The 1D KPZ equation and its universality

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Plan

- The KPZ equation
- Exact solutions

  Height distribution

  Stationary space-time two point correlation function
- A few recent developments

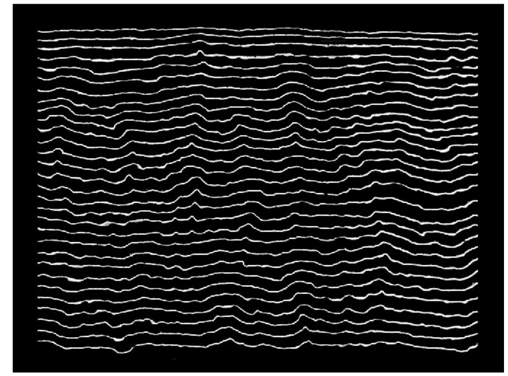
  Dualities

  Free-fermionic structures
- Universality

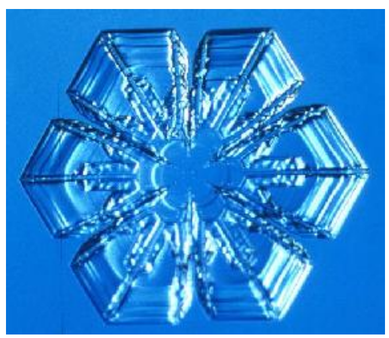
  Brownian motions with oblique reflection

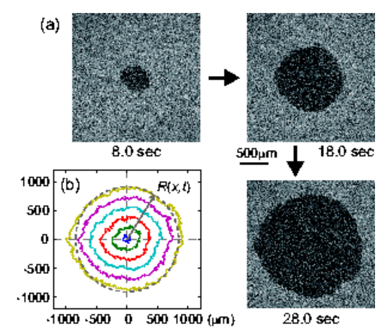
  KPZ in Hamiltonian dynamics

1. Basics of the KPZ equation: Surface growth



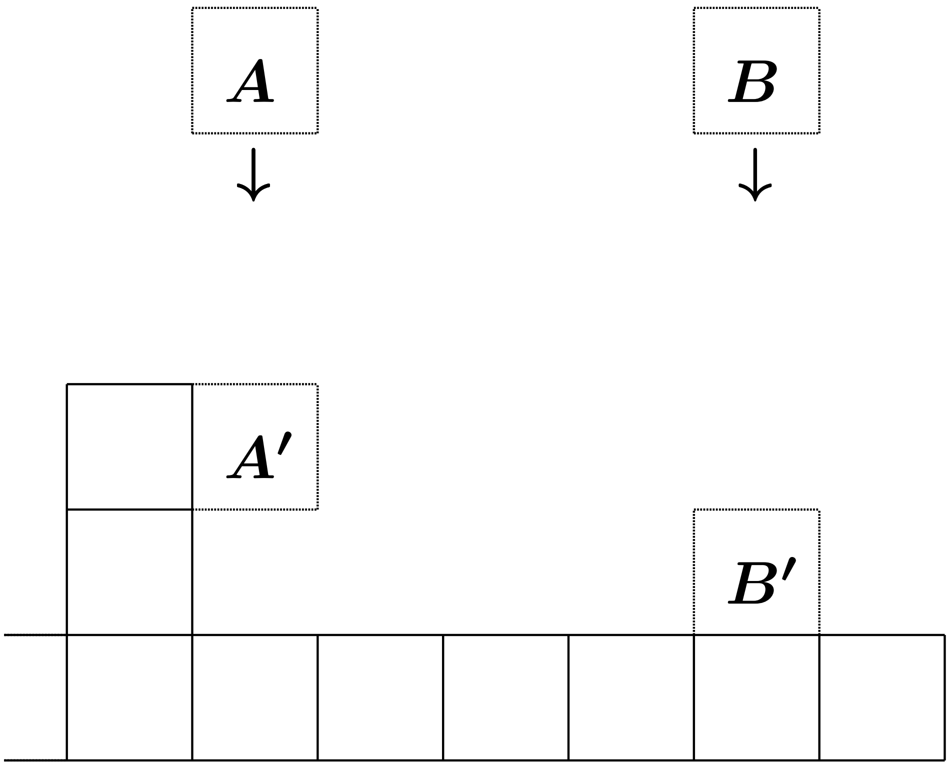
- Paper combustion, bacteria colony, crystal growth, etc
- Non-equilibrium statistical mechanics
- Stochastic interacting particle systems
- Connections to integrable systems, representation theory, etc





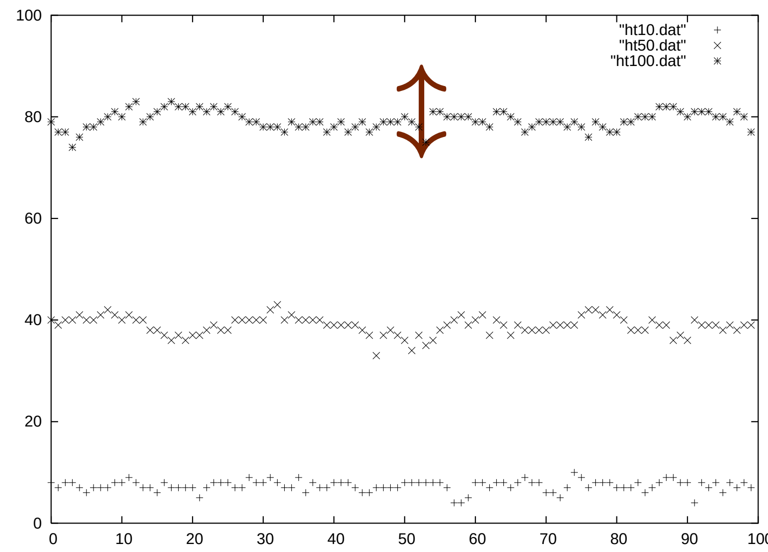
Simulation models

Ex: ballistic deposition



Height fluctuation

$$O(t^{\beta}), \beta = 1/3$$



Flat

KPZ equation

$h(x,t)$: height at position $x \in \mathbb{R}$ and at time $t \geq 0$

1986 Kardar Parisi Zhang (not Knizhnik-Polyakov-Zamolodchikov)

$$\partial_t h(x,t) = \frac{1}{2}\lambda(\partial_x h(x,t))^2 + \nu \partial_x^2 h(x,t) + \sqrt{D}\eta(x,t)$$

where η is the Gaussian noise with mean 0 and covariance

$$\langle \eta(x,t)\eta(x',t')\rangle = \delta(x-x')\delta(t-t')$$

- Dynamical RG analysis: $\to \beta = 1/3$ (KPZ class)
- A simplest nonequilibrium model with nonlinearity, noise and ∞-degrees of freedom
- By a simple scaling we can and will do set $\nu = \frac{1}{2}, \lambda = D = 1$.

Most Famous(?) KPZ

- MBT-70 ／ KPz 70

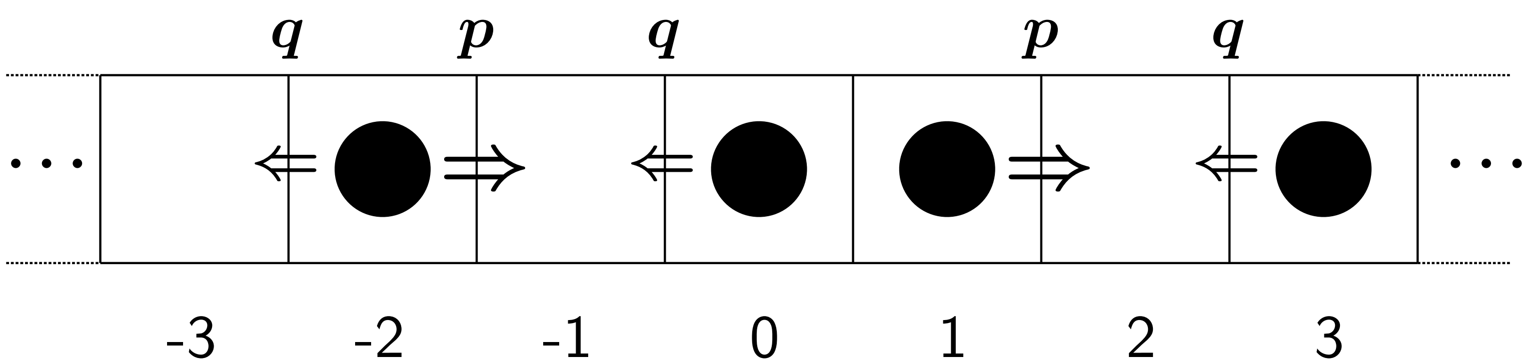
  Tank developed in 1960s by US and West Germany. MBT(MAIN BATTLE TANK)-70 is the US name and KPz(KampfPanzer)-70 is the German name.

New most famous KPZ in Japan(?)

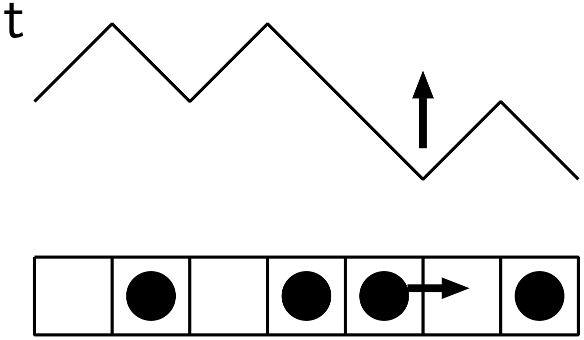
A sushi restaurant franchise with character "kappa" (an imaginary creature) [address: kpz.jp]

A discrete model: ASEP

ASEP = asymmetric simple exclusion process



- TASEP(Totally ASEP, $p = 0$ or $q = 0$)
- $N(x,t)$: Integrated current at $(x, x+1)$ upto time t
  $\Leftrightarrow$ height for surface growth
- In a certain weakly asymmetric limit
  ASEP $\Rightarrow$ KPZ equation



2. Exact solutions: Cole-Hopf transformation

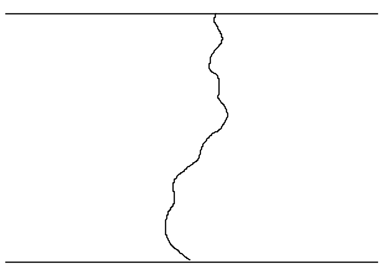
If we set

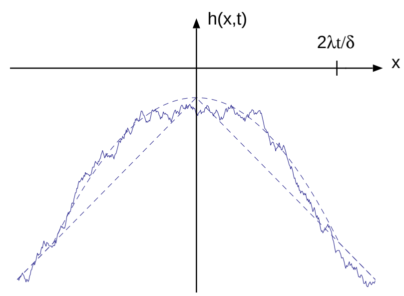
$$Z(x,t) = \exp\left(h(x,t)\right)$$

this quantity (formally) satisfies

$$\frac{\partial}{\partial t} Z(x,t) = \frac{1}{2}\frac{\partial^2 Z(x,t)}{\partial x^2} + \eta(x,t) Z(x,t)$$

This can be interpreted as a (random) partition function for a directed polymer in random environment η.





The polymer from the origin: $Z(x,0) = \delta(x) = \lim_{\delta\to 0} c_\delta e^{-|x|/\delta}$ corresponds to narrow wedge for KPZ.

Replica approach

Dotsenko, Le Doussal, Calabrese

Feynmann-Kac expression for the partition function,

$$Z(x,t) = \mathbb{E}_x \left(e^{\int_0^t \eta(b(s),t-s)ds} Z(b(t),0) \right)$$

Because η is a Gaussian variable, one can take the average over the noise η to see that the replica partition function can be written as (for narrow wedge case)

$$\langle Z^N(x,t) \rangle = \langle x | e^{-H_N t} | 0 \rangle$$

where H_N is the Hamiltonian of the (attractive) δ-Bose gas,

$$H_N = -\frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} - \frac{1}{2} \sum_{j \neq k}^{N} \delta(x_j - x_k).$$

We are interested not only in the average $\langle h \rangle$ but the full distribution of h. We expand the quantity of our interest as

$$\langle e^{-e^{h(0,t)+\frac{t}{24}-\gamma_t s}} \rangle = \sum_{N=0}^{\infty} \frac{\left(-e^{-\gamma_t s}\right)^N}{N!} \left\langle Z^N(0,t) \right\rangle e^{N \frac{\gamma_t^3}{12}}$$

Using the integrability (Bethe ansatz) of the δ-Bose gas, one gets explicit expressions for the moment $\langle Z^N \rangle$ and see that the generating function can be written as a Fredholm determinant. But for the KPZ, $\langle Z^N \rangle \sim e^{N^3}$! Note that the δ-Bose gas is exactly solvable but is in general not a free fermion model.

Explicit determinantal formula

Thm (2010 TS Spohn, Amir Corwin Quastel)

For the initial condition $Z(x,0) = \delta(x)$ (narrow wedge for KPZ)

$$\langle e^{-e^{h(0,t)+\frac{t}{24}-\gamma_t s}} \rangle = \det(1 - K_{s,t})_{L^2(\mathbb{R}_+)}$$

where $\gamma_t = (t/2)^{1/3}$ and $K_{s,t}$ is

$$K_{s,t}(x,y) = \int_{-\infty}^{\infty} \mathrm{d}\lambda \frac{\mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)}{e^{\gamma_t(s-\lambda)}+1}$$

A determinant for non-free-fermion model?

Why Fermi distribution?

Explicit formula for the height distribution

Thm

$$h(x,t) = -x^2/2t - \tfrac{1}{12}\gamma_t^3 + \gamma_t \xi_t$$

where $\gamma_t = (t/2)^{1/3}$. The distribution function of ξ_t is

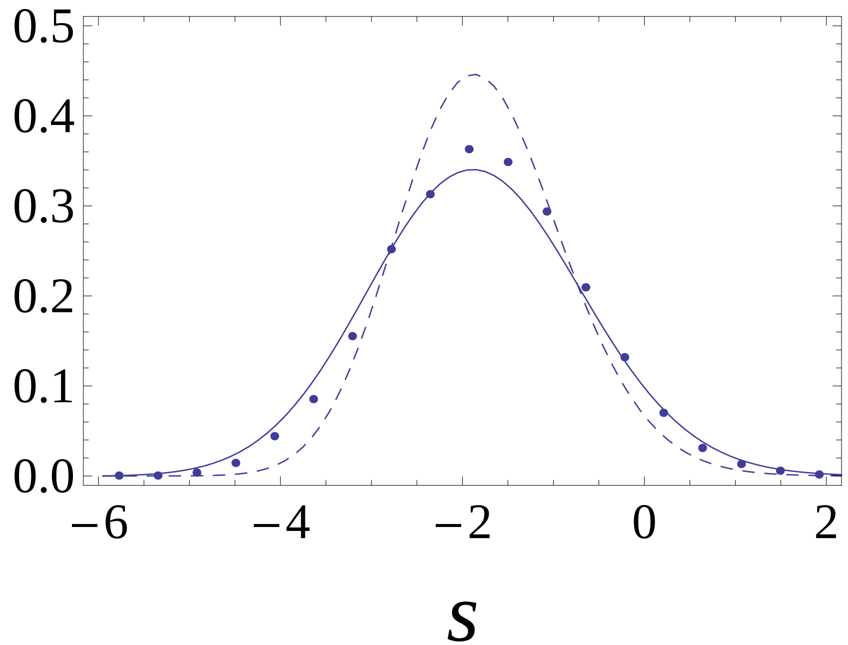
$$F_t(s) = \mathbb{P}[\xi_t \le s] = 1 - \int_{-\infty}^{\infty} \exp\big[-\mathrm{e}^{\gamma_t(s-u)} \big]$$

$$\times \Big(\det(1 - P_u(B_t - P_{\mathrm{Ai}})P_u) - \det(1 - P_u B_t P_u) \Big) \mathrm{d}u$$

where $P_{\mathrm{Ai}}(x,y) = \mathrm{Ai}(x)\mathrm{Ai}(y)$, P_u is the projection onto $[u,\infty)$ and the kernel B_t is

$$B_t(x,y) = \int_{-\infty}^{\infty} \mathrm{d}\lambda \frac{\mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)}{e^{\gamma_t \lambda} - 1}$$

Finite time KPZ distribution and TW



—: exact KPZ density $F'_t(s)$ at $\gamma_t = 0.94$

− −: Tracy-Widom density

- In the large t limit, F_t tends to the GUE Tracy-Widom distribution F_2 from random matrix theory.

Tracy-Widom distributions

For GUE (Gaussian unitary ensemble) with density $P(H)dH \propto e^{-\mathrm{Tr}H^2}dH$ for H: $N \times N$ hermitian matrix, the joint eigenvalue density is (with $\Delta(x)$ Vandelmonde)

$$\frac{1}{Z}\Delta(x)^2 \prod_i e^{-x_i^2}$$

GUE Tracy-Widom distribution

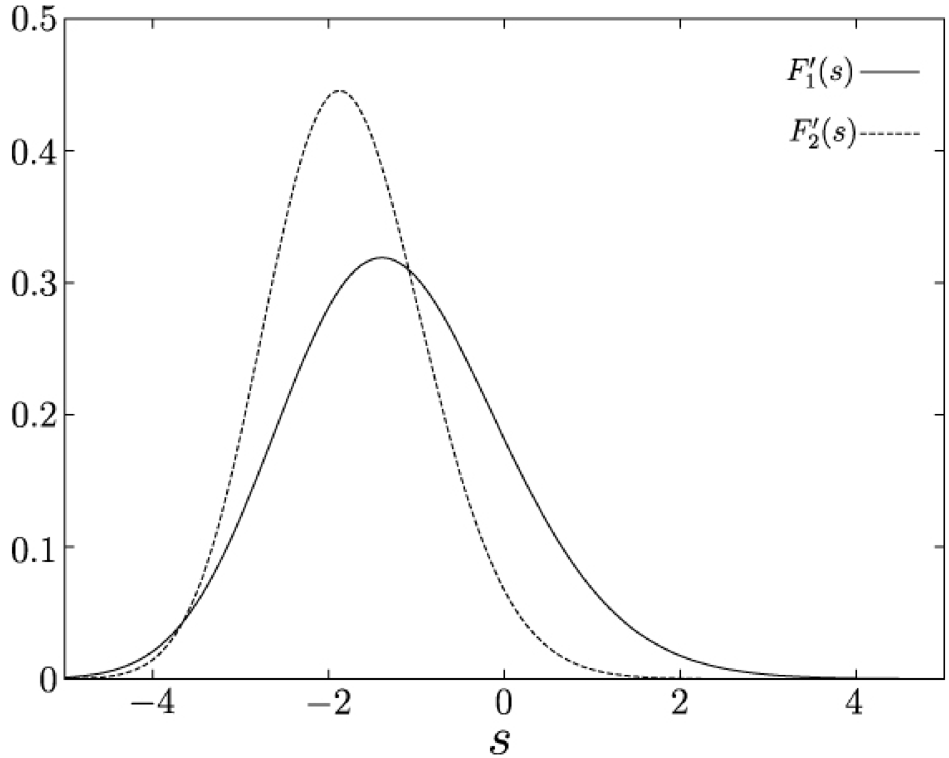
$$\lim_{N\to\infty} \mathbb{P}\left[\frac{x_{\max} - \sqrt{2N}}{2^{-1/2}N^{-1/6}} < s\right] = F_2(s) = \det(1 - P_s K_2 P_s)$$

where P_s: projection onto $[s, \infty)$ and K_2 is the Airy kernel

$$K_2(x, y) = \int_0^\infty \mathrm{d}\lambda \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)$$

There is also GOE TW (F_1) for GOE (Gaussian orthogonal ensemble, real symmetric matrices, for flat surface)

Probability densities of Tracy-Widom distributions



F_2'(GUE), F_1'(GOE)

Stationary 2pt correlation

Not only the height/current distributions but correlation functions show universal behaviors.

- For the KPZ equation, the Brownian motion is stationary.

$$h(x,0) = B(x)$$

  where $B(x), x \in \mathbb{R}$ is the two sided BM.

- Two point correlation

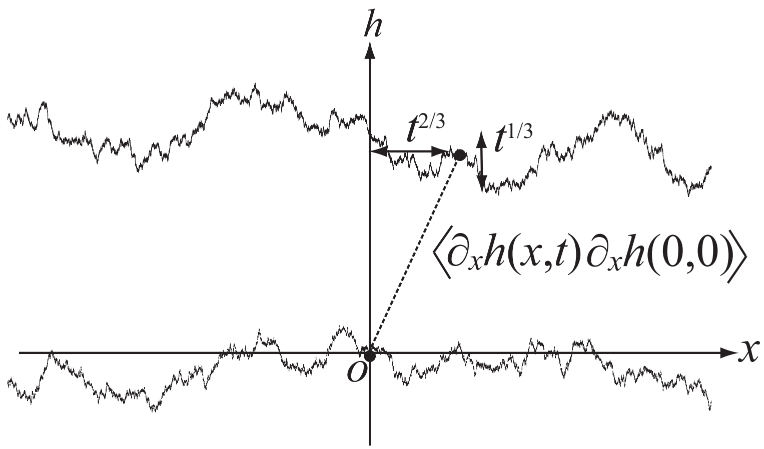
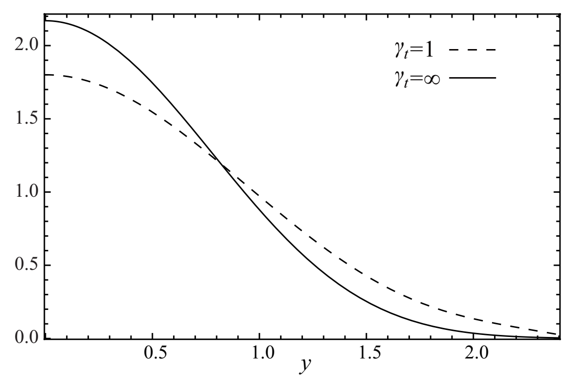


Figure from the formula

Imamura TS (2012)

$$\langle \partial_x h(x,t) \partial_x h(0,0) \rangle = \frac{1}{2}(2t)^{-2/3} g_t''(x/(2t)^{2/3})$$

The figure can be drawn from the exact formula (which is a bit involved though).



Stationary 2pt correlation function $g_t''(y)$ for $\gamma_t := (\frac{t}{2})^{\frac{1}{3}} = 1$.
The solid curve is the scaling limit $g''(y)$.

3.1 Dualities for asymmetric processes

2012-2015 Borodin-Corwin-TS Rigorous replica approach

- For ASEP the n-point function like $\langle \prod_i q^{N(x_i,t)} \rangle$ satisfies the n particle dynamics of the same process (Duality). This is a discrete generalization of δ-Bose gas for KPZ. One can apply the replica approach to get a Fredholm det expression for generating function for $N(x,t)$.

- Rigorous replica: the one for KPZ (which is not rigorous) can be thought of as a shadow of the rigorous replica for ASEP.

- Stationary case(Borodin Corwin Ferrari Veto (2014)), Flat case (Quastel et al (2014), Generalized models (q-Hahn, six-vertex, ...), Plancherel theorem,...

- For ASEP, the duality is related to $U_q(sl_2)$ symmetry.

More general formulation

- Dualities have been an important tool in statistical mechanics (e.g. Kramers-Wannier duality for Ising model).
- For symmetric processes, the duality has been used to study its various properties.

  For symmetric simple exclusion process (SSEP), the n-point function satisfies the n-body problem. This is related to the $SU(2)$ symmetry.

  Another well-known example with duality is the Kipnis-Marchioro-Pressutti (KMP) model of stochastic energy transfer. Its duality is related to the $SU(1,1)$ symmetry.

- As explained, the duality for ASEP is useful to study its current distribution. Its duality is related to $U_q(sl_2)$.
- Carinci Giardina Redig TS (2014,2015) presented a general scheme to construct a duality from a (deformed) symmetry of the process. As an application they have constructed a new process with $U_q(su(1,1))$ symmetry and an asymmetric version of the KMP process.

3.2 A determinantal structure for a finite temperature polymer

2001 O'Connell Yor

Semi-discrete directed polymer in random media

$B_i, 1 \le i \le N$: independent Brownian motions

Energy of the polymer π

$$E[\pi] = B_1(s_1) + B_2(s_1, s_2) + \cdots + B_N(s_{N-1}, t)$$

with $B_j(s,t) = B_j(t) - B_j(s),\ j = 2, \cdots, N$ for $s < t$

Partition function ($\beta = 1/k_B T$: inverse temperature)

$$Z_N(t) = \int_{0<s_1<\cdots<s_{N-1}<t} e^{\beta E[\pi]} ds_1 \cdots ds_{N-1}$$

In continuous limit, this becomes the polymer for KPZ equation.

Zero-temperature limit

In the $T \to 0$ (or $\beta \to \infty$) limit

$$f_N(t) := \lim_{\beta\to\infty} F_N(t) = \max_{0<s_1<\cdots<s_{N-1}<t} E[\pi]$$

2001 Baryshnikov Connection to random matrix theory

$$\text{Prob}\,(f_N(1) \le s) = \int_{(-\infty,s]^N} \prod_{j=1}^{N} dx_j \cdot P_{\text{GUE}}(x_1, \cdots, x_N),$$

$$P_{\text{GUE}}(x_1, \cdots, x_N) = \prod_{j=1}^{N} \frac{e^{-x_j^2/2}}{j!\sqrt{2\pi}} \cdot \prod_{1\le j<k\le N} (x_k - x_j)^2$$

where $P_{\text{GUE}}(x_1, \cdots, x_N)$ is the probability density function of the eigenvalues in the Gaussian Unitary Ensemble (GUE)

A generalization to finite β

Thm Imamura TS (2015)

$$\mathbb{E}\left(e^{-\frac{e^{-\beta u}Z_N(t)}{\beta^{2(N-1)}}}\right) = \int_{\mathbb{R}^N} \prod_{j=1}^{N} dx_j f_F(x_j - u) \cdot W(x_1, \cdots, x_N; t)$$

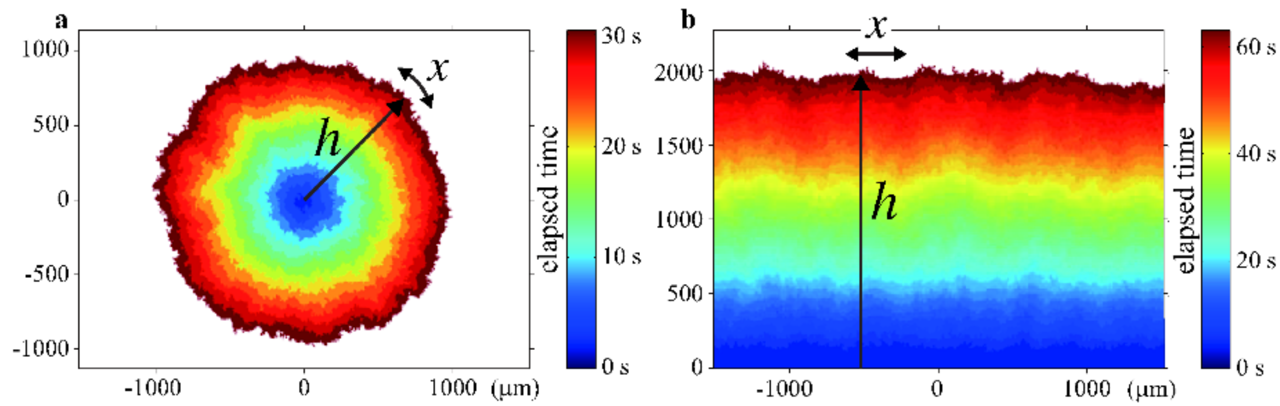
$$W(x_1, \cdots, x_N; t) = \prod_{j=1}^{N} \frac{1}{j!} \prod_{1 \le j < k \le N} (x_k - x_j) \cdot \det\left(\psi_{k-1}(x_j; t)\right)_{j,k=1}^{N}$$

where $f_F(x) = 1/(e^{\beta x} + 1)$ is the Fermi distribution function and

$$\psi_k(x; t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dw e^{-iwx - w^2 t/2} \frac{(iw)^k}{\Gamma(1 + iw/\beta)^N}$$

Proof by generalizing Warren's process on the Gelfand-Tsetlin cone

4. Universality1: Expeirments by Takeuchi-Sano



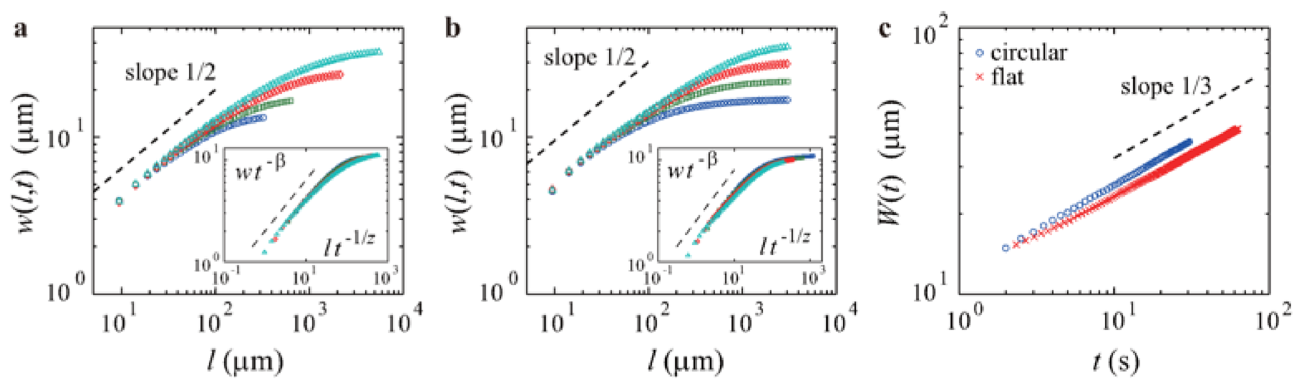


Figure 2 | Family-Vicsek scaling. a,b, Interface width $w(l, t)$ against the length scale l at different times t for the circular (a) and flat (b) interfaces. The four data correspond, from bottom to top, to $t = 2.0$ s, 4.0 s, 12.0 s and 30.0 s for the panel a and to $t = 4.0$ s, 10.0 s, 25.0 s and 60.0 s for the panel b. The insets show the same data with the rescaled axes. c, Growth of the overall width $W(t) \equiv \sqrt{\langle [h(x,t) - \langle h \rangle]^2 \rangle}$. The dashed lines are guides for the eyes showing the exponent values of the KPZ class.

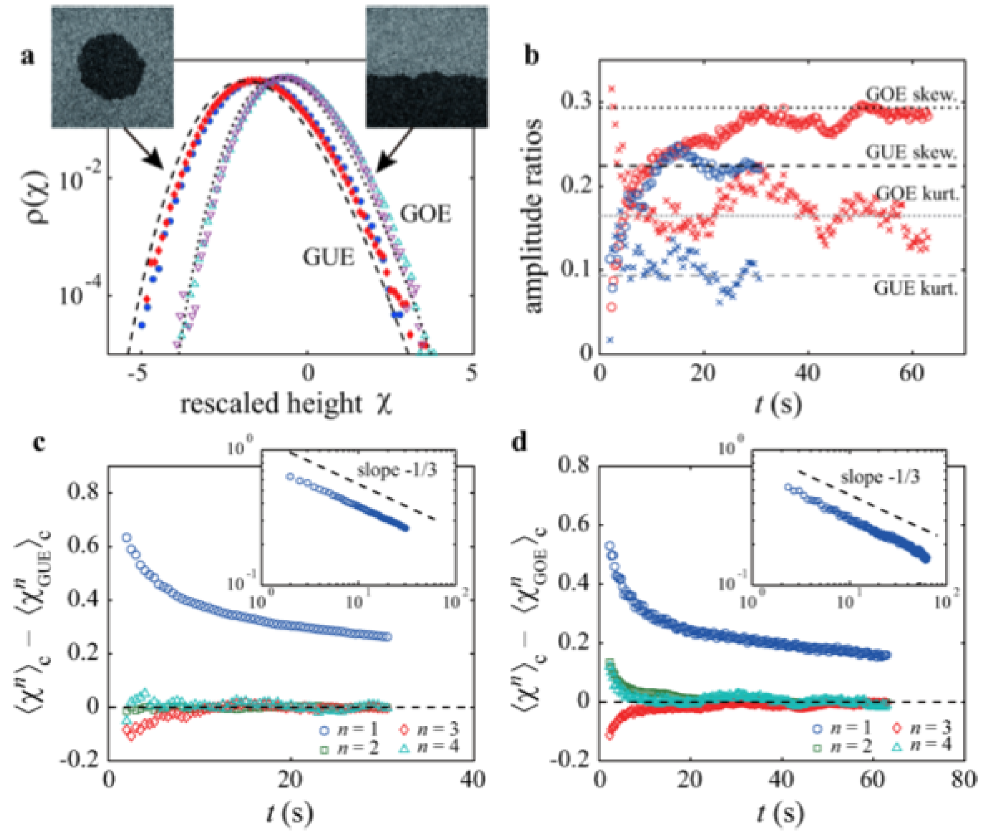


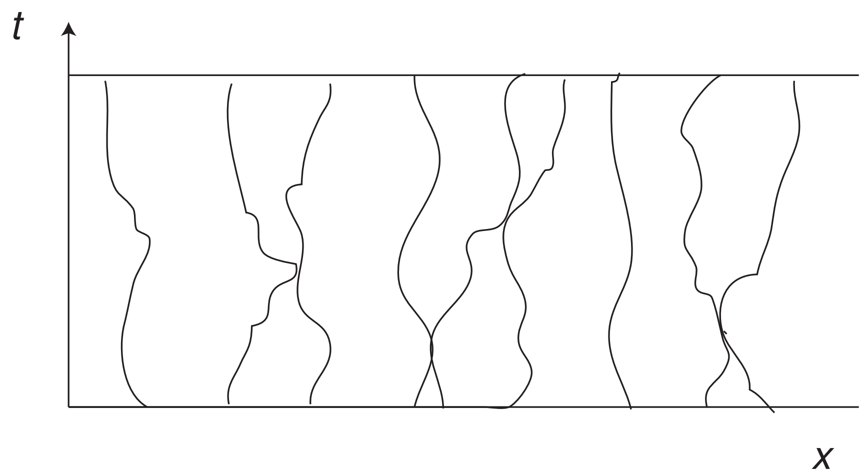
Figure 3 | Universal fluctuations. **a**, Histogram of the rescaled local height $\chi = (h - v_\infty t)/(\Gamma t)^{1/3}$. The blue and red solid symbols show the histograms for the circular interfaces at $t = 10$ s and 30 s; the light blue and purple open symbols are for the flat interfaces at $t = 20$ s and 60 s, respectively. The dashed and dotted curves show the GUE and GOE TW distributions, respectively. Note that for the GOE TW distribution χ is multiplied by $2^{-2/3}$ in view of the theoretical prediction[31]. **b**, The skewness (circle) and the kurtosis (cross) of the distribution of the interface fluctuations for the circular (blue) and flat (red) interfaces. The dashed and dotted lines indicate the values of the skewness and the kurtosis of the GUE and GOE TW distributions[31]. **c**, **d**, Differences in the cumulants between the experimental data $\langle \chi^n \rangle_c$ and the corresponding TW distributions $\langle \chi^n_{\mathrm{GUE}} \rangle_c$ for the ciruclar interfaces (c) and $\langle \chi^n_{\mathrm{GOE}} \rangle_c$ for the flat interfaces (d). The insets show the same data for $n = 1$ in logarithmic scales. The dashed lines are guides for the eyes with the slope $-1/3$.

Takeuchi Sano TS Spohn, Sci. Rep. 1,34(2011)

Universality 2: Brownian motion with oblique reflection

TS Spohn (2014)

We consider a system of Brownian motions in one-dimension in which the jth particle is reflected by the $(j+1)$th particle with weight p and also by the $(j-1)$th particle with weight q, where $j \in \mathbb{N}$ and $p \geq 0, q \geq 0, p+q=1$.



Time evolution equation

Let us denote the position of the m particles by $y(t) = (y_1(t), \ldots, y_m(t))$ with $y_1(t) \leq \ldots \leq y_m(t)$.

The probability density of the position evolves by

$$\partial_t f = \frac{1}{2} \Delta_y f$$

with the boundary conditions for coinciding positions,

$$(p\partial_j - q\partial_{j+1}) f \big|_{y_j = y_{j+1}} = 0$$

This represents the oblique reflection of particles.

Diffusive particle systems in KPZ class

- By using the duality again, one can prove that this interacting Brownian motions with oblique reflection is in the KPZ universality class.

- The above system with oblique reflection is obtained from interacting Brownian motions with the potential V,

$$dx_j(t) = -\big(pV'(x_j(t)-x_{j+1}(t))+qV'(x_j(t)-x_{j-1}(t))\big)dt+dB_j$$

  by taking the $\epsilon \to 0$ limit of the the scaled potential $V_\epsilon(u) = V(u/\epsilon)$.

- Possible realization by colloidal particles? (Bechinger, Seifert(?))

Universality 3: Beijeren-Spohn Conjecture

- The scaled KPZ 2-pt function would appear in rather generic 1D multi-component systems

  This would apply to (deterministic) 1D Hamiltonian dynamics with three conserved quantities, such as the Fermi-Pasta-Ulam chain with $V(x) = \frac{x^2}{2} + \alpha\frac{x^3}{3!} + \beta\frac{x^4}{4!}$.

  There are two sound modes with velocities $\pm c$ and one heat mode with velocity 0. The sound modes would be described by KPZ; the heat mode by $\frac{5}{3}$—Levy.

- Now there have been several attempts to confirm this by numerical simulations. Mendl, Spohn, Dhar, Beijeren, Lepri, Saito, $\cdots$

- Possibly applicable to quantum systems as well.

Mendl Spohn

MD simulations for shoulder potential

$$V(x) = \infty\ (0 < x < \tfrac{1}{2}), 1(\tfrac{1}{2} < x < 1),\ 0(x > 1)$$

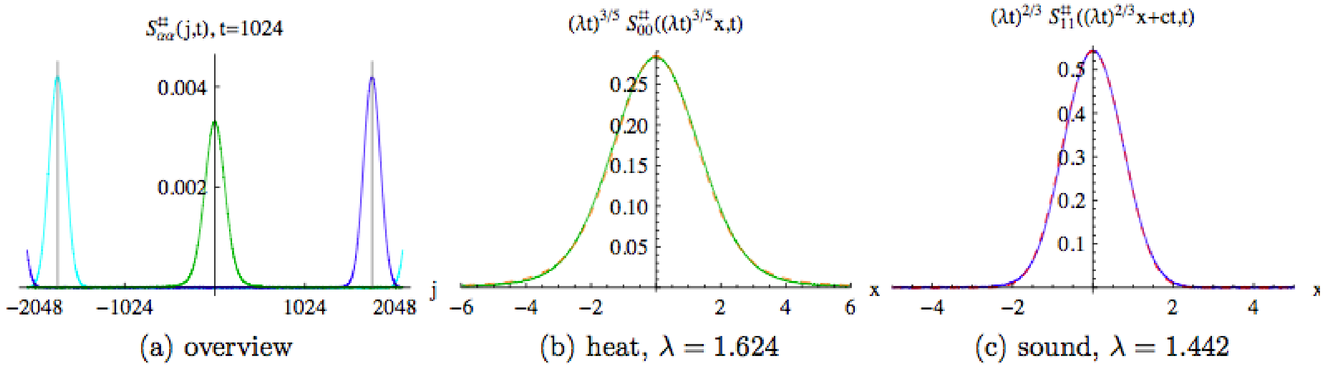


Figure 1: (Color online) MD simulation of an equal mass chain with shoulder potential as defined in Eq. (2.2) and parameters $N = 4096$, $p = 1.2$, $\beta = 2$, at $t = 1024$. (a) Diagonal matrix entries, $S^{\sharp}_{\alpha\alpha}(j,t)$, of the correlator. The gray vertical lines show the sound speed predicted from theory. The tails of the sound peaks reappear on the opposite side due to periodic boundary conditions. (b) Rescaled heat and (c) right sound peak. The theoretical scaling exponents are used and λ is fitted numerically to minimize the L^1-distance between simulation and prediction. The dashed orange curve is the predicted $\frac{5}{3}$-Levy distribution $f_{\mathrm{L},5/3}$ and the dashed red curve shows f_{KPZ}.

Stochastic model

The conjecture would hold also for stochastic models with more than one conserved quantities.

Arndt-Heinzel-Rittenberg(AHR) model (1998)

- Rules

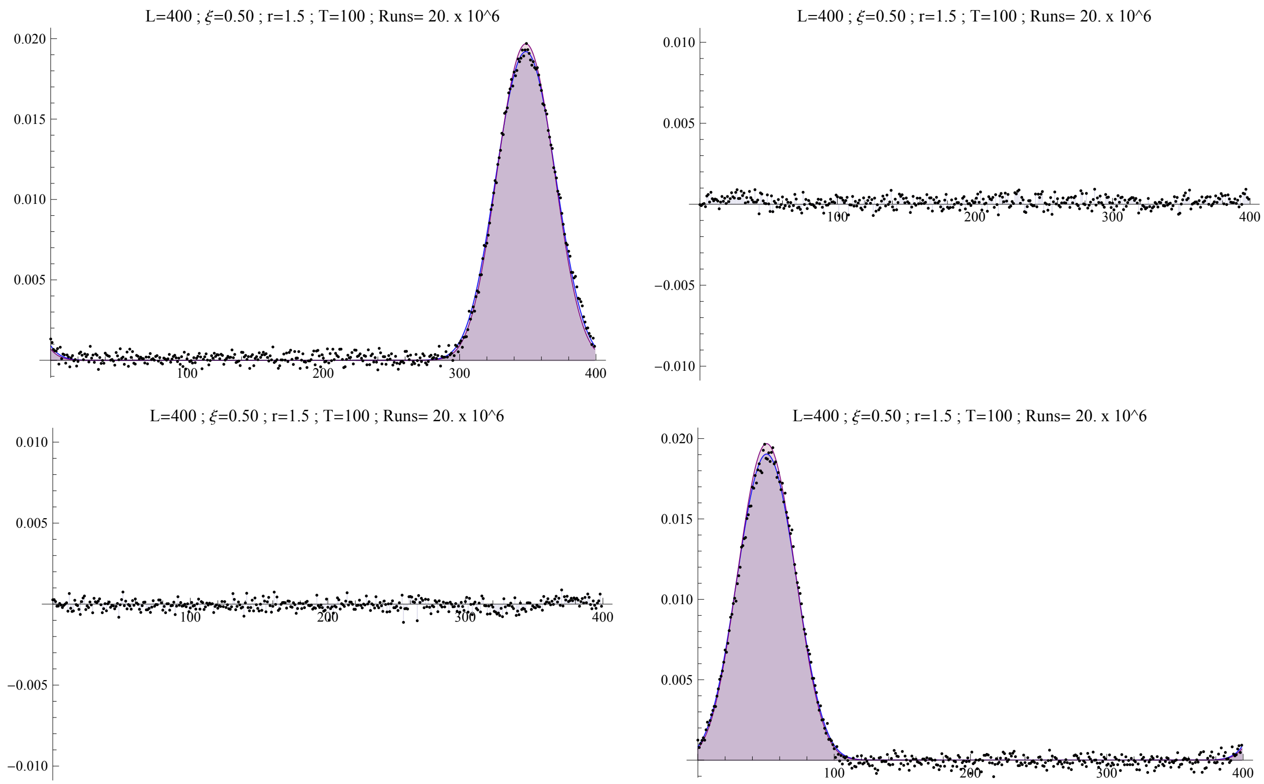
$$+\,\mathbf{0} \xrightarrow{\boldsymbol{\alpha}} \mathbf{0}\,+$$

$$\mathbf{0}\,- \xrightarrow{\boldsymbol{\alpha}} -\,\mathbf{0}$$

$$+\,- \xrightarrow{\mathbf{1}} -\,+$$

- Two conserved quantities (numbers of $+$ and $-$ particles).
- Exact stationary measure is known in a matrix product form.

2013 Ferrari TS Spohn



The KPZ 2pt correlation describes those for the two modes.

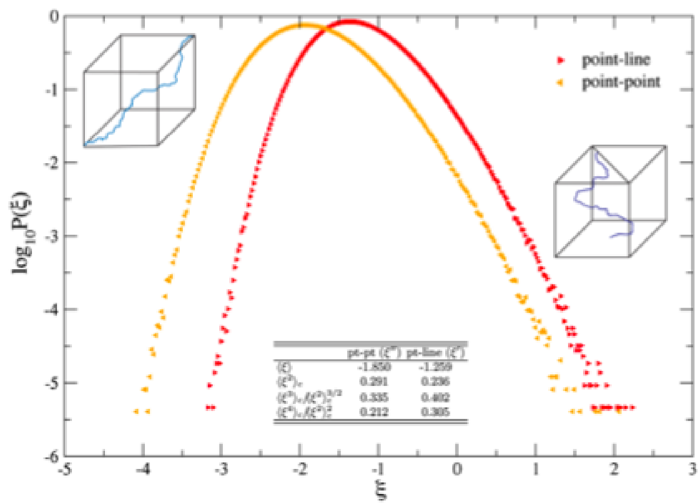
Proving the conjecture for this process seems already difficult.

KPZ in higher dimension?

In higher dimensions, there had been several conjectures for exponents. There are almost no rigorous results.

2012 Halpin-Healy

New extensive Monte-Carlo simulations in 2D on the distributions.



	pt-pt (ξ^∞)	pt-line (ξ^∞)
$\langle\xi\rangle$	-1.850	-1.259
$\langle\xi^2\rangle_c$	0.291	0.236
$\langle\xi^3\rangle_c/\langle\xi^2\rangle_c^{3/2}$	0.335	0.402
$\langle\xi^4\rangle_c/\langle\xi^2\rangle_c^2$	0.212	0.305

FIG. 4 (color online). Universal PDFs: 2 + 1 DPRM *point-point* and *point-line* geometries. Table inset: Distribution moments.

New universal distributions?

5. Summary

- KPZ equation is a model equation to describe surface growth but is of great importance from wider perspective.
- One can write down explicit formulas for its height distribution and the stationary space-time two point correlation function.
- The understanding the mechanism of the exact solvability has deepened considerably. The duality and free fermionic structure have been playing important roles.
- There is a strong universality associated with the KPZ equation. There would be many other experimental relevance. The appearance of KPZ universality seems much wider than considered before. Understanding its nature is an outstanding challenge for the future.

"Derivation"

- Diffusion $$\partial_t h(x,t) = \tfrac{1}{2}\partial_x^2 h(x,t)$$

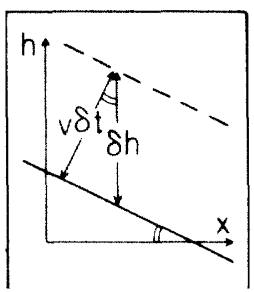
  Not enough: no fluctuations in the stationary state

- Add noise: Edwards-Wilkinson equation

$$\partial_t h(x,t) = \tfrac{1}{2}\partial_x^2 h(x,t) + \eta(x,t)$$

  Not enough: does not give correct exponents

- Add nonlinearity $(\partial_x h(x,t))^2 \Rightarrow$ KPZ equation



$$\begin{aligned}\partial_t h &= v\sqrt{1+(\partial_x h)^2}\\ &\simeq v + (v/2)(\partial_x h)^2 + \ldots\end{aligned}$$

The KPZ equation is not well-defined

- With $\eta(x,t)$" $=$ "$dB(x,t)/dt$, the equation for Z can be written as (Stochastic heat equation)

$$dZ(x,t) = \frac{1}{2}\frac{\partial^2 Z(x,t)}{\partial x^2}dt + Z(x,t) \times dB(x,t)$$

Here $B(x,t)$ is the cylindrical Brownian motion with covariance $dB(x,t)dB(x',t) = \delta(x-x')dt$.

- Interpretation of the product $Z(x,t) \times dB(x,t)$ should be Stratonovich $Z(x,t) \circ dB(x,t)$ since we used usual calculus. Switching to Ito by $Z(x,t) \circ dB(x,t) = Z(x,t)dB(x,t) + dZ(x,t)dB(x,t)$, we encounter $\delta(0)$.

- On the other hand SHE with Ito interpretation from the beginning

$$dZ(x,t) = \frac{1}{2}\frac{\partial^2 Z(x,t)}{\partial x^2}dt + Z(x,t)dB(x,t)$$

is well-defined. For this Z one can define the "Cole-Hopf" solution of the KPZ equation by $h = \log Z$.

So the well-defined version of the KPZ equation may be written as

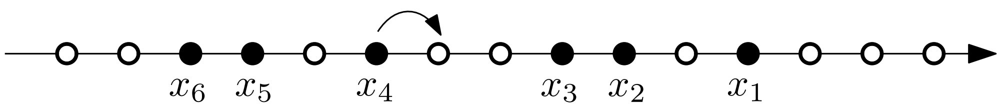
$$\partial_t h(x,t) = \tfrac{1}{2}(\partial_x h(x,t))^2 + \tfrac{1}{2}\partial_x^2 h(x,t) - \infty + \eta(x,t)$$

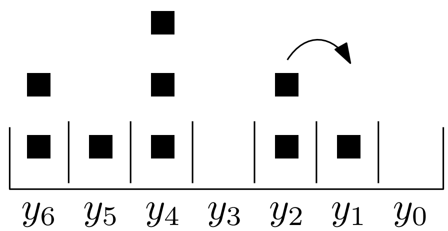
- Hairer found a way to define the KPZ equation without but equivalent to Cole-Hopf (using ideas from rough path and renormalization).

q-TASAEP and q-TAZRP

- q-TASEP 2011 Borodin-Corwin

  A particle i hops with rate $1 - q^{x_{i-1}-x_i-1}$.





- q-TAZRP 1998 TS Wadati

  The dynamics of the gaps $y_i = x_{i-1} - x_i - 1$ is a version of totally asymmetric zero range process in which a particle hops to the right site with rate $1 - q^{y_i}$. The generator of the process can be written in terms of q-boson operators.

- $N(x,t)$: Integrated current for q-TAZRP

Various generalizations and developments

- Flat case (replica) (Le Doussal, Calabrese)

  The limiting distribution is GOE TW F_1 (Geometry dependence)

- Multi-point case (replica) (Dotsenko)

- Stochastic integrability...Connections to quantum integrable systems

  quantum Toda lattice, XXZ chain, Macdonald polynomials...

Polymer and Toda lattice

O'Connell

Semi-discrete finite temperature directed polymer $\cdots$ quantum Toda lattice

Partition function

$$Z_t^N(\beta) = \int_{0<t_1<\ldots<t_{N-1}<t} \exp \beta \left(\sum_{i=1}^{N} (B_i(t_i) - B_i(t_{i-1}) \right)$$

$B_i(t)$: independent Brownian motions

Macdonald process

2011 Borodin, Corwin

- Measure written as

$$\frac{1}{Z} P_\lambda(a) Q_\lambda(b)$$

  where P, Q are Macdonald polynomials.

- A generalization of Schur measure
- Includes Toda, Schur and KPZ as special and limiting cases
- Non-determinantal but expectation value of certain "observables" can be written as Fredholm determinants.

More precisely, for the case with m particles, we consider $y(t) = (y_1(t), \ldots, y_m(t))$ with $y_1(t) \leq \ldots \leq y_m(t)$ which satisfies

$$y_j(t) = y_j + B_j(t) - p\Lambda^{(j,j+1)}(t) + q\Lambda^{(j-1,j)}(t)$$

where $\Lambda^{(0,1)}(t) = \Lambda^{(m,m+1)}(t) = 0$ and

$$\Lambda^{(j,j+1)}(\cdot) = L^{y_{j+1}-y_j}(\cdot, 0)$$

is the local time for $y_{j+1}(\cdot) - y_j(\cdot)$.

Set

$$\mathbb{W}_m^+ = \{y \in \mathbb{R}^m | y_1 \leq \ldots \leq y_m\}$$
$$\mathbb{W}_m^- = \{y \in \mathbb{R}^m | y_1 \geq \ldots \geq y_m\}$$

Generator

Let $\boldsymbol{f} : \mathbb{W}_{\boldsymbol{m}}^{+} \to \mathbb{R}$ be a $\boldsymbol{C^2}$-function and define

$$\boldsymbol{f(y,t) = \mathbb{E}_y\big(f(y(t))\big)}$$

with $\mathbb{E}_{\boldsymbol{y}}$ denoting expectation of the $\boldsymbol{y(t)}$ process starting at $\boldsymbol{y}$. Then

$$\boldsymbol{\partial_t f = \tfrac{1}{2}\Delta_y f}$$

for $\boldsymbol{y} \in (\mathbb{W}_{\boldsymbol{m}}^{+})^{\circ}$ and

$$\boldsymbol{(p\partial_j - q\partial_{j+1})f\big|_{y_j=y_{j+1}} = 0\,,}$$

the directional derivative being taken from the interior of $\mathbb{W}_{\boldsymbol{m}}^{+}$.