Waiting for rare entropic fluctuations in stochastic thermodynamics

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KS, Dhar, arXiv:1504.02187

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4. Mean residence time and integral FT

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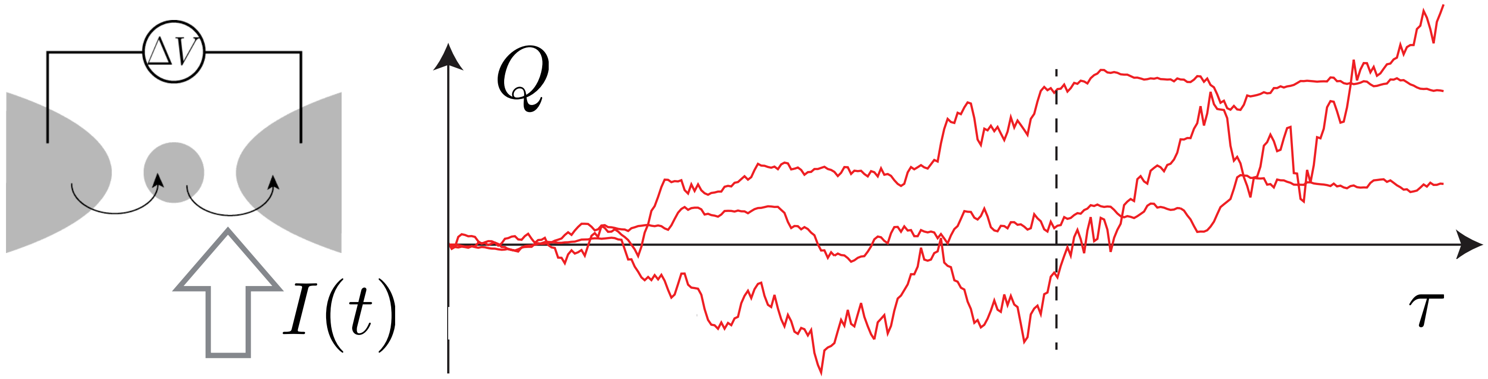
1. Counting statistics

◇ Measuring charge transfer



measure $I(t)$

accumurate $Q = \int_0^\tau dt I(t)$

◇ Statistics given at the "fixed" time

Probability $P(Q)$

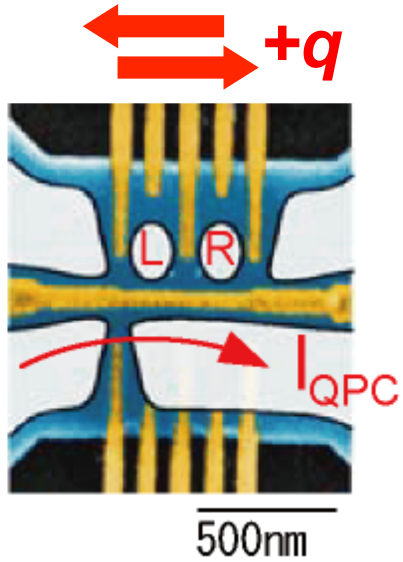
Cumulants $\langle Q^n \rangle_c$

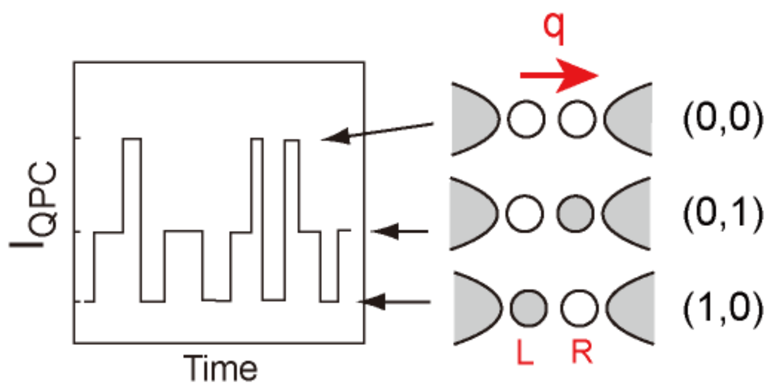
◇ One expects "information" from "fixed time statistics"

Examples of experiment

T. Fujisawa et al., Science (2006)

◇ Classical transport via coupled QDs

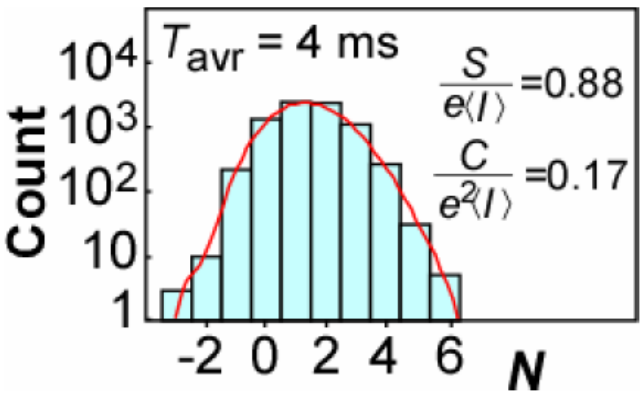




(1,0) → (0,1) forward tunneling

(0,1) → (1,0) backward tunneling

◇ Distribution of transmitted charge



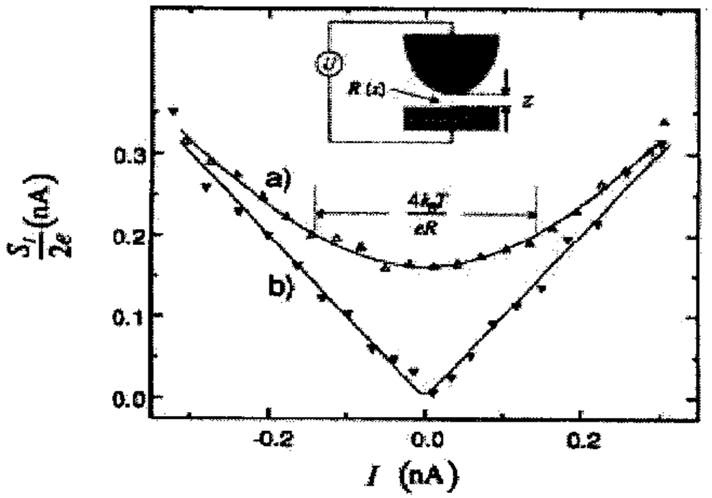
Information from “fixed time statistics”

◇ Zero temperature - Shot Noise -

Average current $I_1 \ \propto T$

Current noise $\quad I_2 \ \propto T(1-T) \sim T \qquad \text{if } T \ll 1$

Fano factor $\quad F = \dfrac{I_2}{I_1} = 2e \qquad$ “Noise is the signal”



◇ Finite temperature ?

Fluctuation relation

Fluctuation relation at the finite temperature

◇ Robust relation derived from time reversal symmetry

- Current context

$$P(-Q) \sim e^{-Q\beta\Delta V} P(Q)$$

- Entropy context (general)

$$P(-S) = e^{-S} P(S)$$

◇ This reproduces linear response results and predicts nonlinear response

Def. $\langle Q^n \rangle_c / \tau := \sum_k L_{n,k} (\beta \Delta V)^k / k!$

- FDT (Kubo formula) $L_{1,1} = L_{2,0}/2$
- Nonlinear response, e.g., $L_{1,2} = L_{2,1}$

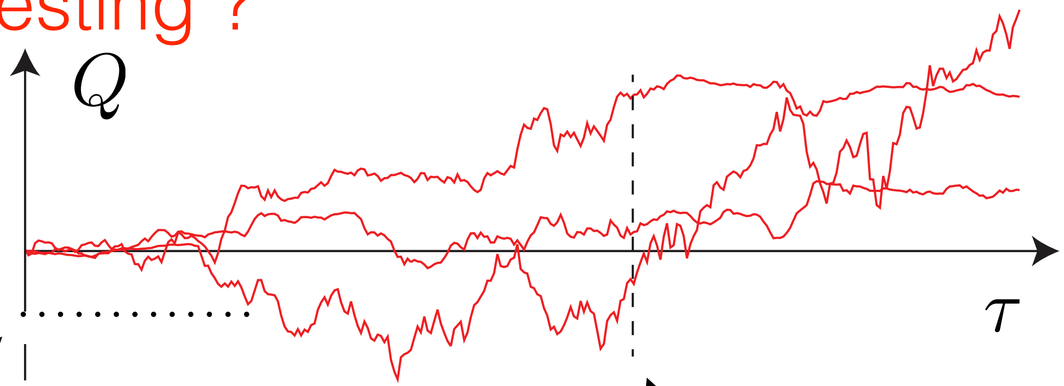
Today's talk

2. From fixed time to fixed Q statistics

◇ So far, statistics at the fixed time

- Questions -

What is fixed Q statistics ?
How formulated ?
Relation between fixed time and fixed Q physics ?
How interesting ?



target Q

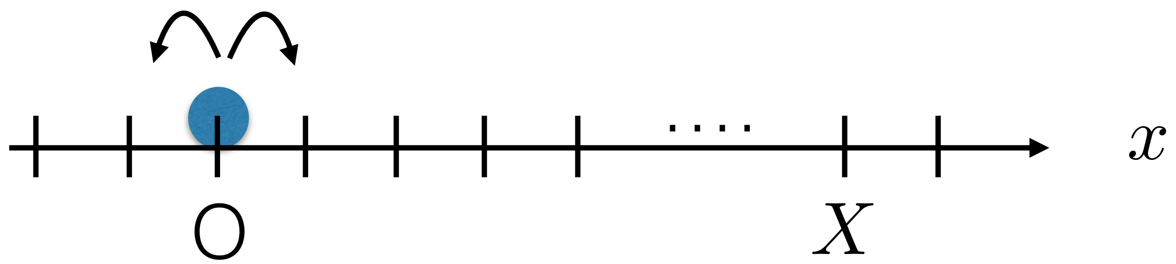
fixed Q statistics fixed time statistics

◇ Mathematically unambiguous statistics

First passage time distribution (FPTD) to get Q

The simplest FPTD: random walk

◇ Biased random walk



◇ Distribution at large time

$$P(x,t) = \frac{1}{\sqrt{2\pi I_2 t}} e^{-\frac{(x-I_1 t)^2}{2I_2 t}} \qquad I_n = \left\langle \left(\int_0^\tau dt\, x(t) \right)^n \right\rangle_c \Big/ \tau \Big|_{\tau\to\infty}$$

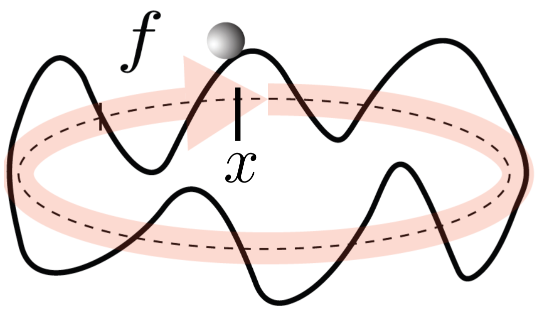
◇ First passage time distribution (FPTD) to reach X

$$\mathcal{F}_{rw}(X,t) = \frac{|X| e^{-\frac{(X-I_1 t)^2}{2I_2 t}}}{\sqrt{2\pi I_2 t^3}} \quad \to e^{-\frac{I_1^2}{2I_2} t - \frac{3}{2}\log t}$$

$$\text{If } I_1 > 0 \qquad \int_0^\infty dt\, \mathcal{F}_{rw}(X,t) \begin{cases} = 1 & \text{for } X > 0 \\ < 1 & \text{for } X < 0 \end{cases}$$

Several models for the FPTD

(a) Driven colloidal particle



Equation of motion

$$\gamma \dot{x} = -\frac{\partial U(x)}{\partial x} + f + \eta(t)$$

$$\langle \eta(t)\eta(t') \rangle = 2\gamma k_B T \delta(t - t')$$

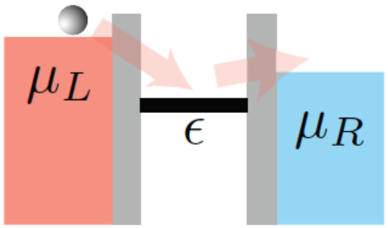
Entropy produced = $\beta Q = \beta \int_0^{\tau} dt (\gamma \dot{x} - \eta(t))$

$$= \beta \left[f \int_0^{\tau} dt\, \dot{x} - (U(x(\tau)) - U(x(0))) \right]$$

Experiments
S. Toyabe et al., Nature Physics(2010)
V. Blickle et al., PRL (2007)

Target: winding number

(b) Charge transfer vi QDs



Entropy produced = $n\beta\Delta V$

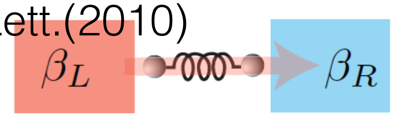
Target: charge transfer

Experiments
T. Fujisawa et al., Science (2006)
B. Kung et al. ,Phys. Rev. X (2012)

(c) Heat transfer

Entropy produced = $Q(1/T_R - 1/T_L)$

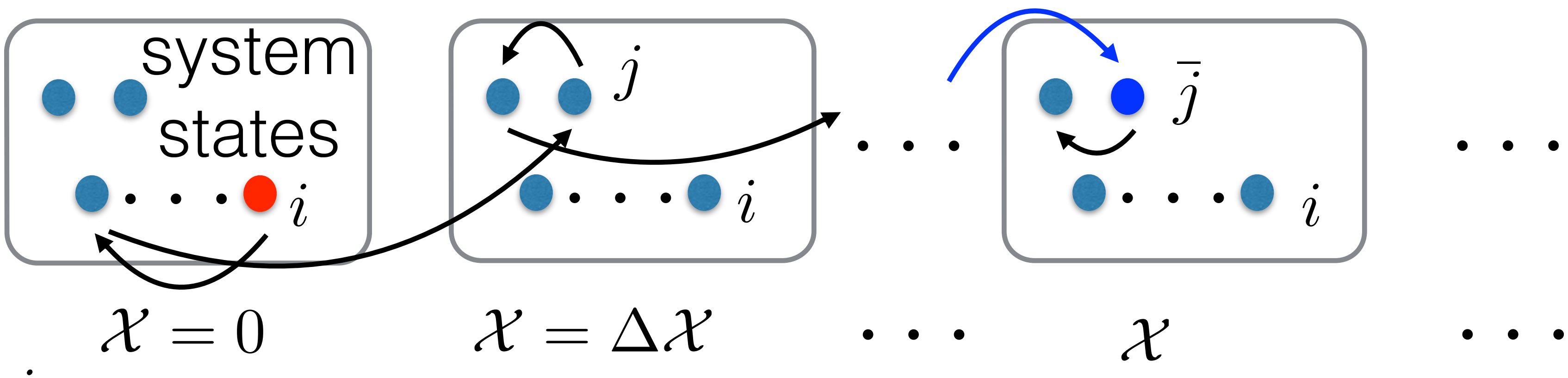
J. R. Gomez-Solano, Europhys Lett.(2010)
S. Ciliberto et al., PRL (2013)



Target: heat transfer

3. Renewal type equation for first passages

◇Framework to reach entropic variable $\mathcal{X}$



i ● : Initial state

$\bar{j}$ ● : "Entrance state" to reach $\mathcal{X}$ for the first time

◇Renewal type of basic relation

$$T_{(\bar{j},\mathcal{X})\leftarrow(i,\mathcal{X}=0)}(t) = \sum_{\bar{j}'} \int_0^t du\, T_{(\bar{j},\mathcal{X}=0)\leftarrow(\bar{j}',\mathcal{X}=0)}(t-u)\, \mathcal{F}_{(\bar{j}',\mathcal{X})\leftarrow(i,\mathcal{X}=0)}(u)$$

transition prob.
$(\bar{j},\mathcal{X}) \leftarrow (i,\mathcal{X}=0)$

FPTD for $\mathcal{X}$ via $\bar{j}'$

◇Laplace transformation

$$\mathcal{F}_{(\bar{j},\mathcal{X})\leftarrow(i,0)}(s) = \sum_{\bar{j}'} [\boldsymbol{T}^{-1}]_{(\bar{j},0)\leftarrow(\bar{j}',0)}(s)\, T_{(\bar{j}',\mathcal{X})\leftarrow(i,0)}(s)$$

Example with driven colloidal system

$$T_{(\bar{j},\mathcal{X})\leftarrow(i,\mathcal{X}=0)}(t) = \sum_{\bar{j}'}\int_0^t du\, T_{(\bar{j},\mathcal{X}=0)\leftarrow(\bar{j}',\mathcal{X}=0)}(t-u)\,\mathcal{F}_{(\bar{j}',\mathcal{X})\leftarrow(i,\mathcal{X}=0)}(u)$$

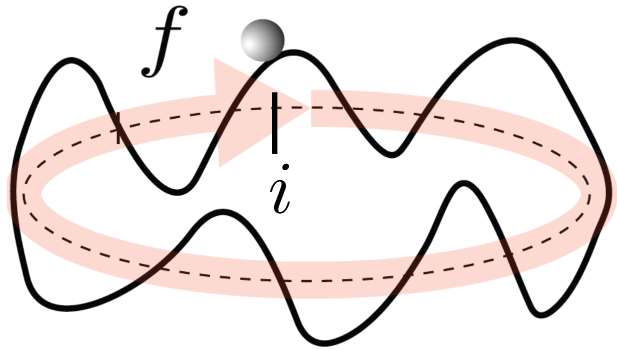
◇ Take winding number as $\mathcal{X}$

$\mathcal{X} = \mathcal{N}$ ⇨ unique entrance state

i

$\bar{j} = i$

$\bar{j}' = i$



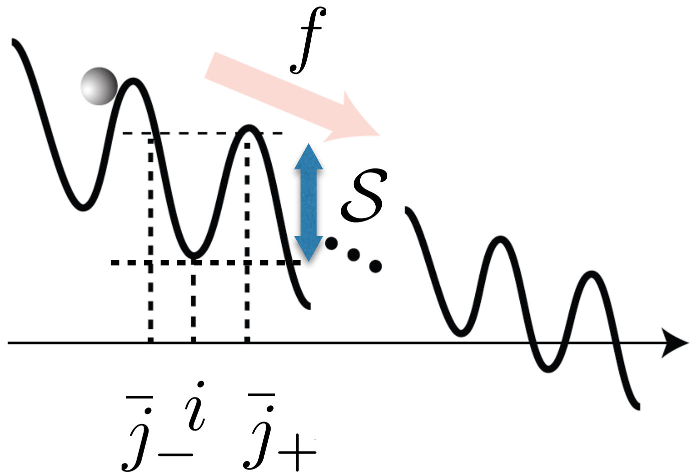
◇ Take bath's entropy as $\mathcal{X}$

$\mathcal{X} = \mathcal{S}$ (negative)

⇨ two entrance states

i

$\bar{j}' = \bar{j}_+$ and $\bar{j}_-$



The FPTD in the driven colloidal particle (model a)

Entropic variable: winding number

◇ Formal solution

$$\mathcal{F}_{(i,\mathcal{N})\leftarrow(i,0)}(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\, \frac{T_{(i,\mathcal{N})\leftarrow(i,0)}(s)}{T_{(i,0)\leftarrow(i,0)}(s)}\, e^{st}$$

$$\mathcal{F}(\mathcal{N},t) = \sum_i \mathcal{F}_{(i,\mathcal{N}\leftarrow(i,0))}(t)\, p_i^{SS}$$

◇ $T_{(i,\mathcal{N})\leftarrow(i,0)}$?

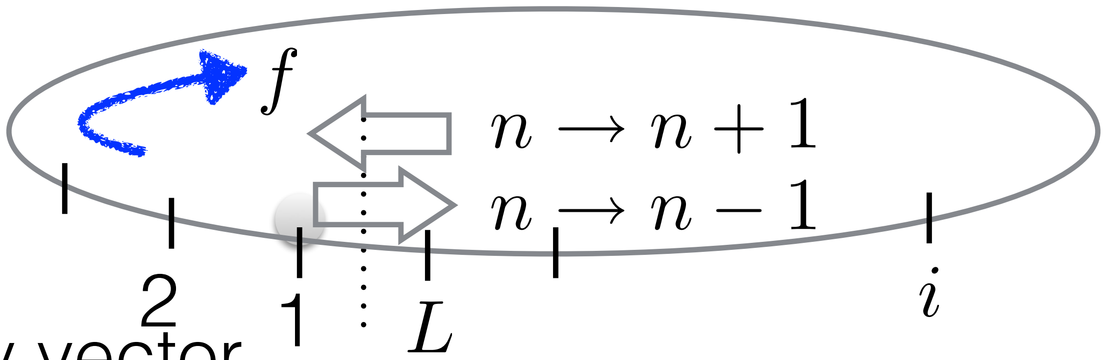
- Master equation

$$\dot{P}_j(t) = \sum_{j'=j\pm1} W_{j\leftarrow j'} P_{j'}(t) + W_{j,j} P_j(t)$$

local detailed balance

$$\frac{W_{i+1\leftarrow i}}{W_{i\leftarrow i+1}} = e^{\beta(U_i - U_{i+1} + f)}$$

- Counting the number of passing through the line: n



- Define the probability vector

$$\boldsymbol{P}(n,t) = \{P_1(n,t), P_2(n,t), \cdots, P_L(n,t)\}$$

$$\frac{\partial \boldsymbol{P}(n,t)}{\partial t} = \boldsymbol{W}_- \boldsymbol{P}(n-1,t) + \boldsymbol{W}_+ \boldsymbol{P}(n+1,t) + \boldsymbol{W}_0 \boldsymbol{P}(n,t)$$

- Solution in Laplace space

$$\boldsymbol{T}(\mathcal{N},s) = \begin{cases} A_+ z_+^{-\mathcal{N}} \boldsymbol{V}^+ & \text{for } \mathcal{N} > 0 \\ A_- z_-^{-\mathcal{N}} \boldsymbol{V}^- & \text{for } \mathcal{N} < 0 \\ \boldsymbol{V}^0 & \text{for } \mathcal{N} = 0 \end{cases}$$

fluctuation relation

$$z_+ z_- = e^{-\beta f L}$$

Two results

1. Asymptotics

$$\mathcal{F}_{(i,\mathcal{N})\leftarrow(i,0)}(t) \sim \int ds\, e^{t\ \left[(\mathcal{N}/t) - \ln z_+(s)\right]}$$

$$s - \mu(\log z_+(s)) = 0$$

→ General expression in terms of cumulants

2. Mean residence time and integral FT in model (a)

1. Asymptotics

$$\mathcal{F}_{\mathrm{asym}}(\mathcal{X},t) = A(\mathcal{X}) \exp\left(-\Gamma t - (3/2)\log t\right)$$

$$\Gamma = \frac{I_1^2}{2I_2} + \frac{I_3 I_1^3}{6I_2^3} + \frac{(3I_3^2 - I_2 I_4)I_1^4}{24 I_2^5} + \cdots \qquad I_k = \langle \mathcal{X}^k \rangle_c / \tau \Big|_{\tau\to\infty}$$

More precisely

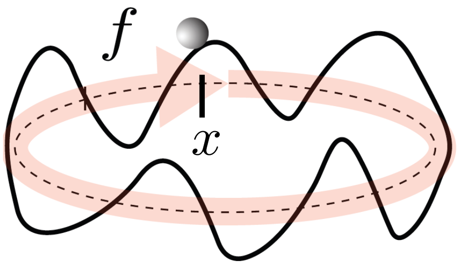
$$\Gamma = \sum_{n=0}^{\infty} \frac{(-I_1)^{n+2}}{(n+2)!}\, q_n(\xi)|_{\xi=0} \qquad q_n(\xi) = \left(\left(\frac{d^2\mu(\xi)}{d\xi^2}\right)^{-1} \frac{d}{d\xi} \right)^n \left(\frac{d^2\mu(\xi)}{d\xi^2}\right)^{-1}$$

$\mu(\xi)$ the cumulant generating function

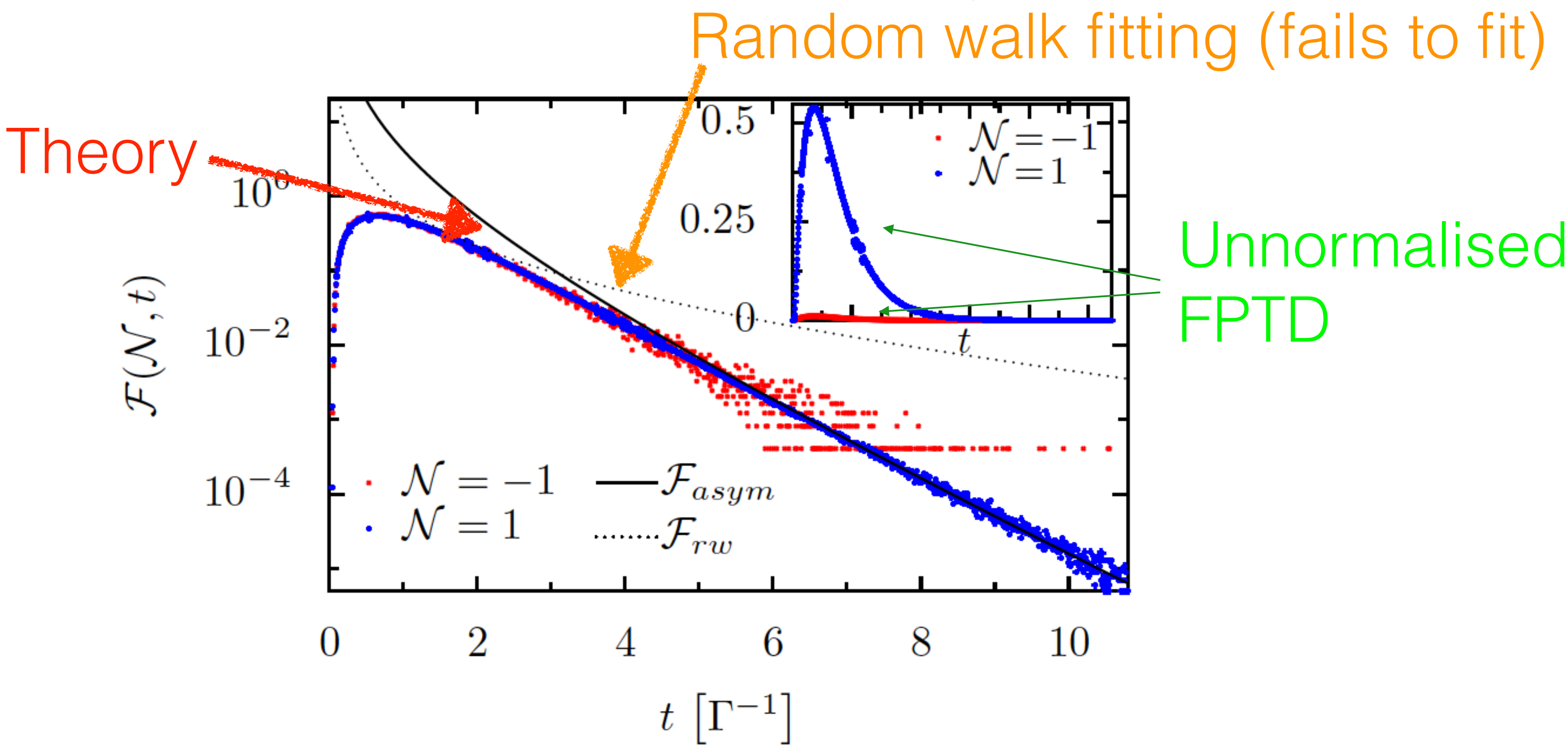
$$\langle \mathcal{X}^n \rangle_c = \partial^n \mu(\xi)/\partial \xi^n \Big|_{\xi=0}$$

1. Asymptotic behavior does not depend on the target values
   (even negative entropy follows the same form)
2. Relaxation rate is written with cumulants
3. First order reproduces random walk picture
   valid for linear response $(\text{small } I_1)$
4. (3/2)log t correction

Numerical demonstration

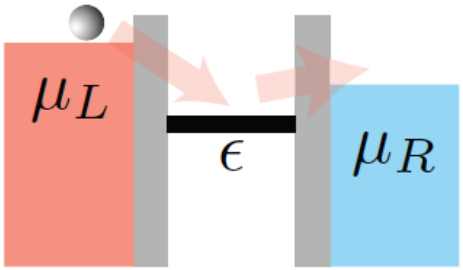


◇ Normalized FPTD for winding number

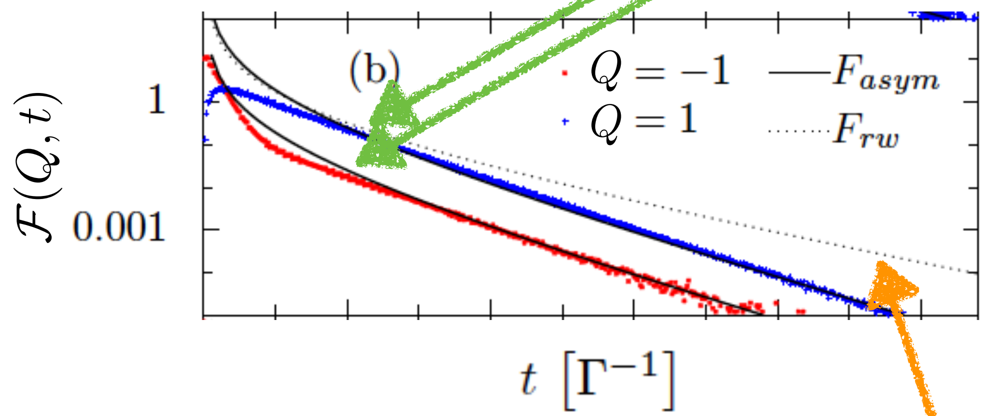


Numerical demonstration

(b) FPTD for charge transfer

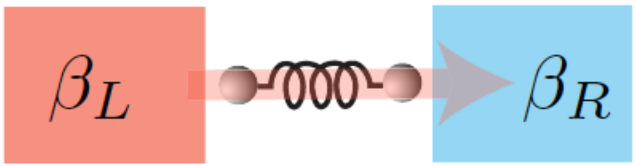


Theory

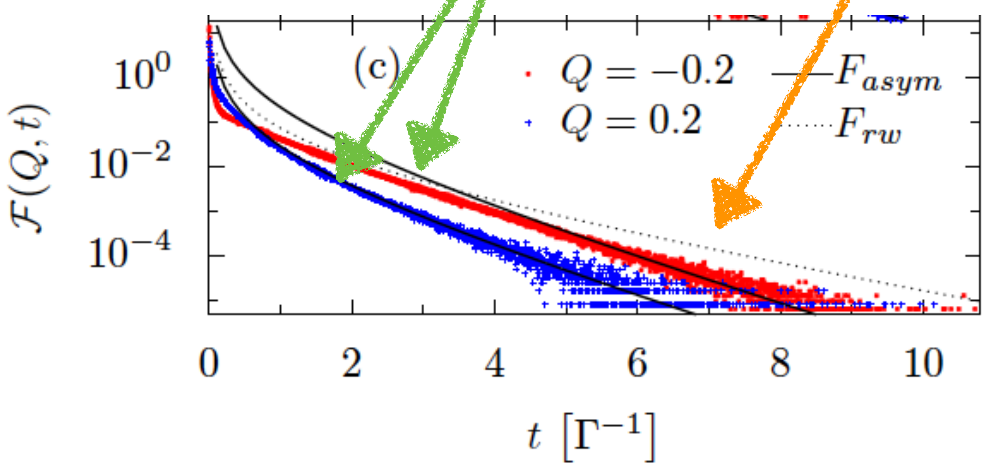


(c) FPTD for heat transfer

random walk fitting



Theory



2. Mean residence time expression and integral FT

Statement:

Model (a): colloidal particle in the ring geometry

◇ Exact expression of mean residence time

$$\int_0^\infty dt\, T_{(i,S=0)\leftarrow(i,S=0)}(t) = \frac{p_i^{SS}}{J}$$

◇ Integral FT in terms of first passage

$$\langle e^{-S_{\rm tot}} \rangle_{{\rm FP}:S} = 1$$

All first passage trajectories to get S (S<0)

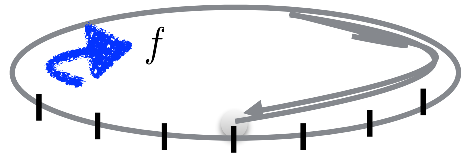
- Usual definition

Total entropy $S_{\rm tot}$ = system's entropy + bath's entropy

$S_{\rm sys}$ S

Formula on mean residence time

◇ Mean residence time



$$\int_0^\infty dt\, T_{(i,S=0)\leftarrow(i,S=0)}(t) = \frac{p_i^{SS}}{J}$$

return probability

steady sate distribution

steady sate current

Remark on this formula

1) In equlibrium case, it diverges, as we know.

2) The formula includes equilibrium result.

3) Residence time is connected to
the steady state as well as steady state current.

Integral FT in terms of first passages

◇ Mean residence formula + basic equation leads to integral FT in terms of first passages

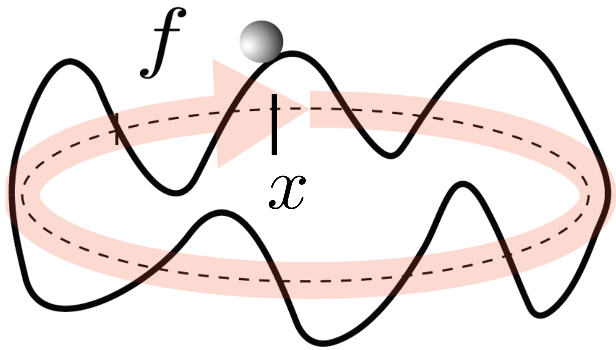
$$T_{(\bar{j},S)\leftarrow(i,S=0)}(t) = \sum_{\bar{j}'=\bar{j}_\pm} \int_0^t du\, T_{(\bar{j},0)\leftarrow(\bar{j}',0)}(t-u)\, \mathcal{F}_{(\bar{j}',S)\leftarrow(i,0)}(u)$$

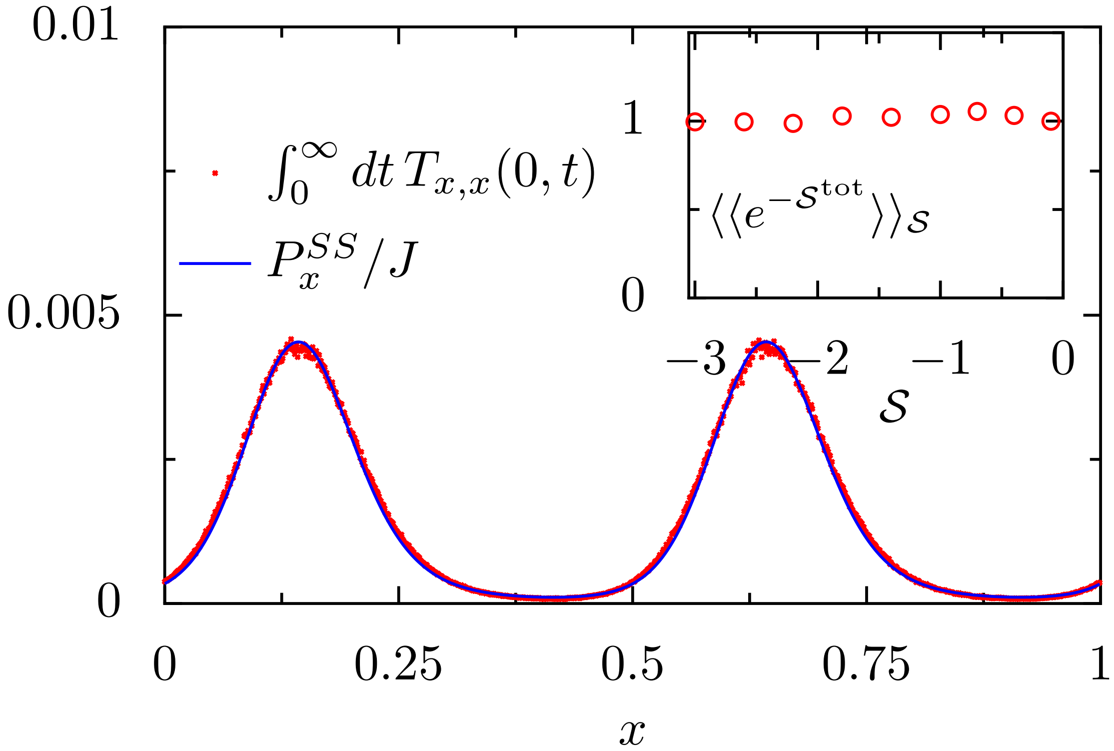
$$S^{\mathrm{tot}}_{j\leftarrow i} = \ln(p_i^{SS}/p_j^{SS}) + S$$

$$\langle e^{-S^{tot}}\rangle_{\mathrm{FP:S}} = \int_0^\infty dt \sum_{\bar{j}} e^{-S^{\mathrm{tot}}_{\bar{j}\leftarrow i}} \mathcal{F}_{(j,S)\leftarrow(i,S=0)}(t)\, p_i^{SS} = 1$$

All first passage trajectories to get S (S<0)

Numerical demonstration

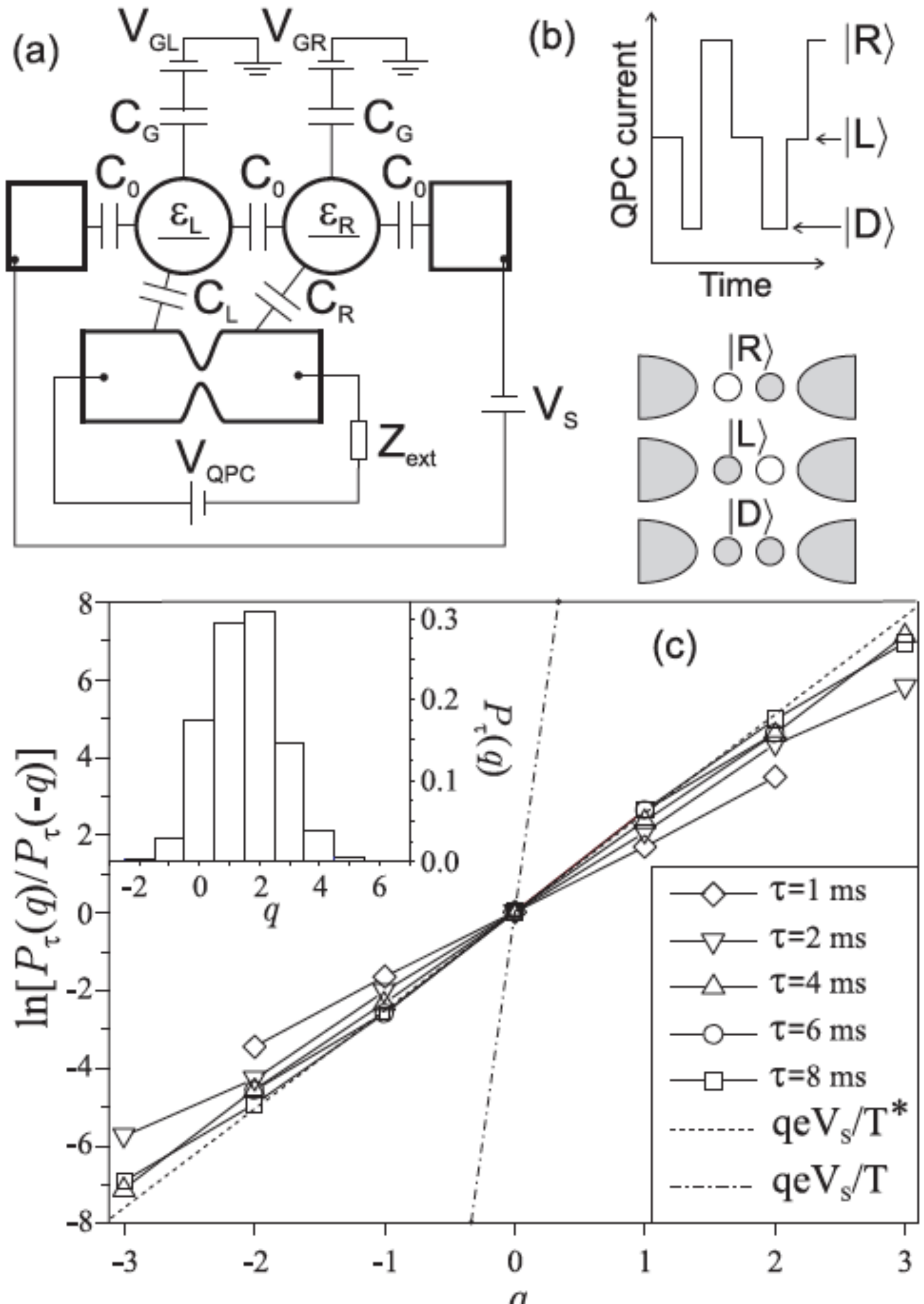




Summary

- ◇ We considered fixed target value statistics
- ◇ The first passage time distribution was studied (FPTD)
- ◇ Basic equation on first passages are considered
- ◇ Asymptotic behaviour has universal expression
- ◇ Exact mean residence time expression was derived
- ◇ Integral FT in terms of first passage exists for the model (a).
  Validity for the other models is open problem

Thank you for attention !



(a)
V_GL
V_GR
C_G
C_G
C_0
C_0
C_0
ε_L
ε_R
C_L
C_R
V_S
Z_ext
V_QPC
(b)
QPC current
Time
|R⟩
|L⟩
|D⟩
|R⟩
|L⟩
|D⟩
(c)
ln[P_τ(q)/P_τ(-q)]
q
P_τ(q)
τ=1 ms
τ=2 ms
τ=4 ms
τ=6 ms
τ=8 ms
qeV_s/T*
qeV_s/T