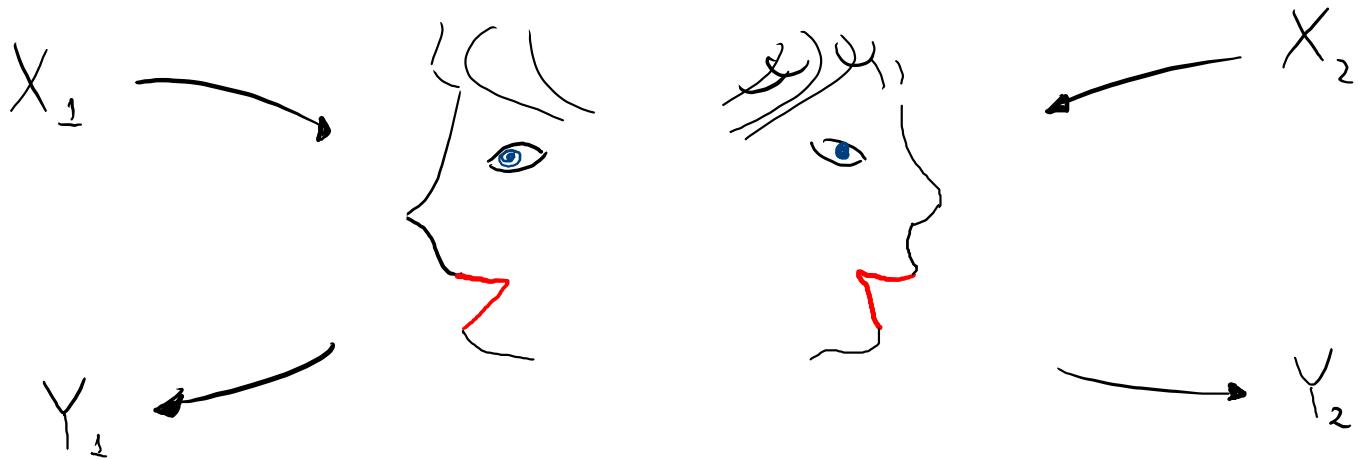


2-headed oracles



$$p(y_1, y_2 | x_1, x_2) \equiv P(Y_1 = y_1, Y_2 = y_2 | X_1 = x_1, X_2 = x_2)$$

- $p \in M_{k_1, k_2}^{e_1, e_2}(\mathbb{R})$
- $\forall x_1, x_2, y_1, y_2 : p(y_1, y_2 | x_1, x_2) \geq 0$
- $\forall x_1, x_2 : \sum_{y_1, y_2} p(y_1, y_2 | x_1, x_2) = 1$

Bipartite phys. system \rightsquigarrow 2-headed oracle

Causality \rightsquigarrow NS-condition:

- oracle cannot be used for $1 \rightarrow 2$ message sending:

$P(Y_2 = y_2 | X_1 = x_1, X_2 = x_2)$ is a func. of $y_2 \not\propto x_2$, only

$$\underbrace{\sum_{y_1} P(y_1, y_2 | x_1, x_2)}$$

- sim. cond. for excluding $1 \leftarrow 2$ message transmissions

Bipartite q. system $\mathcal{H}_A \otimes \mathcal{H}_B$ with (prev. prepared) state $\rho \in S(\mathcal{H}_A \otimes \mathcal{H}_B)$ as a 2-headed oracle:

Alice chooses:

a part. of unity $(A_{y_1}^{x_1})_{y_1}$ on \mathcal{H}_A for each x_1

Bob chooses:

a part. of unity $(B_{y_2}^{x_2})_{y_2}$ on \mathcal{H}_B for each x_2

$$\hookrightarrow P(y_1, y_2 | x_1, x_2) = \text{Tr}(\mathcal{S}(A_{y_1}^{x_1} \otimes B_{y_2}^{x_2}))$$

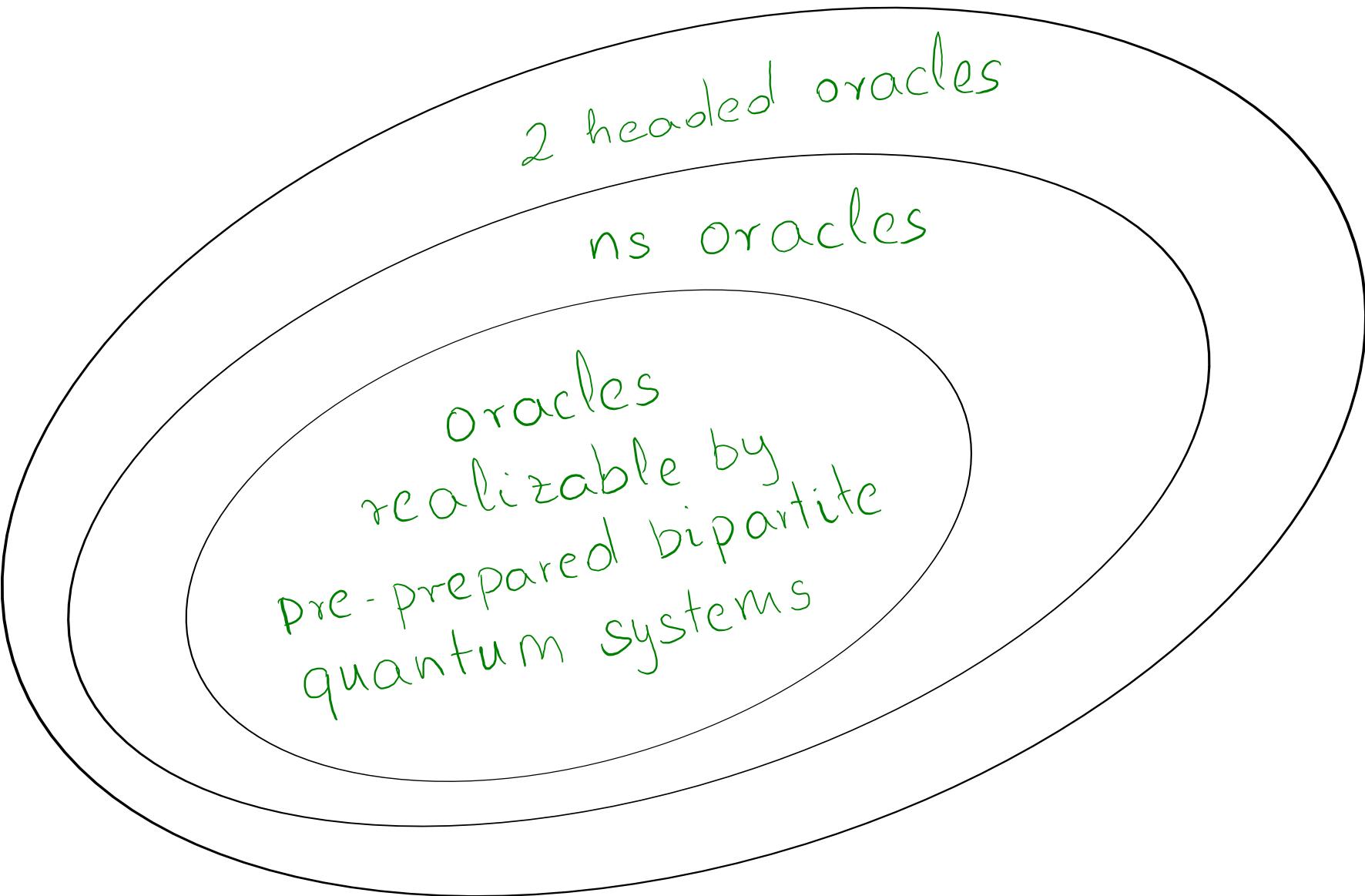
$$P(Y_2 = y_2 \mid X_1 = x_1, X_2 = x_2) = \sum_{y_1} P(y_1, y_2 \mid x_1, x_2) =$$

$$= \sum_{y_1} \text{Tr} \left(\mathcal{S} \left(A_{y_1}^{x_1} \otimes B_{y_2}^{x_2} \right) \right) = \text{Tr} \left(\mathcal{S} \left(\underbrace{\sum_{y_1} A_{y_1}^{x_1}}_{I} \otimes B_{y_2}^{x_2} \right) \right) =$$

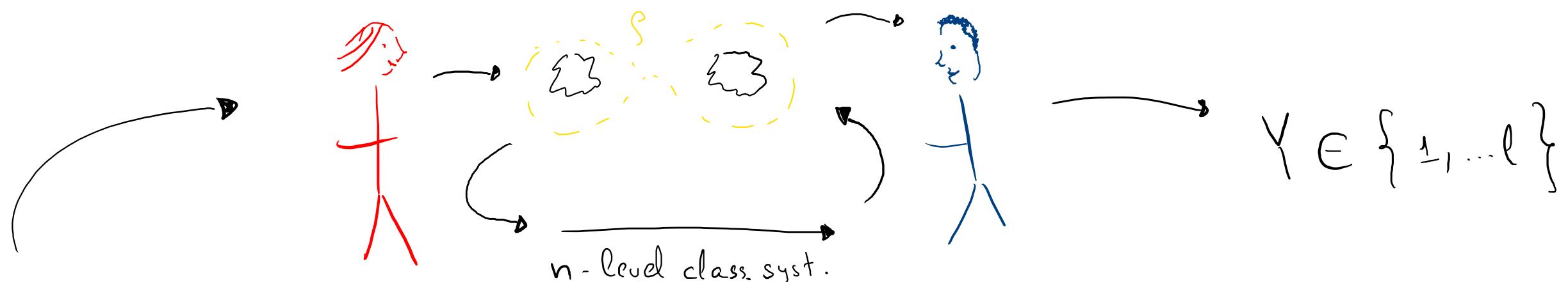
$= \text{Tr} \left(\mathcal{S} \left(I \otimes B_{y_2}^{x_2} \right) \right)$ is a func. of $x_2 \not\propto y_2$ only and

$$\text{Sim. : } P(Y_1 = y_1 \mid X_1 = x_1, X_2 = x_2) = \text{Tr} \left(\mathcal{S} \left(A_{y_1}^{x_1} \otimes I \right) \right)$$

is a func. of $x_1 \not\propto y_1$ only



Can entanglement (i.e. the use of a 2-headed oracle realized by a pre-prepared q. system) improve one-way classical communication over a noiseless classical channel?



$$X \in_R \{1, \dots, k\}$$

(distr. of X and reward = $R(X, Y)$ is public)

Input-output matrix (transition-probabilities) :

$$P(Y=i | X=j) = \sum_{r=1}^n \text{Tr}(g(A_r^{(j)} \otimes B_i^{(r)}))$$

$C_n^e(k,l) :=$ conv. hull of all such \rightarrow matrices
(all choice of partitions $(A_{y_1}^{x_1})_{y_1}$ and $(B_{y_2}^{x_2})_{y_2}$)

More gen.: $C_n^{\text{ent.}}(k,l)$; $C_n^{\text{ns}}(k,l)$.

Triv.: $C_n(k,l) \subset C_n^e(k,l) \subset C_n^{\text{ent.}}(k,l) \subset C_n^{\text{ns}}(k,l)$

Suppose p is an ns -oracle used in this way:



Then $\sum_i P(Y=i | X=i) = \sum_i \sum_{j=1}^n p(j, i | i, j) = \sum_{j=1}^n \sum_i p(j, i | i, j)$

$$\leq \sum_{j=1}^n \sum_i \sum_{j'} p(j', i | i, j) = \sum_{j=1}^n 1 = n$$

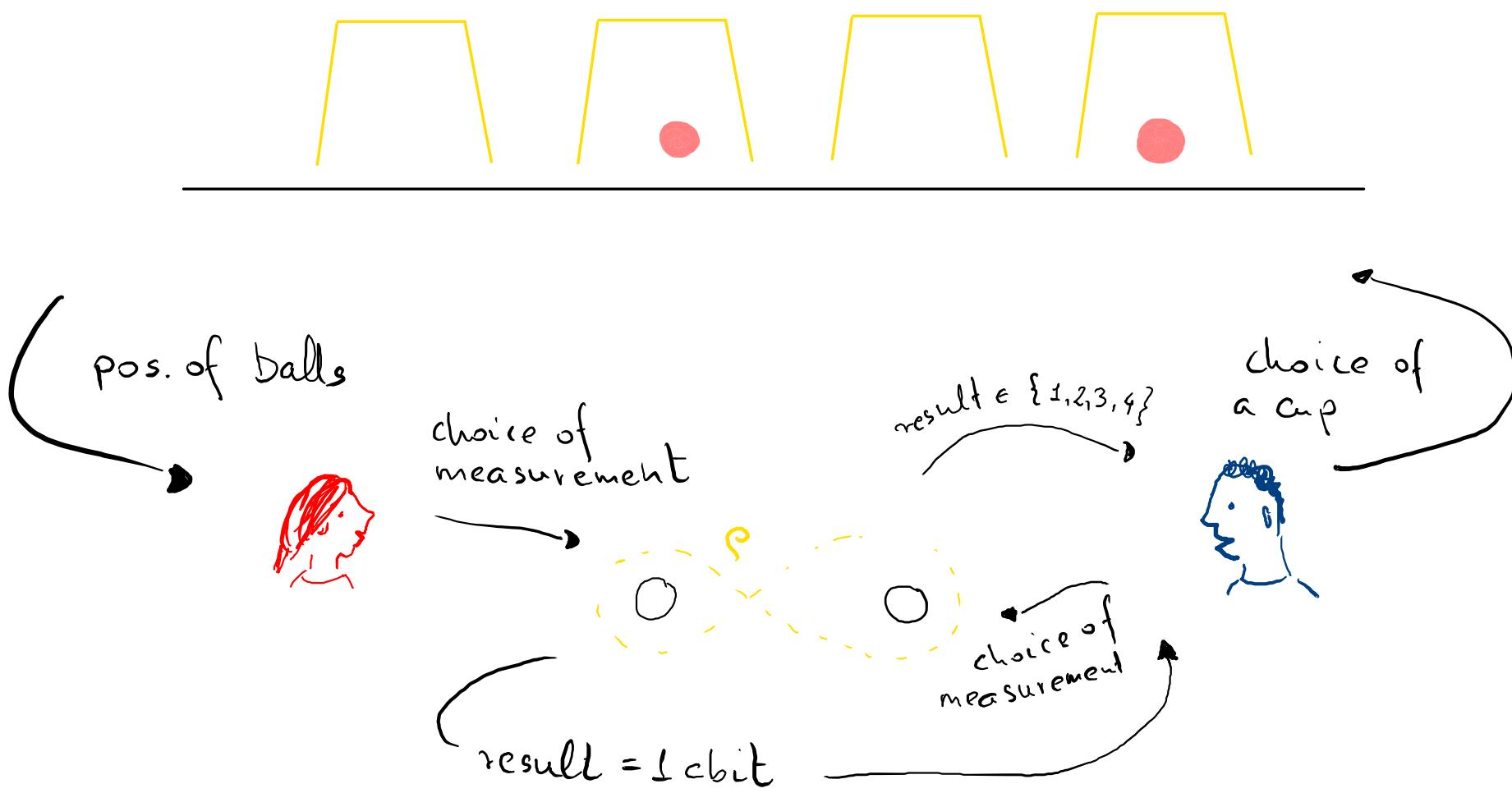
So $\text{st}(C_n^{\text{ns}}) = \text{st}(C_n) = n$. (Easy.)

The hard question: min. of such that ...

- $C_n^P(k, l) \subset C_d(k, l)$ (for a given \mathcal{S})
- $C_n^{\text{ent.}}(k, l) \subset C_d(k, l)$
- $C_n^{\text{ns}}(k, l) \subset C_d(k, l)$

i.e. the signalling dim. of these.

An example: the "4 cups, 2 balls" game, again



$$S \in S(\mathbb{C}^2 \otimes \mathbb{C}^2)$$

$$\rho = | \Psi \rangle \langle \Psi |$$

$$\Psi = \frac{|0,1\rangle - |1,0\rangle}{\sqrt{2}}$$

where

$$|a,b\rangle = |a\rangle \otimes |b\rangle$$

and

$|0\rangle, |1\rangle$ is an ONB of \mathbb{C}^2

WTD for Alice:

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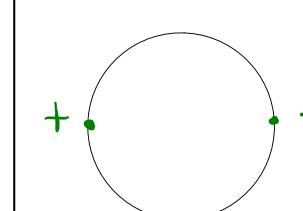
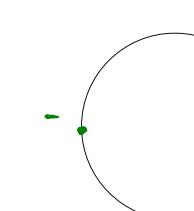
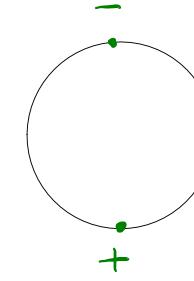
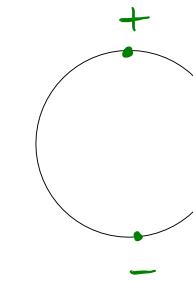
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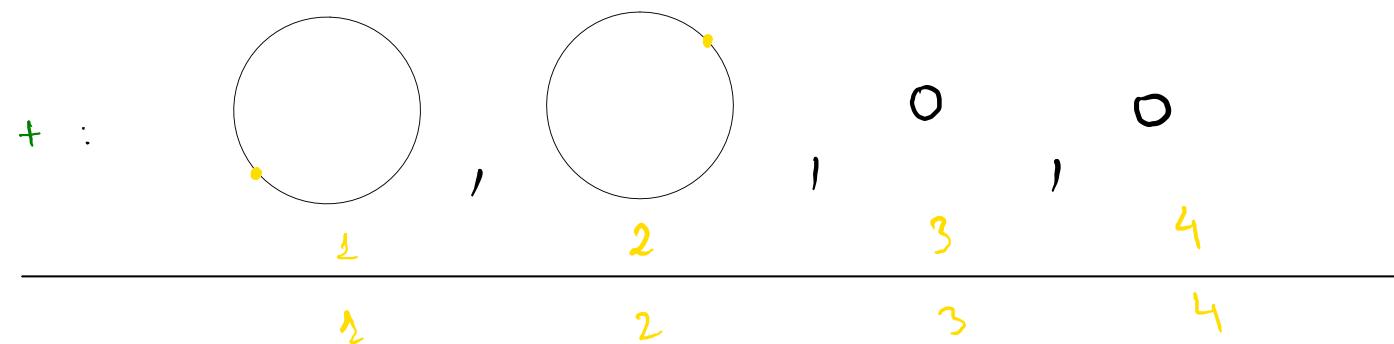
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WTD for Bob:



$$\hookrightarrow p(\text{win}) =$$

$$\frac{1}{6} \left(1 + 1 + \frac{1}{2} \left(1 + \frac{1}{\sqrt{2}} \right) \cdot 4 \right) = \frac{4 + \sqrt{2}}{6} > \frac{5}{6} \quad !! \quad \Rightarrow \quad C_2^P(6,4) \neq C_2(6,4)$$

Conjecture: $\nexists \rho : C_n^{\rho} \subset C_{n^2}$ i.e. $C_n^{\text{ent.}} \subset C_{n^2}$

- for every $d = 2, 3, 4, \dots$ we found a game & ns oracle showing $C_2^{\text{ns}} \not\subset C_d \rightarrow$ sig. dim. of C_n^{ns} ($n \geq 2$) is ∞
- if, on the other hand, the conjecture is true, then "God helps those, who help themselves".
- we can prove the conjecture in a specific case:
we can show that $C_2^{\rho} \subset C_4$ when ρ is max. entangled

Proof : let $\mathcal{S} \in S(\mathfrak{sl}_A \otimes \mathfrak{sl}_B)$ and

- $A_+^{(j)}, A_-^{(j)}$ be a part. of unity on \mathfrak{sl}_A for $\forall j \in \{1, \dots, k\}$
- B_1^+, \dots, B_e^+ and B_1^-, \dots, B_e^- two part. of unity on \mathfrak{sl}_B

Setting $\varepsilon_j^\pm = \text{Tr}_A \left(\left(A_\pm^{(j)} \otimes I \right)^{\frac{1}{2}} \mathcal{S} \left(A_\pm^{(j)} \otimes I \right)^{\frac{1}{2}} \right)$, we have $\varepsilon_j^\pm \geq 0$ and

$$P(Y=i | X=j) = \underbrace{\text{Tr}(\mathcal{S}(A_+^{(j)} \otimes B_i^+))}_{\text{Tr}(\varepsilon_j^+ B_i^+)} + \underbrace{\text{Tr}(\mathcal{S}(A_-^{(j)} \otimes B_i^-))}_{\text{Tr}(\varepsilon_j^- B_i^-)}$$

So

$$P(Y=i | X=j) = \text{Tr} \left[\begin{pmatrix} \mathcal{E}_j^+ & 0 \\ 0 & \mathcal{E}_j^- \end{pmatrix} \begin{pmatrix} \mathcal{B}_i^+ & 0 \\ 0 & \mathcal{B}_i^- \end{pmatrix} \right]$$

$\underbrace{\mathcal{S}_j}_{\tilde{\mathcal{S}}_j}$ $\underbrace{\mathcal{E}_i}_{\tilde{\mathcal{E}}_i}$

where $(\tilde{\mathcal{E}}_i)_{i=1}^l$ is a part. of unity and $\tilde{\mathcal{S}}_j$ ($j=1, \dots, k$) is a density op.

Note that

$$\forall j: \mathcal{E}_j^+ + \mathcal{E}_j^- = S_B \equiv \text{Tr}_A(\rho)$$

→ $\tilde{\mathcal{S}}_j$ are taken from only a certain subset of $S(\mathfrak{sl}_B \oplus \mathfrak{sl}_B)$

Let $T = \left(P(Y=i | X=j) \right)_{i,j} = \left(\text{Tr} \left(\tilde{\beta}_j \tilde{E}_i \right) \right)_{i,j}$. We want

$$\boxed{\begin{array}{ccc|c} & T & & \\ \hline : & & : & \\ I & & I & \end{array}} = \sum_{I=(i_1, i_2, i_3, i_4)} P_I B(I)$$

stochastic matrix
whose rows, apart
from row nr. i_2, i_2, i_3, i_4
are all = 0

By the Supply-Demand THM such $B(I)$ matrices exist iff
the prob. distr. $I \mapsto P_I$, for $\forall j$ col-index satisfies:

$$\forall H \subset \{1, \dots, l\}: \sum_{I \in H^4} P_I \leq \sum_{i \in H} T_{i,j} = \text{Tr} \left(\tilde{\beta}_j \tilde{E}_H \right)$$

where $\tilde{E}_H = \sum_{i \in H} \tilde{E}_i$.

Let's again search for such a prob. distr. in the form

$$P_I = D(\tilde{E}_{i_1}, \tilde{E}_{i_2}, \tilde{E}_{i_3}, \tilde{E}_{i_4}) \quad (I = (i_1, i_2, i_3, i_4)).$$

We are done if we find a D s.t. it is:

- $D(I, I, I, I) = 1$
 - positive : $D(X, Y, Z, V) \geq 0 \quad \forall X, Y, Z, V \geq 0$
 - multi-lin.
 - $D(E, E, E, E) \leq \text{Tr}(\gamma E)$ whenever $\gamma = \begin{pmatrix} \varepsilon^+ & 0 \\ 0 & \varepsilon^- \end{pmatrix}$ for some $\varepsilon_B = \varepsilon^+ + \varepsilon^-$ poz. decom. of ε_B and $0 \leq E \leq I$
- with the above def., $I \mapsto P_I$ is a prob. distr.

Interruption - lemma:

if $\rho_B = \varepsilon^+ + \varepsilon^-$ is a pos. decomp. and $0 \leq E^\pm \leq I$, then

$$|\mathrm{Tr}(\rho_B E^+ E^-)|^2 \leq \mathrm{Tr}\left[\left(\begin{array}{c|c} \varepsilon^+ & 0 \\ \hline 0 & \varepsilon^- \end{array}\right) \left(\begin{array}{c|c} E^+ & 0 \\ \hline 0 & E^- \end{array}\right)\right] = \mathrm{Tr}(\varepsilon^+ E^+) + \mathrm{Tr}(\varepsilon^- E^-)$$

Indeed:

$$|\mathrm{Tr}(\rho_B E^+ E^-)|^2 = |\underbrace{\mathrm{Tr}(\varepsilon^+ E^+ E^-)}_a + \underbrace{\mathrm{Tr}(\varepsilon^- E^+ E^-)}_b|^2$$

$$\text{where } |a|^2 \stackrel{\text{C-Sch}}{\leq} \mathrm{Tr}(\varepsilon^+ (E^+)^2) \mathrm{Tr}(\varepsilon^+ (E^-)^2) \leq \mathrm{Tr}(\varepsilon^+ E^+) \mathrm{Tr}(\varepsilon^+)$$

and sim. $|b|^2 \leq \mathrm{Tr}(\varepsilon^- E^-) \mathrm{Tr}(\varepsilon^-)$. Hence

$$\left| \text{Tr}(\rho_B E^+ E^-) \right|^2 = |a + b|^2 = |a|^2 + |b|^2 + 2 \operatorname{Re}(ab)$$

$$\leq |a|^2 + |b|^2 + 2\sqrt{|a|^2 |b|^2}$$

$$\leq \text{Tr}(\varepsilon^+ E^+) \text{Tr}(\varepsilon^+) + \text{Tr}(\varepsilon^- E^-) \text{Tr}(\varepsilon^-) + 2\sqrt{\text{Tr}(\varepsilon^+ E^+) \text{Tr}(\varepsilon^-) \text{Tr}(\varepsilon^- E^-) \text{Tr}(\varepsilon^+)}$$

$$\leq \text{Tr}(\varepsilon^+ E^+) \text{Tr}(\varepsilon^+) + \text{Tr}(\varepsilon^- E^-) \text{Tr}(\varepsilon^-) + \cancel{2} \frac{\text{Tr}(\varepsilon^+ E^+) \text{Tr}(\varepsilon^-) + \text{Tr}(\varepsilon^- E^-) \text{Tr}(\varepsilon^+)}{\cancel{2}}$$



$$\text{Tr}(\varepsilon^+ E^+) + \text{Tr}(\varepsilon^- E^-)$$

Since $\text{Tr}(\varepsilon^+) + \text{Tr}(\varepsilon^-) = \text{Tr}(\rho_B) = 1$.

$$D(X, Y, Z, V) := \text{Tr}(\rho_B X^+ Y^-) \text{Tr}(\rho_B Z^- V^+) \quad \text{where } (+)$$

$$\hookrightarrow D(E, E, E, E) = \text{Tr}(\rho_B E^+ E^-) \text{Tr}(\rho_B E^- E^+)$$

So by the previous lemma, for $0 \leq E \leq I$:

$$D(E, E, E, E) = \left| \text{Tr}(\rho_B E^+ E^-) \right|^2 \leq \text{Tr} \left[\begin{pmatrix} \varepsilon^+ & 0 \\ 0 & \varepsilon^- \end{pmatrix} E \right] \quad \nexists \text{ pos. decomp. } \rho_B = \varepsilon^+ + \varepsilon^-.$$

Moreover:

- D is multilin.
- $D(I, I, I, I) = 1$.

However, in gen. our D is not positive. Nevertheless:

if ρ is a max. entangled state, then

$$\rho_B = \frac{1}{m} I \quad \text{where } m = \dim \mathcal{H}_B$$

$$\hookrightarrow D(X, Y, Z, V) = \frac{1}{m^2} \text{Tr}(X^+ Y^-) \text{Tr}(Z^- V^+)$$

is evidently positive!

