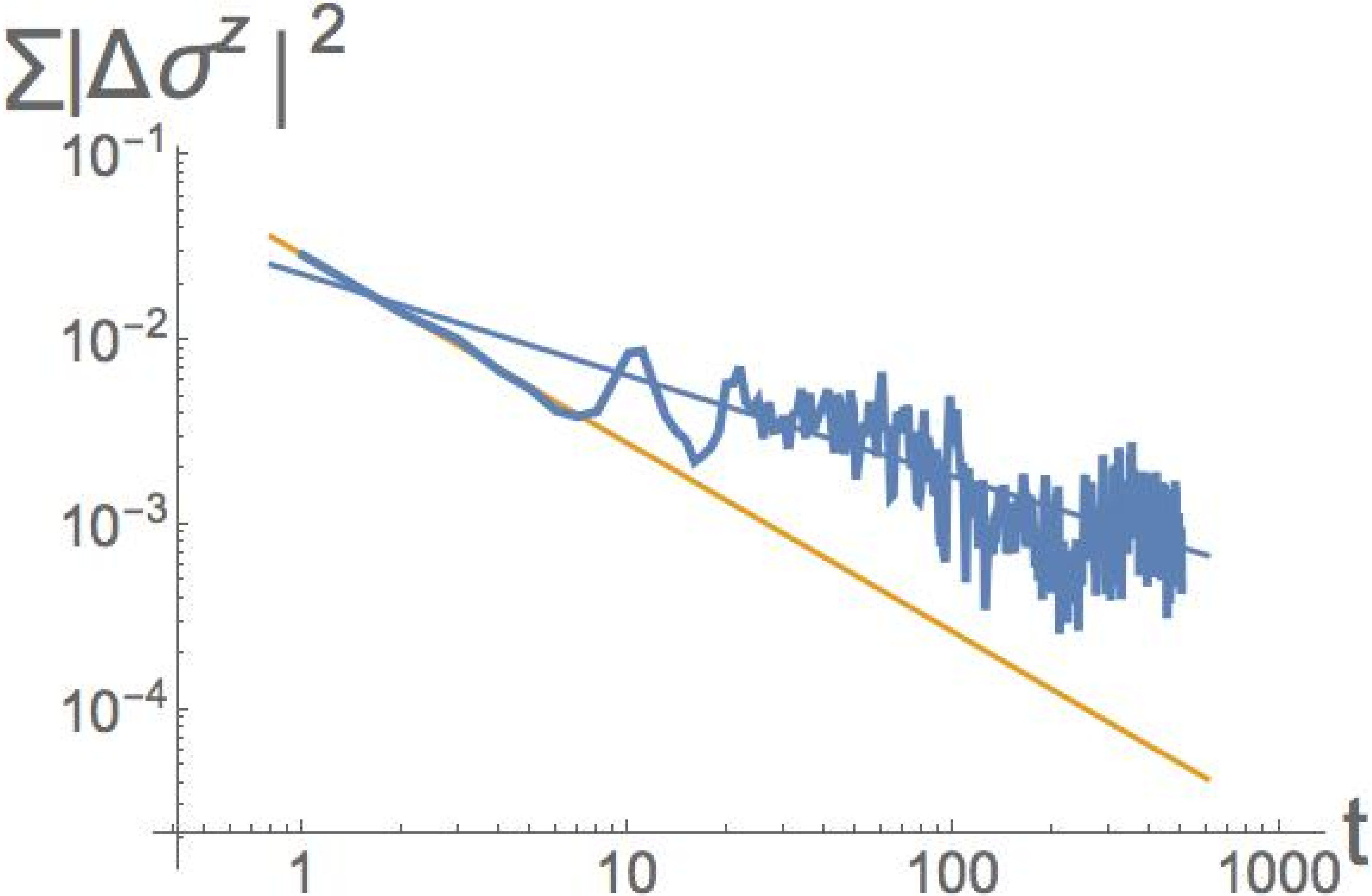
t=0
t=2
t=4
t=6
t=8
t=10
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1
5
10
15
20
25
30
35
40
45
50



t=12
t=14
t=16
t=18
t=20
t=22
t=24
t=26
t=28
t=30
0.3
0.2
0.1
0
1
5
10
15
20
25
30
35
40
45
50

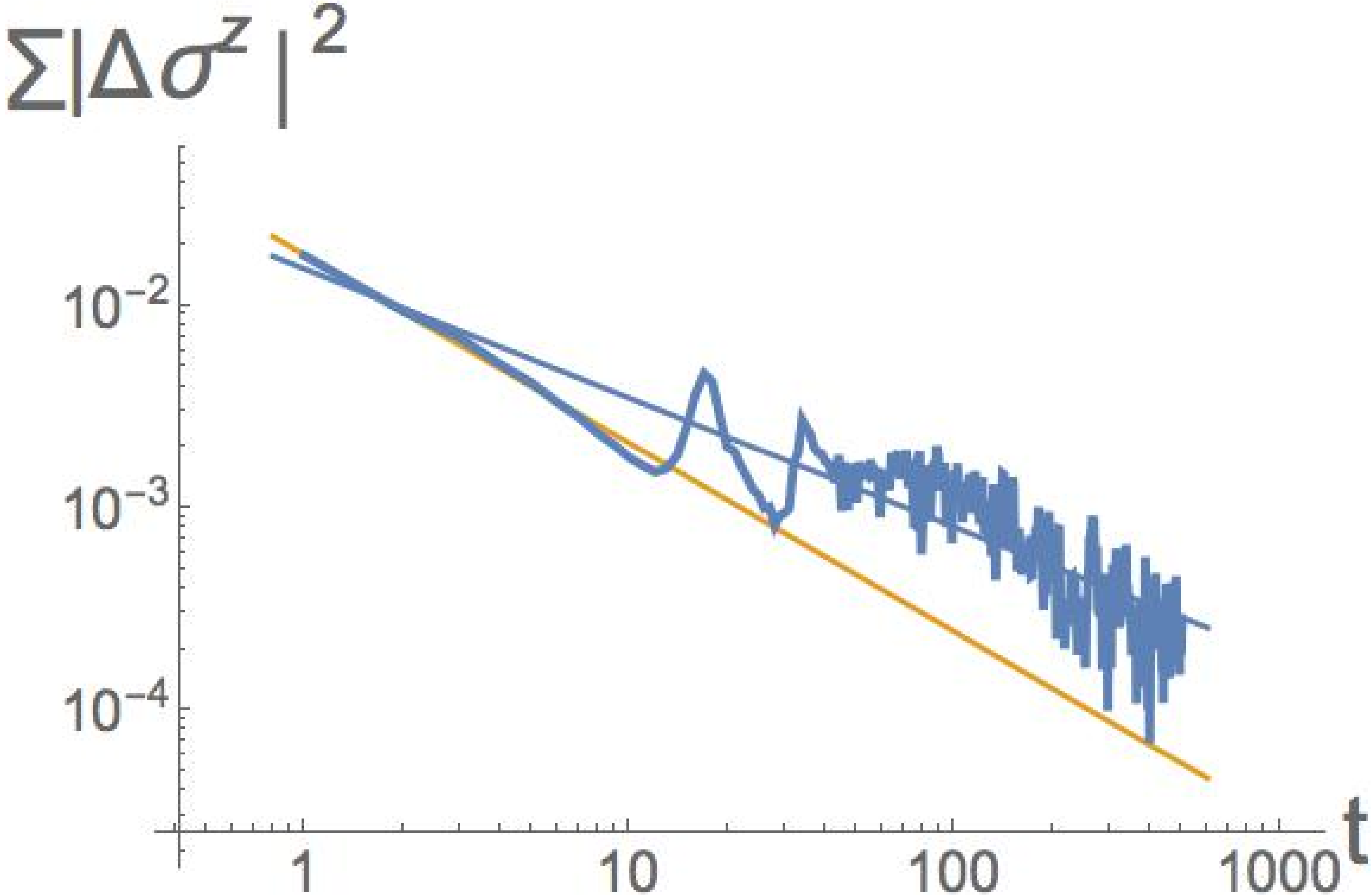
In the all-spinon state for N=30, square deviations of local magnetizations decay almost as an inverse of time initially, then as an inverse power of time with smaller exponent

$$\Sigma|\Delta\sigma^z|^2 = \Sigma_{m=1}^{N} (<\sigma^z_m> - \Sigma_{j=1}^{N} <\sigma^z_j>/N)^2 /N$$



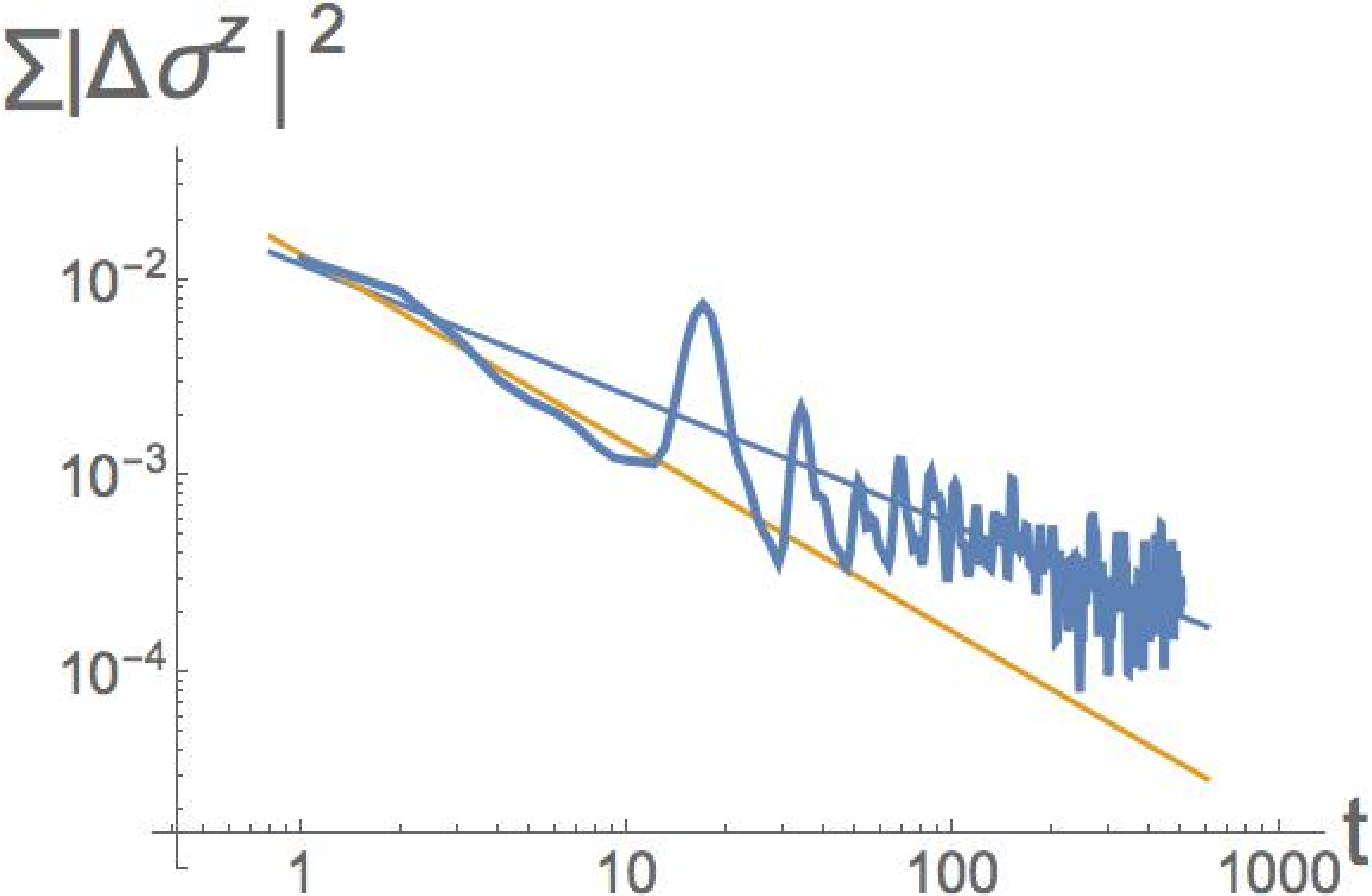
All-spinon state for N=50.

$\Sigma|\Delta\sigma^z|^2 = \Sigma_{m=1}^{N} (<\sigma^z_m> - \Sigma_{j=1}^{N} <\sigma^z_j>/N)^2 /N$

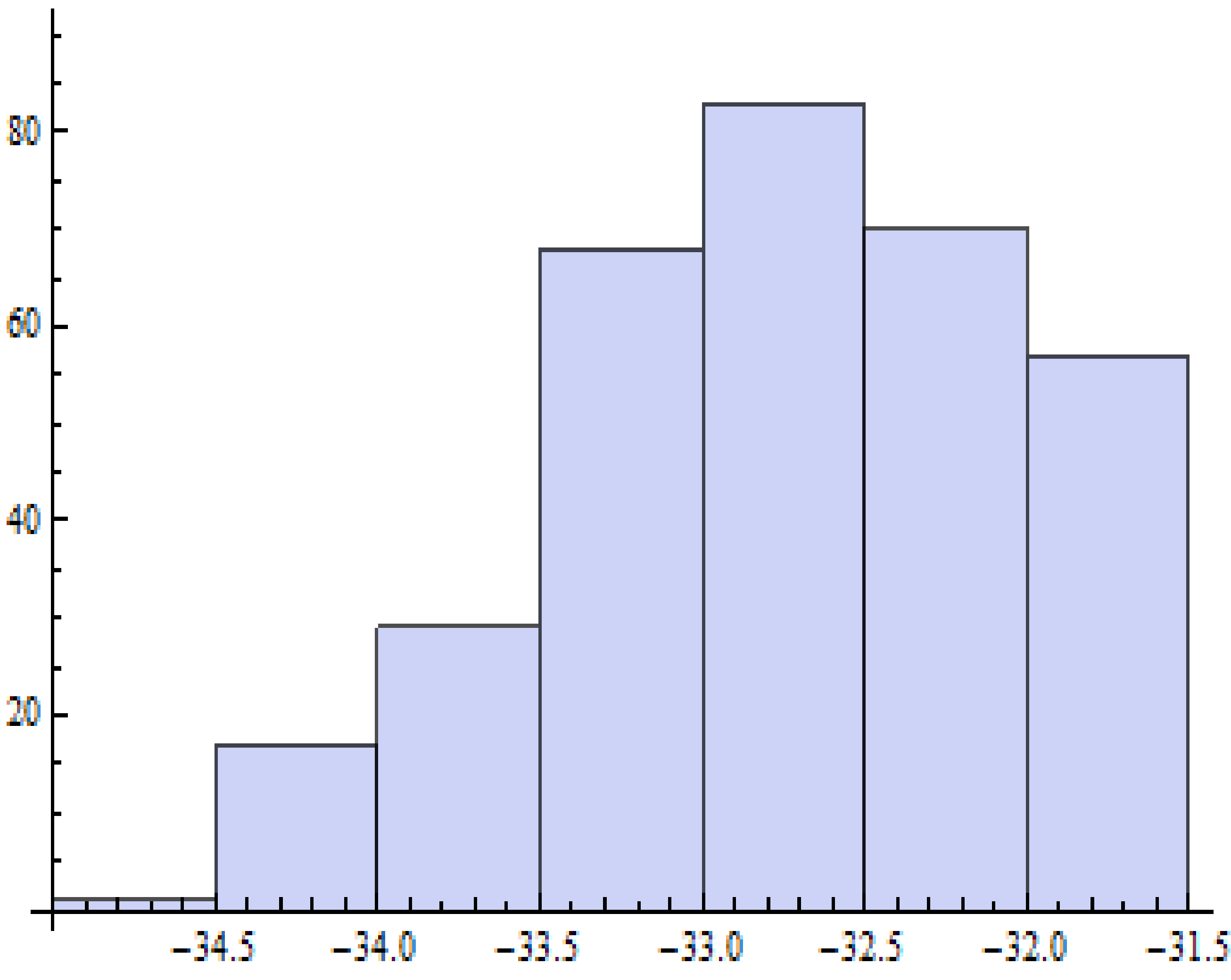


Partial sum of spinon states for N=50, E= -33.5 with $\Delta E=1$.

$$\Sigma|\Delta\sigma^z|^2 = \Sigma_{m=1}^{N} (<\sigma^z_m> - \Sigma_{j=1}^{N} <\sigma^z_j>/N)^2 /N$$

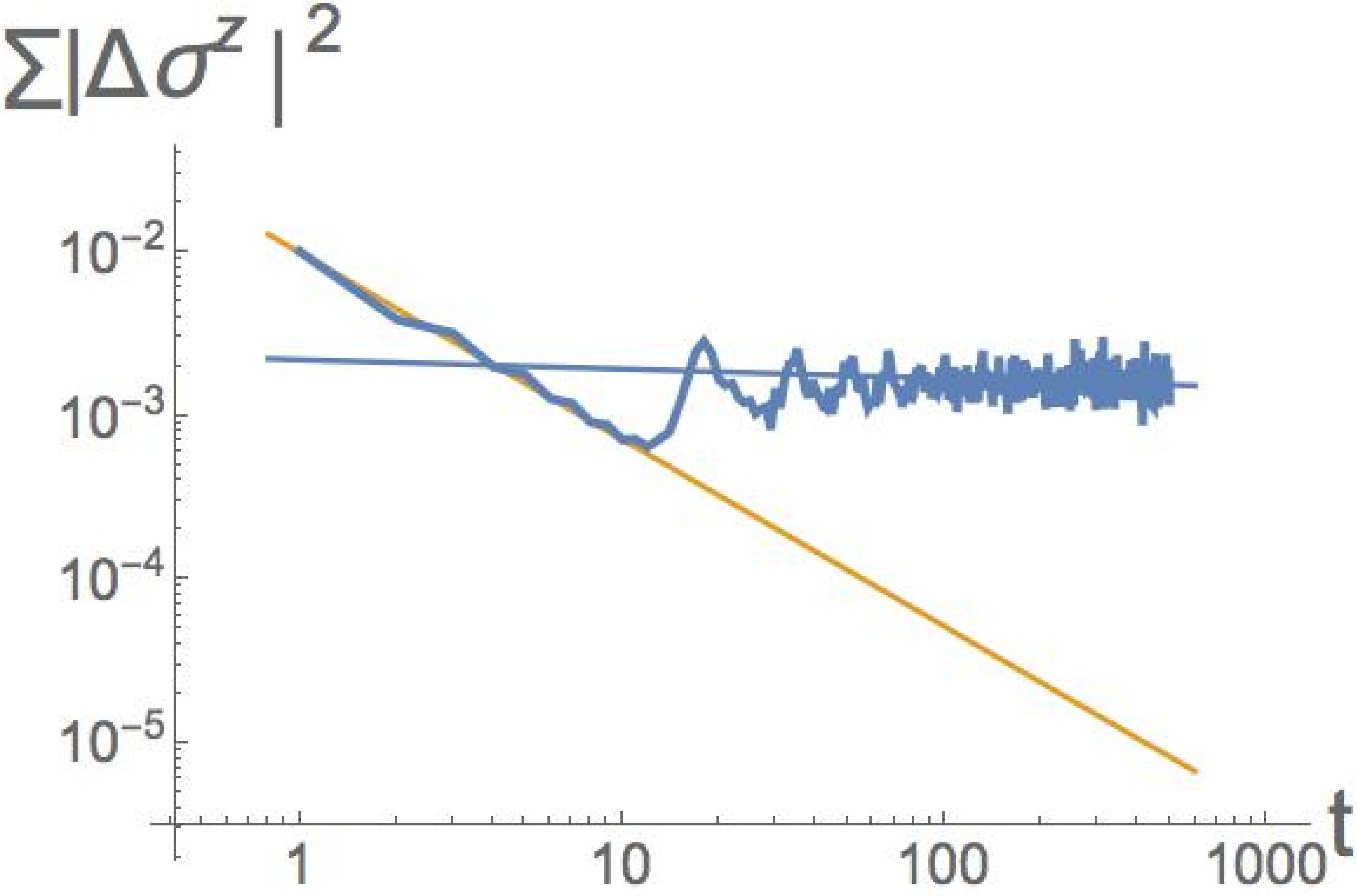


Histogram of spinon energy spectrum for N=50



Sum over Yrast states for N=50.

$$\Sigma|\Delta\sigma^z|^2 = \Sigma_{m=1}^{N}(<\sigma^z_m> - \Sigma_{j=1}^{N} <\sigma^z_j>/N)^2/N$$



How to define relaxation time of $<\sigma^z{}_m>$?

- Two viewpoints:

(1) Power-law decay suggests

there is no definite relaxation time

(2) Traveling time of localized wave suggests

T_R = system size/spinon velocity -> O(N)

(Cf. Lieb-Robinson bound)

Conclusions (part 3)

(1) Power law relaxation for local magnetization $<\sigma^z_m>$ in the XXX chain:

The square deviations of the local magnetization decay as a power of time.

Local magnetization $<\sigma^z_m>$ oscillates in time;
typically, the fluctuations decay to O(1/N^2) or O(1/M).
(M is the number of eigenstates in the sum)

The power law decay may be **universal** for expectation values of local quantities. (We can perform exact dynamics also for other quantities.)

(2) **Power law decay suggests no definite relaxation time in the dynamics**

Equilibration of $<\sigma^z_m>$ is very much slower than that of the fidelity.

Slow due to integrability ?

Conclusions of the talk

- 1D Bose gas (Part 2)

  Exact relaxation dynamics of an initially localized state (density profile)

- XXX chain (Part 3)

Power law relaxation of local magnetization in the quantum Heisenberg chain (the XXX chain) for several initial states such as particular sums of spinon states.

Relaxation is much **slower** than that of fidelity

It shows how an **atypical** local operator should equilibrate in time.

We suggest that other local operators such as the local energy operator should equilibrate in time similarly as the local magnetizations.

Power law relaxation behavior may be universal for equilibration of local quantities in the XXX chain .

Acknowledgement

- We would like to thank for useful comments   on quantum dark solitons:

  A. del Campo (U Mass)

- Thank you for your attention.

Algebraic Bethe ansatz: R-matrix and monodromy matrix

The R-matrix of the XXZ spin chain is given by

$$R_{ij}(u) = \varphi(u+\eta) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{[i,j]} .$$

In terms of $\varphi(x) = \sinh x$ we difine $b(u)$ and $c(u)$ by

$$b(u) = \varphi(u)/\varphi(u+\eta), \quad c(u) = \varphi(\eta)/\varphi(u+\eta).$$

We define the monodromy matix $T(\lambda)$ with inhomogeneity parameters w_j by

$$T_{0,12\cdots L}(\lambda; \{w_j\}_L) = R_{0L}(\lambda - w_L) \cdots R_{02}(\lambda - w_2) R_{01}(\lambda - w_1)$$

We denote the matrix elements of the monodromy matrix as

$$T_{0,12\cdots L}(\lambda; \{w_j\}_L) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} .$$