Relation between classical stochastic dynamics and quantum annealing

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Quantum annealing

(Classical) Combinatorial optimization

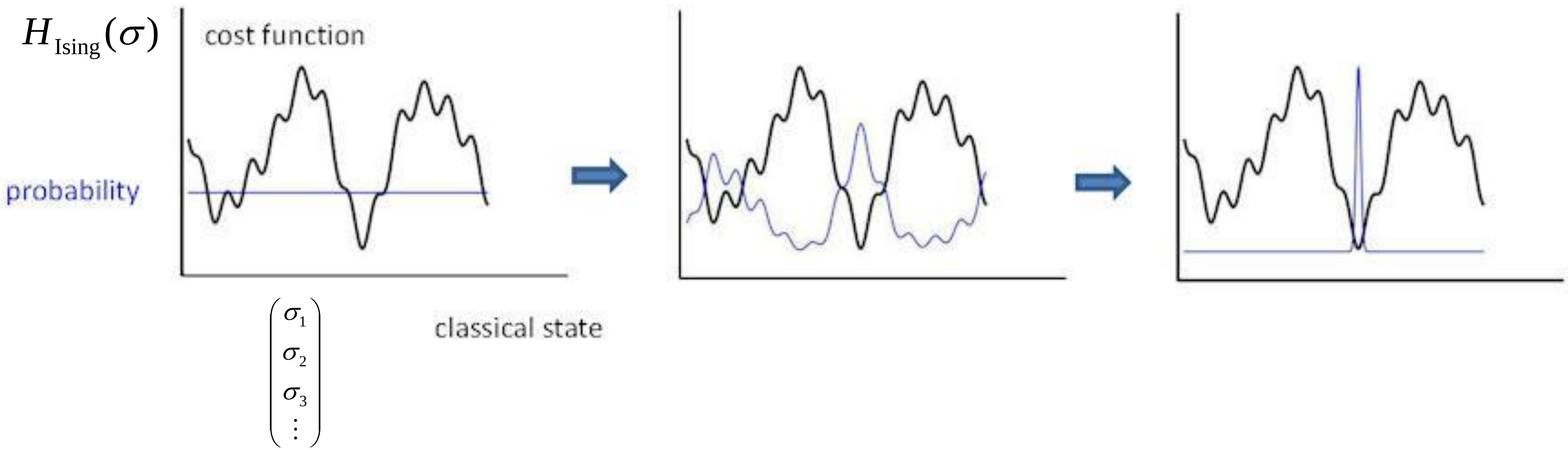
Ground state of Ising model

$$H_{\text{Ising}} = -\sum J_{ij}\sigma_i\sigma_j - \sum h_i\sigma_i \quad (\sigma_i = \pm 1)$$

- Travelling salesman problem
- Machine learning / Artificial intelligence
- Protein / polymer folding
- Power grid optimization
- Medical problems

Quantum Annealing (QA)

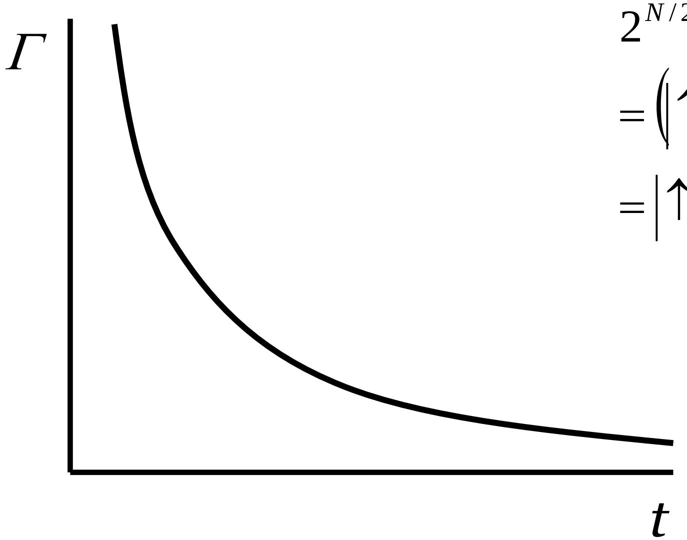
- Metaheuristic: Generic, approximate algorithm
- Search by quantum fluctuations



Quantum annealing
Formulation

$$H_{\text{Ising}} = -\sum J_{ij}\sigma_i^z\sigma_j^z - \sum h_i\sigma_i^z$$

$$H(t) = H_{\text{Ising}} + H_{\text{quantum}} = -\sum J_{ij}\sigma_i^z\sigma_j^z - \sum h_i\sigma_i^z - \Gamma(t)\sum \sigma_i^x$$



Ground state of H_{quantum}

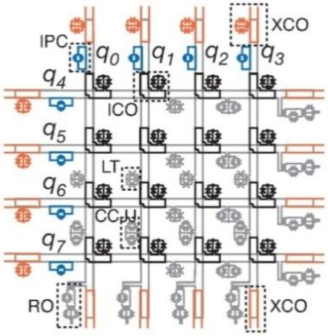
$$2^{N/2}|\rightarrow\rangle_1 \otimes |\rightarrow\rangle_2 \otimes\cdots\otimes |\rightarrow\rangle_N$$
$$= \left(|\uparrow\rangle_1 + |\downarrow\rangle_1\right)\otimes\left(|\uparrow\rangle_2 + |\downarrow\rangle_2\right)\otimes\cdots\otimes\left(|\uparrow\rangle_N + |\downarrow\rangle_N\right)$$
$$= |\uparrow_1\uparrow_2\uparrow_3\cdots\uparrow_N\rangle + |\uparrow_1\uparrow_2\uparrow_3\cdots\downarrow_N\rangle + \cdots + |\downarrow_1\downarrow_2\downarrow_3\cdots\downarrow_N\rangle$$

Schrödinger dynamics

$$|\uparrow\downarrow\downarrow\uparrow\uparrow\ldots\uparrow\downarrow\rangle$$

Ground state of H_{Ising}

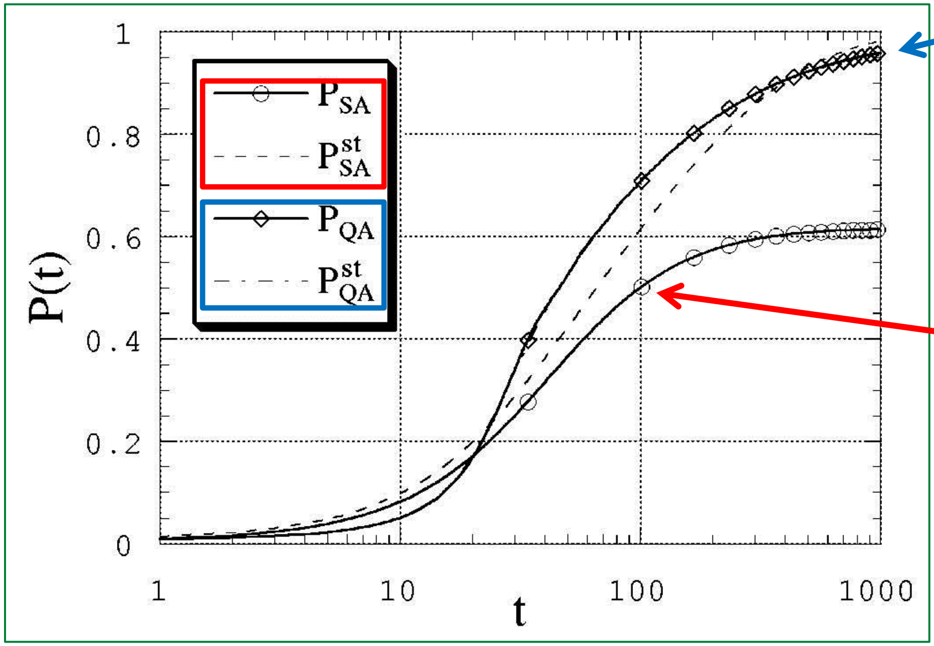
D-Wave Machine



Washington Chip V7
1,152 qubits
128,000 Josephson couplings

Master eqn vs. Schrödinger eqn

Random J_{ij} (Spin glass) with 8 spins



$$\Gamma(t) = \frac{3}{\sqrt{t}}$$

Schrödinger eqn.

$$T(t) = \frac{3}{\sqrt{t}}$$

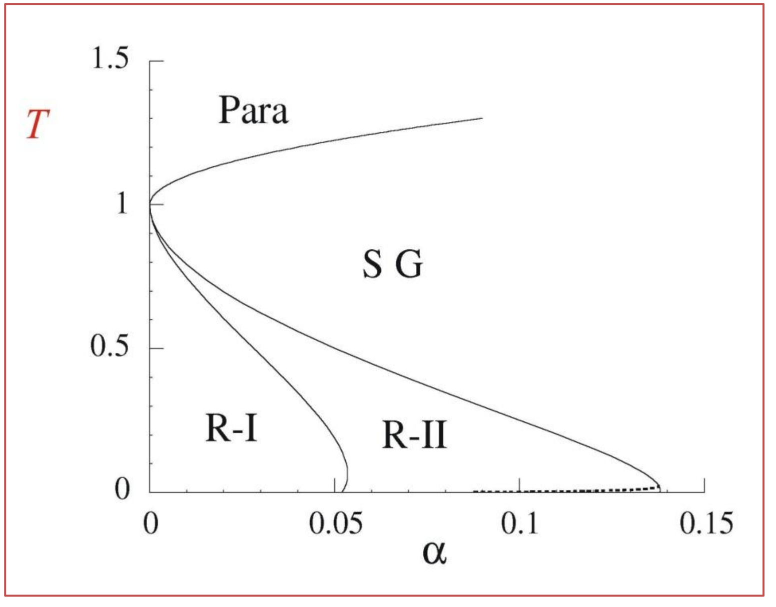
Master eqn.

Kadowaki & Nishimori (1998)

T vs Γ: Hopfield model

$$H = -\sum J_{ij}\sigma_i\sigma_j \quad (\text{Finite } T)$$

$$J_{ij} = \sum_{\mu=1}^{p} \xi_i^\mu \xi_j^\mu$$

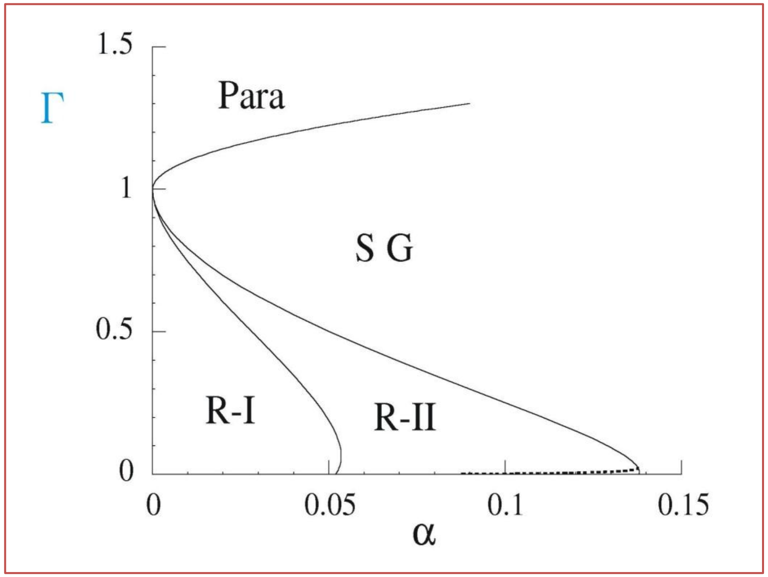


$$\alpha = \frac{p}{N}$$

Amit, Gutfreunt, Sompolinsky (1985)

T vs *Γ*: Hopfield model

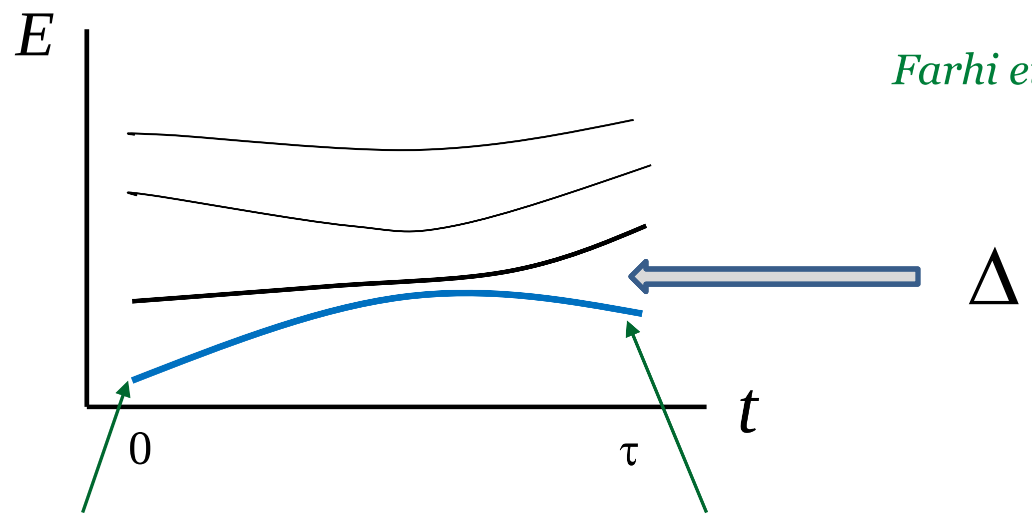
$$H = -\sum J_{ij}\sigma_i^z\sigma_j^z - \Gamma\sum\sigma_i^x \quad (T=0)$$



Nishimori & Nonomura (1996)

Adiabatic quantum computation

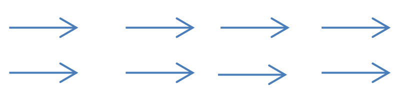
Farhi et al (2001)

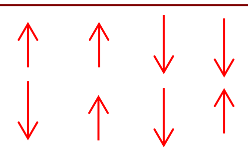


Trivial initial state

Non-trivial final state

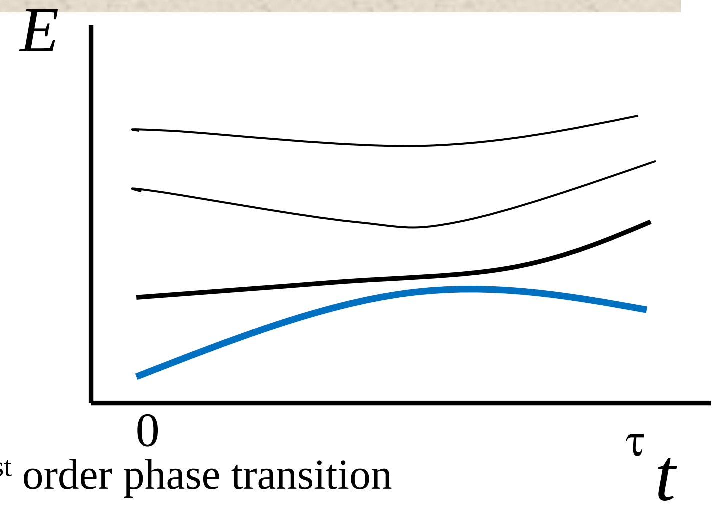
$$H(t) = -\left(1 - \frac{t}{\tau}\right)\sum \sigma_i^x - \frac{t}{\tau}\sum J_{ij}\sigma_i^z\sigma_j^z$$





Computational complexity

Finite-size analysis



1st order phase transition

2nd order phase transition

Adiabatic theorem $\tau \propto \Delta^{-2}$

Gap scaling $\Delta \propto \begin{cases} e^{-aN} \\ N^{-b} \end{cases}$

Complexity $\tau \propto \begin{cases} e^{2aN} & \text{(hard)} \\ N^{2b} & \text{(easy)} \end{cases}$

QA and classical stochastic dynamics

Classical to quantum mapping

Simulating SA by QA

cf: Castelnovo et al (2005)

Classical dynamics of Ising model (*T*:fixed)

$$H_0(\sigma),\ \sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_N\}$$

$$\frac{dP_\sigma(t)}{dt} = \sum_\sigma W_{\sigma\sigma'} P_{\sigma'}(t), \quad W\psi^{(R,n)} = -\lambda_n \psi^{(R,n)}, \quad \lambda_0(=0), -\lambda_1, -\lambda_2, \ldots$$

$$P(t) = P_{\mathrm{eq}} + a_1 e^{-\lambda_1 t} + a_2 e^{-\lambda_2 t} + \ldots$$

$$e^{-t/\tau_{\mathrm{relax}}}, \quad \tau_{\mathrm{relax}} = \frac{1}{\lambda_1}$$

$$T = T_c : \quad \tau_{\mathrm{relax}} \propto N^a \text{ (2nd order)}, \quad e^{bN} \text{ (1st order)}$$

$$\lambda_1 \propto N^{-a} \text{ (2nd order)}, \quad e^{-bN} \text{ (1st order)}$$

Construction of quantum Hamiltonian

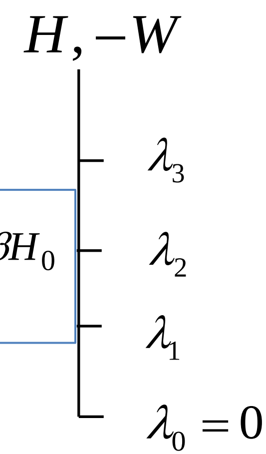
$$H_{\sigma\sigma'} = -e^{\frac{1}{2}\beta H_0(\sigma)} W_{\sigma\sigma'} e^{-\frac{1}{2}\beta H_0(\sigma')}$$

$$H_{\sigma\sigma'} = H_{\sigma'\sigma} \ (\leftarrow \text{detailed balance}) \quad \Rightarrow \quad \text{Hamiltonian}$$

$$W\psi^{(R,n)} = -\lambda_n \psi^{(R,n)}$$

$$H\phi^{(n)} = \lambda_n \phi^{(n)} \quad (\phi^{(n)} = e^{\frac{1}{2}\beta H_0} \psi^{(R,n)})$$

$$\phi^{(0)} = e^{-\frac{1}{2}\beta H_0}$$



$$\Delta = \lambda_1 (= 1/\tau_{\text{relax}}) = N^{-a} \text{(2nd)}, \quad e^{-bN} \text{(1st)}$$

Classical phase transition = Quantum phase transition

Example: 1*d* Ising model

$$H_0(\sigma) = -\sum \sigma_j \sigma_{j+1}$$

$$H = -\frac{1}{2}\tanh\beta \sum \sigma_j^z \sigma_{j+1}^z - \frac{1}{2\cosh 2\beta}\sum (\cosh^2\beta - \sinh^2\beta\, \sigma_{j-1}^z \sigma_{j+1}^z)\sigma_j^x$$

cf. $H_{\mathrm{TFIM}} = -\sum \sigma_j^z \sigma_{j+1}^z - \Gamma \sum \sigma_j^x$ Phase transition at $\Gamma=1$

Exactly solvable ⇒ Glauber's solution

Tsuda, Knysh, and Nishimori, Phys. Rev. E 91, 012104 (2015)

Quantum to classical mapping

Simulating QA by SA

$$W_{\sigma\sigma'} = -e^{-\frac{1}{2}H_0(\sigma)} H_{\sigma\sigma'} e^{\frac{1}{2}H_0(\sigma')}$$

$$\text{Restriction}: \quad H_{\sigma\sigma'} \le 0 \ (\sigma \ne \sigma') \ (\leftarrow W_{\sigma\sigma'} \ge 0)$$

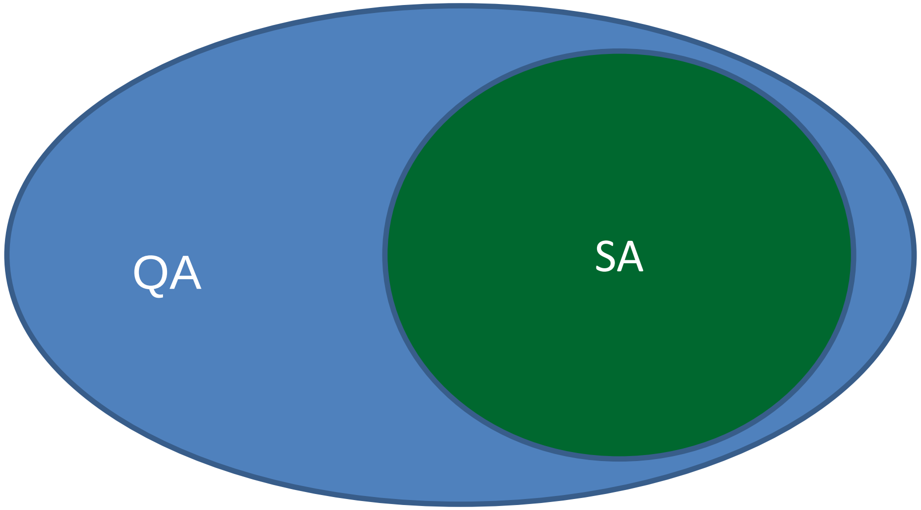
$$\text{Eigenvalue shift}: H\phi^{(0)} = 0$$

$$\text{Perron-Frobenius}: \phi_\sigma^{(0)} > 0 \qquad \sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_N\}$$

$$\text{Define Ising}: H_0(\sigma) = -2\ln \phi_\sigma^{(0)} \qquad \text{cf. SA to QA}: \phi_\sigma^{(0)} = e^{-\frac{1}{2}\beta H_0(\sigma)}$$

$$\text{Non-local}: H_0(\sigma) = c - \sum h_j \sigma_j - \sum J_{ij}\sigma_i\sigma_j - \ldots - J_N \sigma_1\sigma_2\ldots\sigma_N$$

Relation of simulated annealing and quantum annealing



Conclusion

- QA is effective in solving combinatorial optimization.
- Classical to quantum mapping:
  Same spatial dimension, short-range to short-range
- Quantum to classical mapping:
  Same spatial dimension, short-range to long-range