Generalized Gibbs Ensembles in Interacting Integrable Models

Fabian Essler (Oxford)

E. Ijjevski, J. de Nardis, B. Wouters, J.-S. Caux, F.H.L. Essler and T. Prosen, arXiv:1507.02993

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This talk will be about **exact**, but **not rigorous**, results for quantum quenches in integrable models.

Outline

A. Quantum Quenches in isolated systems.

B. Steady state and Generalized Gibbs Ensembles (GGE).

C. "Micro-canonical" viewpoint.

D. Integrable models and local conservation laws.

E. Failure of the "Minimal GGE" in interacting models.

F. Quasi-local conservation laws.

G. GGE for the spin-1/2 XXZ chain.

Quantum Quenches in isolated many-particle systems

A. Consider an **isolated** quantum system in the **thermodynamic limit**; Hamiltonian H(h) (short-ranged), h e.g. bulk magnetic field

B. Prepare the system in the ground state $|\psi\rangle$ of $H(h_0)$

C. At time t=0 change the Hamiltonian to H(h)

D. (Unitary) time evolution $|\psi(t)\rangle = \exp(-iH(h)t)\,|\psi\rangle$

E. Goal: study time evolution of local (in space) observables

$$\langle\psi(t)|\mathcal{O}(x)|\psi(t)\rangle\ ,\ \langle\psi(t)|\mathcal{O}_1(x)\mathcal{O}_2(y)|\psi(t)\rangle\ ,\ \langle\psi(t)|\mathcal{O}_1(x,t_1)\mathcal{O}_2(y,t_2)|\psi(t)\rangle$$

Local Relaxation

Given that we are considering an **isolated** system, does the system relax in some way ?

- It can never relax as a whole.

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Initial state $|\psi\rangle$ after the quench is a **pure state**

$$|\psi(t)\rangle = \exp(-iH(h)t)\,|\psi\rangle = \Sigma_n \exp(-iE_n t)\,\langle n|\psi\rangle\,|n\rangle.$$

Can **always** choose observables O that never relax, e.g.

$$O=O^\dagger = |1\rangle\langle 2|+|2\rangle\langle 1|$$

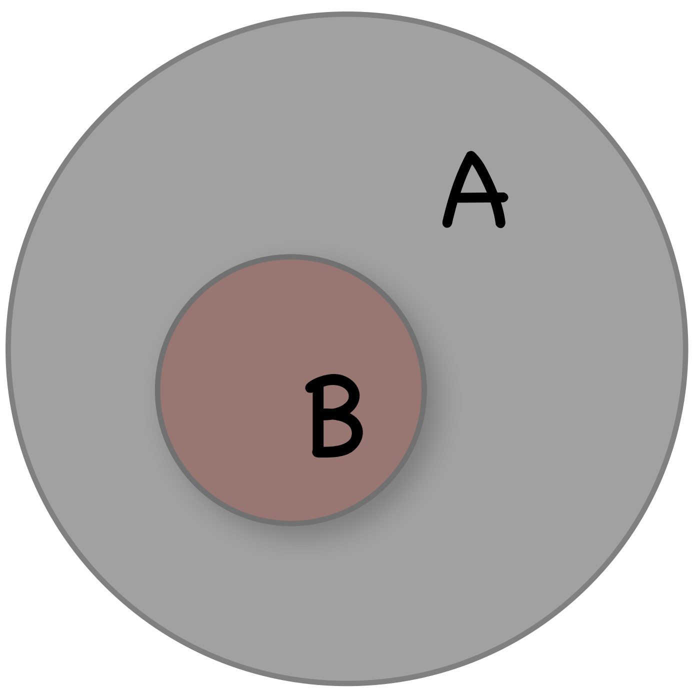
$$\langle\psi(t)|O|\psi(t)\rangle = A\,\cos([E_1-E_2]t+\varphi)$$

Local Relaxation

Given that we are considering an **isolated** system, does the system relax in some way ?

- It can never relax as a whole.

- It can relax **locally** (in space).



- Entire System: $A \cup B$
- Take A infinite, B finite
- Ask questions only about B:

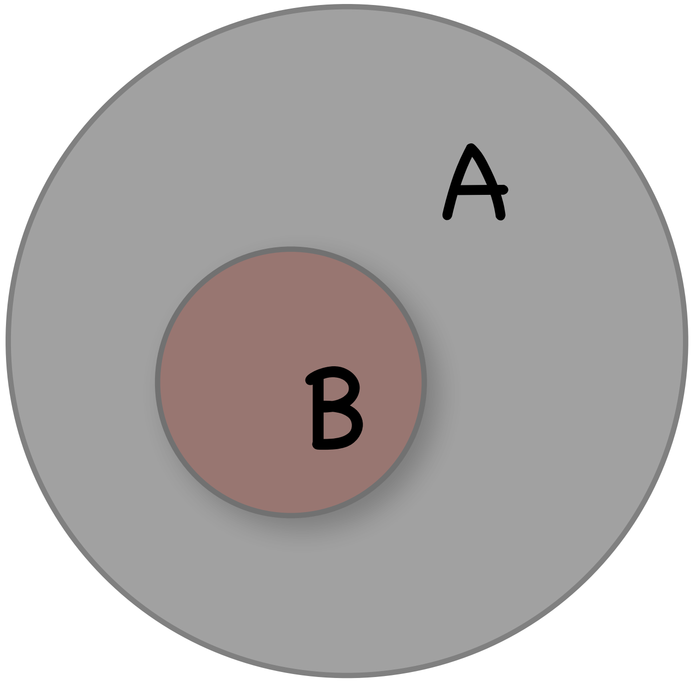
Expectation values of **local** ops:

$$\langle \Psi(t) | O_B(x) | \Psi(t) \rangle$$

Physical Picture: A acts like a bath for B.

Subsystems and Reduced Density Matrices

$|\psi\rangle$ = initial (pure) state of the entire system $A \cup B$ (A infinite)



Density matrix: $\rho(t)=|\psi(t)\rangle\langle\psi(t)|$

Reduced density matrix: $\rho_B(t)=\mathrm{tr}_A\, \rho(t)$

Nonequilibrium Steady State

For the initial states we are interested in $\lim_{t\to\infty} \rho_B(t) = \rho_B(\infty)$ exists for any finite subsystem B in the thermodynamic limit.

$\Longleftrightarrow$ $\langle\psi(t)|\mathcal{O}_B(x)|\psi(t)\rangle$ become time-independent for all **local operators**.

How to characterize the steady state?

Conservation laws

Isolated system $\rightarrow$ energy conserved $[H, e^{-iHt}] = 0$

No other conserved quantities $\rightarrow$ system **thermalizes**

cf previous talks,
Deutsch '91, Srednicki '94,....

Define a Gibbs Ensemble: $$\rho_{\mathrm{GE}} = \frac{1}{Z_{\mathrm{GE}}} e^{-\beta_{\mathrm{eff}} H}$$

fix effective temperature: $$e = \lim_{L\to\infty} \frac{\langle \Psi(0)|H|\Psi(0)\rangle}{L} = \lim_{L\to\infty} \frac{1}{L} \mathrm{Tr}\left(\rho_{\mathrm{GE}} H\right)$$

Reduced density matrix: $$\rho_{\mathrm{GE,B}} = \mathrm{Tr}_A\left(\rho_{\mathrm{GE}}\right)$$

Thermalization: $$\lim_{t\to\infty} \rho_B(t) = \rho_{\mathrm{GE,B}}$$

Further conserved quantities: system does not **thermalize**

$$[I_\alpha, H] = 0 \Rightarrow \langle \Psi(t) | I_\alpha | \Psi(t) \rangle = \text{const.}$$

Define a Generalized Gibbs Ensemble:

M. Rigol et. al. '07

$$\rho_{\text{GGE}} = \frac{1}{Z_{\text{GGE}}} e^{-\sum_\alpha \lambda_\alpha I_\alpha}$$

fix Lagrange multipliers:

$$e_\alpha = \lim_{L\to\infty} \frac{\langle \Psi(0) | I_\alpha | \Psi(0) \rangle}{L}$$
$$= \lim_{L\to\infty} \frac{1}{L} \text{Tr}\left(\rho_{\text{GGE}} I_\alpha\right)$$

Reduced density matrix: $\rho_{\text{GGE,B}} = \text{Tr}_A\left(\rho_{\text{GGE}}\right)$

Non-thermal Steady State:

$$\lim_{t\to\infty} \rho_{\text{B}}(t) = \rho_{\text{GGE,B}}$$

Barthel&Schollwöck '08
Cramer, Eisert et al '08

"Microcanonical" viewpoint

Caux&Essler '13

Cassidy, Clark & Rigol '11

Construct simultaneous **eigenstate** $|\Phi\rangle$ of all I_a such that

$$\lim_{L\to\infty} \frac{\langle\Phi|I_\alpha|\Phi\rangle}{L} = \lim_{L\to\infty} \frac{\langle\Psi(0)|I_\alpha|\Psi(0)\rangle}{L}$$

This macro-state is described by "particle/hole" densities

Ideal Fermi gas:

Hamiltonian in mtm space:
$$H = \sum_p \epsilon(p)\hat{n}(p)$$

particle/hole densities:
$$\rho^p(k) = \frac{1}{2\pi} - \rho^h(k) = \frac{\langle\Phi|\hat{n}(k)|\Phi\rangle}{2\pi}$$

non-equilibrium steady state:
$$\rho^p(k) = \frac{\langle\Psi(0)|\hat{n}(k)|\Psi(0)\rangle}{2\pi}$$

specific "representative state" in large, finite volume:

$$|\Phi\rangle_L = \prod_j c^\dagger(k_j)|0\rangle\ ,\quad \rho^p(k_j) = \frac{1}{L(k_{j+1}-k_j)}\ ,\quad k_j = \frac{2\pi n_j}{L}$$

General integrable models:

Bound states: generally many particle species

Corresponding particle/hole densities related non-trivially:

$$\rho_n^p(\lambda) + \rho_n^h(\lambda) = a_n(\lambda) - \sum_{m=1}^{\infty} \int d\mu \ K_{nm}(\lambda - \mu) \rho_m^p(\mu)$$

Bethe Ansatz equations

known (model-specific) functions

particle densities fixed by “GTBA equations”

$$\eta_n(\lambda) = \frac{\rho_n^h(\lambda)}{\rho_n^p(\lambda)}$$

$$\ln\left[\eta_n(\lambda)\right] = g_n(\lambda) + \sum_m \int d\mu \ K_{nm}(\lambda - \mu) \ \ln\left[1 + \frac{1}{\eta_m^p(\lambda)}\right]$$

determined by “overlaps” $\langle \Psi(0) | \psi_n \rangle$

Descriptions of the stationary state

For finite sub-systems B in the thermodynamic limit

$$\rho_{\mathrm{GGE,B}} = \mathrm{Tr}_{\bar{B}}\left(|\Phi\rangle\langle\Phi|\right) = \rho_{\mathrm{diag,B}}$$

Globally they are all different.

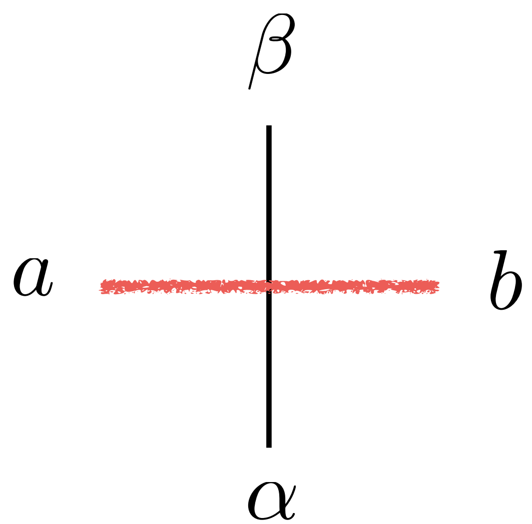
Relaxation to GGEs has been shown in integrable models that can be mapped to free fermions or free bosons.

What about interacting integrable models?

Local conservation laws in integrable models

$$H = J\sum_j S^x_j S^x_{j+1} + S^y_j S^y_{j+1} + \Delta S^z_j S^z_{j+1}, \quad 1 \leq \Delta = \cosh\eta$$

Transfer matrix formulation (6-vertex model):



vertex weight

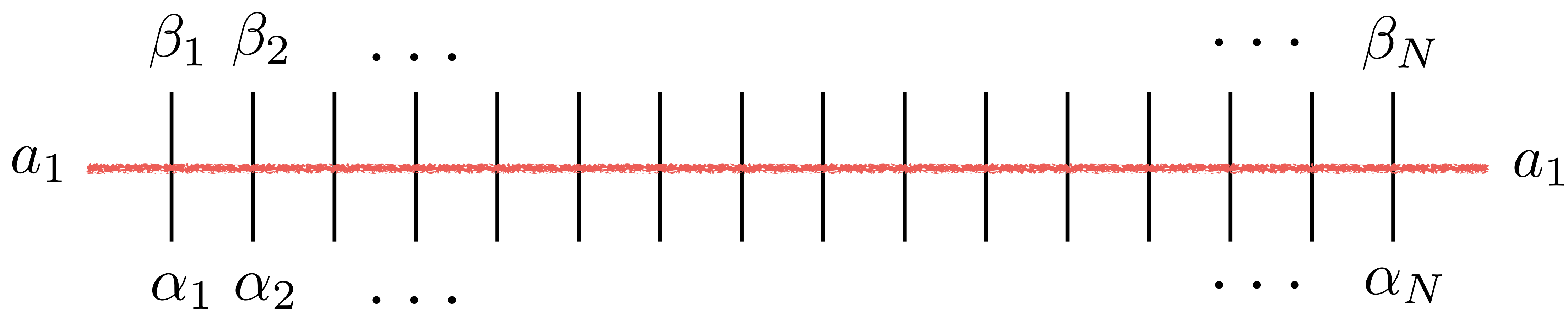
$$(L(\lambda))^{ab}_{\alpha\beta} \qquad a,b = 1,2\ ,\ \alpha,\beta = \uparrow,\downarrow$$

$$L(\lambda) = \frac{1}{\sinh\eta}\left[\sinh\lambda\cosh\left(\eta S^z\right)\sigma^0 + \cosh\lambda\sinh\left(\eta S^z\right)\sigma^z + \sinh\eta\left(S^-\sigma^+ + S^+\sigma^-\right)\right]$$

"auxiliary" space "quantum" space

Define a **transfer matrix** by

$$\tau(\lambda)^{\beta_1\ldots\beta_N}_{\alpha_1\ldots\alpha_N} = (L(\lambda))^{a_1 a_2}_{\alpha_1\beta_1}\,(L(\lambda))^{a_2 a_3}_{\alpha_2\beta_2}\ldots(L(\lambda))^{a_N a_1}_{\alpha_N\beta_N}$$



Transfer matrices **commute** $\qquad [\tau(\lambda), \tau(\mu)] = 0$

and generate **local conservation laws**

$$H^{(n)} = \frac{i}{n!}\frac{\partial^n}{\partial\lambda^n}\bigg|_{\lambda=\frac{i\eta}{2}} \ln\left[\tau(-i\lambda)\right]$$

Structure:

$$H^{(n)} = \sum_{j=1}^{N} H^{(n)}_{j,j+1,\ldots,j+n}$$

$$H^{(n)}_{j,j+1,\ldots,j+n} = \sum_{k=1}^{n+1} \sum_{\alpha_1,\ldots,\alpha_k=0}^{3} f_{\alpha_1\alpha_2\ldots\alpha_k}\ \sigma_j^{\alpha_1}\ldots\sigma_{j+k}^{\alpha_k}$$

Hamiltonian: $H^{(1)} \propto H$

Lieb '67, Sutherland '70
Baxter '72, ...

“Minimal” GGE

Fagotti&Essler ‘13

GGE density matrix

$$\rho_{\mathrm{GGE}} = \frac{1}{Z_{\mathrm{GGE}}} \exp\left(- \sum_{l=1} \lambda_l H^{(l)} \right) .$$

Can be viewed as **thermal** density matrix of integrable Hamiltonian

cf Klümper& Sakai ‘02

$$\mathcal{H} = \sum_{l=1} \frac{\lambda_l}{\lambda_1} H^{(l)}$$

Can use (Quantum Transfer Matrix) formalism developed for finite temperature correlators

Boos, Göhmann, Klümper et al ’04-’10

Boos, Miwa, Jimbo, Smirnov, Takeyama ’06-’09

to study GGE expectation values!

Neat trick: circumvent determining $\{\lambda_j\}$ by using generating function

$$X_{\frac{1}{2}}(\lambda) = \left(\frac{\sinh(\eta + i\lambda)}{\sinh(-\eta + i\lambda)}\right)\tau\Big(-\frac{\eta}{2} + i\lambda\Big)\tau'\Big(\frac{\eta}{2} + i\lambda\Big)$$

"Initial data" encoded in the function

$$\Omega_{\frac{1}{2}}(\lambda) = \langle\Psi(0)|X_{\frac{1}{2}}(\lambda)|\Psi(0)\rangle$$

1. Steady state described by system of nonlinear integral eqns.
2. Initial data enters only through $\Omega_{1/2}(\lambda)$.
3. Closed form expressions for $\Omega_{1/2}(\lambda)$ for various product states.
4. Numerically exact expressions for matrix-product states.

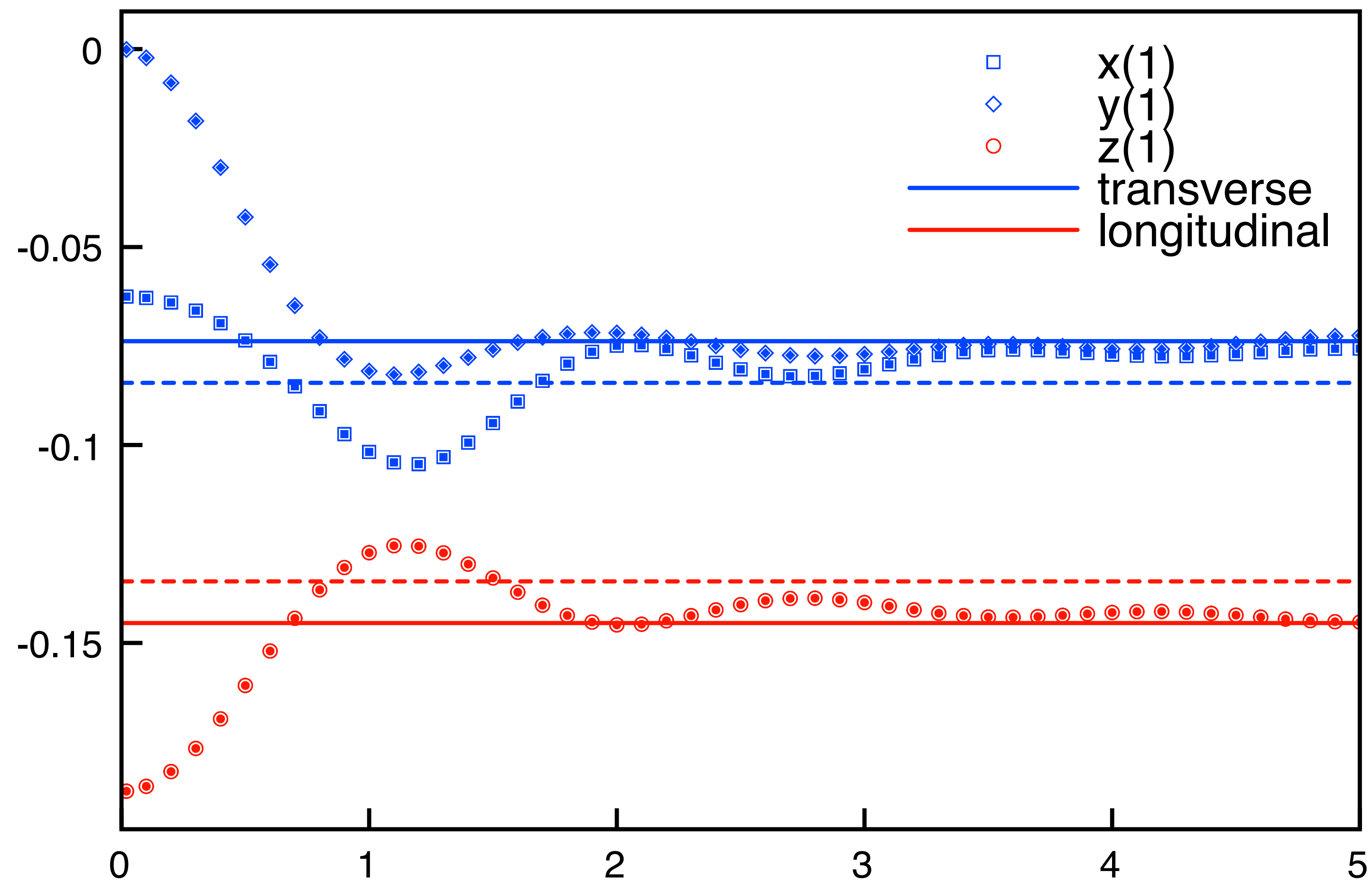
→ Explicit results for short-distance spin-spin correlation functions in "minimal GGE"

Fagotti et al '14

Comparsion to numerics (TDMRG):

Fagotti, Collura, Essler &Calabrese ‘14

$$\langle \Psi(t)|\sigma_j^\alpha \sigma_{j+1}^\alpha|\Psi(t)\rangle$$



mGGE

thermal

time

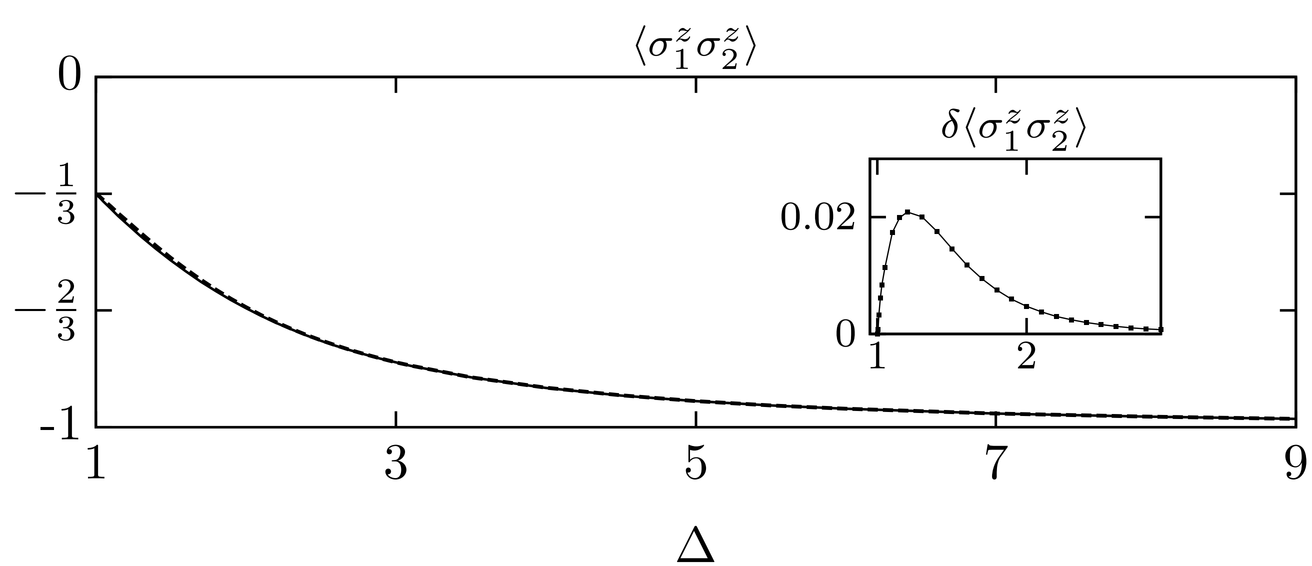
initial state $|\mathbf{N\acute{e}el}_\alpha\rangle = e^{-i\alpha\sum_j \sigma_j^x}|\uparrow\downarrow\uparrow\downarrow\dots\rangle$ rotated Néel state

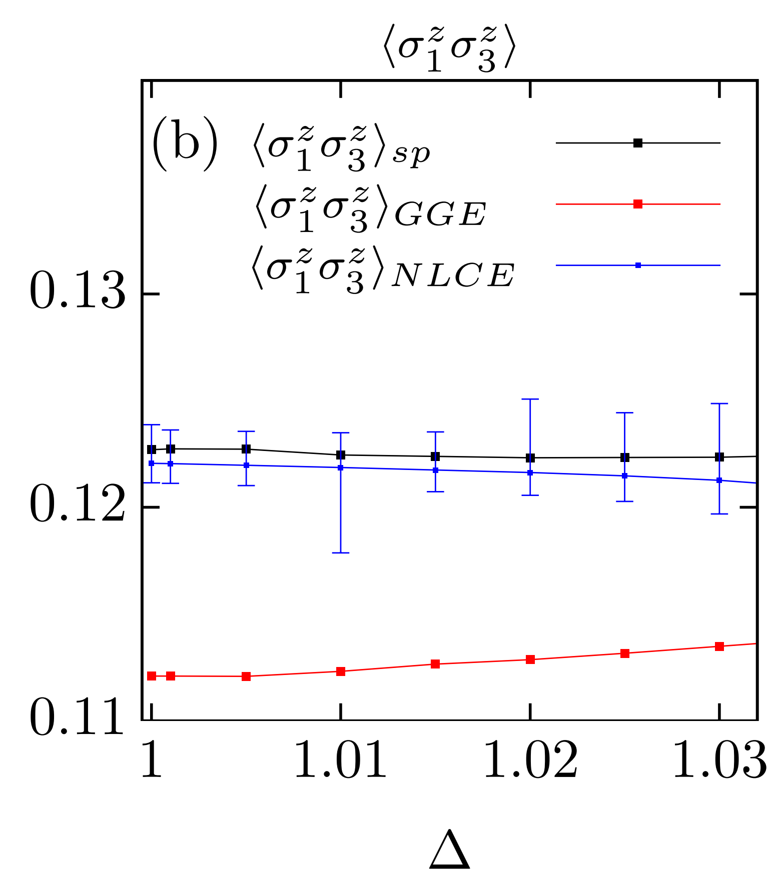
Failure of the "Minimal GGE"

Caux&Essler '13 — "Quench Action" approach to non-equilibrium dynamics in integrable models.

Wouters et al '14
Poszgay et al '14 — stationary state for quenches from Néel and Majumdar-Ghosh initial states.

Results for spin-spin correlators are different from Minimal GGE!



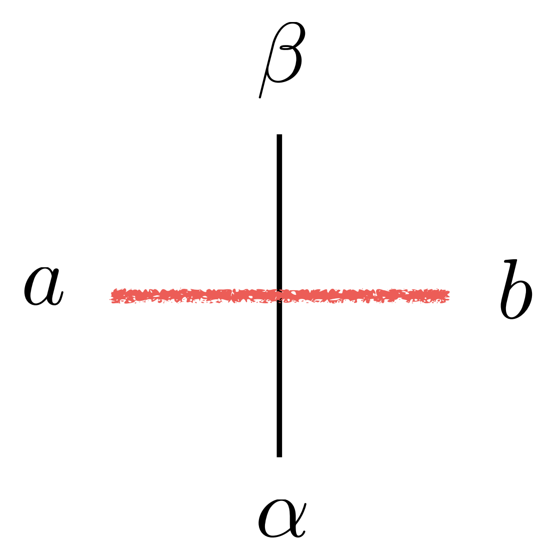


What is going on?

- There must be additional conservation laws! FHLE
- GGE concept fails! several groups

Quasi-local conservation laws

Commuting family of transfer matrices is in fact **much larger:**



$$(\mathcal{L}_s(\lambda))^{ab}_{\alpha\beta}\,,\;\; a,b=1,\ldots,2s+1$$
$$\alpha,\beta=\uparrow,\downarrow$$

$$\mathcal{L}_s(\lambda)=\frac{1}{\sinh\eta}\left[\sinh\lambda\cosh\left(\eta S^z\right)\sigma^0+\cosh\lambda\sinh\left(\eta S^z\right)\sigma^z+\sinh\eta\left(S^-\sigma^+ + S^+\sigma^-\right)\right]$$

where now S^α act on a spin-s representation of $SU(2)_q$

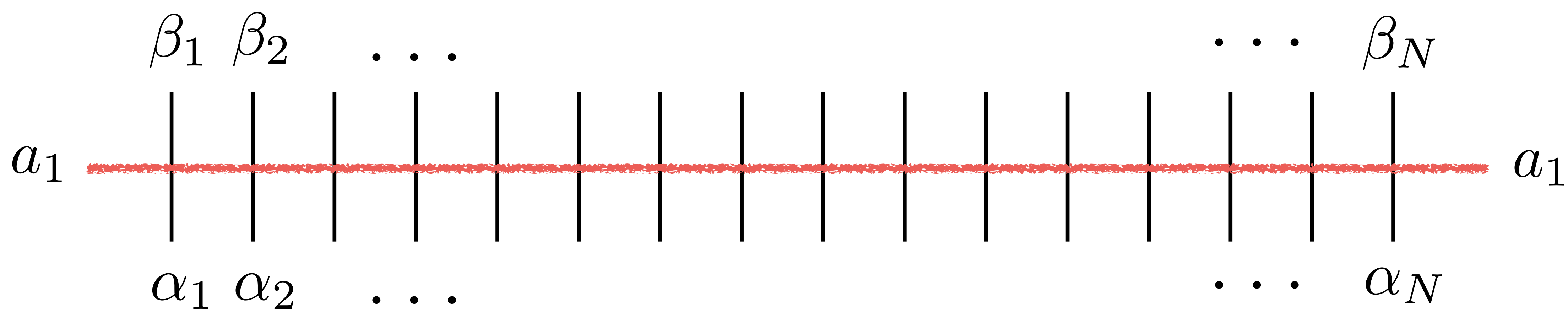
$$[S^+,S^-]=[2S^z]_q\;,\;\;[S^z,S^\pm]=\pm S^\pm\;,\;\;[x]_q\equiv\frac{\sinh(\eta x)}{\sinh\eta}$$

$$S^z|n\rangle=n|n\rangle\;,\;\;S^\pm|n\rangle=\sqrt{[s+1\pm n]_q[s\mp n]_q}|n\pm 1\rangle \qquad n=-s,\ldots,s$$

s=1/2 reduces to previously considered case.

Define a **2-parameter family** of transfer matrices by

$$\tau_s(\lambda)^{\beta_1 \ldots \beta_N}_{\alpha_1 \ldots \alpha_N} = (\mathcal{L}_s)^{a_1 a_2}_{\alpha_1 \beta_1} \, (\mathcal{L}_s)^{a_2 a_3}_{\alpha_2 \beta_2} \ldots (\mathcal{L}_s)^{a_N a_1}_{\alpha_N \beta_N}$$



Transfer matrices **commute** $\quad [\tau_S(\lambda), \tau_{S'}(\mu)] = 0$

Generating function for **conservation laws:**

$$X_s(\lambda) = \left(\frac{\sinh((s+\frac{1}{2})\eta + i\lambda)}{\sinh(-(s+\frac{1}{2})\eta + i\lambda)} \right) \tau_s\big(-\frac{\eta}{2} + i\lambda\big) \tau_s'\big(\frac{\eta}{2} + i\lambda\big)$$

Conservation laws:

$$H_s^{(n+1)} \equiv \frac{1}{n!} \frac{\partial^n}{\partial \lambda^n}\bigg|_{\lambda=0} X_s(\lambda), \;\; s = \frac{1}{2}, 1, \frac{3}{2}, \ldots$$

For s=1/2 these are **local** in the sense previously defined.

For s>1/2 these are **quasi-local:**

$$H_{s\geq 1}^{(n)} = \sum_j f^{(2)}_{\alpha_1\alpha_2} \sigma_j^{\alpha_1} \sigma_{j+1}^{\alpha_2} + f^{(3)}_{\alpha_1\alpha_2\alpha_3} \sigma_j^{\alpha_1} \sigma_{j+1}^{\alpha_2} \sigma_{j+2}^{\alpha_3}$$
$$+ f^{(4)}_{\alpha_1\alpha_2\alpha_3\alpha_4} \sigma_j^{\alpha_1} \sigma_{j+1}^{\alpha_2} \sigma_{j+2}^{\alpha_3} \sigma_{j+3}^{\alpha_4} + \ldots$$

$f^{(k)}_{\alpha_1\alpha_2\ldots\alpha_k}$ decay sufficiently fast with k s.t. conservation laws are extensive

Recall that our Hilbert space has dimension 2^N

Inner product on the space of operators:

$$(A,B) \equiv \langle A^\dagger B\rangle \ , \ \langle A\rangle \equiv 2^{-N}\mathrm{Tr}(A)$$

An operator Q is quasi-local if

(1) $(Q - \langle Q\rangle, Q - \langle Q\rangle) \propto N$

(2) (Q, B_k) asymptotically independent of N for all B_k that act non-trivially only on k sites

"Complete" Generalized Gibbs Ensemble

Density matrix:

$$\rho_{\mathrm{GGE}} = \frac{1}{Z_{\mathrm{GGE}}} \exp\left(- \sum_{n,s} \lambda_{s,n} H_s^{(n)} \right)$$

$\lambda_{s,n}$ fixed by initial conditions

$$\lim_{N\to\infty} \frac{\langle \Psi(0)| H_s^{(n)} |\Psi(0)\rangle}{N} = \lim_{N\to\infty} \frac{1}{N} \mathrm{Tr}\left(\rho_{\mathrm{GGE}}\; H_s^{(n)} \right)$$

We have shown that

1. Initial conditions fix a unique steady state

$$\{\rho_n^{p,h}(\lambda)|n=1,2,...\}$$

($H^{(n)}{}_s$ with higher s fix hole densities of longer strings)

Set of conservation laws is complete!

2. **Explicit** determination of the steady state requires

$$\Omega_s(\lambda)=\langle\Psi(0)|X_s(\lambda)|\Psi(0)\rangle$$

At the moment possible for simple (low entanglement) initial states and s not too large (say 100 for product states).

3. For simple initial states we can calculate short-distance spin-spin correlation functions in the steady-state to extremely high precision.

Conclusions

1. Have constructed the GGE describing the steady state after a quench to an **interacting**, integrable model, the XXZ chain.

2. Involves **quasi-local** conservation laws (QLCL).

3. There are many more QLCL than the "traditional" local ones.

4. Construction readily generalizable to other integrable models.

5. Notion of "truncated GGE" appears to remain viable.

Fagotti&Essler '13

"Diagonal Ensemble"

energy eigenstates: $H|\psi_n\rangle = E_n|\psi_n\rangle$

density matrix: $$\rho_{\rm diag} = \sum_n |\langle\Psi(0)|\psi_n\rangle|^2 \; |\psi_n\rangle\langle\psi_n|$$