Note on high-temperature expansion for fermions *

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(Dated: Last updated on March 5, 2020)

We derive the high temperature expansion formulae for fermions with finite chemical potential, as an extension of the results by Kapusta and Gale.

I. INTRODUCTION OF HIGH-TEMPERATURE EXPANSION

High-temperature expansion in terms of normalized mass m/T of pressure P or equivalently the grand potential density $F_{\rm eff} = \Omega/V = -P$ is useful, for example, to discuss the chiral phase transition.

Pressure of free bosons and fermions is given as

$$P^B = d_B \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3\sqrt{p^2+m^2}} \frac{1}{e^{\sqrt{p^2+m^2}/T} - 1} , \tag{1}$$

$$P^F = \frac{d_F}{2} \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3\sqrt{p^2+m^2}} \left[\frac{1}{e^{(\sqrt{p^2+m^2}-\mu)/T} + 1} + \frac{1}{e^{(\sqrt{p^2+m^2}+\mu)/T} + 1} \right] . \tag{2}$$

where T, m and μ are the temperature, mass and chemical potential, respectively, and $d_{B,F}$ represents the degrees of freedom; for example $d_B = 3$ for pions ($\pi^{0,\pm}$) and $d_F = 4N_cN_f$ for quarks with N_c colors and N_f flavors.

High-temperature expansion in terms of m/T is not just a Taylor expansion of the functions in the pressure integral. Since the mass as well as T and μ provide scales of p, we have non-analytic term proportional to $m^4 \log m$ in pressure. In order to systematically study the small mass region, Kapusta developed a method which enables us to include the singularity. We can find the explicit expression for the high-temperature expansion to $(m/T)^4$ for bosons and fermions at zero chemical potential in Ref. [1], less number of terms are shown for fermions with finite chemical potential.

In this note, we explain how to obtain the high-temperature expansion for fermions at finite chemical potential. The fermion pressure to $(m/T)^4$ is found to be,

$$P^F/d_F = \frac{7}{8}\frac{\pi^2}{90}T^4 + \frac{1}{24}\mu^2T^2 + \frac{\mu^4}{48\pi^2} - \frac{m^2}{16\pi^2}\left[\frac{\pi^2}{3}T^2 + \mu^2\right] - \frac{m^4}{32\pi^2}\left[\log\left(\frac{m}{\pi T}\right) - \frac{3}{4} + \gamma_{\rm E} - H^\nu\left(\frac{\mu}{T}\right)\right] + \mathcal{O}\left(m^6\right) , \tag{3}$$

$$h_5^F(y,\nu) = \frac{7\zeta(4)}{32} + \frac{\zeta(2)}{16}\nu^2 + \frac{\nu^4}{192} - \frac{y^2}{32}\left[\zeta(2) + \frac{\nu^2}{2}\right] - \frac{y^4}{128}\left[\log\left(\frac{y}{\pi}\right) - \frac{3}{4} + \gamma_{\rm E} - H^\nu(\nu)\right] + \mathcal{O}(y^6) , \tag{4}$$

$$H^\nu(\nu) = 2\left(\frac{\nu}{\pi}\right)^2 \sum_{l=1}^{\infty} \frac{1}{(2l-1)[(2l-1)^2 + (\nu/\pi)^2]} = 2\sum_{k=1}^{\infty}(-1)^{k-1}\left(\frac{\nu}{\pi}\right)^{2k}\left[1 - \frac{1}{2^{2k+1}}\right]\zeta(2k+1) , \tag{5}$$

where $\gamma_{\rm E} = 0.5772156649\cdots$ is the Euler's constant and $\zeta(n)$ is the zeta function; $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, $\zeta(3) = 1.2020569031595942854\cdots$, $\zeta(5) = 1.0369277551433699263\cdots$, $\zeta(7) = 1.0083492773819228268\cdots$, $\zeta(9) = 1.0020083928260822144\cdots$. The first three terms show the massless results (Stefan-Boltzmann limit), and the terms proportional to m^2 and m^4 show the modification of pressure by finite mass.

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*Prepared as a part of the lecture note in "Dense Matter 2015" (Helmholtz International Summer School & Dubna International Advanced School on Theoretical Physics / DIAS-TH), Jun.29-Jul.11, 2015, Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Russia.

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II. HIGH-TEMPERATURE EXPANSION

A. h functions

Following Ref. [1], we introduce h functions. Pressure is represented by the function h_5,

$$P^{B,F} = \frac{4T^4 d_{B,F}}{\pi^2}\, h_5^{B,F}\left(y=\frac{m}{T}, \nu=\frac{\mu}{T}\right) , \tag{6}$$

where $h_n^{B,F}$ are given by the following integral [1],

$$h_n^B(y) = \frac{1}{(n-1)!}\int_0^\infty \frac{x^{n-1}dx}{\sqrt{x^2+y^2}} \frac{1}{e^{\sqrt{x^2+y^2}}-1} , \tag{7}$$

$$h_n^F(y,\nu) = \frac{1}{2(n-1)!}\int_0^\infty \frac{x^{n-1}dx}{\sqrt{x^2+y^2}} \left\{ \frac{1}{e^{\sqrt{x^2+y^2}-\nu}+1} + \frac{1}{e^{\sqrt{x^2+y^2}+\nu}+1} \right\} . \tag{8}$$

These functions are found to satisfy the following recursion formulae,

$$\frac{dh_{n+1}}{dy} = -\frac{y}{n} h_{n-1} . \tag{9}$$

This recursion is derived by using the relation $df/dy = y/x\, df/dx$ for a function of $\omega = \sqrt{x^2+y^2}$. Thus if we know massless integrals, $h_3(y=0)$ and $h_5(y=0)$, and the $n=1$ function, $h_1(y,\nu)$, h_5 is obtained by using the recursion formulae.

$$h_3(y) = h_3(0) - \frac{1}{2}\int_0^y y' dy' h_1(y') , \tag{10}$$

$$h_5(y) = h_5(0) - \frac{1}{4}\int_0^y y' dy' h_3(y') = h_5(0) - \frac{y^2}{8} h_3(0) + \frac{1}{8}\int_0^y y' dy' \int_0^{y'} y'' dy'' h_1(y'') . \tag{11}$$

B. Massless integrals

At zero chemical potentials, massless integrals are obtained by using the expansion of the boson and fermion distribution functions,

$$h_n^B(0) = \frac{1}{(n-1)!}\int_0^\infty \frac{x^{n-2}dx}{e^x-1} = \frac{1}{(n-1)!}\sum_{\ell=1}^\infty \int_0^\infty x^{n-2}e^{-\ell x}\,dx = \frac{1}{(n-1)!}\sum_{\ell=1}^\infty \frac{(n-2)!}{\ell^n} = \frac{\zeta(n-1)}{n-1} \quad (n \geq 2) , \tag{12}$$

$$h_n^F(0,0) = \frac{1}{(n-1)!}\int_0^\infty \frac{x^{n-2}dx}{e^x+1} = \frac{1}{(n-1)!}\sum_{\ell=1}^\infty (-1)^{\ell-1} \int_0^\infty x^{n-2}e^{-\ell x}\,dx = \frac{\zeta(n-1)}{n-1}\left(1-\frac{1}{2^{n-2}}\right) \quad (n \geq 2) . \tag{13}$$

Explicit values for $n = 3, 5$ are given as

$$h_3^B(0) = \zeta(2)/2 = \pi^2/12 , \quad h_5^B(0) = \zeta(4)/4 = \pi^4/360 , \tag{14}$$

$$h_3^F(0,0) = \zeta(2)/4 = \pi^2/24 , \quad h_5^F(0,0) = 7\zeta(4)/32 = 7\pi^4/2880 . \tag{15}$$

Massless fermion integrals at finite ν contains several terms.

$$h_{2n+1}^F(0,\nu) = \frac{1}{2(2n)!}\int_0^\infty x^{2n-1}\,dx \left(\frac{1}{e^{x-\nu}+1} + \frac{1}{e^{x+\nu}+1}\right)$$

$$= \frac{\nu^{2n}}{4n\,(2n)!} + \frac{1}{2n}\sum_{k=0}^{n-1} \frac{\nu^{2k}}{(2k)!}\zeta(2(n-k)) \left[1 - \frac{1}{2^{2(n-k)-1}}\right] , \tag{16}$$

$$h_3^F(0,\nu) = \frac{\zeta(2)}{4} + \frac{\nu^2}{8} , \tag{17}$$

$$h_5^F(0,\nu) = \frac{7}{8}\frac{\zeta(4)}{4} + \frac{\zeta(2)\nu^2}{16} + \frac{\nu^4}{8\cdot 4!} , \tag{18}$$

$$\tag{19}$$

This relation is obtained by using the property of the Fermi distribution function,

$$\begin{aligned}
&\int_0^\infty f(x)dx\left(\frac{1}{e^{x-\nu}+1}+\frac{1}{e^{x+\nu}+1}\right)\\
&=\int_0^\nu f(\nu-x)dx-\int_0^\nu dx\,\frac{f(\nu-x)+f(x-\nu)}{e^x+1}+\int_0^\infty dx\,\frac{f(x+\nu)+f(x-\nu)}{e^x+1}\ , \qquad (20)
\end{aligned}$$

For $f(x)=x^{2n-1}$, the second term vanishes and we get Eq. (16).

C. High-temperature expansion for bosons

$n=1$ integrals requires special care, since it is divergent at small x. Following Ref. [1], we use the following identities for bosons to separate the singular contribution,

$$\frac{1}{\omega}\,\frac{1}{e^\omega-1}=\frac{1}{\omega^2}-\frac{1}{\omega}+2\sum_{l=1}^{\infty}\frac{1}{\omega^2+(2\pi l)^2}\ . \qquad (21)$$

We can obtain this identity by considering the contour integral,

$$I^B(\omega)=\oint_{\mathcal{C}}\frac{dz}{2\pi}\,\frac{1}{e^{iz}-1}\,\frac{1}{z^2+\omega^2}\ , \qquad (22)$$

where $\mathcal{C}$ represents the contour integral around the poles of the first term, $z=2\pi l$ with integer l. By evaluating the residues at $z=2\pi l$, we get $\sum_l 1/[\omega^2+(2\pi l)^2]$. The integral can be evaluated in a different way. By replacing the integral on the upper and lower contours, we get

$$I^B(\omega)=-\oint_{\mathcal{C}_{\mathrm{U}}+\mathcal{C}_{\mathrm{D}}}\frac{dz}{2\pi}\,\frac{1}{e^{iz}-1}\,\frac{1}{z^2+\omega^2}=\frac{1}{2\omega}+\frac{1}{\omega}\,\frac{1}{e^\omega-1}\ , \qquad (23)$$

where two poles of the second function, $z=\pm i\omega$, contribute the integral.

We substitute the identity Eq. (21) into h_1 in Eq. (7), then we get

$$\begin{aligned}
h_1^B(y)&=\int_0^\infty\frac{dx}{\omega}\,\frac{1}{e^\omega-1}=\lim_{L\to\infty}\int_0^{2\pi L}dx\left[\frac{1}{\omega^2}-\frac{1}{2\omega}+2\sum_{l=1}^{\infty}\frac{1}{\omega^2+(2\pi l)^2}\right]\quad(\omega=\sqrt{x^2+y^2})\\
&=\frac{\pi}{2y}+\lim_{L\to\infty}\left\{-\frac{1}{2}\,\mathrm{arcsinh}\,(2\pi L/y)+\sum_{l=1}^{\infty}\frac{2}{\omega_l}\left[\frac{\pi}{2}-\arctan(\omega_l/2\pi L)\right]\right\}\quad(\omega_l=\sqrt{y^2+(2\pi l)^2})\\
&=\frac{\pi}{2y}+\frac{1}{2}\log\frac{y}{4\pi}+\lim_{L\to\infty}\left\{\frac{1}{2}\sum_{l=1}^{L}\left(\frac{2\pi}{\omega_l}-\frac{1}{l}\right)+\frac{1}{2}\left(\sum_{l=1}^{L}\frac{1}{l}-\log\,L\right)\right\}\\
&\quad+\lim_{L\to\infty}\left\{-\sum_{l=1}^{L}\frac{2}{\omega_l}\arctan(\omega_l/2\pi L)+\sum_{l=L+1}^{\infty}\frac{2}{\omega_l}\arctan(2\pi L/\omega_l)\right\}\ . \qquad (24)
\end{aligned}$$

We have introduced the UV cutoff $2\pi L$, and take the limit $L\to\infty$. The sum over l is divided into two parts, $l\le L$ and $l>L$, and the relation $\pi/2-\arctan x=\arctan(1/x)$. The last line in Eq. (24) is found to vanish.

$$\begin{aligned}
&\lim_{L\to\infty}\left\{-\sum_{l=1}^{L}\frac{2}{\omega_l}\arctan(\omega_l/2\pi L)+\sum_{l=L+1}^{\infty}\frac{2}{\omega_l}\arctan(2\pi L/\omega_l)\right\}\\
&=-\frac{1}{\pi}\int_0^1\frac{d\theta}{\theta}\arctan\theta+\frac{1}{\pi}\int_1^\infty\frac{d\theta}{\theta}\arctan(1/\theta)=0\ . \qquad (25)
\end{aligned}$$

We have replaced the sum by the integral over $\theta = l/L = \lim_{L\to\infty} \omega_l/2\pi L$. The function $h_1^B(y)$ is found to be

$$h_1^B(y) = \frac{\pi}{2y} + \frac{1}{2}\log\frac{y}{4\pi} + \frac{\gamma_{\rm E}}{2} + \frac{1}{2}\sum_{l=1}^{\infty}\left(\frac{2\pi}{\omega_l} - \frac{1}{l}\right) \tag{26}$$

$$= \frac{\pi}{2y} + \frac{1}{2}\log\frac{y}{4\pi} + \frac{\gamma_{\rm E}}{2} - H^B(y)\,, \tag{27}$$

$$H^B(y) \equiv -\frac{1}{2}\sum_{l=1}^{\infty}\left(\frac{2\pi}{\omega_l(y)} - \frac{1}{l}\right) = \frac{1}{4}\left(\frac{y}{2\pi}\right)^2\zeta(3) - \frac{3}{16}\left(\frac{y}{2\pi}\right)^4\zeta(5) + \frac{5}{32}\left(\frac{y}{2\pi}\right)^6\zeta(7) - \cdots$$

$$= \sum_{k=1}^{\infty}\frac{1}{k!}(-1)^{k-1}\frac{(2k-1)!!}{2^{k+1}}\left(\frac{y}{2\pi}\right)^{2k}\zeta(2k+1)\,. \tag{28}$$

Or the following expression would be more useful for numerical summation,

$$H^B(y) = \frac{\zeta(3)}{4}\left(\frac{y}{2\pi}\right)^2 - \frac{1}{4}\left(\frac{y}{2\pi}\right)^4\sum_{l=1}^{\infty}\frac{2l+\omega_l'(y)}{\omega_l'(y)(l+\omega_l'(y))^2} \quad (\omega_l' = \sqrt{l^2+(y/2\pi)^2}). \tag{29}$$

By using massless integrals, $h_3^B(0)$ and $h_5^B(0)$, and $h_1^B(y)$, the high-temperature expansion of $h_3^B(y)$ and $h_5^B(y)$ are found to be

$$h_3^B(y) = \frac{\zeta(2)}{2} - \frac{\pi y}{4} - \frac{y^2}{8}\left(\log\frac{y}{4\pi} - \frac{1}{2} + \gamma_{\rm E}\right) + \frac{\zeta(3)}{32}\left(\frac{y^2}{2\pi}\right)^2 + \mathcal{O}(y^6)\,, \tag{30}$$

$$h_5^B(y) = \frac{\zeta(4)}{4} - \frac{\zeta(2)}{16}y^2 + \frac{\pi}{48}y^3 + \frac{y^4}{128}\left(\log\frac{y}{4\pi} - \frac{3}{4} + \gamma_{\rm E}\right) - \frac{\zeta(3)}{32}\frac{y^6}{96\pi^2} + \mathcal{O}(y^8)\,. \tag{31}$$

D. High-temperature expansion for fermions at finite chemical potential

Fermion integrals are obtained in a similar but a little more complicated manner. We use the following identities for fermions,

$$\frac{1}{2\omega}\left[\frac{1}{e^{\omega-\nu}+1} + \frac{1}{e^{\omega+\nu}+1}\right] = \frac{1}{2\omega} - \sum_{l=-\infty}^{\infty}\frac{1}{\omega^2+[\pi(2l-1)-i\nu]^2}\,. \tag{32}$$

We can obtain this identity by considering the contour integral,

$$I^F(\omega) = -\oint_{\mathcal{C}}\frac{dz}{2\pi}\frac{1}{e^{iz-\nu}+1}\frac{1}{z^2+\omega^2} = \sum_{l}\frac{1}{\omega^2+[\pi(2l-1)-i\nu]^2}\,, \tag{33}$$

where $\mathcal{C}$ represents the contour integral surrounding the poles of the first term, $z = \pi(2l-1) - i\nu$ with integer l. The integral can be evaluated in a different way. By replacing the integral to that on the upper and lower contours, we get

$$I^F(\omega) = \oint_{\mathcal{C}_{\rm U}+\mathcal{C}_{\rm D}}\frac{dz}{2\pi}\frac{1}{e^{iz-\nu}+1}\frac{1}{z^2+\omega^2} = \frac{1}{2\omega} - \frac{1}{2\omega}\left[\frac{1}{e^{\omega-\nu}+1} + \frac{1}{e^{\omega+\nu}+1}\right]\,, \tag{34}$$

where two poles of the second function, $z = \pm i\omega$, contribute the integral.

We substitute the identity Eq. (32) into h_1 in Eq. (8), and we get

$$
\begin{aligned}
h_1^F(y,\nu) =& \frac{1}{2}\int_0^\infty \frac{dx}{\omega}\left[\frac{1}{e^{\omega-\nu}+1}+\frac{1}{e^{\omega+\nu}+1}\right] \quad (\omega=\sqrt{x^2+y^2}) \\
=& \lim_{L\to\infty}\int_0^{2\pi L} dx\left[\frac{1}{2\omega}-\sum_{l=-\infty}^{\infty}\frac{1}{\omega^2+[\pi(2l-1)-i\nu]^2}\right] \\
=& -\frac{1}{2}\log\frac{y}{\pi}-\frac{1}{2}\gamma_{\rm E}-\frac{1}{2}\sum_{l=1}^{\infty}\left[\frac{\pi}{\omega_l}+\frac{\pi}{\omega_l^*}-\frac{2}{2l-1}\right] \quad (\omega_l=\sqrt{y^2+[\pi(2l-1)-i\nu]^2}) \qquad (35)\\
=& -\frac{1}{2}\log\frac{y}{\pi}-\frac{1}{2}\gamma_{\rm E}+\frac{1}{2}H^F(y,\nu)\ , \qquad (36)\\
H^F(y,\nu) \equiv& -\sum_{l=1}^{\infty}\left[\frac{\pi}{\omega_l}+\frac{\pi}{\omega_l^*}-\frac{2}{2l-1}\right]=H^\nu(\nu)+\mathcal{O}(y^2)\ , \qquad (37)\\
H^\nu(\nu) =& -\sum_{l=1}^{\infty}\left[\frac{1}{2l-1-i\nu/\pi}+\frac{1}{2l-1+i\nu/\pi}-\frac{2}{2l-1}\right]=2\left(\frac{\nu}{\pi}\right)^2\sum_{l=1}^{\infty}\frac{1}{(2l-1)[(2l-1)^2+(\nu/\pi)^2]} \\
=& \frac{7}{4}\zeta(3)\left(\frac{\nu}{\pi}\right)^2-\frac{31}{16}\zeta(5)\left(\frac{\nu}{\pi}\right)^4+\frac{127}{64}\zeta(7)\left(\frac{\nu}{\pi}\right)^6+\cdots \\
=& 2\sum_{k=1}^{\infty}(-1)^{k-1}\left(\frac{\nu}{\pi}\right)^{2k}\left(1-\frac{1}{2^{2k+1}}\right)\zeta(2k+1)\ . \qquad (38)
\end{aligned}
$$

By using massless integrals, $h_3^F(0,\nu)$ and $h_5^F(0,\nu)$, and $n=1$ integral $h_1^F(y,\nu)$, the high-temperature expansion of $h_3^F(y,\nu)$ and $h_5^F(y,\nu)$ are found to be

$$
\begin{aligned}
h_3^F(y,\nu) =& \frac{\zeta(2)}{4}+\frac{\nu^2}{8}+\frac{y^2}{8}\left(\log\frac{y}{\pi}-\frac{1}{2}+\gamma_{\rm E}-H^\nu(\nu)\right)+\mathcal{O}(y^4)\ , \qquad (39)\\
h_5^F(y,\nu) =& \frac{7}{8}\frac{\zeta(4)}{4}+\frac{\zeta(2)\nu^2}{16}+\frac{\nu^4}{8\cdot 4!}-\frac{y^2}{64}\left(2\zeta(2)+\nu^2\right)-\frac{y^4}{128}\left(\log\frac{y}{\pi}-\frac{3}{4}+\gamma_{\rm E}-H^\nu(\nu)\right)+\mathcal{O}(y^6)\ . \qquad (40)
\end{aligned}
$$

Fermion pressure is obtained via the relation $P^F = 4d_F\, T^4\, h_5^F(m/T,\mu/T)/\pi^2$, and we obtain Eq. (3).

III. EXPECTED PHASE BOUNDARY

We shall now apply the high-temperature expansion formula Eq. (40) to guess the phase boundary in the Nambu-Jona-Lasinio (NJL) model [2]. We follow the notation in Ref. [3]. In the mean field treatment of the NJL model, the grand potential density (effective potential) is given as

$$
\begin{aligned}
F_{\rm eff} =& \Omega/V = -P^F + F_{\rm vac} = -\frac{4d_F T^4}{\pi^2}\, h_f^F\left(\frac{m}{T},\frac{\mu}{T}\right)+F_{\rm vac}\ , \qquad (41)\\
F_{\rm vac} =& \frac{\Lambda^2}{2}\sigma^2 - d_F\int\frac{d^3k}{(2\pi)^3}\frac{E(k)}{2} \\
=& \frac{\Lambda^2 m^2}{2G^2}-\frac{d_F\Lambda^2}{2}I\left(\frac{m}{\Lambda}\right) \quad \text{(chiral limit)}\ , \qquad (42)\\
I(x) =& \frac{1}{16\pi^2}\left[\sqrt{1+x^2}(2+x^2)-x^4\log\frac{1+\sqrt{1+x^2}}{x}\right] \\
=& \frac{1}{2\pi^2}\left[1+x^2+\frac{x^4}{2}\left(\log\frac{x}{2}+\frac{1}{4}\right)+\mathcal{O}(x^6)\right]\ , \qquad (43)
\end{aligned}
$$

where $F_{\rm vac}$ is the vacuum effective potential, and $E(k)=\sqrt{k^2+(m_0+G\sigma)^2}$ is the quark energy. The first term in Eq. 42 comes from the bosonization, and the second term shows the zero-point energy, which is negative for fermions. The constituent quark mass is given as $m = m_0 + G\sigma$.

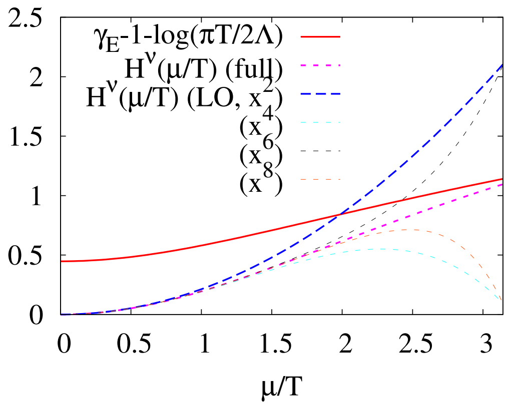

FIG. 1: Contributions to c_4. Solid line shows $c_4^T = \gamma_{\rm E} - 1 - \log(\pi T/2\Lambda)$ at $T = T_c/\sqrt{1+3\mu^2/\pi^2T^2}$ where c_2 vanishes. Other lines show $H^\nu(\mu/T)$. The fourth order coefficient vanishes when two lines cross.

We consider the chiral limit ($m_0 = 0$), then $F_{\rm eff}$ is found to be

$$F_{\rm eff}(m;T,\mu) = F_{\rm eff}(0;T,\mu) + \frac{c_2(T,\mu)}{2}m^2 + \frac{c_4(T,\mu)}{24}m^4 + \mathcal{O}(m^6)\ , \tag{44}$$

$$c_2(T,\mu) = -\frac{d_F}{24}\left[\frac{3}{\pi^2}\Lambda^2\left(1-\frac{8\pi^2}{d_F G^2}\right) - \left(T^2+\frac{3}{\pi^2}\mu^2\right)\right]\ , \tag{45}$$

$$c_4(T,\mu) = \frac{3d_F}{4\pi^2}\left[\gamma_{\rm E} - 1 - \log\left(\frac{\pi T}{2\Lambda}\right) - H^\nu(\mu/T)\right]\ . \tag{46}$$

In vacuum, the chiral symmetry is broken due to negative c_2. This is achieved in the case where the coupling is strong, $G^2 > G_c^2 = 8\pi^2/d_F$. At $\mu = 0$, c_2 increases with increasing T and becomes zero at

$$T = T_c = \frac{\Lambda}{\pi}\sqrt{3\left(1-\frac{G_c^2}{G^2}\right)} < \frac{\sqrt{3}\Lambda}{\pi}\ . \tag{47}$$

At finite μ, c_2 is expected to be zero at [4]

$$T^2 + \frac{3}{\pi^2}\mu^2 = T_c^2\ . \tag{48}$$

As long as c_4 is positive, zero c_2 implies the second order phase transition. If we extrapolated the results to $T = 0$ (it is too much..), we get the transition chemical potential at $T = 0$ as $\mu_c = \pi T_c/\sqrt{3} \simeq 290$ MeV. This is close to the one third of the nucleon mass, and the present estimate may not be too crazy.

At around the empirical values, e.g. $T_c = 160$ MeV and $\Lambda = 600$ MeV, c_4 is positive at $(T,\mu) = (T_c, 0)$, suggesting that the phase transition is the second order in the present setup. Since H^ν is a increasing function of $\nu = \mu/T$, c_4 decreases with increasing ν. In the leading order approximation of H^ν, c_4 becomes zero at around $\nu = 2$ on the phase boundary. The simultaneous disappearance of c_2 and c_4 implies the tricritical point.

To say the truth, c_4 does not vanish on the phase boundary in the conversion radius, $\nu < \pi$, when we use numerically obtained H^ν. Even though, the results of high-temperature expansion clearly suggest the existence of the tricritical point.

IV. SUMMARY

We have shown the high-temperature expansion formulae for fermions with finite chemical potential, as an extension of the results by Kapusta and Gale [1]. The suggested phase boundary from the expansion seems to catch some of the characteristic features of the QCD phase transition. It also suggest the existence of the tricritical point in the chiral limit (critical point with small finite quark mass); the fourth order coefficient decreases with increasing μ/T.

Acknowledgement

This work was supported in part by the Grants-in-Aid for Scientific Research from JSPS (Nos. 23340067, 24340054, 24540271, 15K05079), the Grants-in-Aid for Scientific Research on Innovative Areas from MEXT (Nos. 23105713, 24105001, 24105008), and by the Yukawa International Program for Quark-Hadron Sciences.

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