On the growth of interfaces: dynamical scaling and beyond

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Overview:

1. Physical ageing & interface growth
2. Interface growth & KPZ universality class
3. Interface growth on semi-infinite substrates
4. A spherical model of interface growth: the (first) Arcetri model
5. Linear responses and extensions of dynamical scaling
6. Form of the scaling functions & LSI
7. Conclusions
1. Physical ageing & interface growth

known & practically used since prehistoric times (metals, glasses)
systematically studied in physics since the 1970s
⇒ discovery: ageing effects reproducible & universal!
occur in widely different systems
(structural glasses, spin glasses, polymers, simple magnets, . . .)

**Three defining properties of ageing:**

1. slow relaxation (non-exponential!)
2. **no** time-translation-invariance (TTI)
3. dynamical scaling without fine-tuning of parameters

Cooperative phenomenon, far from equilibrium

**Question**: what can be learned about intrisically irreversible systems by studying their ageing behaviour?
\[ t = t_1 \]

\[ t = t_2 > t_1 \]

magnet \( T < T_c \)

\[ \longrightarrow \text{ordered cluster} \]

magnet \( T = T_c \)

\[ \longrightarrow \text{correlated cluster} \]

growth of ordered/correlated domains, of typical linear size

\[ L(t) \sim t^{1/z} \]

dynamical exponent \( z \) : determined by equilibrium state
Interface growth

deposition (evaporation) of particles on a substrate

→ height profile $h(t, r)$

slope profile $u(t, r) = \nabla h(t, r)$

$p = \text{deposition prob.}$

$1 - p = \text{evap. prob.}$

Questions:

* average properties of profiles & their fluctuations?
* what about their relaxational properties?
* are these also examples of physical ageing?

? does dynamical scaling always exist? ? are there extensions?
Analogies between magnets and growing interfaces

Common properties of critical and ageing phenomena:

* collective behaviour,
  very large number of interacting degrees of freedom
* algebraic large-distance and/or large-time behaviour
* described in terms of universal critical exponents
* very few relevant scaling operators
* justifies use of extremely simplified mathematical models
  with a remarkably rich and complex behaviour
* yet of experimental significance

see talks by T. Sasamoto and K. Takeuchi at this conference
Magnets
thermodynamic equilibrium state
order parameter \( \phi(t, r) \)
phase transition, at critical temperature \( T_c \)
variance:
\[
\langle (\phi(t, r) - \langle \phi(t) \rangle)^2 \rangle \sim t^{-2\beta/(\nu z)}
\]
relaxation, after quench to \( T \leq T_c \)
autocorrelator
\[
C(t, s) = \langle \phi(t, r) \phi(s, r) \rangle_c
\]
Interfaces
growth continues forever
height profile \( h(t, r) \)
same generic behaviour throughout
roughness:
\[
w(t)^2 = \langle (h(t, r) - \bar{h}(t))^2 \rangle \sim t^{2\beta}
\]
relaxation, from initial substrate:
autocorrelator
\[
C(t, s) = \langle (h(t, r) - \bar{h}(t))(h(s, r) - \bar{h}(s)) \rangle
\]
ageing scaling behaviour:

when \( t, s \to \infty \), and \( y := t/s > 1 \) fixed, expect, with \( \left\{ \begin{array}{l}
\text{waiting time } s \\
\text{observation time } t > s
\end{array} \right. \)

\[
C(t, s) = s^{-b} f_C(t/s) \quad \text{and} \quad f_C(y) \overset{y \to \infty}{\sim} y^{-\lambda_C/z}
\]
b, \( \beta \), \( \nu \) and dynamical exponent \( z \) : universal & related to stationary state
autocorrelation exponent \( \lambda_C \) : universal & independent of stationary exponents
Magnets

exponent value $b = \begin{cases} 0 & ; \ T < T_c \\ 2\beta/\nu z & ; \ T = T_c \end{cases}$

Interfaces

exponent value $b = -2\beta$

models:

(a) gaussian field

$\mathcal{H}[\phi] = -\frac{1}{2} \int \text{d} \mathbf{r} (\nabla \phi)^2$

(b) Ising model

$\mathcal{H}[\phi] = -\frac{1}{2} \int \text{d} \mathbf{r} \left[ (\nabla \phi)^2 + \tau \phi^2 + \frac{g}{2} \phi^4 \right]$

such that $\tau = 0 \leftrightarrow T = T_c$

dynamical Langevin equation (Ising):

$$\partial_t \phi = -D \frac{\delta \mathcal{H}[\phi]}{\delta \phi} + \eta$$

$$= D \nabla^2 \phi + \tau \phi + g \phi^3 + \eta$$

$\eta(t, \mathbf{r})$ is the usual white noise, $\langle \eta(t, \mathbf{r}) \eta(t', \mathbf{r}') \rangle = 2T \delta(t - t') \delta(\mathbf{r} - \mathbf{r}')$

phase transition exactly solved $d = 2$

relaxation exactly solved $d = 1$

Onsager ’44, Glauber ’63, ...

Sasamoto & Spohn ’10

Calabrese & Le Doussal ’11, ...

(a) Edwards-Wilkinson (EW):

$$\partial_t h = \nu \nabla^2 h + \eta$$

(b) Kardar-Parisi-Zhang (KPZ):

$$\partial_t h = \nu \nabla^2 h + \frac{\mu}{2} (\nabla h)^2 + \eta$$
2. Interface growth & KPZ class

deposition (evaporation) of particles on a substrate $\rightarrow$ height profile $h(t,r)$
generic situation: RSOS (restricted solid-on-solid) model

\begin{align*}
\partial_t h &= \nu \nabla^2 h + \frac{\mu}{2} (\nabla h)^2 + \eta \\
\partial_t h &= \nu \nabla^2 h + \eta
\end{align*}

some universality classes:

\begin{align*}
\text{(a) KPZ} & \quad \partial_t h = \nu \nabla^2 h + \frac{\mu}{2} (\nabla h)^2 + \eta \\
\text{(b) EW} & \quad \partial_t h = \nu \nabla^2 h + \eta
\end{align*}

$\eta$ is a gaussian white noise with $\langle \eta(t,r)\eta(t',r') \rangle = 2\nu T \delta(t-t')\delta(r-r')$
**Family-Viscek scaling** on a spatial lattice of extent $L^d$: 

$$\overline{h}(t) = L^{-d} \sum_j h_j(t)$$

**two-time correlator:**

$$C(t, s; r) = \langle (h(t, r) - \langle \overline{h}(t) \rangle) (h(s, 0) - \langle \overline{h}(s) \rangle) \rangle = s^{-b} F_C \left( \frac{t}{s}, \frac{r}{s^{1/z}} \right)$$

with ageing exponent: $b = -2\beta$

**rigorous bound:** $\lambda_C \geq (d + zb)/2$

**KPZ class**, to all orders in perturbation theory $\lambda_C = d$, if $d < 2$

$w^2(t; L) = \frac{1}{L^d} \sum_{j=1}^{L^d} \langle (h_j(t) - \overline{h}(t))^2 \rangle = L^{2\alpha} f (tL^{-z}) \sim \begin{cases} L^{2\alpha} ; & \text{if } tL^{-z} \gg 1 \\ t^{2\beta} ; & \text{if } tL^{-z} \ll 1 \end{cases}$

$\beta$: growth exponent, $\alpha$: roughness exponent, $\alpha = \beta z$

**autocorrelation exponent**

$$F_C(y, 0) \sim y^{-\lambda_c/z}$$

**rigorous bound**

$$\lambda_C \geq (d + zb)/2$$

**KPZ class**, to all orders in perturbation theory $\lambda_C = d$, if $d < 2$
1D relaxation dynamics, starting from an initially flat interface

observe all 3 properties of ageing: slow dynamics, no TTI, dynamical scaling

confirm simple ageing for the 1D KPZ universality class

confirm expected exponents $b = -2/3$, $\lambda_C / z = 2/3$

pars pro toto

Kallabis & Krug 96; Krech 97; Bustingorry et al. 07-10; Chou & Pleimling 10;
D’Aquila & Täuber 11/12; mh, Noh, Pleimling 12 . . .
**Experiment** : **universality** of interface exponents, KPZ class

<table>
<thead>
<tr>
<th>model/system</th>
<th>$d$</th>
<th>$z$</th>
<th>$\beta$</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>KPZ</strong></td>
<td>1</td>
<td>3/2</td>
<td>1/3</td>
<td>1/2</td>
</tr>
<tr>
<td>Ag electrodeposition</td>
<td>1</td>
<td>$\approx$ 1/3</td>
<td>$\approx$ 1/2</td>
<td></td>
</tr>
<tr>
<td>slow paper cumbustion</td>
<td>1</td>
<td>1.44(12)</td>
<td>0.32(4)</td>
<td>0.49(4)</td>
</tr>
<tr>
<td>liquid crystal (flat)</td>
<td>1</td>
<td>1.34(14)</td>
<td>0.32(2)</td>
<td>0.43(6)</td>
</tr>
<tr>
<td>liquid crystal (circular)</td>
<td>1</td>
<td>1.44(10)</td>
<td>0.334(3)</td>
<td>0.48(5)</td>
</tr>
<tr>
<td>cell colony growth</td>
<td>1</td>
<td>1.56(10)</td>
<td>0.32(4)</td>
<td>0.50(5)</td>
</tr>
<tr>
<td>(almost) isotrope colloïds</td>
<td>1</td>
<td></td>
<td>0.37(4)</td>
<td>0.51(5)</td>
</tr>
<tr>
<td>autocatalytic reaction front</td>
<td>1</td>
<td>1.45(11)</td>
<td>0.34(4)</td>
<td>0.50(4)</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1.63(3)</td>
<td>0.2415(15)</td>
<td>0.393(4)</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1.63(2)</td>
<td>0.241(1)</td>
<td>0.393(3)</td>
</tr>
<tr>
<td>CdTe/Si(100) film</td>
<td>2</td>
<td>1.61(5)</td>
<td>0.24(4)</td>
<td>0.39(8)</td>
</tr>
<tr>
<td><strong>EW</strong></td>
<td>2</td>
<td>0(log)</td>
<td>0(log)</td>
<td></td>
</tr>
<tr>
<td>sedimentation/electrodispersion</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Experimental results from **several groups**, since 1999 (**mainly** since 2010)
3. Interface growth on semi-infinite substrates

properties of growing interfaces near to a boundary?
→ crystal dislocations, face boundaries . . .

Experiments: Family-Vicsek scaling not always sufficient
→ distinct global and local interface fluctuations

\{ anomalous scaling, growth exponent \( \beta \) larger than expected

grainy interface morphology, facetting

! analyse simple models on a semi-infinite substrate!
frame co-moving with average interface deep in the bulk
characterise interface by

\[
\begin{align*}
\text{height profile} & \quad \langle h(t, r) \rangle \\
\text{width profile} & \quad w(t, r) = \left\langle \left[ h(t, r) - \langle h(t, r) \rangle \right]^2 \right\rangle^{1/2}
\end{align*}
\]

\[ h \to 0 \text{ as } |r| \to \infty \]
specialise to $d = 1$ space dimensions; boundary at $x = 0$, bulk $x \to \infty$

cross-over for the phenomenological growth exponent $\beta$ near to boundary

bulk behaviour $w \sim t^\beta$
‘surface behaviour’ $w_1 \sim t^{\beta_1}$?

cross-over, if causal interaction with boundary

experimentally observed, e.g. for semiconductor films

values of growth exponents (bulk & surface):

$\beta = 0.25$ $\beta_{1,\text{eff}} \simeq 0.32$ Edwards-Wilkinson class

$\beta \simeq 0.32$ $\beta_{1,\text{eff}} \simeq 0.35$ Kardar-Parisi-Zhang class
simulations of RSOS models:
well-known bulk adsorption processes (& immediate relaxation)

description of immediate relaxation if particle is adsorbed at the boundary
explicit boundary interactions \( h_1(t) = \partial_x h(t, x)|_{x=0} \)

\[
\begin{align*}
(\partial_t - \nu \partial_x^2) h(t, x) - \frac{\mu}{2} (\partial_x h(t, x))^2 - \eta(t, x) &= \nu (\kappa_1 + \kappa_2 h_1(t)) \delta(x) \\
\end{align*}
\]

height profile \( \langle h(t, x) \rangle = t^{1/\gamma} \Phi \left( x t^{-1/z} \right) \), \( \gamma = \frac{z}{z - 1} = \frac{\alpha}{\alpha - \beta} \)

EW & exact solution, \( h(t, 0) \sim \sqrt{t} \) self-consistently

KPZ
Scaling of the width profile:

EW & exact solution $\lambda^{-1} = 4tx^{-2}$

bulk \hspace{1cm} boundary

**same** growth scaling exponents in the bulk and near to the boundary

large **intermediate scaling regime** with effective exponent (slopes)

agreement with **RG** for non-disordered, local interactions

? ageing behaviour near to a boundary ?
4. A spherical model of interface growth: the Arcetri model

\[ \text{KPZ} \rightarrow \text{intermediate model} \rightarrow \text{EW} \]

preferentially exactly solvable, and this in \( d \geq 1 \) dimensions

**Inspiration:** mean spherical model of a ferromagnet

| Berlin & Kac 52 |
| Lewis & Wannier 52 |

Ising spins \( \sigma_i = \pm 1 \)

spherical spins \( S_i \in \mathbb{R} \)

spherical constraint \( \langle \sum_i S_i^2 \rangle = N \)

Obey \( \sum_i \sigma_i^2 = N = \# \text{ sites} \)

Hamiltonian \( \mathcal{H} = -J \sum_{(i,j)} S_i S_j - \lambda \sum_i S_i^2 \)

Lagrange multiplier \( \lambda \)

Exponents non-mean-field for \( 2 < d < 4 \) and \( T_c > 0 \) for \( d > 2 \)

Kinetics from Langevin equation

\[ \partial_t \phi = -D \frac{\delta \mathcal{H}[\phi]}{\delta \phi} + \zeta(t) \phi + \eta \]

Time-dependent Lagrange multiplier \( \zeta(t) \) fixed from spherical constraint

All equilibrium and ageing exponents exactly known, for \( T < T_c \) and \( T = T_c \)

Ronca 78, Coniglio & Zannetti 89, Cugliandolo, Kurchan, Parisi 94, Godrèche & Luck ’00, Corberi, Lippiello, Fusco, Gonnella & Zannetti 02-14
consider **RSOS/ASEP**-adsorption process:

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**rigorous**: continuum limit gives KPZ

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Bertini & Giacomin 97

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use *not* the heights $h_n(t) \in \mathbb{N}$ on a discrete lattice, but rather the slopes $u_n(t) = \frac{1}{2} (h_{n+1}(t) - h_{n-1}(t)) = \pm 1$

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RSOS

---

? let $u_n(t) \in \mathbb{R}$, & impose a spherical constraint $\sum_n \langle u_n(t)^2 \rangle = \mathcal{N}$

---

? consequences of the ‘hardening’ of a soft EW-interface by a ‘spherical constraint’ on the $u_n$?
Arcetri model: precise formulation & simple ageing

slope \( u(t, x) = \partial_x h(t, x) \) obeys Burgers’ equation, replace its non-linearity by a mean spherical condition

\[
\partial_t u_n(t) = \nu (u_{n+1}(t) + u_{n-1}(t) - 2u_n(t)) + \zeta(t)u_n(t) + \frac{1}{2} (\eta_{n+1}(t) - \eta_{n-1}(t))
\]

\[
\sum_n \langle u_n(t)^2 \rangle = N \quad \langle \eta_n(t)\eta_m(s) \rangle = 2T\nu \delta(t - s)\delta_{n,m}
\]

Extension to \( d \geq 1 \) dimensions:

define gradient fields \( u_a(t, r) := \nabla_a h(t, r), \)

\[
\partial_t u_a(t, r) = \nu \nabla_r \cdot \nabla_r u_a(t, r) + \zeta(t)u_a(t, r) + \nabla_a \eta(t, r)
\]

\[
\sum_r \sum_{a=1}^d \langle u_a(t, r)^2 \rangle = dN^d
\]

interface height: \( \hat{u}_a(t, q) = i \sin q_a \hat{h}(t, q) \quad ; \quad q \neq 0 \) in Fourier space
exact solution:

\[
\hat{h}(t, q) = \hat{h}(0, q) e^{-2t\omega(q)} \sqrt{\frac{1}{g(t)}} + \int_0^t d\tau \, \hat{\eta}(\tau, q) \sqrt{\frac{g(\tau)}{g(t)}} e^{-2(t-\tau)\omega(q)}
\]

in terms of the auxiliary function \( g(t) = \exp \left( -2 \int_0^t d\tau \, \tilde{z}(\tau) \right), \) which satisfies Volterra equation

\[
g(t) = f(t) + 2T \int_0^t d\tau \, g(\tau)f(t-\tau), \quad f(t) := d \frac{e^{-4t} l_1(4t)}{4t} \left( e^{-4t} l_0(4t) \right)^{d-1}
\]

* for \( d = 1 \), identical to ‘spherical spin glass’, with \( T = 2T_{SG} \):

  hamiltonian \( \mathcal{H} = -\frac{1}{2} \sum_{i,j} J_{ij} S_i S_j \); \( J_{ij} \) random matrix, its eigenvalues distributed according to Wigner’s semi-circle law

  \[\text{Cugliandolo & Dean 95}\]

* also related to distribution of first gap of random matrices \[\text{Perret & Schehr 15/16}\]

* for \( 2 < d < 4 \), scaling functions identical to the ones of the critical bosonic pair-contact process with diffusion, with rates

  \[
  \Gamma[2A \rightarrow (2 + k)A] = \Gamma[2A \rightarrow (2 - k)A] = \mu, \quad k = 1, 2
  \]

  \[\text{Howard & Täuber 97; Houchmandzadeh 02; Paessens & Schütz 04; Baumann, mh, Pleimling, Richert 05}\]
phase transition: long-range correlated surface growth for $T \leq T_c$

$$\frac{1}{T_c(d)} = \frac{1}{2} \int_0^\infty dt \ e^{-dt} t^{-1} I_1(t) I_0(t)^{d-1} ; \quad T_c(1) = 2, \ T_c(2) = \frac{2\pi}{\pi - 2}$$

Some results: always simple ageing

1. $T = T_c$, $d < 2$:
   - rough interface, width $w(t) = t^{(2-d)/4} \implies \beta = \frac{2-d}{4} > 0$
   - ageing exponents $a = b = \frac{d}{2} - 1$, $\lambda_R = \lambda_C = \frac{3d}{2} - 1$; $z = 2$

   **exponents $z, \beta, a, b$ same as EW, but exponent $\lambda_C = \lambda_R$ different**

2. $T = T_c$, $d > 2$:
   - smooth interface, width $w(t) = \text{cste.} \implies \beta = 0$
   - ageing exponents $a = b = \frac{d}{2} - 1$, $\lambda_R = \lambda_C = d$; $z = 2$

   **same asymptotic exponents as EW, but scaling functions are distinct**

3. $T < T_c$:
   - rough interface, width $w^2(t) = (1 - T/T_c)t \implies \beta = \frac{1}{2}$
   - ageing exponents $a = \frac{d}{2} - 1$, $b = -1$, $\lambda_R = \lambda_C = \frac{d-2}{2}$; $z = 2$
Illustration: Shape of the height Fluctuation-Dissipation Ratio, \( T = T_c \)

\[ X(t, s) := TR(t, s) \frac{\partial C(t, s)}{\partial s} = X \left( \frac{t}{s} \right) \xrightarrow{t/s \to \infty} X_\infty = \begin{cases} 
\frac{d}{d+2} ; & 0 < d < 2 \\
\frac{d}{4} ; & 2 < d
\end{cases} \]

lim FDR \( X_\infty \) is universal

\[ X_{\infty} \]

\( X_{\infty} \) is distinct from \( X_{\text{EW}, \infty} = 1/2 \) for all \( d > 0 \)

green line: \( X_{\text{EW}} \) for \( d = 4 \)
Summary of results in the (first) Arcetri model:

Captures at least some qualitative properties of growing interfaces.

* phenomenology of relaxation analogous to domain growth in simple magnets $\Rightarrow$ dynamical scaling form of simple ageing

* existence of a critical point $T_c(d) > 0$ for all $d > 0$ as a magnet

* at $T = T_c$, rough interface for $d < 2$, smooth interface for $d > 2$; upper critical dimension $d^* = 2$

* at $T = T_c$, $d < 2$, the stationary exponents $(\beta, z)$ are those of EW, but the non-stationary ageing exponents are different

  explicit example for expectation from field-theory renormalisation group in domain growth of independent exponents $\lambda_{C,R}$

  different from EW and KPZ classes, where $\lambda_C = d$ for all $d < 2$ [Krech 97]

* at $T = T_c$, $d > 2$, distinct from EW, although all exponents agree

* for $d = 1$, equivalent to $p = 2$ spherical spin glass

* at $T = T_c$ and $2 < d < 4$, same ageing behaviour as at the multicritical point of the bosonic pair-contact process with diffusion (BPCPD)

* for $T < T_c$, distinct universality class
5. Linear responses and extensions of dynamical scaling

extend Family-Viscek scaling to two-time responses:

analogue: TRM integrated response in magnetic systems

two-time integrated response:

* sample \( A \) with deposition rates \( p_i = p \pm \epsilon_i \), up to time \( s \),
* sample \( B \) with \( p_i = p \) up to time \( s \);

then switch to common dynamics \( p_i = p \) for all times \( t > s \)

\[
\chi(t, s; r) = \int_0^s \mathrm{d}u \ R(t, u; r) = \frac{1}{L} \sum_{j=1}^{L} \left\langle \frac{h^{(A)}_{j+r}(t; s) - h^{(B)}_{j+r}(t)}{\epsilon_j} \right\rangle = s^{-a} F_\chi \left( \frac{t}{s}, \frac{|r|^z}{s} \right)
\]

with \( a \): ageing exponent

expect for \( y = t/s \gg 1 \): \( F_R(y, 0) \sim y^{-\lambda_R/z} \) autoresponse exponent

? Values of these exponents ?
Effective action of the KPZ equation:

\[ \mathcal{J} [\phi, \tilde{\phi}] = \int dt dr \left[ \tilde{\phi} \left( \partial_t \phi - \nu \nabla^2 \phi - \frac{\mu}{2} \left( \nabla \phi \right)^2 \right) - \nu T \tilde{\phi}^2 \right] \]

\[ \Rightarrow \text{Very special properties of KPZ in } d = 1 \text{ spatial dimension!} \]

Exact critical exponents: \( \beta = 1/3, \alpha = 1/2, z = 3/2, \lambda_C = 1 \)  

kpz 86; Krech 97

related to precise symmetry properties:

A) tilt-invariance (Galilei-invariance)

kept under renormalisation!

\[ \Rightarrow \text{exponent relation } \alpha + z = 2 \]

holds for any dimension \( d \)

Forster, Nelson, Stephen 77

Medina, Hwa, Kardar, Zhang 89

B) time-reversal invariance

special property in 1D, where also \( \alpha = \frac{1}{2} \)

Lvov, Lebedev, Paton, Procaccia 93
Frey, Täuber, Hwa 96
Special KPZ symmetry in 1D: let \( v = \frac{\partial \phi}{\partial r}, \, \tilde{\phi} = \frac{\partial}{\partial r} (\tilde{p} + \frac{v}{2T}) \)

\[
\mathcal{J} = \int dt \, dr \left[ \tilde{p} \frac{\partial t}{\alpha} v - \frac{v}{4T} (\partial_r v)^2 - \frac{\mu}{2} v^2 \partial_r \tilde{p} + v T (\partial_r \tilde{p})^2 \right]
\]

is invariant under time-reversal

\[
t \mapsto -t, \quad v(t, r) \mapsto -v(-t, r), \quad \tilde{p} \mapsto \tilde{p}(-t, r)
\]

⇒ fluctuation-dissipation relation for \( t \gg s \)

\[
TR(t, s; r) = -\partial_r^2 C(t, s; r)
\]

distinct from the equilibrium FDT \( TR(t - s) = \partial_s C(t - s) \) Kubo

Combination with ageing scaling, gives the ageing exponents:

\[
\lambda_R = \lambda_C = 1 \quad \text{and} \quad 1 + a = b + \frac{2}{z}
\]
relaxation of the integrated response, 1D

observe all 3 properties of ageing:

- slow dynamics
- no TTI
- dynamical scaling

exponents $a = -1/3$, $\lambda_R/z = 2/3$, as expected from FDR

N.B.: numerical tests for 2 models in KPZ class
Simple ageing is also seen in space-time observables

correlator \( C(t, s; r) = s^{2/3} F_C \left( \frac{t}{s}, \frac{r^{3/2}}{s} \right) \)

integrated response \( \chi(t, s; r) = s^{1/3} F_{\chi} \left( \frac{t}{s}, \frac{r^{3/2}}{s} \right) \)

confirm \( z = 3/2 \)
Question: Are there model-independent results on the form of universal scaling functions?

‘Natural’ starting point: try to draw analogies with conformal invariance at equilibrium

⇒ ‘normally’ works for sufficiently ‘local’ theories

What about time-dependent critical phenomena?

Theorem: Consideration of the ‘deterministic part’ of the Janssen-de Dominicis action permits to reconstruct the full time-dependent responses and correlators, from the dynamical symmetries of the ‘deterministic part’.

essential tool: Bargman superselection rule of ‘deterministic part’
Time-dependent critical phenomena & ageing

Characterised by dynamical exponent $z : t \mapsto tb^{-z}$, $r \mapsto rb^{-1}$

? Can one extend to local dynamical scaling, with $z \neq 1$?

For $z = 2$, example of the Schrödinger group:

$\begin{align*}
    t &\mapsto \frac{\alpha t + \beta}{\gamma t + \delta}, \\
    r &\mapsto \frac{Dr + vt + a}{\gamma t + \delta}; \quad \alpha \delta - \beta \gamma = 1
\end{align*}$

⇒ study ageing phenomena as paradigmatic example

**essential**: (i) absence of TTI & (ii) Galilei-invariance

Transformation $t \mapsto t'$ with $\beta(0) = 0$ and $\dot{\beta}(t') \geq 0$ and

$t = \beta(t')$, $\phi(t) = \left(\frac{d\beta(t')}{dt'}\right)^{-x/z} \left(\frac{d \ln \beta(t')}{dt'}\right)^{-2\xi/z} \phi'(t')$

out of equilibrium, have 2 distinct scaling dimensions, $x$ and $\xi$.

mean-field for magnets: expect $\\begin{cases} 
    \xi = 0 \text{ in ordered phase } T < T_c \\
    \xi \neq 0 \text{ at criticality } T = T_c
\end{cases}$

NB: if TTI (equilibrium criticality), then $\xi = 0$. 

Dynamical symmetry I: Schrödinger algebra \( \mathfrak{sch}(d) \)

Dynamical symmetries of Langevin equation (deterministic part !)

Schrödinger operator in \( d \) space dimensions:

\[
S = 2\mathcal{M} \partial_t - \partial_r \cdot \partial_r
\]

(free) Schrödinger/heat equation

(noiseless) Edwards-Wilkinson equation

\[
\begin{align*}
[S, \mathcal{Y}_{\pm 1/2}] &= [S, \mathcal{M}_0] = [S, X_{-1}] = [S, \mathcal{R}] = 0 \\
[S, X_0] &= -S \\
[S, X_1] &= -2tS + 2\mathcal{M}\left(x - \frac{d}{2}\right)
\end{align*}
\]

Infinitesimal change:

\[
\delta \phi = \varepsilon \mathcal{X} \phi, \quad \mathcal{X} \in \mathfrak{sch}(d), |\varepsilon| \ll 1
\]

Lemma: If \( S\phi = 0 \) and \( x = x_\phi = \frac{d}{2} \), then \( S(\mathcal{X} \phi) = 0 \).

\( \mathfrak{sch}(d) \) maps solutions of \( S\phi = 0 \) onto solutions.
Dynamical symmetry II: ageing algebra $\text{age}(d)$

1D Schrödinger operator: 

$$S = 2\mathcal{M} \partial_t - \partial_r^2 + 2\mathcal{M} \left(x + \xi - \frac{1}{2}\right) t^{-1}$$

Generalised ‘Schrödinger equation’:

Extra potential term arises in several models, without time-translations (e.g. 1D Glauber-Ising, spherical & Arcetri models)

If time-translations ($X_{-1} = -\partial_t$) are included, then $\xi = 0$

$$[S, Y_{\pm 1/2}] = [S, M_0] = 0$$

$$[S, X_0] = -S$$

$$[S, X_1] = -2tS$$

Infinitesimal change: $\delta \phi = \varepsilon \mathcal{X} \phi$, $\mathcal{X} \in \text{age}(d), |\varepsilon| \ll 1$

Lemma: If $S\phi = 0$, then $S(\mathcal{X} \phi) = 0.$

Niederer '74; mh & Stoimenov '11

$\text{age}(d)$ maps solutions of $S\phi = 0$ onto solutions.
Example for the $t^{-1}$-term in Langevin eq.: Arcetri model

Continuous slopes $u_i \in \mathbb{R}^d$, constraint $\sum_{i \in \Lambda} u_i^2 = dN$

For $d > 0$ phase transition $T_c(d) > 0$, exponents not mean-field if $d < 2$

Spherical constraint: $\langle \sum_{i \in \Lambda} u_i^2 \rangle = dN$

Langevin equation, with Lagrange multiplier $\zeta(t)$ & centered gaussian noise $\eta_i(t)$

\[
\frac{\partial u_a(t, r)}{\partial t} = \nu \Delta u_a(t, r) + \zeta(t) u_a(t, r) + \partial_a \eta(t, r) , \quad \langle \eta(t, r) \eta(s, r') \rangle = 2\nu T \delta(t - s) \delta(r - r')
\]

Set $g(t) := \exp \left( 2 \int_0^t dt' \zeta(t') \right)$, spherical constraint gives Volterra eq.

\[
g(t) = f(t) + 2T \int_0^t d\tau \, f(t - \tau) g(\tau) , \quad f(t) = \frac{d e^{-4t} I_1(4t)}{4t} \left( e^{-4t} I_0(4t) \right)^{d-1}
\]

Find for $T \leq T_c$ : $g(t) \xrightarrow{t \to \infty} t^{-F} \iff \zeta(t) \sim \frac{F}{2} t^{-1}$

Quite analogous to spherical model of a ferromagnet

Godrèche & Luck 00
Picone & MH 04
Schrödinger- & ageing-covariant two-point functions

Two-point function

\[ R = R(t, s; r_1 - r_2) := \langle \phi_1(t, r_1) \tilde{\phi}_2(s, r_2) \rangle \]

Each \( \phi_i \) characterized by (i) scaling dimensions \( x_i, \xi_i \) (ii) mass \( \mathcal{M}_i \)

* from Schrödinger-invariance

\[
R(t, s, r) = r_0 \delta_{x_1, x_2} s^{-1-a} \left( \frac{t}{s} - 1 \right)^{-1-a} \exp \left[ -\frac{\mathcal{M}_1}{2} \frac{r^2}{t-s} \right]
\]

* from ageing-invariance

\[
R(t, s; r) = r_0 s^{-1-a} \left( \frac{t}{s} \right)^{1+a'-\frac{\lambda_R}{2}} \left( \frac{t}{s} - 1 \right)^{-1-a'} \exp \left( -\frac{\mathcal{M}_1}{2} \frac{r^2}{t-s} \right)
\]

with

\[ 1 + a = \frac{x_1 + x_2}{2}, \quad a' - a = \xi_1 + \xi_2, \quad \lambda_R = 2(x_1 + \xi_1), \quad \mathcal{M}_1 + \mathcal{M}_2 = 0 \]

Bargman rule

can derive **causality condition** \( t > s \)

\( \Rightarrow \) \( R \) is physically a **response function**.
1D KPZ: find $R(t, s) = \langle \psi(t)\tilde{\psi}(s) \rangle$ from ‘logarithmic partner’ of order parameter $(\psi, \phi)$

scaling dimensions become Jordan matrices $\begin{pmatrix} x & x' \\ 0 & x \end{pmatrix}$, $\begin{pmatrix} \xi & \xi' \\ 0 & \xi \end{pmatrix}$ and similarly for response fields

* good collapse $\Rightarrow$ no logarithmic corrections $\Rightarrow$ $x' = \tilde{x}' = 0$

* no logarithmic factors for $y \gg 1 \Rightarrow \xi' = 0$

$\Rightarrow$ only $\tilde{\xi}' = 1$ remains

$$f_R(y) = y^{-\lambda R/z} \left(1 - \frac{1}{y}\right)^{-1-a'} \left[ h_0 - g_0 \ln \left(1 - \frac{1}{y}\right) - \frac{1}{2}f_0 \ln^2 \left(1 - \frac{1}{y}\right) \right]$$

find integrated autoresponse $\chi(t, s) = \int_0^s du \ R(t, u) = s^{1/3} f_\chi(t/s)$

$$f_{\chi}(y) = y^{1/3} \left\{ A_0 \left[ 1 - \left(1 - \frac{1}{y}\right)^{-a'} \right] + \left(1 - \frac{1}{y}\right)^{-a'} \left[ A_1 \ln \left(1 - \frac{1}{y}\right) + A_2 \ln^2 \left(1 - \frac{1}{y}\right) \right] \right\}$$

with free parameters $A_0, A_1, A_2$ and $a'$ — for the 1D KPZ class, use $\frac{\lambda R}{z} - a = 1$
non-log LSI with $a = a'$: deviations $\approx 20\%$

non-log LSI with $a \neq a'$: works up to $\approx 5\%$

log LSI: works better than $\approx 0.1\%$

<table>
<thead>
<tr>
<th>$R$</th>
<th>$a'$</th>
<th>$A_0$</th>
<th>$A_1$</th>
<th>$A_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle \phi \phi \rangle$ – LSI</td>
<td>$-0.500$</td>
<td>$0.662$</td>
<td>$0$</td>
<td>$0$</td>
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<tr>
<td>$\langle \phi \psi \rangle$ – $L^1$LSI</td>
<td>$-0.500$</td>
<td>$0.663$</td>
<td>$-6 \cdot 10^{-4}$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\langle \psi \psi \rangle$ – $L^2$LSI</td>
<td>$-0.8206$</td>
<td>$0.7187$</td>
<td>$0.2424$</td>
<td>$-0.09087$</td>
</tr>
</tbody>
</table>

logarithmic LSI fits data at least down to $y \simeq 1.01$, with $a' - a \approx -0.4873$ (can we make a conjecture?)
7. Conclusions

* long-time dynamics of growing interfaces naturally evolves towards dynamical scaling & ageing
* phenomenology very similar to ageing phenomena in simple magnets
* subtleties in the precise scaling forms & space-dependent profiles
* shape of two-time response functions compatible with extended forms of dynamical scaling, according to LSI
* in certain cases logarithmic contributions in the scaling functions (but \textit{without} logarithmic corrections to scaling) :

\[ \implies \text{implications for interpretation of numerical data for the 2D KPZ, where } \lambda^C,\text{eff} \neq \lambda^R,\text{eff} \neq 2 \quad ? \]

\text{Halpin-Healy et al. 14, Ódor et al. 14}

proving dynamical symmetries can remain a delicate affair
Arcetri model, exact solution:

\[
\omega(q) = \sum_{a=1}^{d}(1 - \cos q_a), \quad q \neq 0
\]

\[
\hat{h}(t, q) = \hat{h}(0, q)e^{-2t\omega(q)}\sqrt{\frac{1}{g(t)}} + \int_0^t d\tau \, \hat{\eta}(\tau, q)\sqrt{\frac{g(\tau)}{g(t)}} e^{-2(t-\tau)\omega(q)}
\]

in terms of the auxiliary function \(g(t) = \exp\left(-2\int_0^t d\tau \, \hat{z}(\tau)\right)\),

which satisfies Volterra equation

\[
g(t) = f(t) + 2T \int_0^t d\tau \, g(\tau)f(t-\tau) , \quad f(t) := d \frac{e^{-4t}I_1(4t)}{4t} \left(e^{-4t}I_0(4t)\right)^{d-1}
\]

* for \(d = 1\), identical to ‘spherical spin glass’, with \(T = 2T_{SG}\) :

hamiltonian \(\mathcal{H} = -\frac{1}{2} \sum_{i,j} J_{ij}S_iS_j\); \(J_{ij}\) random matrix, its eigenvalues distributed according to Wigner’s semi-circle law

* also related to distribution of first gap of random matrices \(\text{Perret \\ & Schehr 15/16}\)

a further auxiliary function: \(F_r(t) := \prod_{a=1}^{d} e^{-2tI_{r_a}(2t)} \)

\(I_n\) : modified Bessel function

for initially uncorrelated heights and initially flat interface

\(\text{Cugliandolo \\ & Dean 95}\)
height autocorrelator:
\[ C(t, s) = \langle h(t, r) h(s, r) \rangle_c = \frac{2F_0(t+s)}{\sqrt{g(t)g(s)}} + \frac{2T}{\sqrt{g(t)g(s)}} \int_0^s d\tau \ g(\tau) F_0(t+s-2\tau) \]

interface width:
\[ w^2(t) = C(t, t) = \frac{2F_0(2t)}{g(t)} + \frac{2T}{g(t)} \int_0^t d\tau \ g(\tau) F_0(2t-2\tau) \]

slope autocorrelator:
\[ A(t, s) = \sum_{a=1}^d \langle u_a(t, r) u_a(s, r) \rangle_c = \frac{2f((t+s)/2)}{\sqrt{g(t)g(s)}} + \int_0^s d\tau \ \frac{2Tg(\tau)}{\sqrt{g(t)g(s)}} f((t+s)/2 - \tau) \]

height response:
\[ R(t, s; r) = \left. \frac{\delta \langle h(t, r) \rangle}{\delta j(s, 0)} \right|_{j=0} = \Theta(t-s) \sqrt{g(s)/g(t)} F_r(t-s) \]

slope autoresponse:
\[ Q(t, s; 0) = \Theta(t-s) \sqrt{g(s)/g(t)} f((t-s)/2) \]

* correspondence of 1D \( A/I \) model with spherical spin glass:
  - spins \( S_i \leftrightarrow \) slopes \( u_n \)

spin glass autocorrelator:
\[ C_{SG}(t, s) = \frac{1}{N} \sum_{i=1}^N \langle S_i(t) S_i(s) \rangle = A(t, s) \]

spin glass response:
\[ R_{SG}(t, s) = \sum_{i=1}^N \left. \frac{\delta \langle S_i(t) \rangle}{\delta h_i(s)} \right|_{h=0} = 2Q(t, s) \]

* kinetics of heights \( h_n(t) \) in model \( A/I \) driven by phase-ordering of the spherical spin glass \( \equiv \) 3D kinetic spherical model
Relationship with the critical diffusive bosonic pair-contact process (BPCPD)

* each site of a hypercubic lattice is occupied by $n_i \in \mathbb{N}_0$ particles
* single particles hop to a nearest-neighbour site with diffusion rate $D$
* on-site reactions, with rates $\Gamma[2A \rightarrow (2 + k)A] = \Gamma[2A \rightarrow (2 - k)A] = \mu$
  \[ k \text{ is either } 1 \text{ or } 2 \]

* control parameter $\alpha := \frac{k^2 \mu}{D}$

$\Rightarrow$ for $d > 2$, particles cluster on a few sites only, if $\alpha > \alpha_C$  

Figure: 2D section of BPCPD in $d = 3$; height of columns $\sim$ particle number
$\Rightarrow$ fluctuations grow with $t$ when $\alpha > \alpha_C$ & are bounded for $\alpha < \alpha_C$
bosonic creation operator \( a^\dagger(t, r) \), commutator \([a(t, r), a^\dagger(t', r')] = \delta(r - r')\)  
\(\implies\) average particle number is constant! 

\[
n(t, r) = \langle a^\dagger(t, r)a(t, r) \rangle = \langle a(t, r) \rangle = \rho_0 = \text{cste}.
\]

**clustering transition** at \( \alpha = \alpha_C \), characterised by changes in the variance.

\[
\bar{C}(t, s) := \langle a^\dagger(t, r)a(s, r) \rangle - \rho_0^2 \xrightarrow{t, s \to \infty} \langle n(t, r)n(s, r) \rangle - \rho_0^2 = s^{-b}f_C(t/s)
\]

\[
\bar{R}(t, s) := \left. \frac{\delta \langle a(t, r) \rangle}{\delta j(s, r)} \right|_{j=0} = s^{1-a}f_R(t/s)
\]

obey simple ageing for \( \alpha \leq \alpha_C \). Precisely at the clustering transition \( \alpha = \alpha_C \), for \( 2 < d < 4 \), the scaling functions are **identical**:

- **BPCPD** : \( b + 1 = a = d/2 - 1 \)
- **Arcetri** : \( b = a = d/2 - 1 \)

\[
f_{R,\text{BPCPD}}(y) = (y - 1)^{d-2} = f_{R,\text{Arc}}(y)
\]

\[
f_{C,\text{BPCPD}}(y) = (y + 1)^{-d/2}_2F_1 \left( \frac{d}{2}, \frac{d}{2}; \frac{d}{2} + 1; \frac{2}{1+y} \right) = f_{C,\text{Arc}}(y)
\]

**N.B.** : for \( d > 4 \), Arcetri \( \neq \) BPCPD \( \neq \) EW, although all exponents, up to \( b \), agree.