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Minimal Model of Stochastic Athermal Systems: Origin of Non-Gaussian Noise

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Outline of my talk

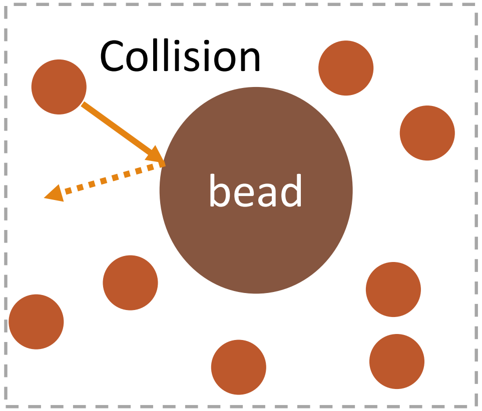
- Background
  - Experimental development for small systems
  - Why the Langevin Eq. is important in physics?
  - Toward the non-Gaussian type Langevin Eq.
- Main 1 : Systematic derivation of non-Gaussian Langevin Eq. & application to granular physics

  KK, T.G. Sano, T. Sagawa, H. Hayakawa, Phys. Rev. Lett. **114**, 090601 (2015)

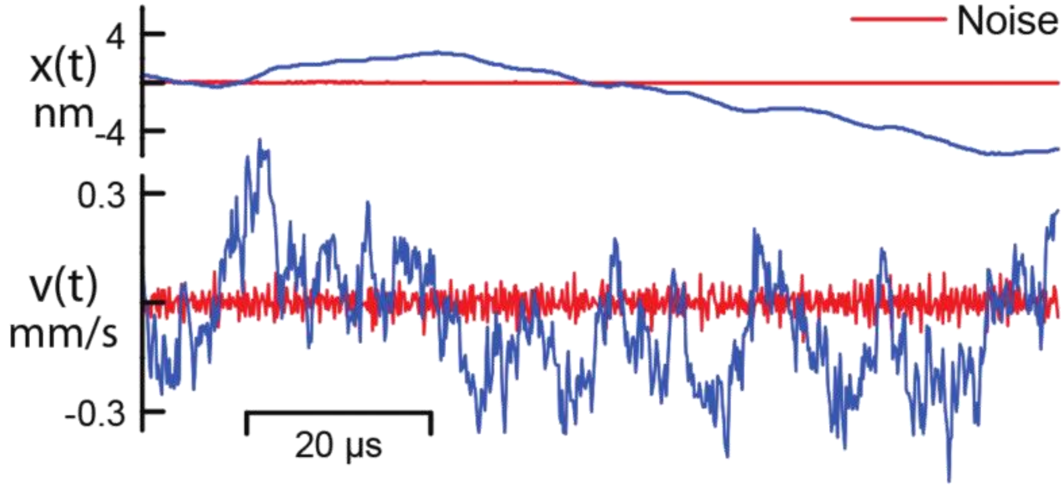
- Main 2 : Full-order solution of non-Gaussian Langevin Eq. with non-linear friction

  KK, T.G. Sano, T. Sagawa, H. Hayakawa, J. Stat. Phys. **160**, 1294 (2015)

Experimental development: Quantitative description of fluctuation

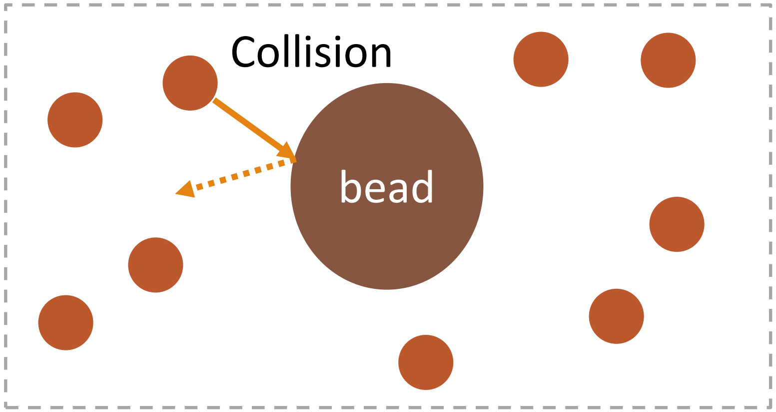


S. Kheifets *et al*., Science 343 (2014)



- Brownian motion: The motion of a small tracer in water
- Precise observation of fluctuation (e.g., instantaneous velocity)
  ←Thanks to the experimental development
- Modeling dynamics of small systems is an important issue

Minimal model of the dynamics of a small bead: the Langevin Equation



Fluctuation-dissipation relation

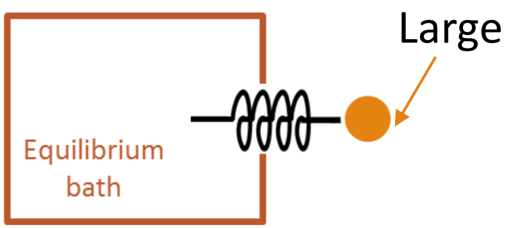
$$M\frac{d\hat{V}}{dt} = -\gamma\hat{V} + \sqrt{2\gamma T}\hat{\xi}^{\mathrm{G}}$$

inertia

White Gaussian noise

- Fundamental equation for a small bead in water (μm)
- Applicable to physics, chemistry, biology, economics
- Basis toward Thermodynamics for small systems

Why is the Langevin Equation important for mesoscopic systems?



$$M\frac{d\hat{V}}{dt} = -\gamma\hat{V} + \sqrt{2\gamma T}\hat{\xi}^{\mathrm{G}}$$

①Universality : Massive particle

- Mori-Zwanzig (Projector method)
- System size expansion
  (Perturbation of master equations,
  as will be discussed later…)

$$\frac{d\hat{V}}{dt} = -F(\hat{V}) + \sqrt{2\sigma^2}\hat{\xi}^{\mathrm{G}}$$

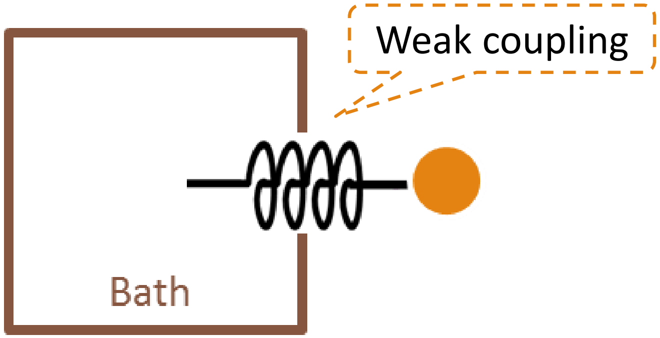
$$\Rightarrow P_{\mathrm{SS}} \propto e^{-\int_0^V \frac{dx F(x)}{\sigma^2}}$$

②Simplicity: Analytically solvable

- Specificity of the Gaussian noise
- Steady dist. is analytically obtained even for non-linear friction

⇒ **Minimal model with universality & simplicity**

Derivation of the Langevin Equation: the system size expansion (SSE)



$$\frac{d\hat{v}}{dt} = \varepsilon \hat{\eta}(t;\hat{v})$$

ε ・・・small parameter

$\hat{\eta}(t;\hat{v})$ ・・・Markov jump noise (ε-independent)

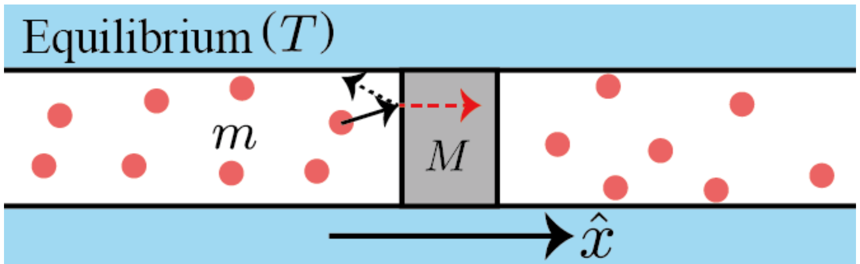
$\varepsilon \to 0$

$$M\frac{d\hat{V}}{dt} = -\gamma\hat{V} + \sqrt{2\gamma T}\hat{\xi}^{\mathrm{G}}$$

- Large system size limit (i.e., Ω-expansion)
- A single stochastic bath described by Markov jump noise
  →Weak coupling (e.g., ε =mass ratio $\propto 1/\Omega$)
- Universality in the weak coupling limit

⇨ Utilizing the kinetic theory and this expansion, the Langevin Eq. can be derived microscopically.

Example of the System Size Expansion (SSE): Rayleigh Piston



$$\frac{\partial P(V,t)}{\partial t} = \rho S \int_{-\infty}^{\infty} dv_x |v_x - V| \times$$
$$\{P(V',t)\phi_{\mathrm{eq}}(v_x') - P(V,t)\phi_{\mathrm{eq}}(v_x)\}$$

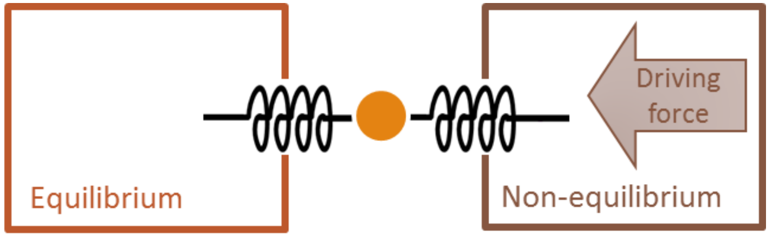
$\varepsilon \to 0$

$$M\frac{d\hat{V}}{dt} = -\gamma\hat{V} + \sqrt{2\gamma T}\hat{\xi}^{\mathrm{G}}$$

- Piston's motion in rarefied gas (linearized Boltzmann Eq.)
  - Markov jump process
    - Collision→discontinuous velocity jump of Piston
  - Strong environmental correlation (not even white noise)
    - V = large → energy outflow to gas
    - V = small → energy inflow from gas

- Massive piston limit ($\varepsilon \equiv \sqrt{m/M} \ll 1$)
  1. Correlation is renormalized only into viscosity
  2. Fluctuation is reduced to white noise
  3. Fluctuation is reduced to Gaussian (the CLT)

My interest: Athermal fluctuation Non-Gaussian Langevin Eq.



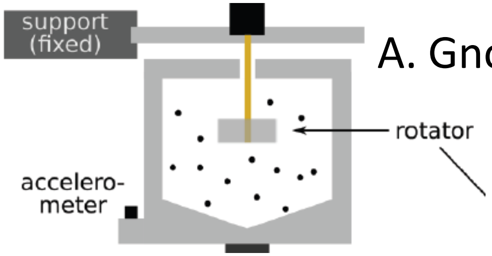
$$\frac{d\hat{V}}{dt} = -\gamma\hat{V} + \sqrt{2\gamma T}\hat{\xi}^{G} + \hat{\xi}^{NG}$$

White non-Gaussian noise

- **Athermal fluctuation**
  - Originating from non-eq. environments
  - Characterized by white non-Gaussian noise
  - Granular, electrical, biological systems
- **The conventional SSE is not valid…**

(even if higher-orders are taken into account…)

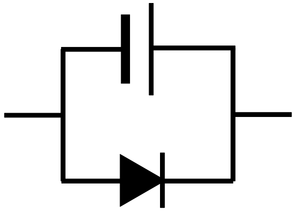
Example 1: Granular noise



A. Gnoli EPL 2014

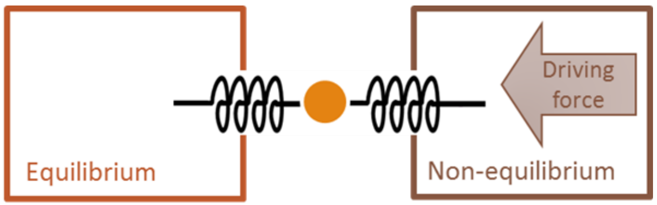
- Granular gas in non-eq.
- Non-eq. fluctuation

Example 2: Avalanche noise



- Reverse voltage on diodes
- Chain-reaction

The aim of this talk



$$\frac{d\hat{V}}{dt} = -\gamma\hat{V} + \sqrt{2\gamma T}\hat{\xi}^{G} + \hat{\xi}^{NG}$$

◆Main 1: Microscopic derivation of NGL Eq.

1. Systematic Derivation to leading order
2. Application to a granular system
3. What does non-Gaussianity means?

$$\frac{d\hat{V}}{dt} = -F(\hat{V}) + \sqrt{2\gamma T}\hat{\xi}^{G} + \hat{\xi}^{NG}$$

White but non-Gaussian

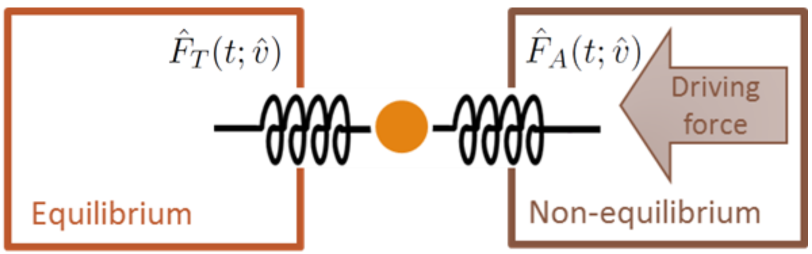
Solution : $P_{SS}(V) = ??$

◆Main 2 : Analytical simplicity

1. NGL Eq. with non-linear friction
2. Steady distribution for non-linear systems
3. Full-order perturbative solution

NGL Eq. has universality & simplicity, which is required as a minimal model.

Main 1:
Asymptotic derivation of NGL Eq.



$$\frac{d\hat{v}}{dt} = \hat{F}_T(t;\hat{v}) + \hat{F}_A(t;\hat{v})$$

$$\begin{cases} \hat{F}_T(t;\hat{v}) = -\gamma\hat{v} + \sqrt{2\gamma T}\hat{\xi}^{\mathrm{G}} \\ \hat{F}_A(t;\hat{v}) = \boxed{\varepsilon}\hat{\eta}_A(t;\hat{v}) \end{cases}$$

γ ・・・Thermal viscous coefficient

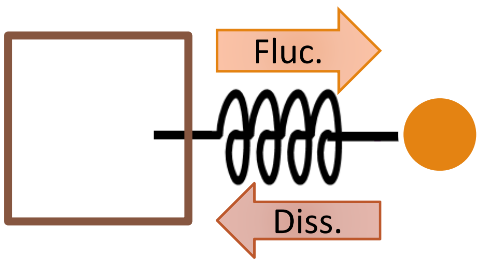
γ_A・・・Athermal viscous coefficient $= O(\varepsilon)$

- Idea: System attached to multiple baths
  (We apply the SSE twice)
- Assumption
  1. Weak coupling : $\varepsilon \to 0$
  2. Coexistence of both noise : $T = \varepsilon^2 \mathcal{T}$ (ε-independent)
  3. Large thermal viscosity :
     $\gamma \gg \gamma_A \Leftrightarrow \gamma$ is ε-independent
- Assumption 1→Environmental correlation disappear (Reduction to white noise)
- Assumption 3→the CLT is violated

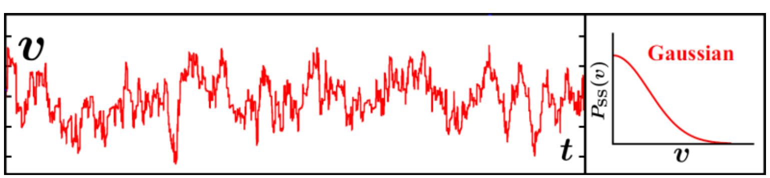
⇒ The NGL Eq. is derived systematically to leading order approximation

Separation of origins of the fluctuation and dissipation

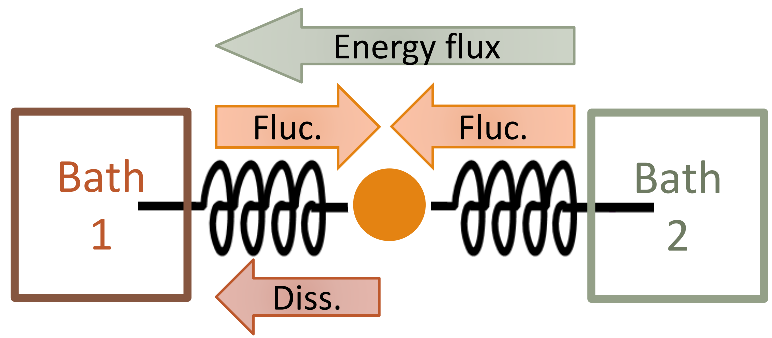
A SINGLE BATH (GAUSSIAN)



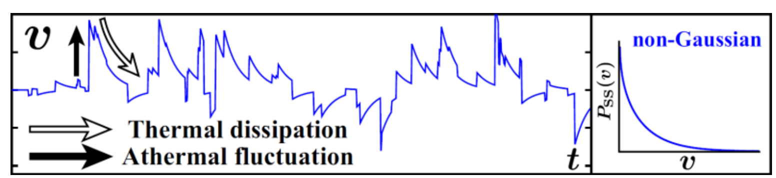
- Expansion for a single bath
- The same origin in terms of the fluctuation and dissipation



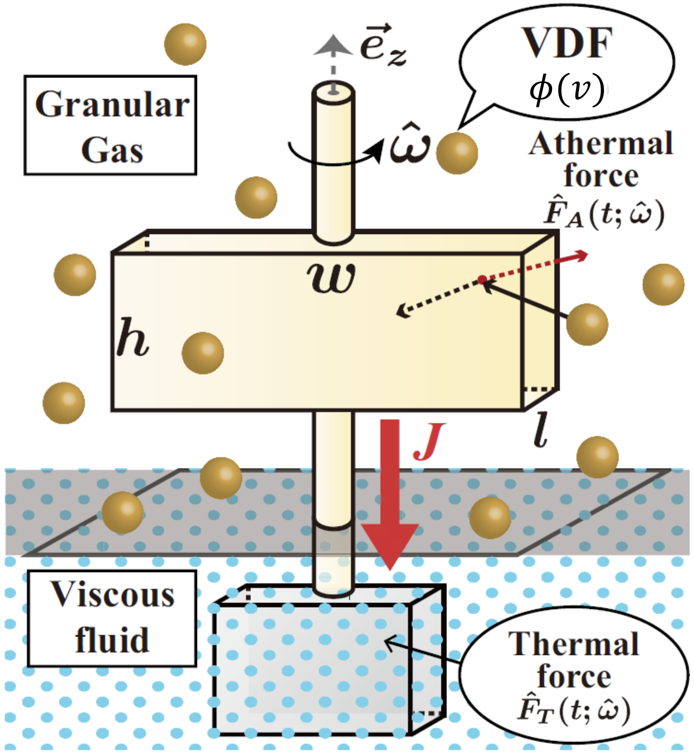
DOUBLE BATH (NON-GAUSSIAN)



- Expansion for double baths
- Different origins in terms of the fluctuation and dissipation
- Fluctuation irrelevant to relaxation becomes non-Gaussian noise



Example:
Granular rotor (model)



$$K_{2n} \propto \int_0^\infty dv v^{n+3}\phi(v), \quad \hat{\Omega} = \frac{\hat{\omega}}{\varepsilon}$$

- A non-eq. Rayleigh Piston
- Linearized Boltzmann Eq.

$$\frac{\partial P(\omega,t)}{\partial t} = \gamma\left[\frac{\partial}{\partial\omega}\omega + \frac{T}{I}\frac{\partial^2}{\partial\omega^2}\right]P(\omega,t)$$

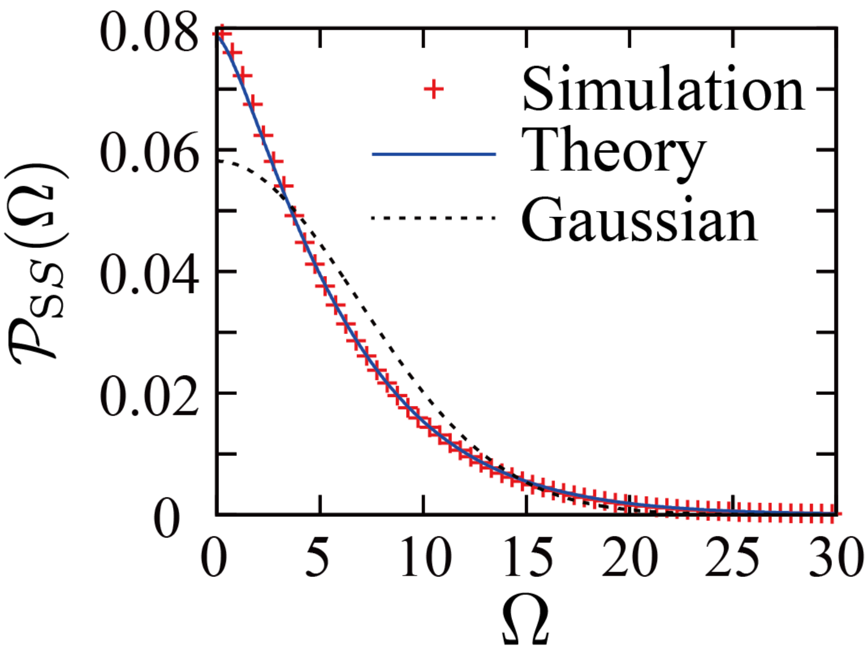
$$+\int_{-\infty}^{\infty} dy[P(\omega-y,t)W(y;\omega-y) - P(\omega,t)W(y;\omega)]$$

- Assumption

1. Weak coupling : $\varepsilon \equiv m/M \ll 1$
2. Coexistence of both noise : $T = \varepsilon^2 \mathcal{T}$ (ε-independent)
3. Strong thermal viscosity : $\gamma \gg \gamma_A$

$$\Rightarrow \frac{d\hat{\Omega}}{dt} = -\gamma\hat{\Omega} + \sqrt{2\gamma\mathcal{T}}\hat{\xi}^{\mathrm{G}} + \hat{\xi}^{\mathrm{NG}}, \quad \hat{\Omega} \equiv \frac{\hat{\omega}}{\varepsilon}$$

Rotor's steady distribution



Results based on the MD is presented in T.G. Sano's poster "No. 52."

- When the granular velocity dist. is exponential: $\phi(v) \propto e^{-v/v_0}$
- The exact distribution of the rotor's angular velocity Ω

$$\mathcal{P}_{\rm SS}(\tilde{\Omega}) = \int_{-\infty}^{\infty} \frac{ds}{2\pi} \frac{e^{[-is\tilde{\Omega} - v_0 s^2/\tilde{v}(1+s^2)]}}{(1+s^2)^{3v_0/2\tilde{v}}}$$

- Numerical data are consistent with our result but not with the Gaussian approximation

Can we extract information from the fluctuation?

- Fluctuation includes some useful information
  （Fluctuation dissipation theorem：2^{nd} cumulant＝temperature）
- What does non-Gaussianity（skewness, kurtosis）mean microscopically?
  （e.g., shot noise→elementary charge）

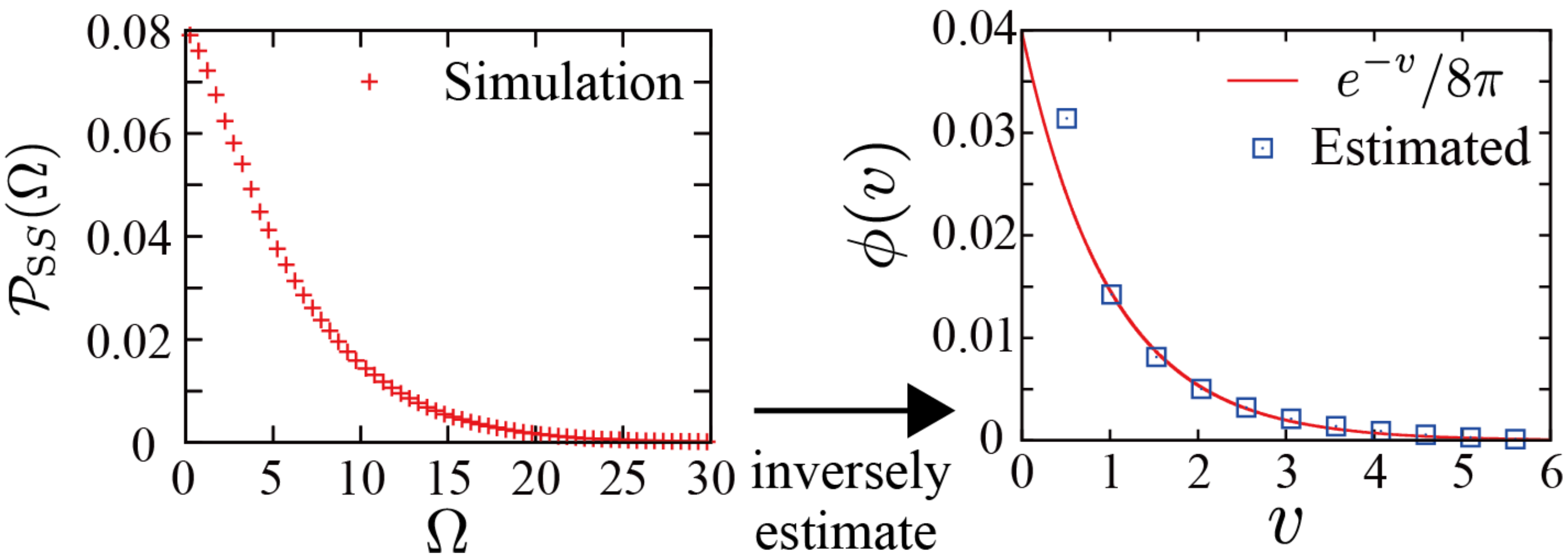
Isotropic granular rotor

- Isotropic granular velocity dist. $\phi(v)$
- Rotor's dist. function $P_{SS}(\Omega)$ $\Longleftrightarrow$ Granular dist. $\phi(v)$

$$\phi(v) = \int_0^\infty \frac{ds}{\pi|v|}\left[a - \frac{bs^2}{2} - cs^3\frac{d}{ds}\log\tilde{P}_{SS}(s/F_g)\right]\cos(sv)$$

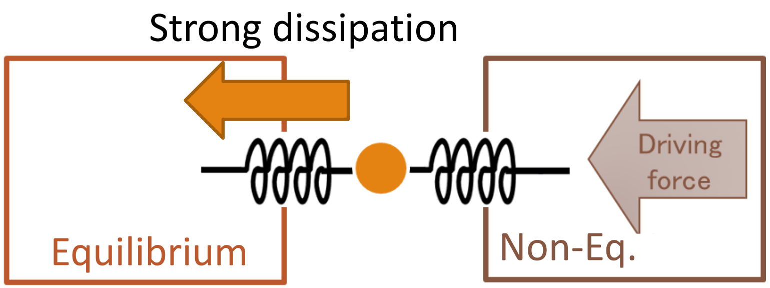
$\tilde{P}_{SS}(s)$・・・Fourier representation of $P_{SS}(\Omega)$,　a, b, c, F_g・・・constants

Numerical demonstration of inverse estimation



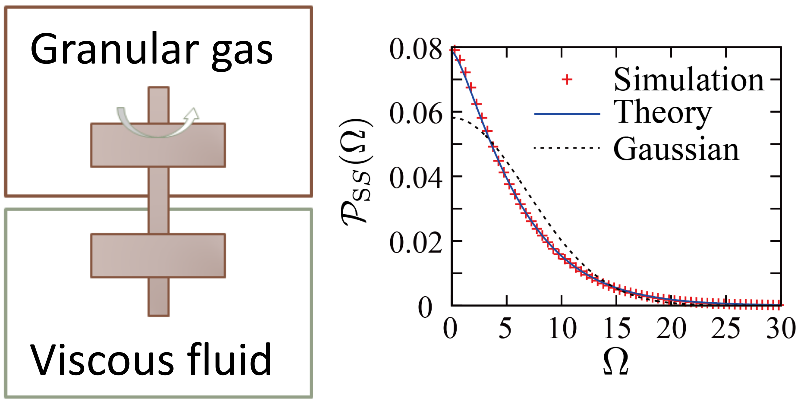
- Inverse estimation of the granular dist. $\phi(v)$ from the rotor's steady dist. $P_{SS}(\Omega)$
- Microscopic information of the environment can be inferred from the observation of the tracer (rotor)

Conclusion of Main 1



◆Derivation of NGL Eq.

1. Weak coupling
2. Coexistence of thermal & athermal fluc.
3. Strong thermal dissipation



◆Granular rotor

◆How to interpret non-Gaussianity

- Microscopic info. on environment
- Angular velocity of granular rotor Ω

$P_{SS}(\Omega) \rightarrow \phi(v)$ (Granular dist.)

KK, T.G. Sano, T. Sagawa, H. Hayakawa, PRL **114**, 090601 (2015).

Main 2: Analytical solution of NGL. Eq.

$$\frac{d\hat{V}}{dt} = -F(\hat{V}) + \hat{\xi}^{\mathrm{NG}} \Rightarrow \text{Solution}: P_{SS}(V) = ??$$

- Main 1 : Microscopic derivation of the NGL Eq.
  →We next solve the NGL Eq. in terms of the steady dist.
- Main 2 : Perturbative formulation
  1. Develop a general & perturbative framework for an arbitrary non-linear NGL Eq. for steady distribution
  2. Full-order solution & its physical meaning (Diagrammatical rep.)

Fundamental equation

$$\frac{d\hat{V}}{dt} = -F(\hat{V}) + \hat{\xi}^{\mathrm{NG}}$$

Transition rate $W(Y)$

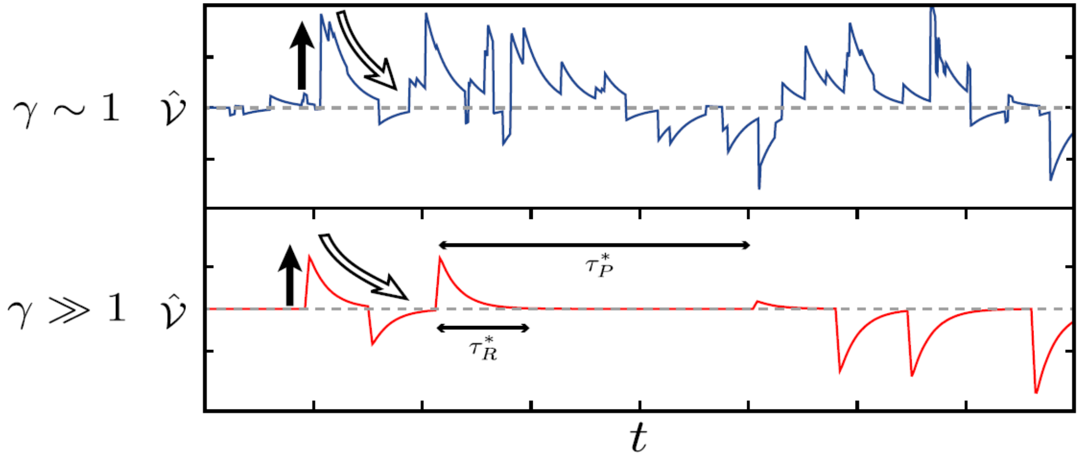
$$\frac{\partial}{\partial v} F(V) P_{\mathrm{SS}}(V) + \int_{-\infty}^{\infty} dY W(Y) \left\{ P_{\mathrm{SS}}(V-Y) - P_{\mathrm{SS}}(V) \right\} = 0$$

$$\frac{is}{2\pi} \int_{-\infty}^{\infty} du \tilde{F}(s-u) \tilde{P}(u) = \Phi(s) \tilde{P}(s)$$

- Master Eq. for the steady dist. (Integro-differential Eq.)
  - $\tilde{P}(s), \tilde{F}(s)$ ･･･Fourier rep. of the steady dist. & friction
  - $\Phi(s) = \int dY (e^{isY} - 1) W(Y)$ ･･･Cumulant function for the non-Gaussian noise
- Construction of the perturbative solution

& Clarification of the meaning of all the perturbative terms

Large frictional limit



- In the large friction limit→Analytically solvable?

$$\gamma \gg 1 \Longleftrightarrow \mu \equiv 1/\gamma \ll 1$$

$$F(V) = \gamma f(V)$$

$$\tilde{P}(s) = 1 + \sum_{n=1}^{\infty} \mu^n \tilde{a}_n(s)$$

- $\mu \ll 1$→The typical number of kicks during relaxation decreases →nth-order perturbation corresponds to the n times kicks process?

Full-order formal solution

$$\mathcal{I}[s;h(s')] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dV(e^{isV}-1)}{f(V)}\int_{-\infty}^{\infty}ds' e^{-is'\mathcal{V}}\frac{\Phi(s')}{is'}h(s')$$

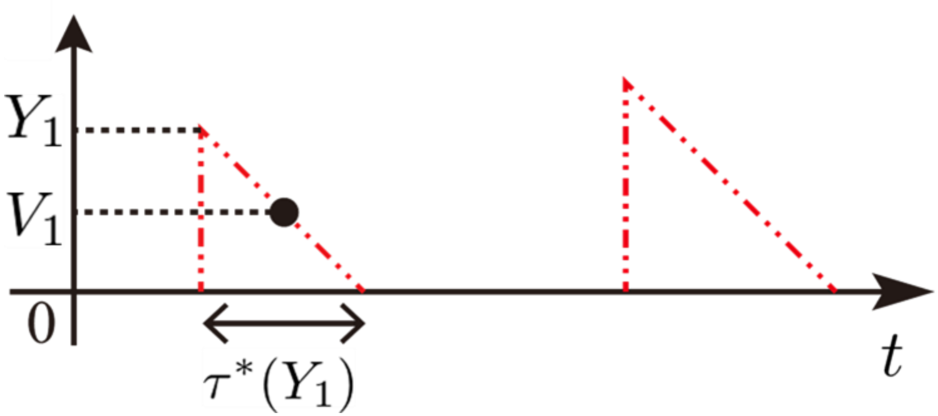
$$\tilde{P}(s) = 1 + \mu\mathcal{I}[s;\mathbf{1}(s')] + \mu^2\mathcal{I}^2[s;\mathbf{1}(s')] + \cdots = [1-\mu\mathcal{I}]^{-1}[s;\mathbf{1}(s')]$$

- We have constructed the full-order perturbative solution
- Too formal to be understood…
  (We have not used the probabilistic property)
- We next transform the terms to clarify their probabilistic meanings…

The 1st-order solution (1): Probabilistic representation

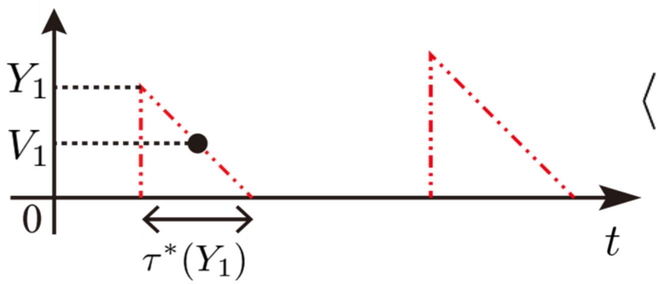
$$\tilde{P}(s) = 1 + \int_{-\infty}^{\infty} dY W(Y) \int_0^Y \frac{\mu dV}{f(V)} [e^{isV} - 1] + O(\mu^2)$$

- The 1st-order solution is represented with the transition rate $W(Y)$.
- $W(Y)$ appears only once→Effect of a trajectory with single kicks?



We have to prove equivalence between These pictures

The 1st-order solution (2): Independent kick model (IKM)



$$\langle h(\hat{V})\rangle \simeq \int_{-\infty}^{\infty} dY W(Y) \int_0^{\tau^*(Y)} dt h(V(t;Y))$$

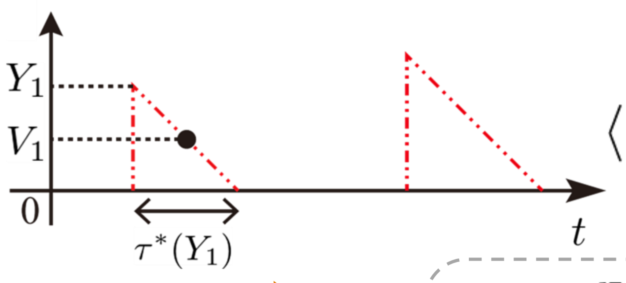
J. Talbot *et al.*, PRL (2011)

- Phenomenological picture (IKM): Effect of a single kick during relaxation
- Derivation: The trajectory is assumed to be decomposed into single-kick shapes, and we calculate an average of an arbitrary quantity $h(V)$

$$\langle h(\hat{V})\rangle_{\mathrm{SS}} = \lim_{T\to\infty}\frac{1}{T}\int_0^T dt h(\hat{V}(t)) \simeq \lim_{T\to\infty}\frac{1}{T}\sum_{i=1}^{\hat{N}(T)}\int_0^{\tau^*(\hat{Y}_1)} dt h(V(t;\hat{Y}_1))$$

$$\simeq \int_{-\infty}^{\infty} dY W(Y) \int_0^{\tau^*(Y)} dt h(V(t;Y))$$

The 1st-order solution (3): Steady distribution for the IKM



$$\langle h(\hat{V})\rangle \simeq \int_{-\infty}^{\infty} dY W(Y) \int_0^{\tau^*(Y)} dt h(V(t;Y))$$

Remark

1. $\dfrac{dV}{dt} = -\dfrac{f(V)}{\mu} \Longleftrightarrow dt = -\mu \dfrac{dV}{f(V)}$ ← Jacobian
2. $h(V) = e^{isV} - 1$ (Fourier rep. for dist.)

$$\tilde{P}(s) - 1 = \langle e^{is\hat{V}} - 1\rangle \simeq \int_{-\infty}^{\infty} dY_1 W(Y_1) \int_0^{Y_1} \frac{\mu dV_1}{f(V_1)} (e^{isV_1} - 1)$$

- This rep. is completely equivalent to the 1st-order solution.
- Equivalence between the 1st-order solution & IKM

The 1st-order solution (4) : Diagrammatic representation

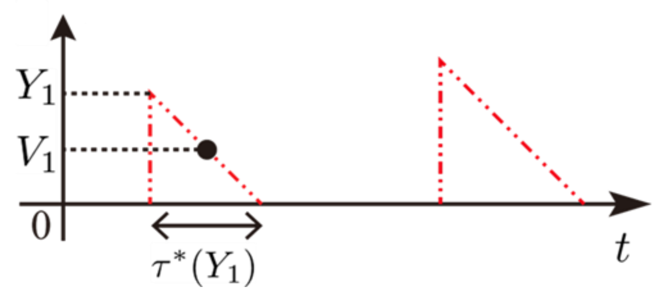
$$\tilde{P}(s) = 1 + \int_{-\infty}^{\infty} dY W(Y) \int_0^Y \frac{\mu dV}{f(V)} [e^{isV} - 1] + O(\mu^2)$$

1st kick — Time-average along the trajectory

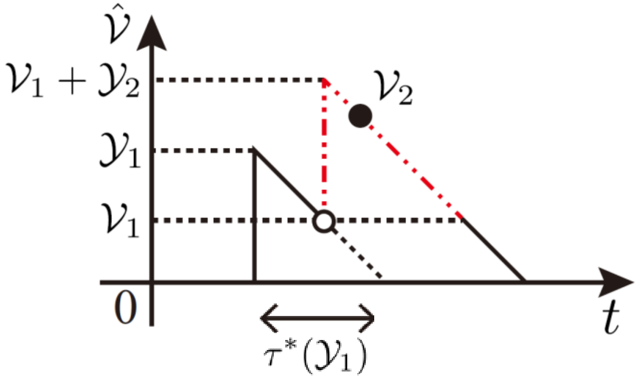
- The 1st-order solution＝Single-kicks trajectory (IKM)
- Introduction of a diagram

$$(\bullet) = \int_{-\infty}^{\infty} dY W(Y) \int_0^Y \frac{\mu dV}{f(V)} [e^{isV} - 1]$$

$$\Rightarrow \quad \tilde{P}(s) = 1 + (\bullet)$$



2^{nd}-order diagrams



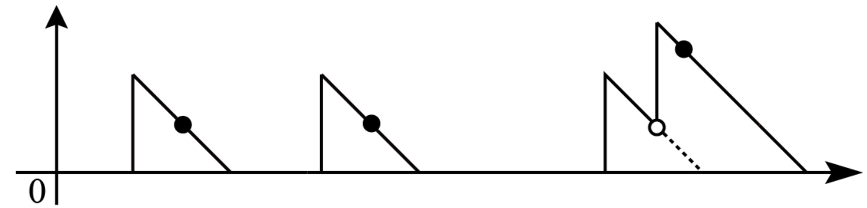
"•" → Variable coupling to $(e^{isV} - 1)$

"∘" → Variable decouple of $(e^{isV} - 1)$

$$(\circ) = \underbrace{\int_{-\infty}^{\infty} dY_1 W(Y_1)}_{\text{1st kick}} \int_0^{Y_1} \underbrace{\frac{\mu dV_1}{f(V_1)}}_{\text{Black line}} = \underbrace{\int_{-\infty}^{\infty} dY_1 W(Y_1) \tau^*(Y_1)}_{\text{Prob. where the 2nd kick happens}}$$

$$(\circ \to \bullet) = \underbrace{\int_{-\infty}^{\infty} dY_1 W(Y_1)}_{\text{1st kick}} \int_0^{Y_1} \underbrace{\frac{\mu dV_1}{f(V_1)}}_{\text{Black line}} \times \underbrace{\int_{-\infty}^{\infty} dY_2 W(Y_2)}_{\text{2nd kick}} \int_{Y_1}^{Y_1+Y_2} \underbrace{\frac{\mu dV_2}{f(V_2)} (e^{isV_2} - 1)}_{\text{Red line; Coupling to } (e^{isV_2} - 1)}$$

2nd-order soluti



Effect of a single-kick trajectory

$$\tilde{P}(s) = 1 + [1 - (\circ)](\bullet) + (\circ \rightarrow \bullet)$$

Prob. of emergence of a single-kick trajectory

Effect of a double-kick trajectory

$$(\bullet) = \int_{-\infty}^{\infty} dY_1 W(Y_1) \int_0^{Y_1} \frac{\mu dV_1}{f(V_1)} (e^{isV_1} - 1)$$

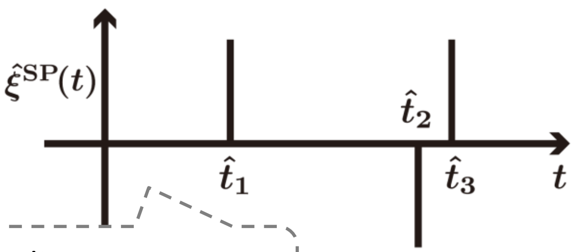
$$(\circ) = \int_{-\infty}^{\infty} dY_1 W(Y_1) \int_0^{Y_1} \frac{\mu dV_1}{f(V_1)}$$

$$(\circ \rightarrow \bullet) = \int_{-\infty}^{\infty} dY_1 W(Y_1) \int_0^{Y_1} \frac{\mu dV_1}{f(V_1)} \times \int_{-\infty}^{\infty} dY_2 W(Y_2) \int_{Y_1}^{Y_1+Y_2} \frac{\mu dV_2}{f(V_2)} (e^{isV_2} - 1)$$

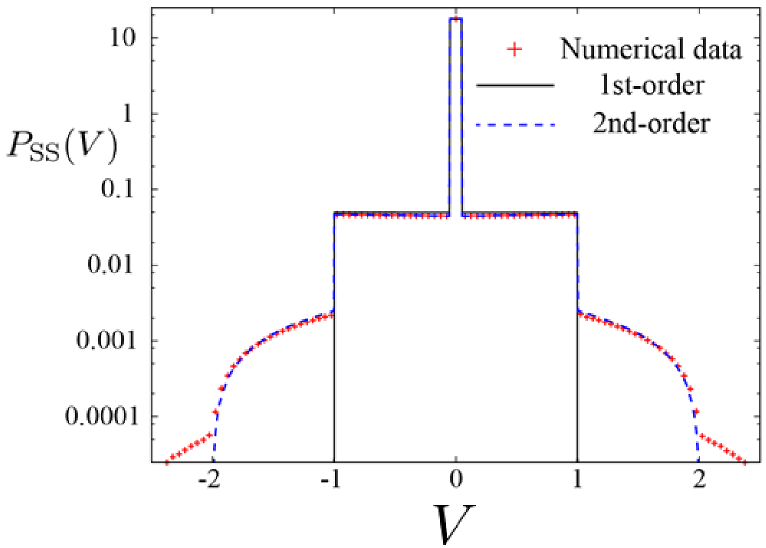
$$= dt$$

- Diagrammatic rep. of the 2nd-order
- Meaning
  - $(\bullet)$ ・・・Contribution of the 1st kick
  - $(\circ)$ ・・・Prob. where the 2nd kick appears
  - $(\circ \rightarrow \bullet)$・・・Contribution of the 2nd kick
- 2nd Solution＝single kick trajectories
  ＋double kick trajectories

Example of the 1st-order solution: Symmetric Poisson + Coulomb friction



Jump distance : $Y_0 = 1$
Transition rate : $\lambda = 1$



- Coulomb friction ＋ Symmetric Poisson noise

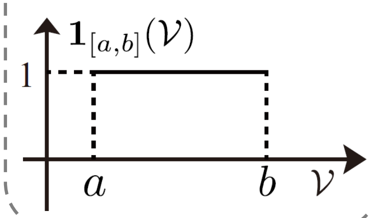
$$\frac{d\hat{V}}{dt} = -\gamma \mathrm{sgn}(\hat{V}) + \hat{\xi}^{SP}$$

- The 1st-order solution:

$$P_{\mathrm{SS}}(V) = [1 - \mu\lambda Y_0]\delta(V) + \frac{\mu\lambda}{2}\mathbf{1}_{[-Y_0, Y_0]}(V)$$

δ-type singularity
＝Coulomb friction is so strong that the system relaxes to $V = 0$ in a finite time

Indicator func.



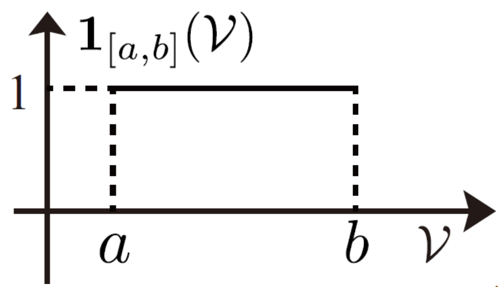
- Valid only for the range $|V| < 1$
(Only the effect of single kicks is taken into account)

Example of the 2nd-order solution

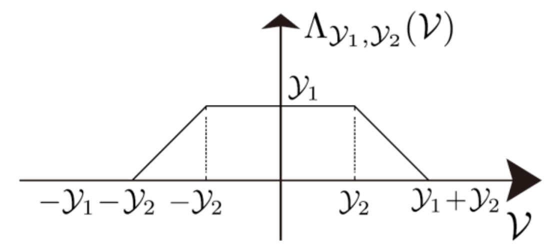
$$P_{\mathrm{SS}}(V) = \left[1-\mu\lambda Y_0 + \frac{\mu^2\lambda^2 Y_0^2}{2}\right]\delta(V) + \frac{\mu\lambda}{2}[1-\mu\lambda Y_0]\mathbf{1}_{[-Y_0,Y_0]}(V) - \frac{\mu^2\lambda^2}{4}\Lambda_{Y_0,0} + \frac{\mu^2\lambda^2}{4}\Lambda_{Y_0,Y_0}(V)$$

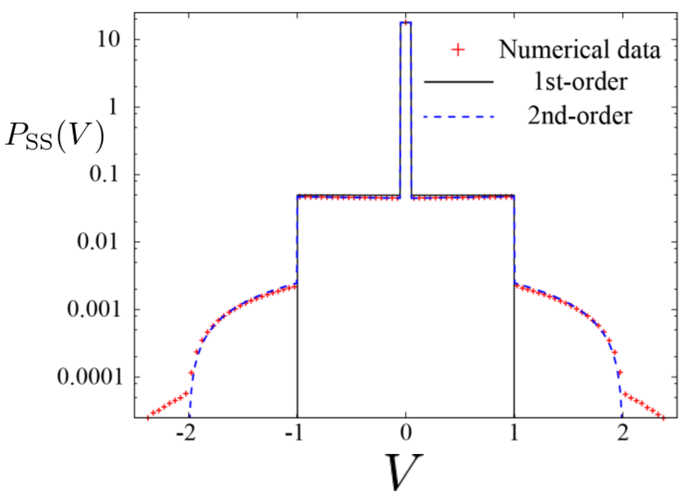
δ-function originating from the Coulomb friction

≒indicator function



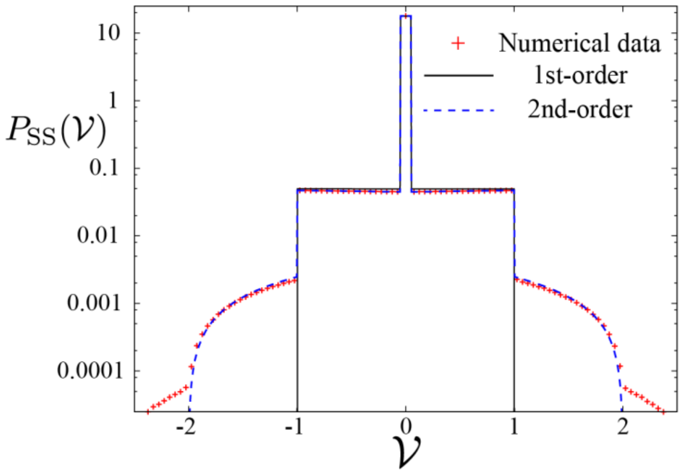
=trapezoid function

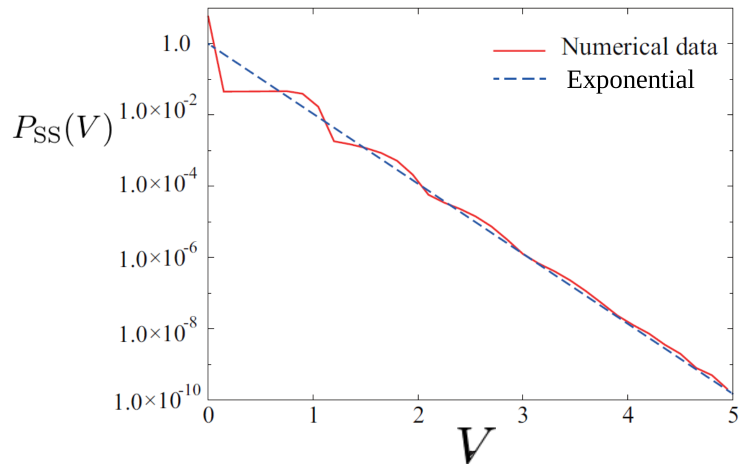




- The 2nd-order solution is only valid for the range $|V| < 2$ (double-kicks are take into account)
- Generally, nth solution is valid for $|V| < n$

Tail of the distribution



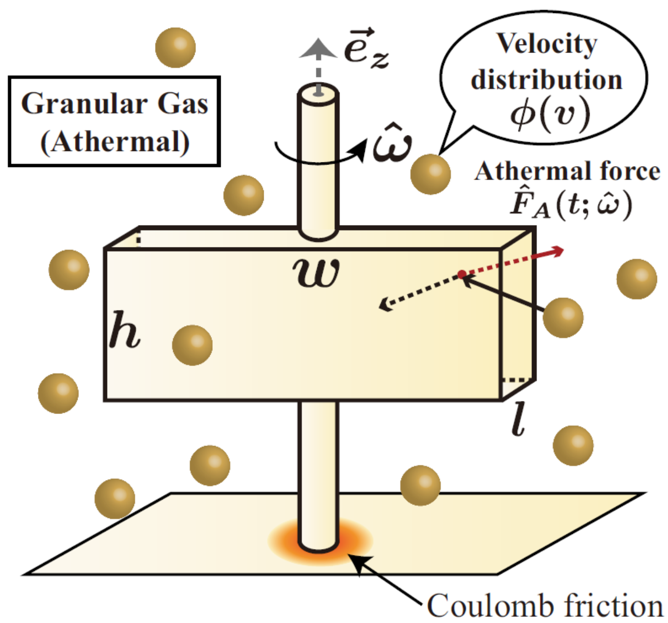


- Tail cannot be reproduced as it is relevant to rare events
  →However, the tail is analytically obtained (exponential function)

$$P_{SS}(V) \sim e^{-a|V|} \quad (|V| \gg 1), \qquad a = \mu \cos a$$

- Overview $=$ δ function (0th solution)
  $+$ indicator function (1st solution)
  $+$ trapezoid function (2nd solution)
  $+$ … $+$ exponential tail (Tail)

Granular rotor under dry friction : Derivation of nonlinear NGL Eq.



◆ Linearized Boltzmann Eq.

$$\frac{\partial P(\omega,t)}{\partial t} = \frac{\partial}{\partial \omega}\gamma \mathrm{sgn}(\omega) P(\omega,t)$$
$$+ \int_{-\infty}^{\infty} dy [P(\omega - y, t) W(\omega - y; y) - P(\omega, t) W(\omega; y)]$$

◆ Assumptions

1. Weak coupling : $\varepsilon \equiv m/M \ll 1$
2. Strong Coulomb friction : $\beta^{-1} \lesssim 1$

$$\beta^{-1} \equiv \frac{\tau_R}{\tau_C} = \frac{\varepsilon \rho S v_0^2}{\gamma R_I}$$

$\tau_R \equiv \varepsilon v_0 / \gamma R_I$: Relaxation time
$\tau_C \equiv 1/\rho S v_0$: Collision interval

- ρ : Density
- I : Inertia Moment
- M : Mass of rotor
- S : Area
- v_0 : granular vel.
- $R_I \equiv \sqrt{I/M}$

◆ Coulombic NGL Eq. is derived:

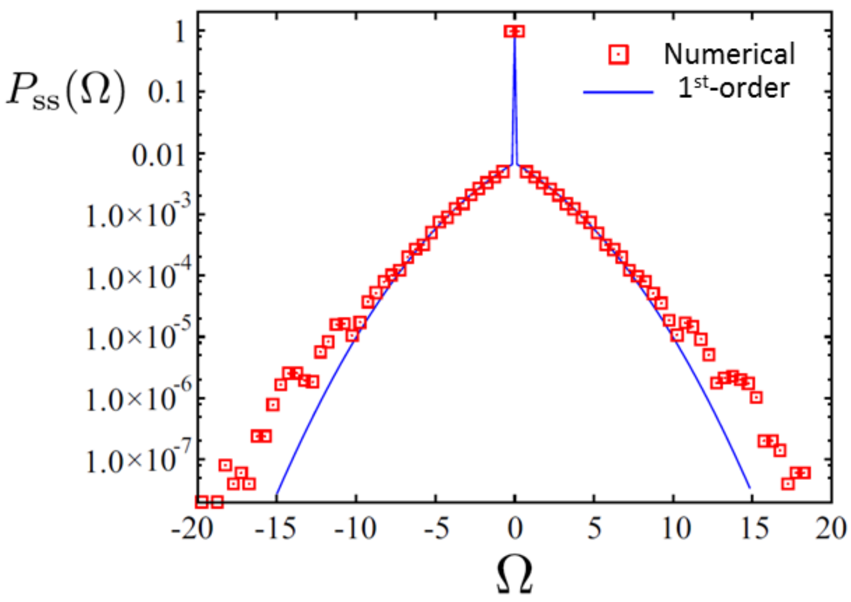
$$\frac{d\hat{\Omega}}{dt} = -\tilde{\gamma}\mathrm{sgn}(\hat{\Omega}) + \hat{\xi}^{\mathrm{NG}}, \quad \hat{\Omega} \equiv \frac{\hat{\omega}}{\varepsilon}$$

Granular rotor under dry friction : The 1st-order solution

$$\mathcal{P}(\Omega) = \left[1 - \mu\rho h v_0 (l\Omega_l^* + w\Omega_w^*)\right]\delta(\Omega) + \mathcal{P}_{\text{smooth}}^{(l)}(\Omega) + \mathcal{P}_{\text{smooth}}^{(w)}(\Omega)$$

$$\mathcal{P}_{\text{smooth}}^{(p)}(\Omega) \equiv \frac{\rho h p v_0}{4\tilde{\gamma}}\left(3 + \frac{|\Omega|}{\Omega_p^*}\right) e^{-|\Omega|/\Omega_p^*}$$

$$\Omega_p^* \equiv \frac{p(1+e)v_0}{2R_I^2} \quad \tilde{\gamma} \equiv \frac{\gamma}{\varepsilon}$$

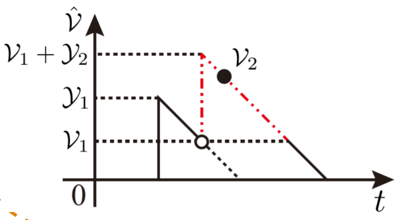


- For $\phi(v) = e^{-v/v_0}/8\pi v_0^3$
- $\beta^{-1} \ll 1 \Leftrightarrow \mu \ll 1$
- Numerical validation
- Valid for the range $|\Omega| \lesssim 3.5$ (Only the single kick effect is taken…)

- $l = 0$
- $M = 0.01$
- $\tilde{\gamma} = 200$
- $\beta^{-1} = 0.035$
- $\Omega_w^* \simeq 3.5$
- $w = \sqrt{12}$

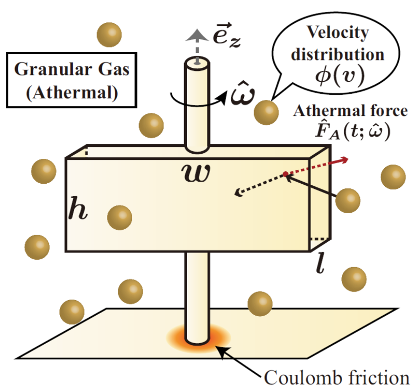
Conclusion of Part 2: Analytical solution of non-linear NGL Eq.

$$\frac{d\hat{V}}{dt} = -\gamma f(\hat{V}) + \hat{\xi}^{\mathrm{NG}}, \quad \mu = 1/\gamma$$



$$\tilde{P}(s) = 1 + \sum_{n=1}^{\infty} \mu^n \tilde{a}_n(s)$$

- Full-order perturbative solution for non-linear NGL Eq.
  →Explicit representation
- Physical meaning : high-order terms corresponds multiple-kicks effect



- Application to granular rotor under Coulomb friction
  - $\beta^{-1} \lesssim 1$: NGL Eq.
  - $\beta^{-1} \ll 1$: 1^{st}-order perturbation
- Numerical validation

KK, T.G. Sano, T. Sagawa, H. Hayakawa, J. Stat. Phys. **160**, 1294 (2015)

Summary of this talk

MAIN 1: MICROSCOPIC DERIVATION OF NGL EQ. (KINETIC THEORY)

◆Perturbative derivation of NGL Eq.
1. Weak coupling
2. Coexistence of thermal & athermal flucs.
3. Strong thermal dissipation

◆Application to granular rotor

◆Information in non-Gaussianity
＝Microscopic info. of environment
（i.e., granular velocity dist.）

KK, T.G. Sano, T. Sagawa, H. Hayakawa, PRL **114**, 090601 (2015)

PART 2: ANALYTICAL SOLUTION OF NGL EQ. WITH NON-LINEAR FRICTION

◆Full-order perturbative formula for the steady dist. under nonlinear friction

◆High-order terms corresponds to multiple kicks during relaxation

◆Diagrammatic representation

◆Application to granular rotor under Coulomb friction

KK, T.G. Sano, T. Sagawa, H. Hayakawa, J. Stat. Phys. **160**, 1294 (2015)

⇒ Universality & Analytical simplicity of NGL Eq.