Operational probabilistic theories and cellular automata: how I learned to stop worrying and love C* algebras

School on Advanced Topics in Quantum Information and Foundations
Quantum Information Unit and the Yukawa Institute for Theoretical Physics, Kyoto University

UNIVERSITÀ DI PAVIA

INFN Istituto Nazionale di Fisica Nucleare

QUit quantum information theory group

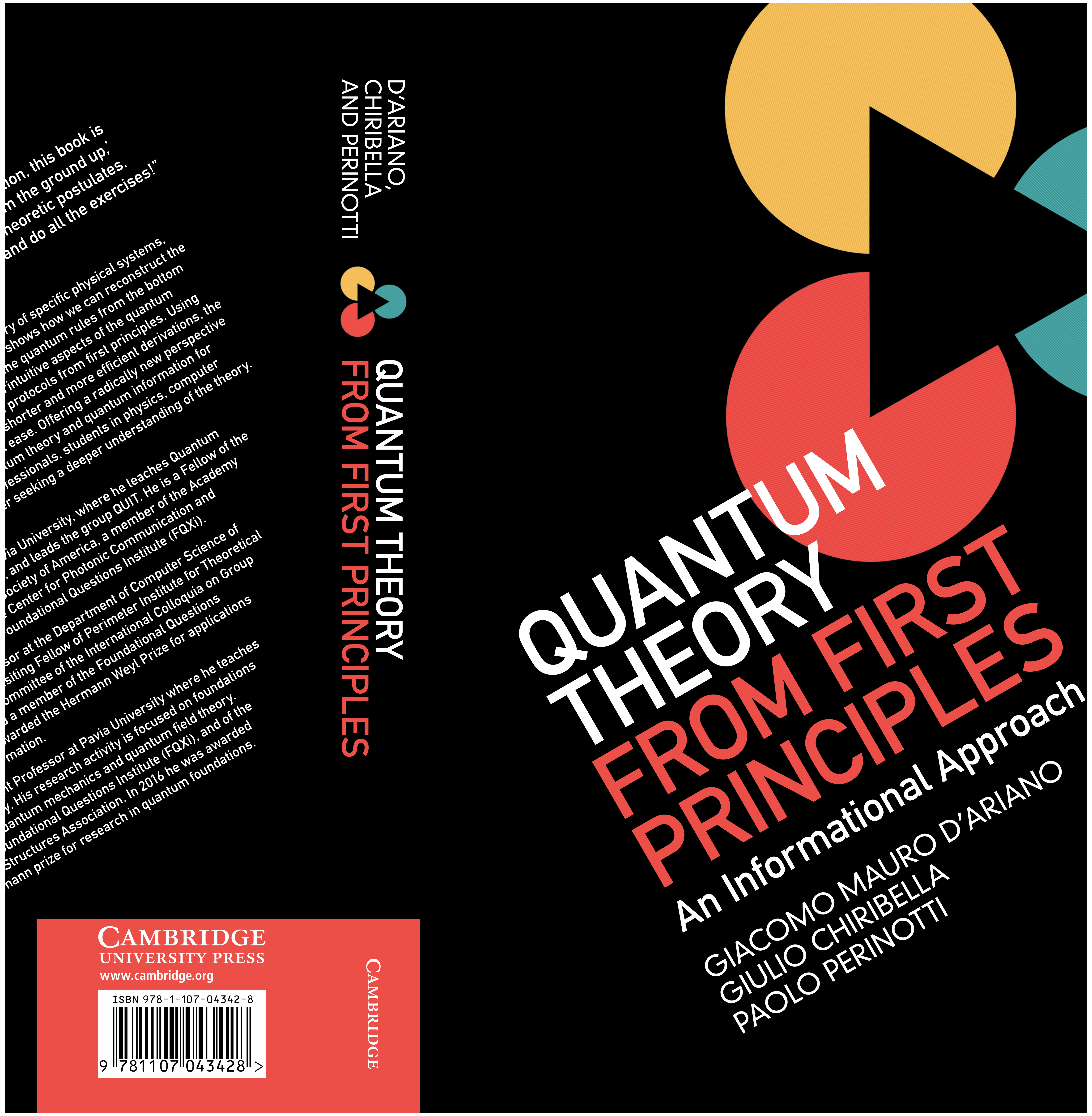
Paolo Perinotti - February 8-12 2021

Lecture 1
Operational Probabilistic Theories

Quantum information theory

Informational derivation

- The mathematical language of quantum theory: systems and processes

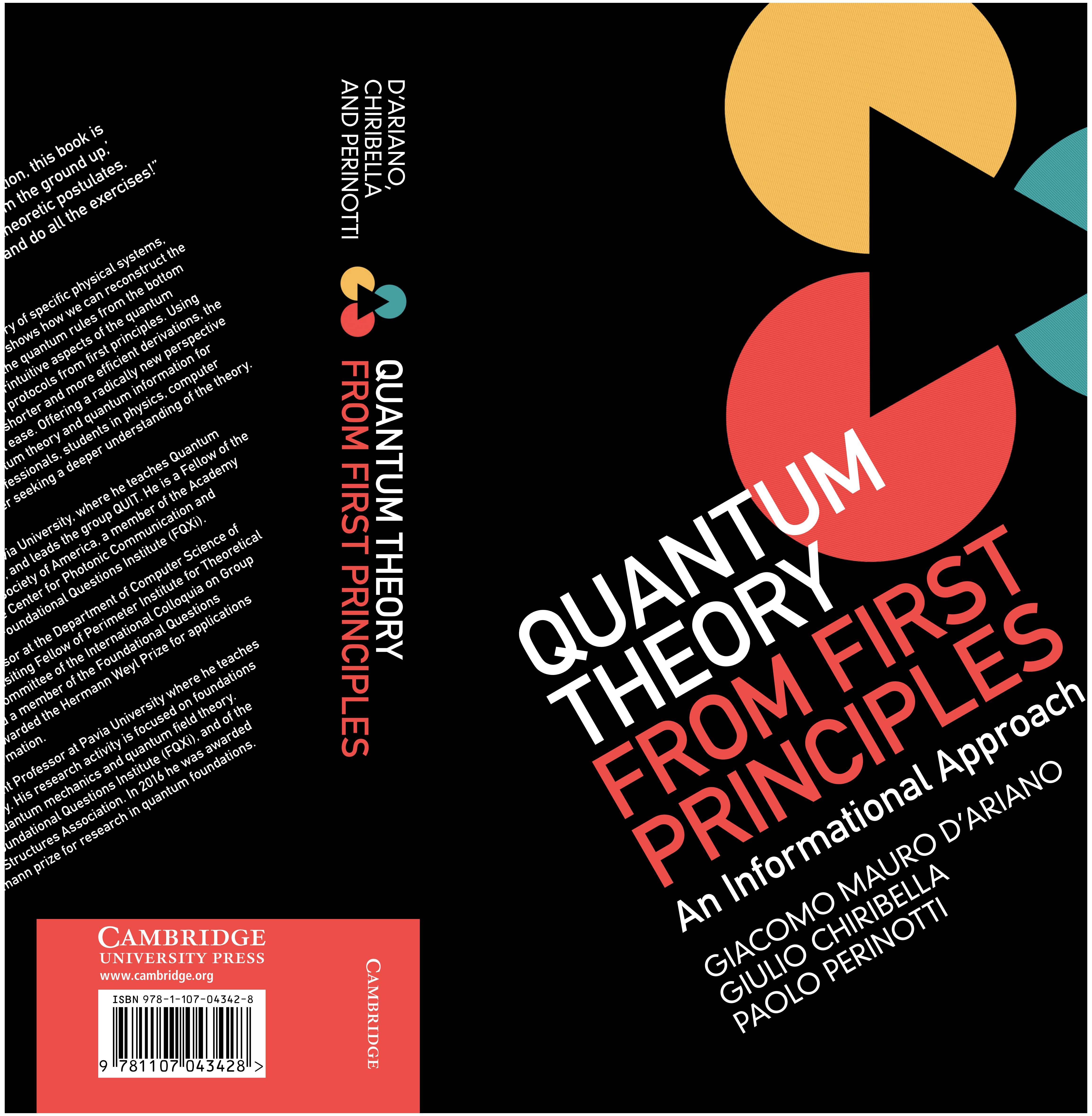


Quantum information theory

Informational derivation

- The mathematical language of quantum theory: systems and processes

- Systems are thought of as elementary memory cells in the first place, rather than elementary constituents of matter



Physical Semantics

How to recover it in a purely information-theoretic framework?

- No notion of space and time
- Is it possible to recover mechanical concepts?
- Is it possible to derive physical laws?
- How?

The classical universe

- Classical mechanics and Laplace's clockwork universe

> We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit these data to analysis, it would embrace in a single formula the movements of the greatest bodies of the universe and those of the tiniest atom; for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes.
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> Pierre Simon Laplace, *A Philosophical Essay on Probabilities*

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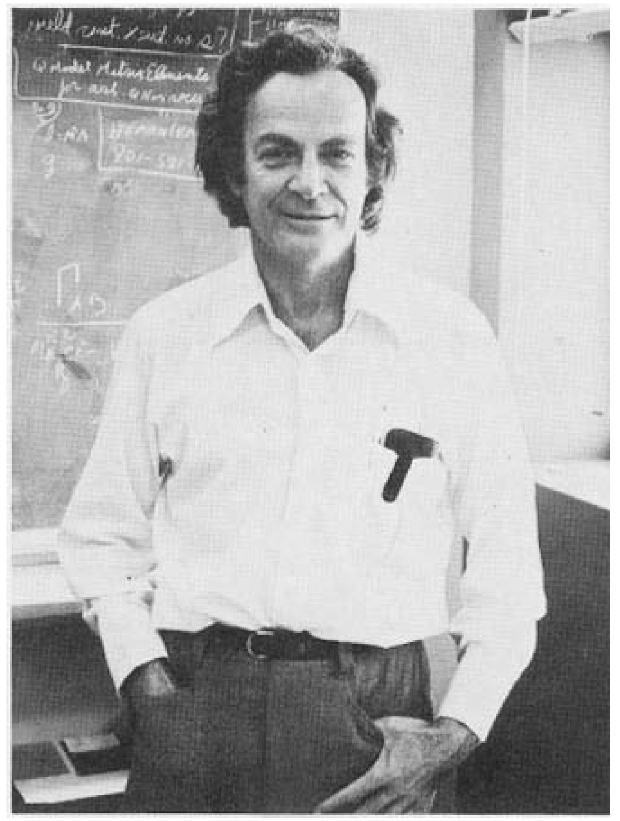
- World view of quantum mechanics?

Digital Universe



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R. Feynman, Int. J. Theo. Ph. **21**, 467 (1982)

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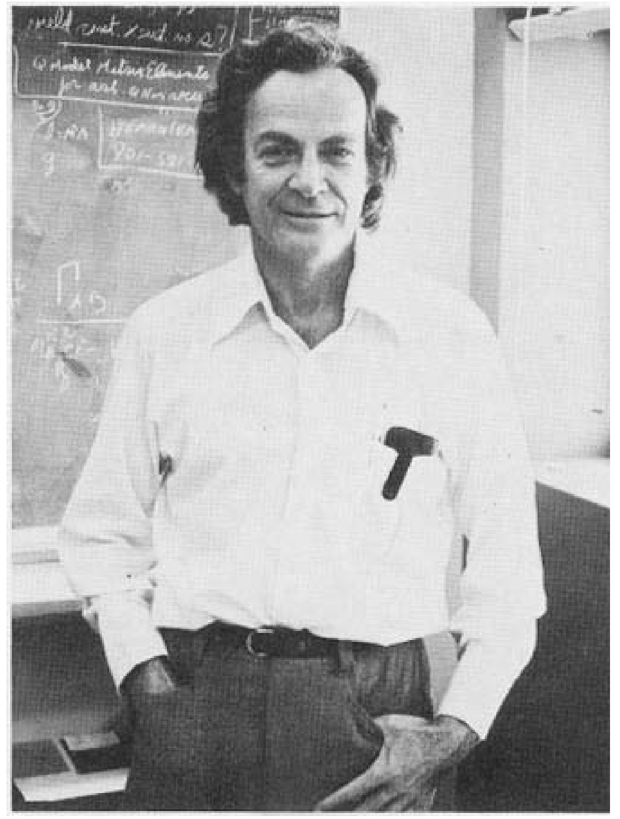

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Physical law ⟷ Algorithm

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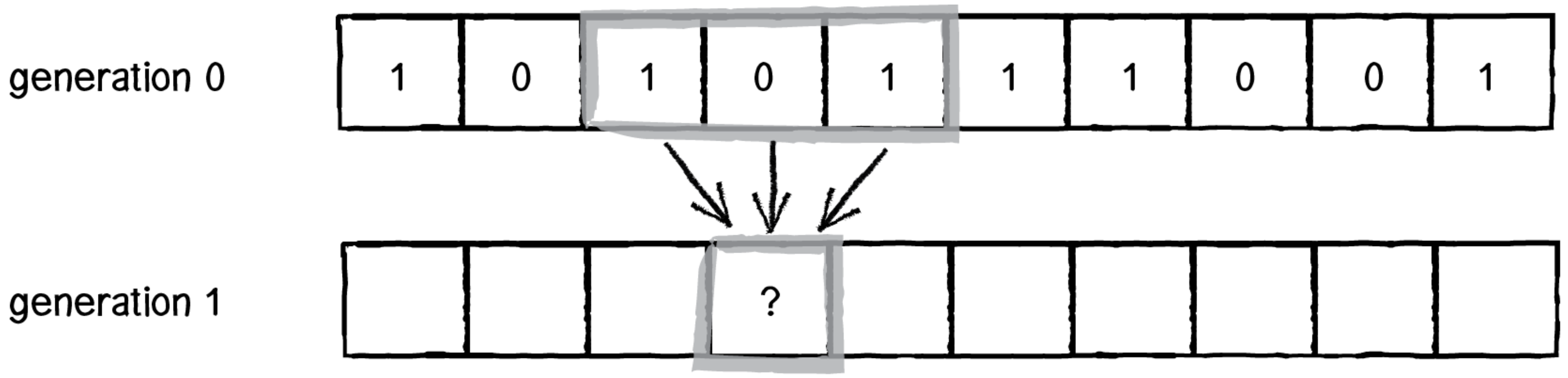
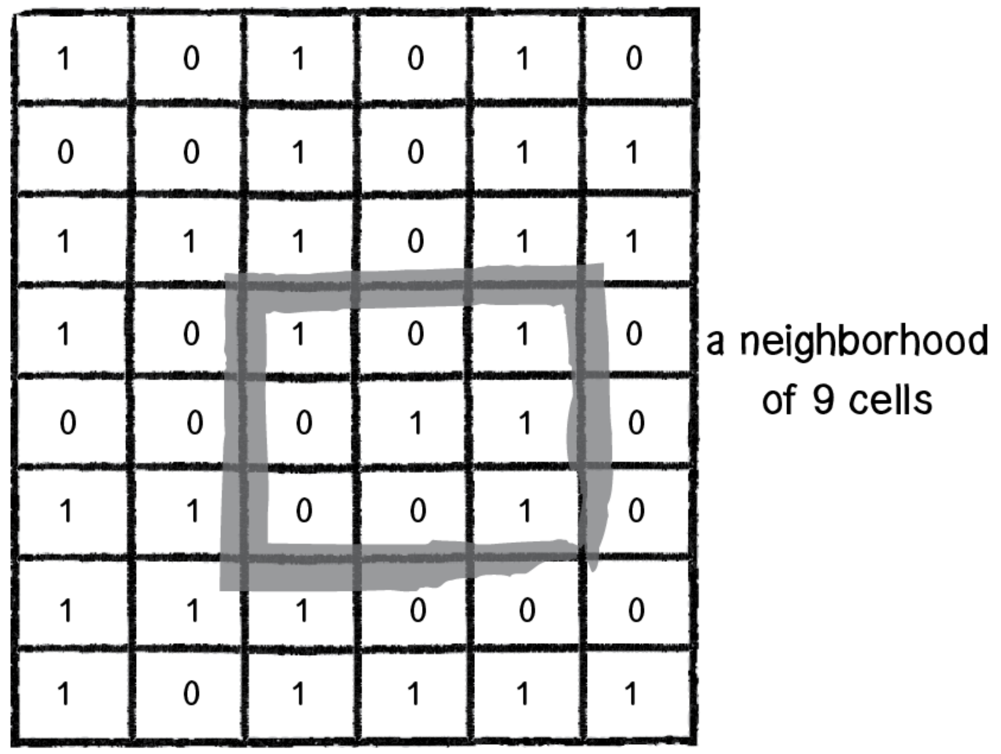
Cellular Automata

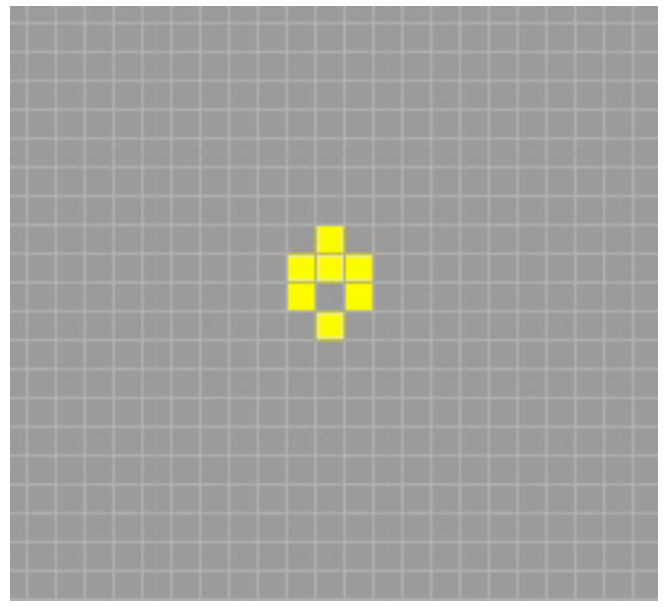
J. Von Neumann and A. W. Burks, “Theory of self-reproducing automata” 1966

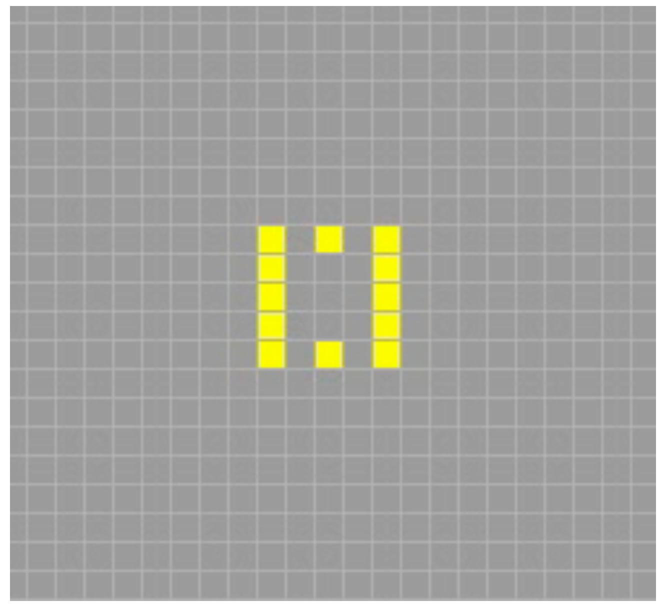
Conway’s game of life

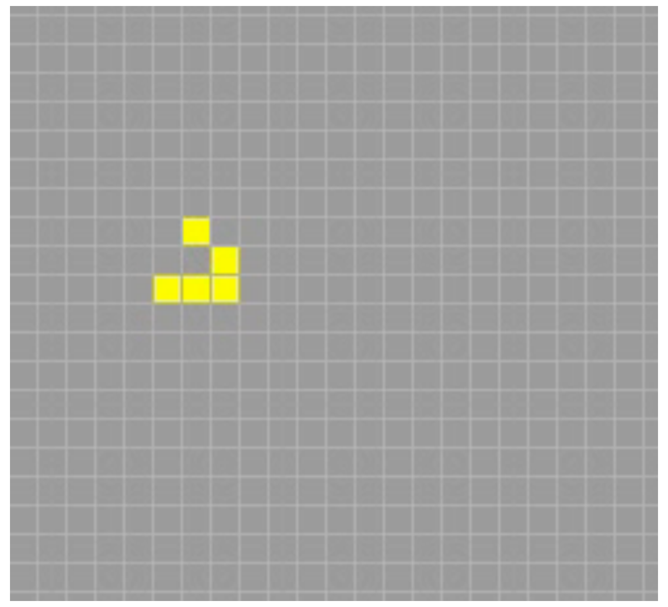


Two-dimensional cellular automata









http://web.stanford.edu/~cdebs/GameOfLife/

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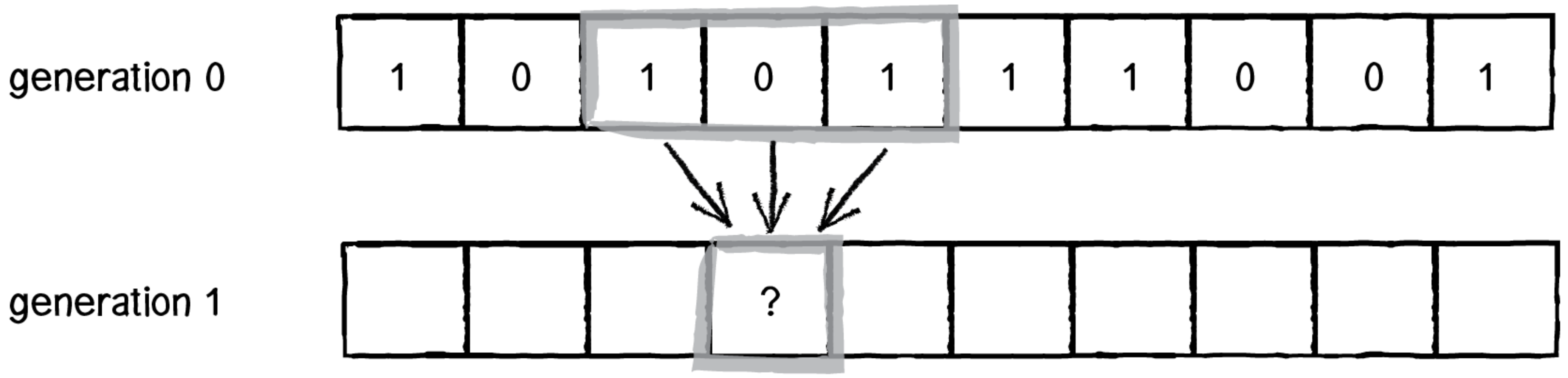

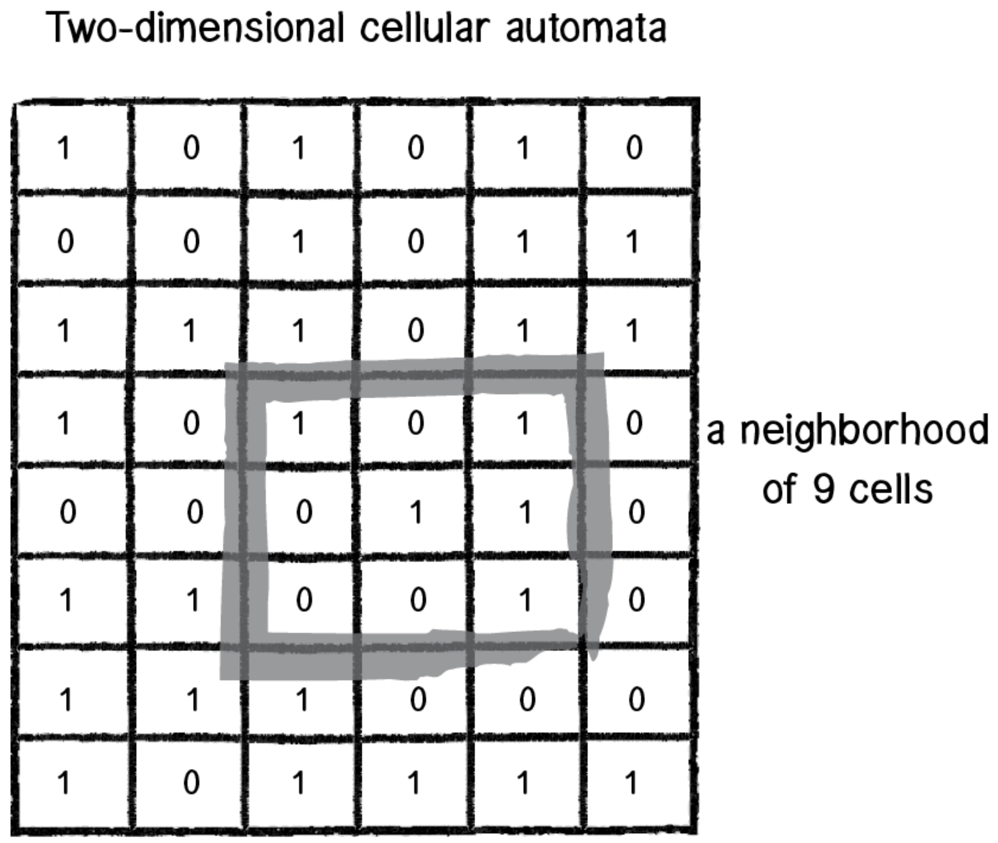

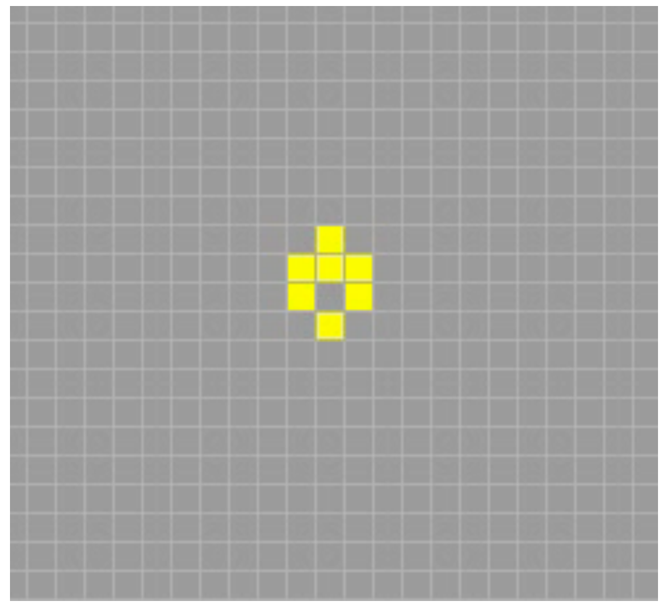

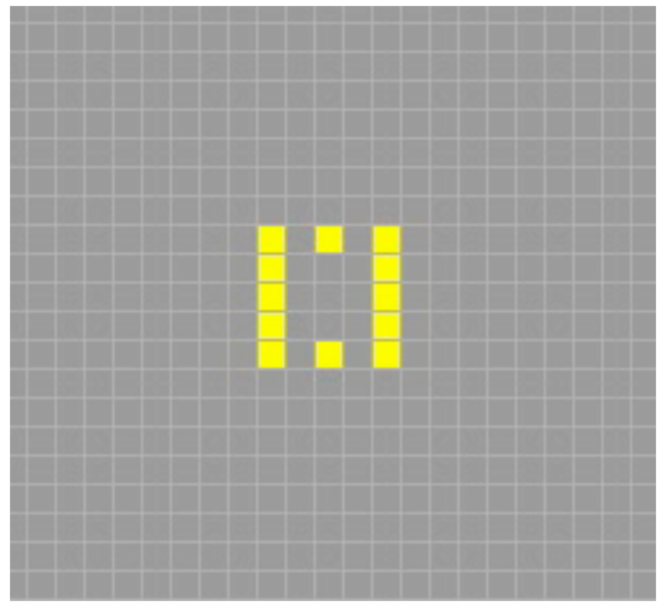

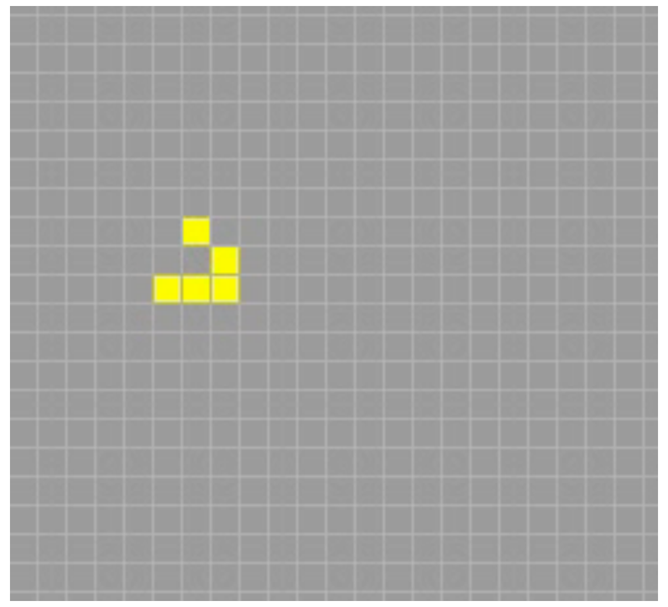

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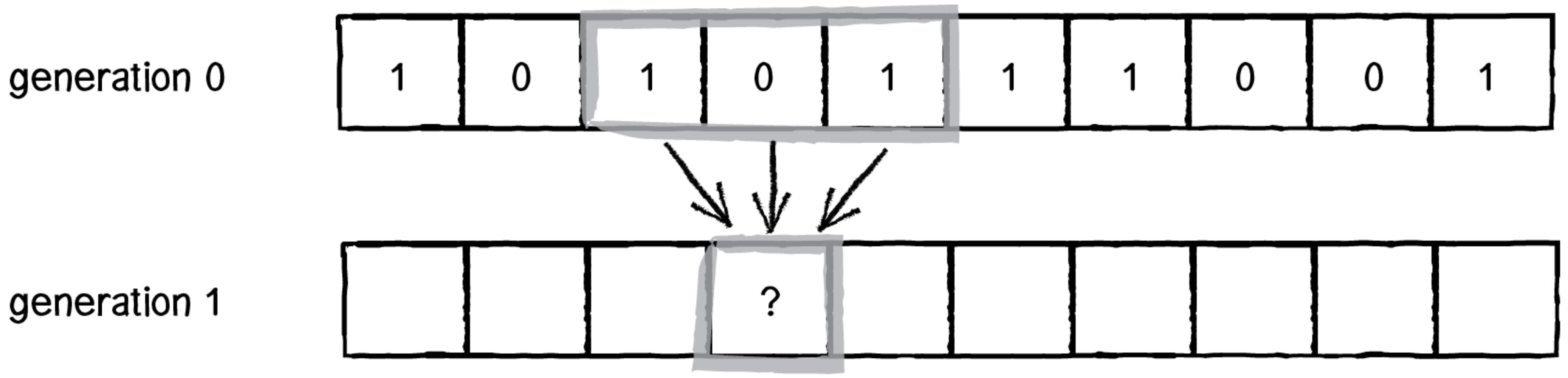

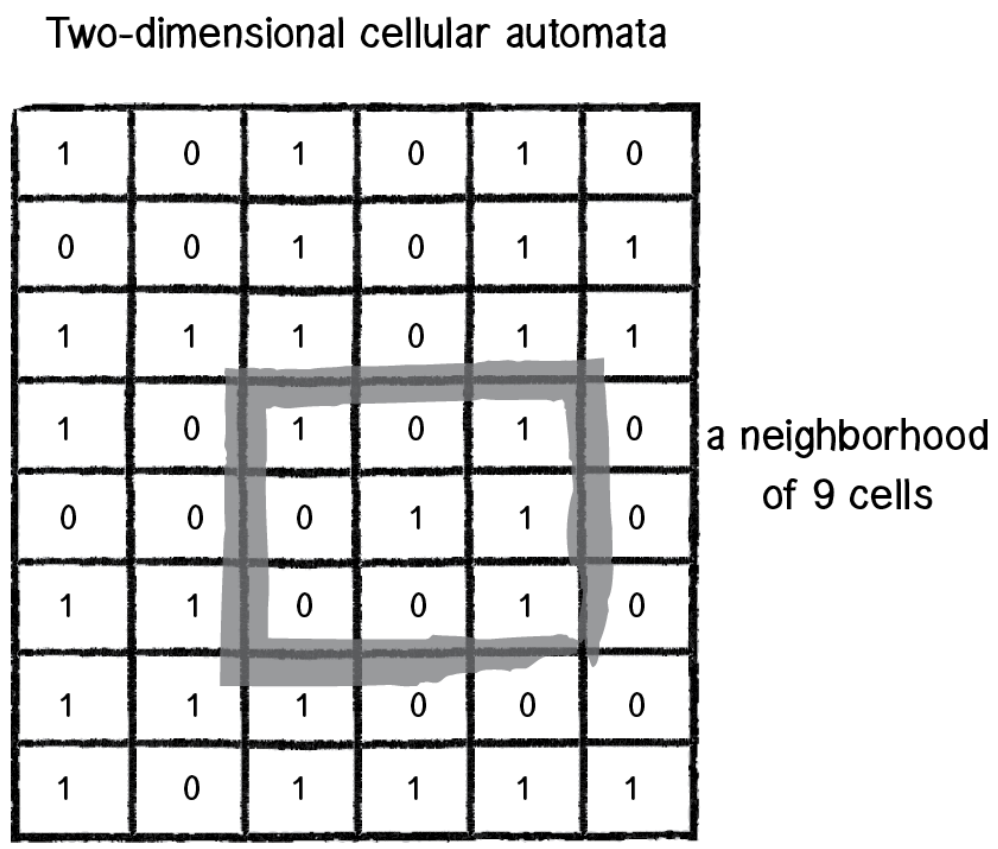

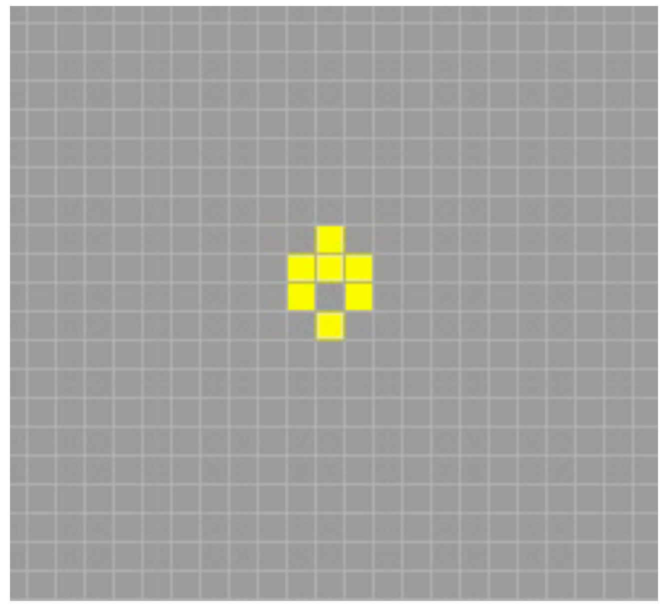

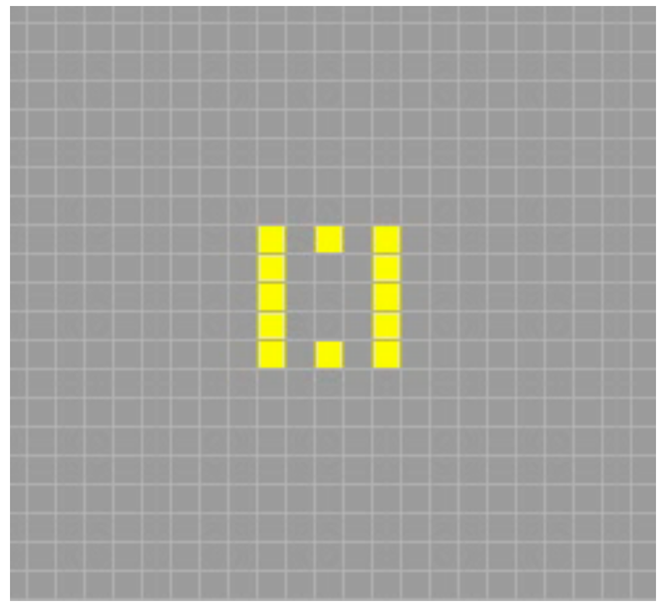

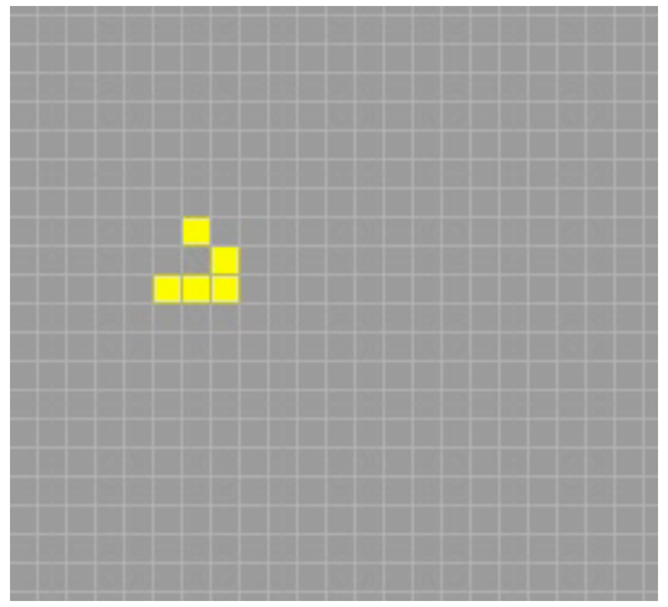

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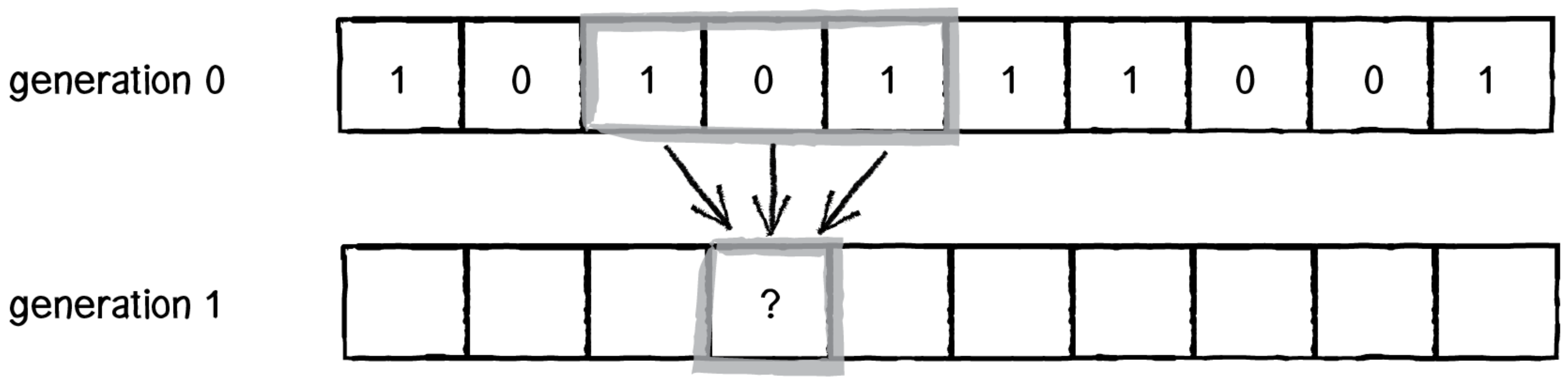

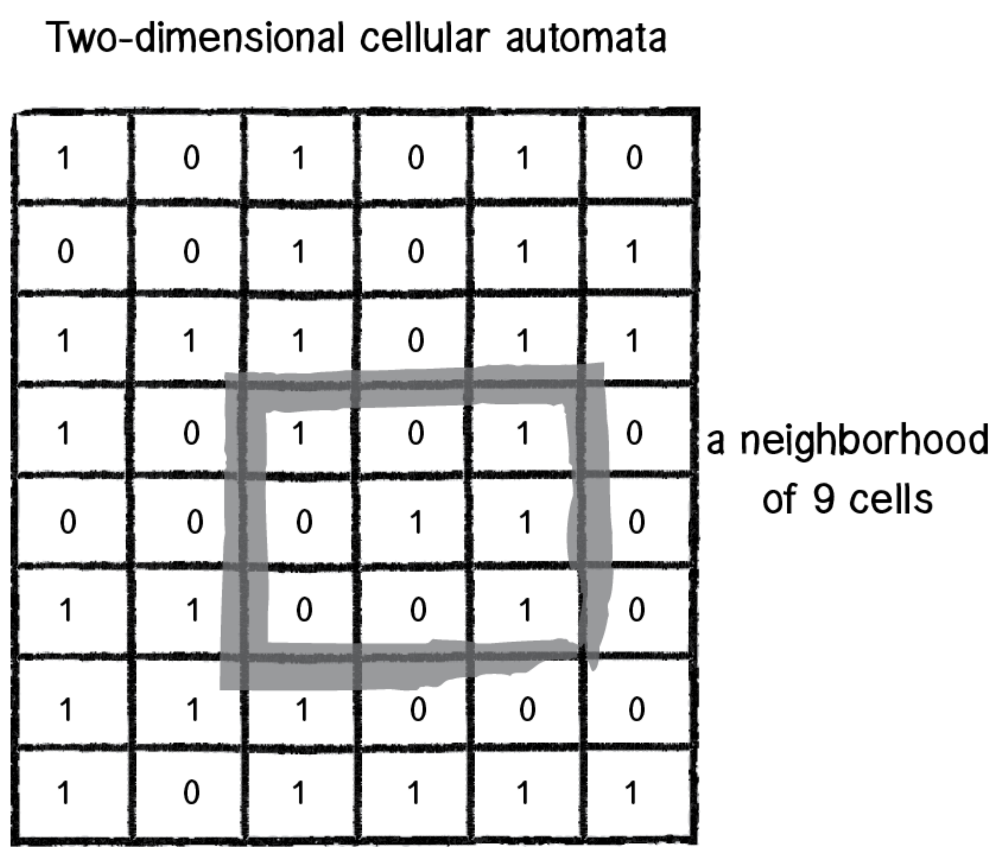

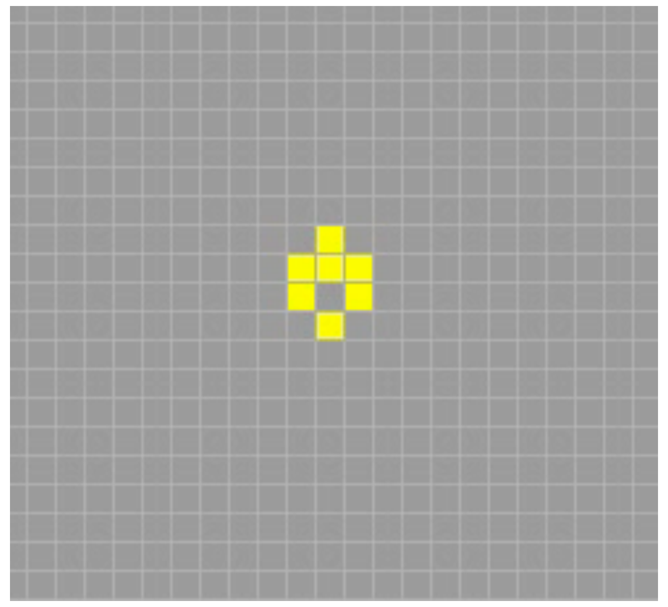

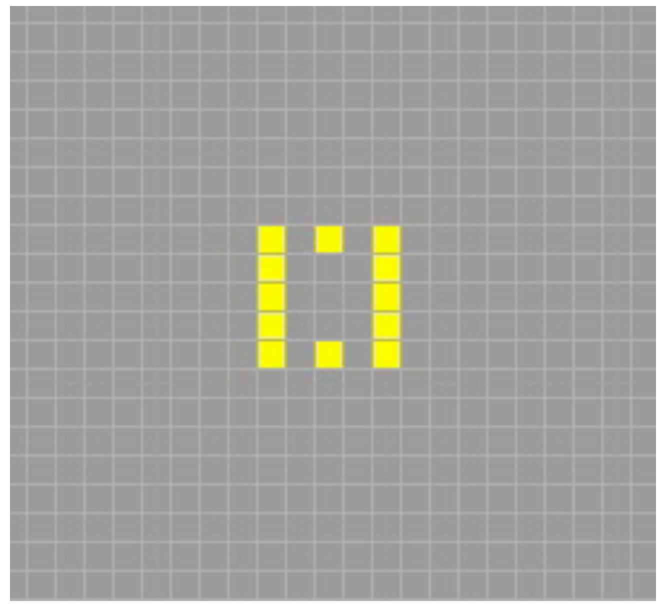

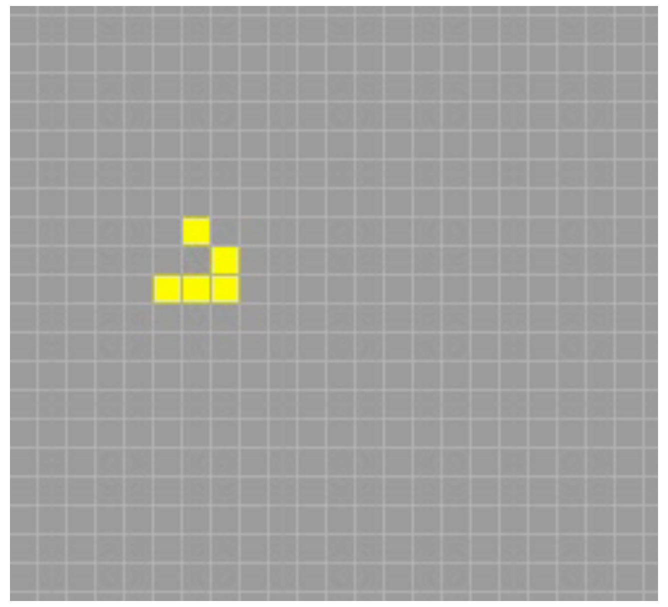

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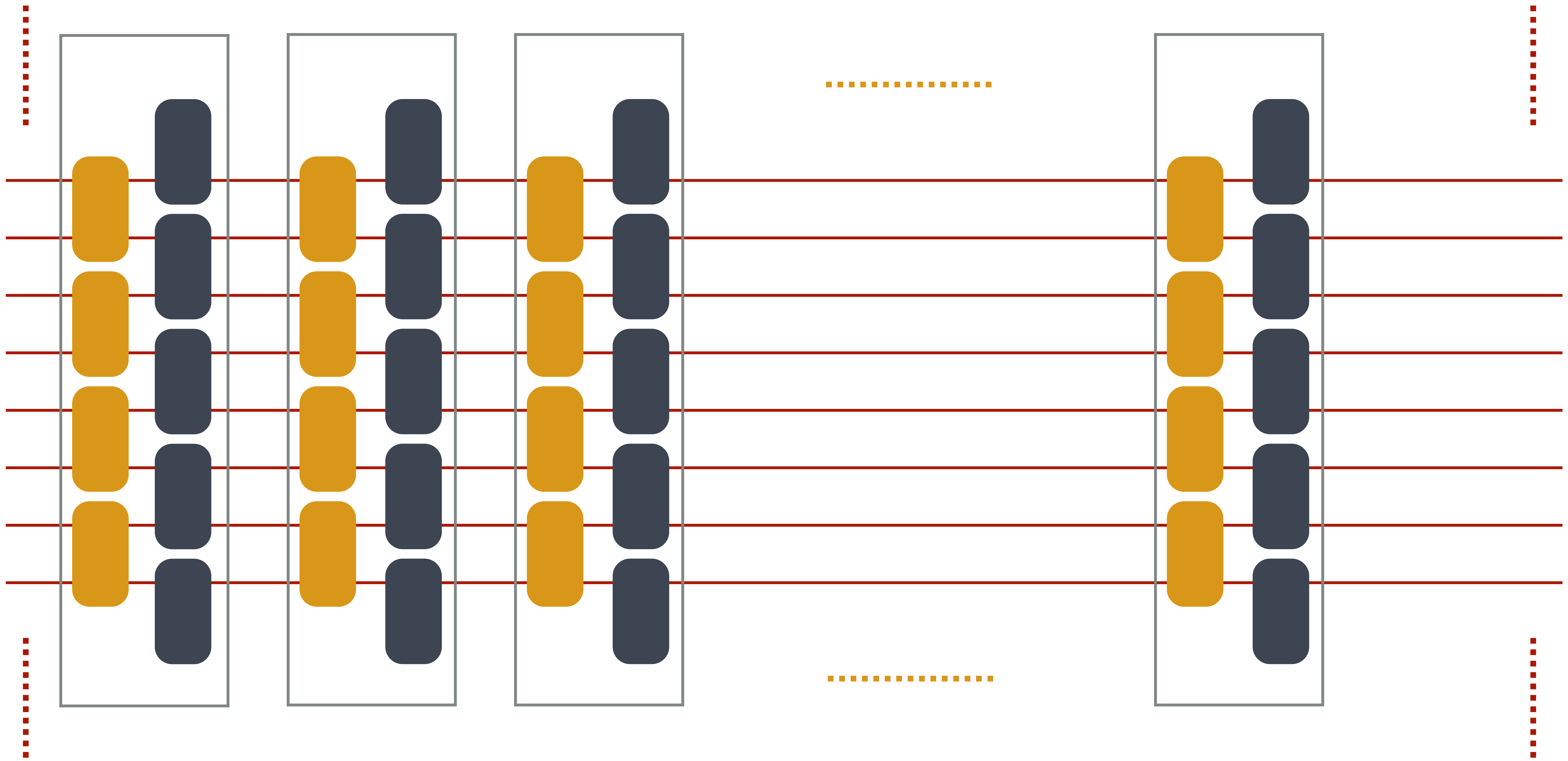
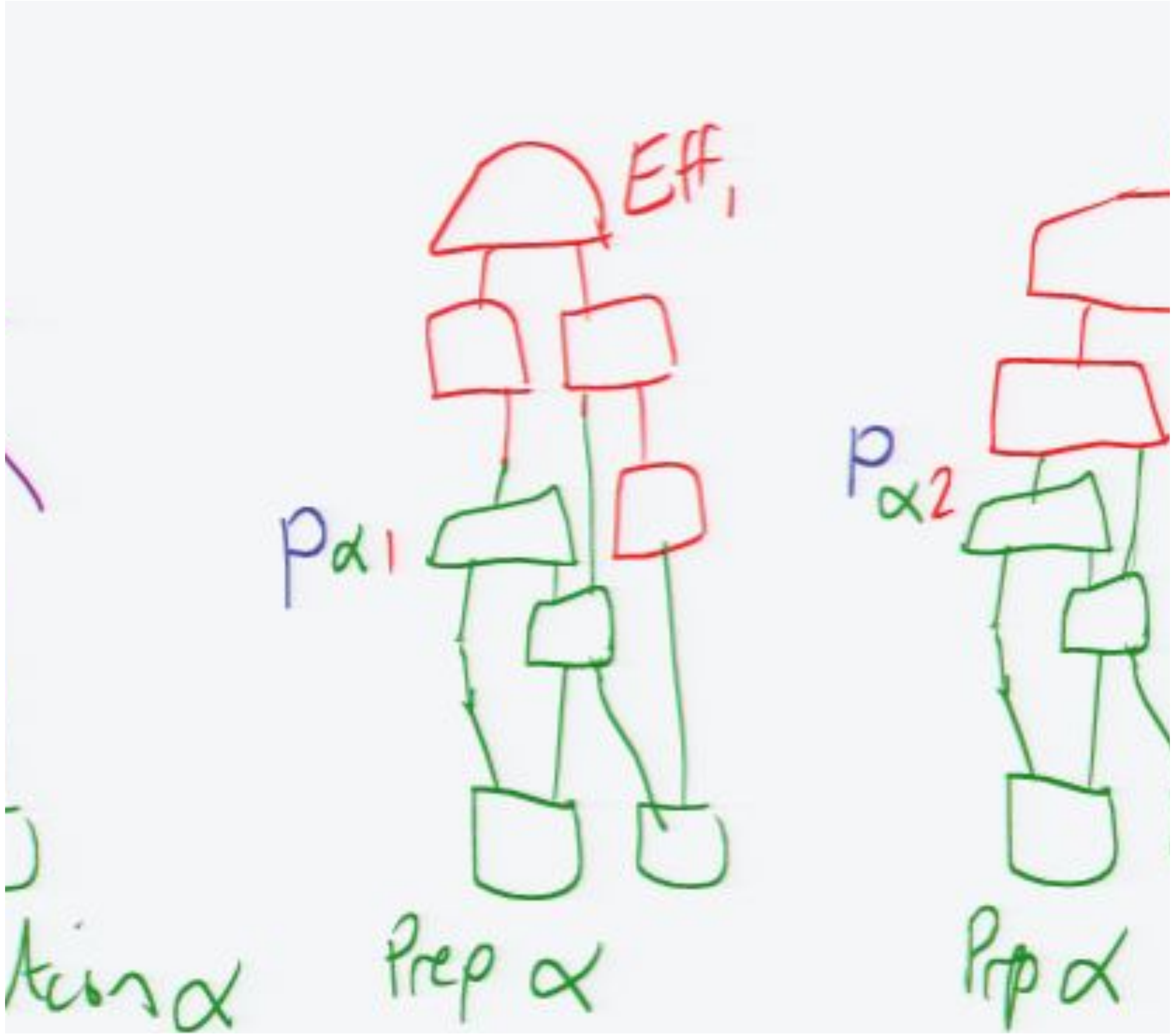
Quantum cellular automata



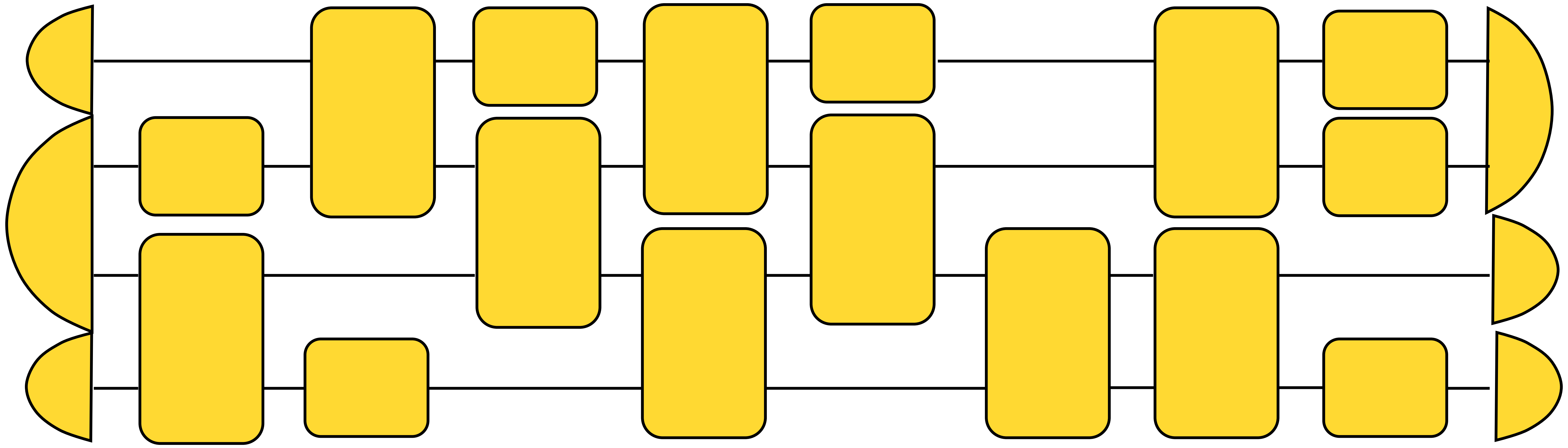
B. Schumacher and R. F. Werner, arXiv:quant-ph/0405174 (2004).

Overview

- Operational language
- Probabilistic structure
- Examples
- Main properties
    - Causality
    - Local discriminability
    - Purification



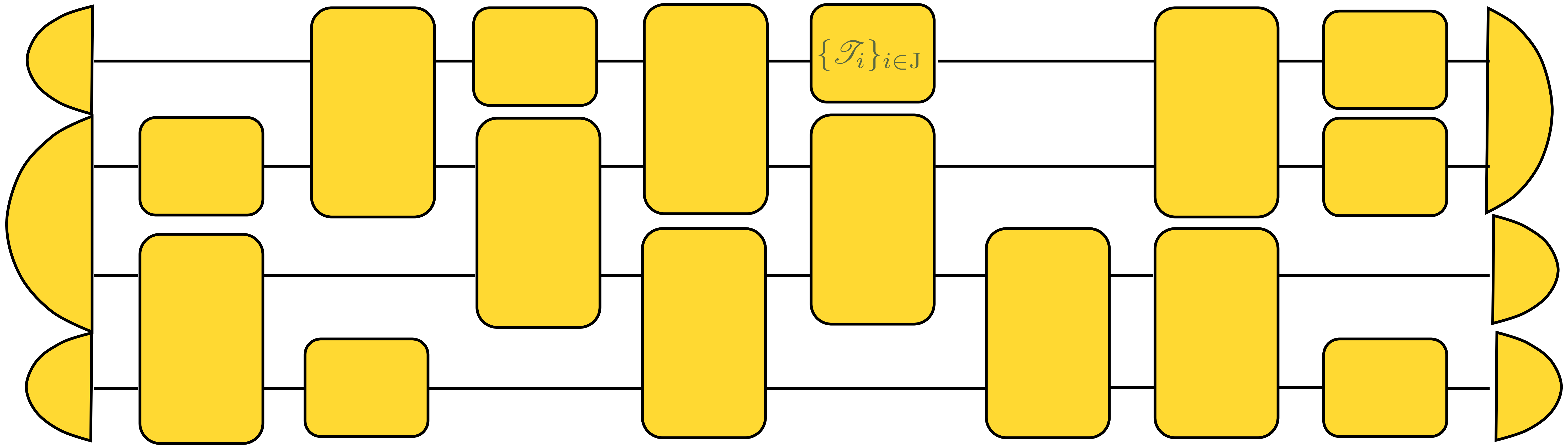
Operational Language



- Operational theory: tests with composition rules

G. Chiribella, G. M. D'Ariano, and PP, Phys. Rev. A **81**, 062348 (2010)

Operational Language



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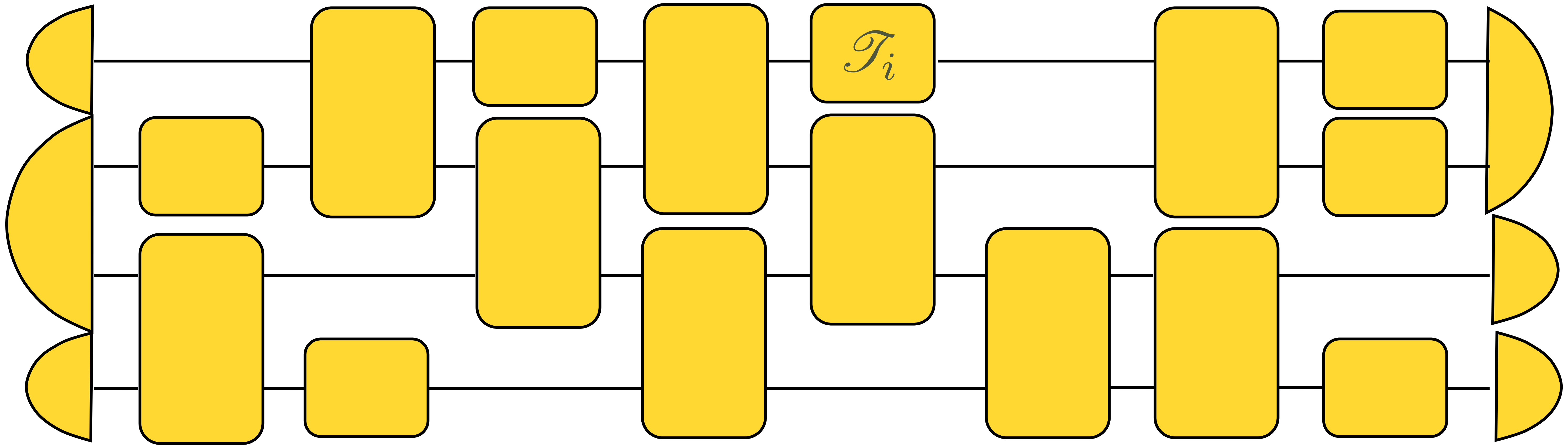

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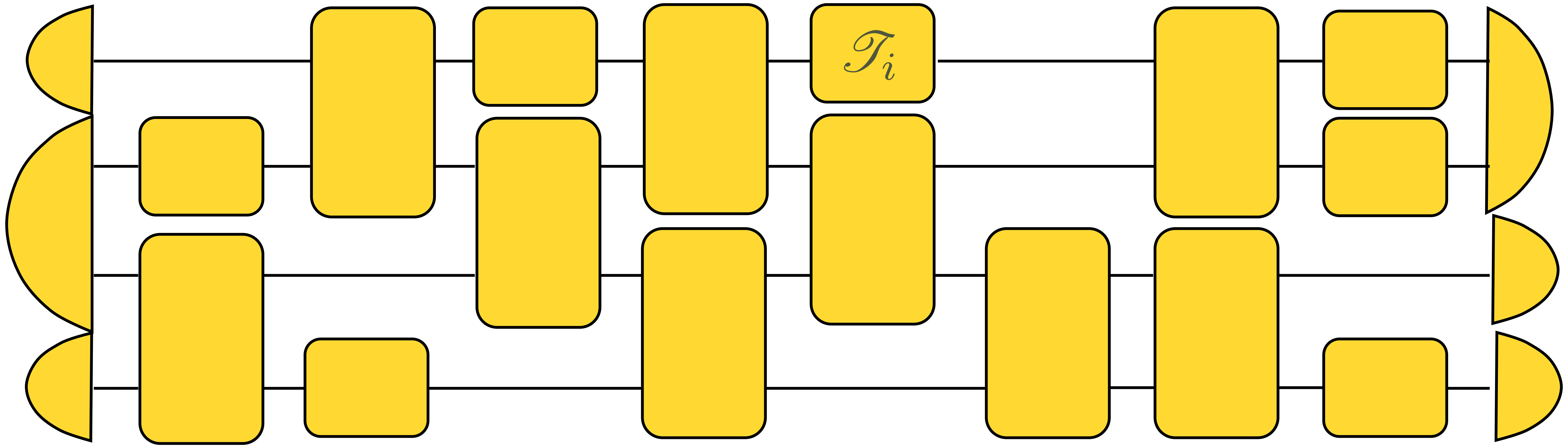
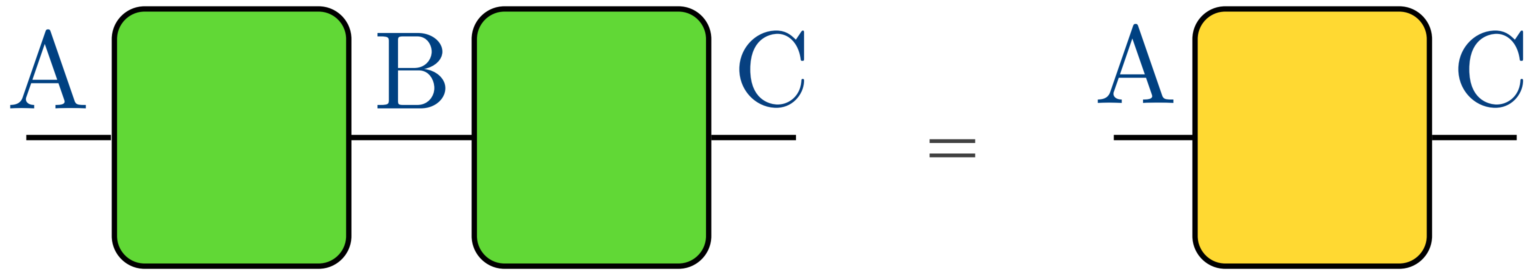


- Operational theory: tests with composition rules

Sequential



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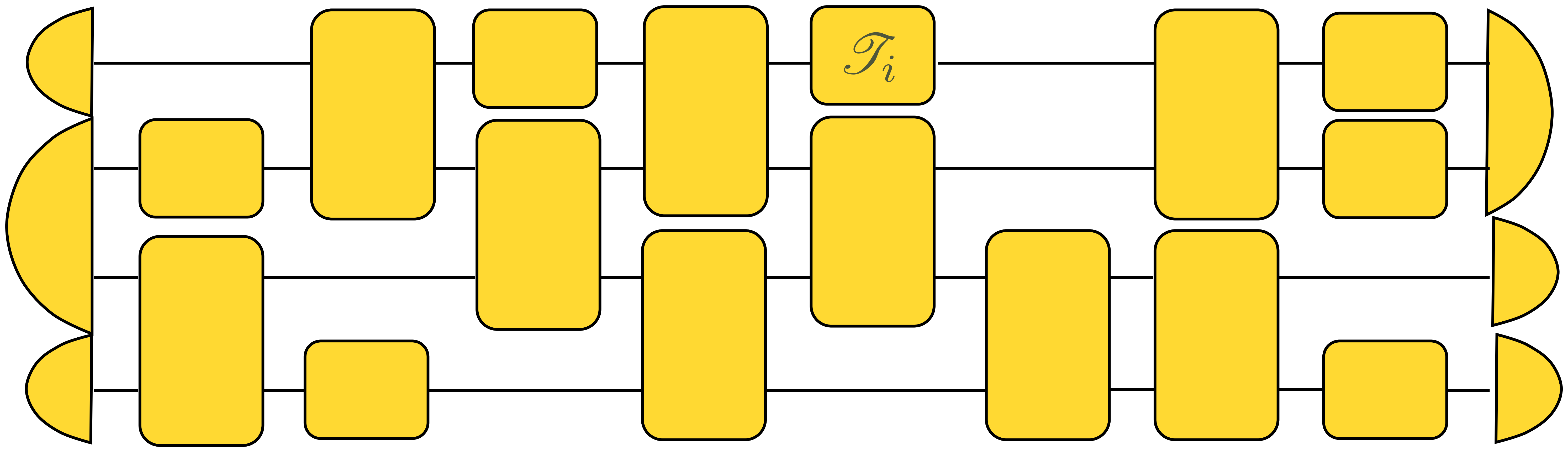

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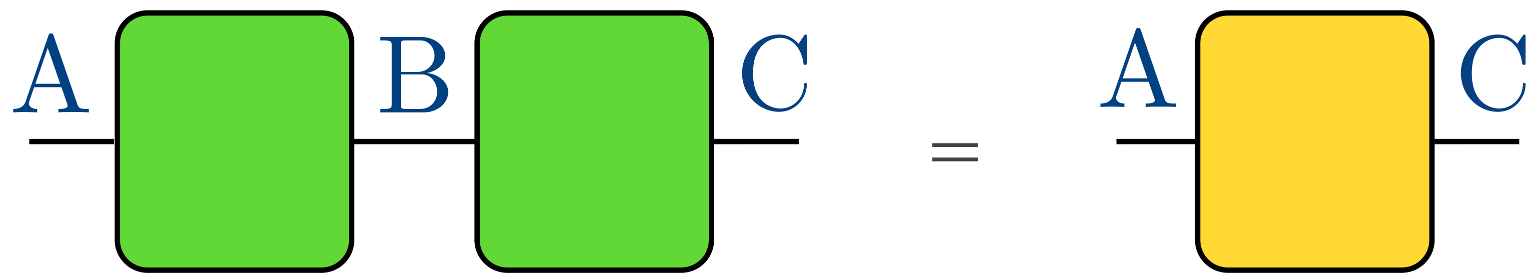
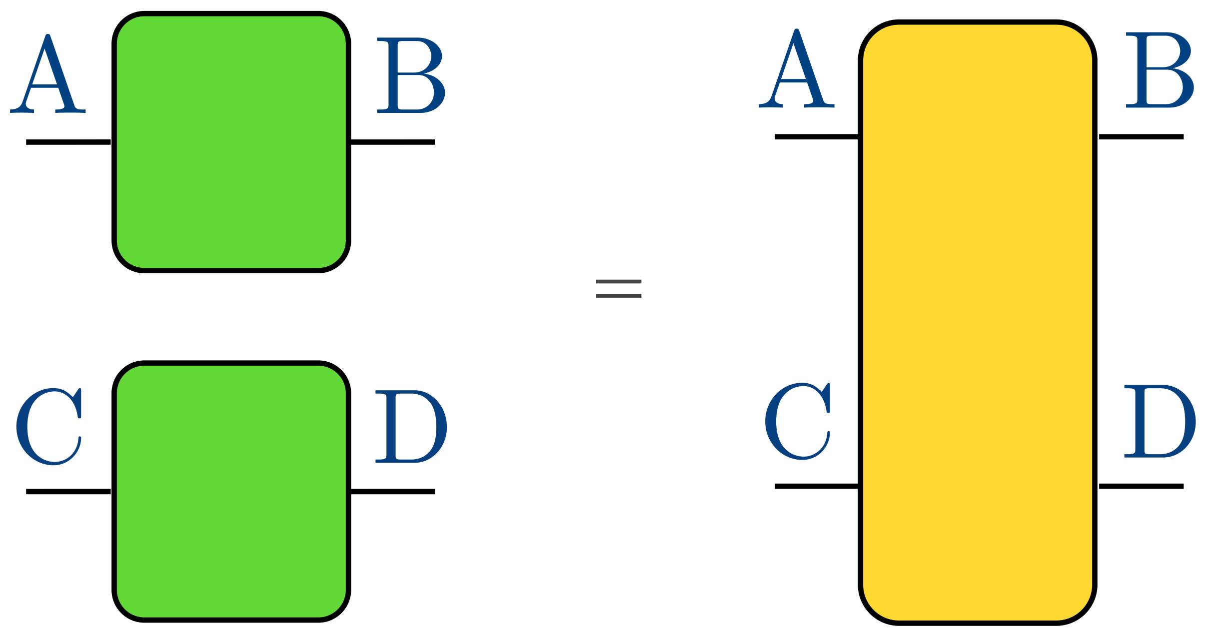


Parallel



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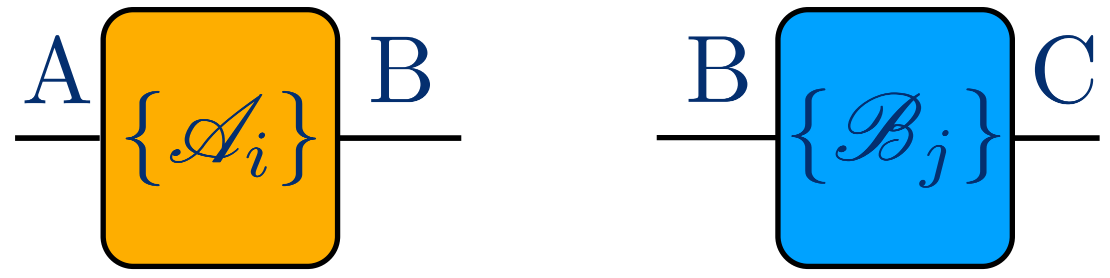
Sequential composition



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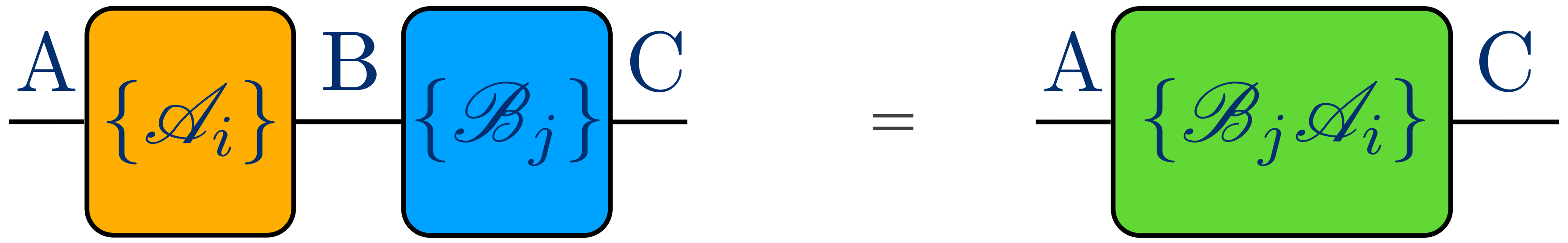
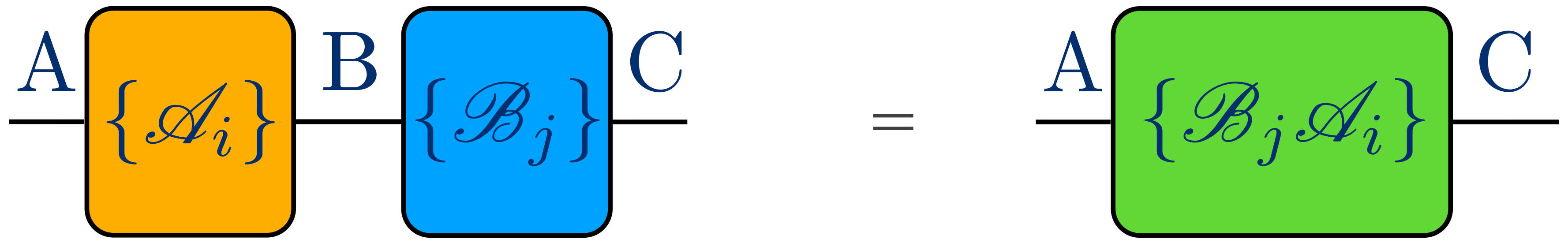


$$i \in I, \; j \in J, \; \Rightarrow \; (i, j) \in I \times J$$

Sequential composition



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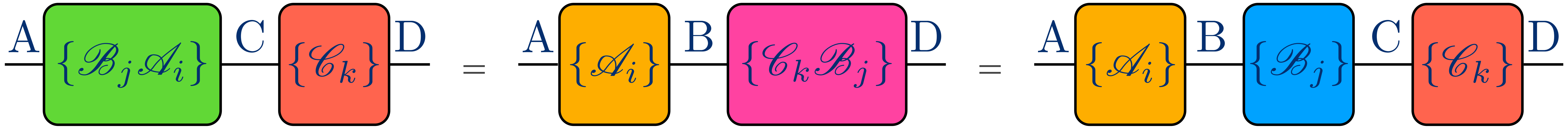
- Properties
  - Associativity



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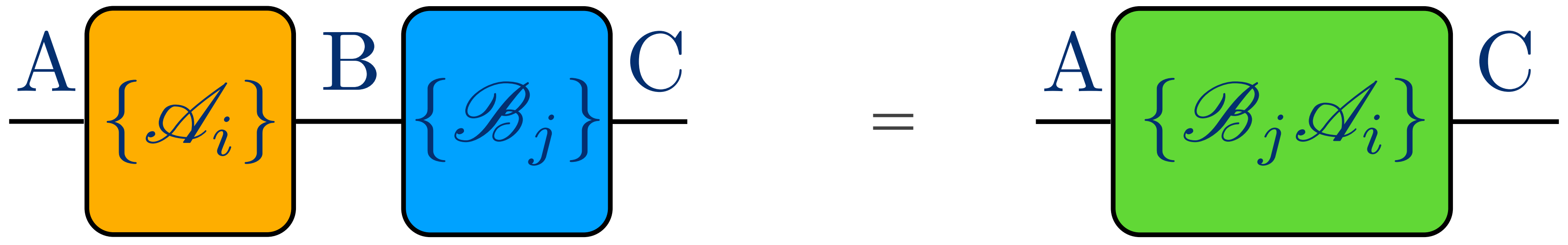

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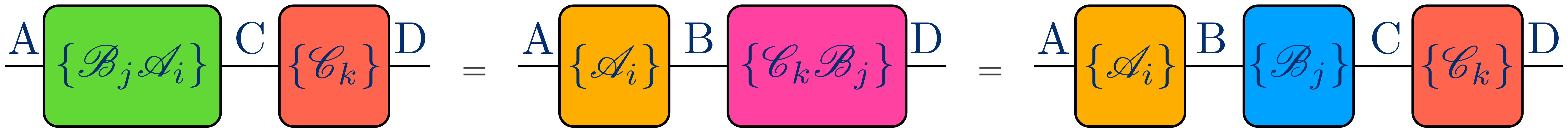
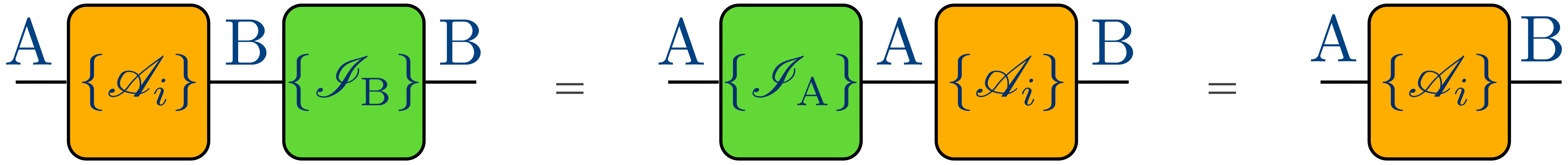
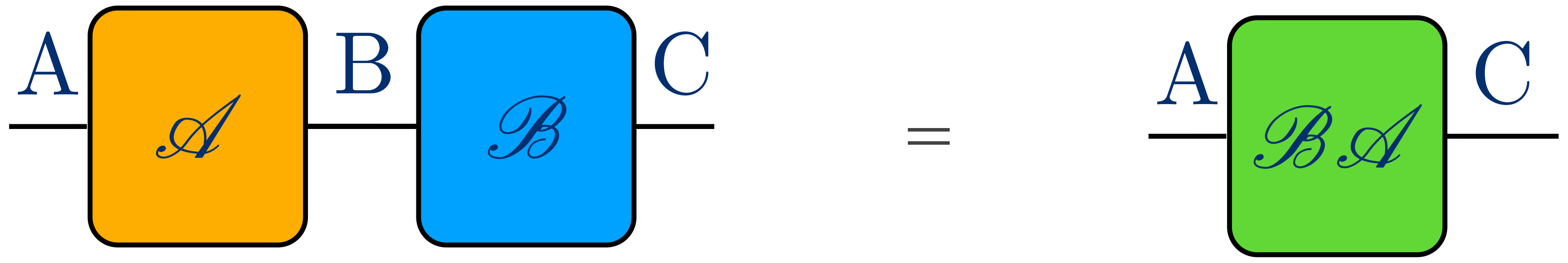


  - Unit



Events



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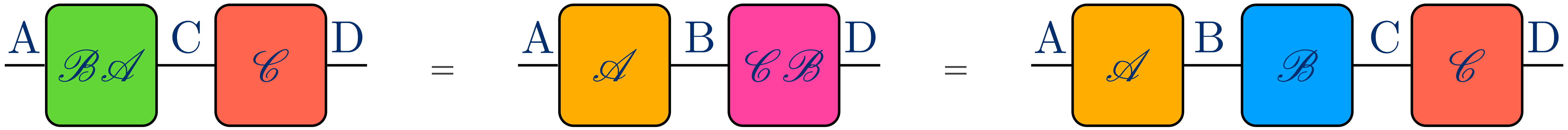
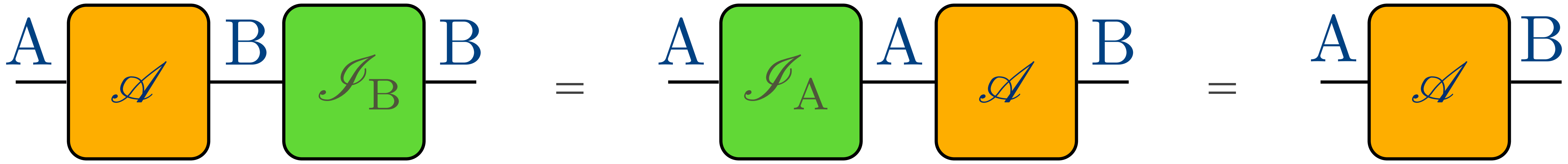
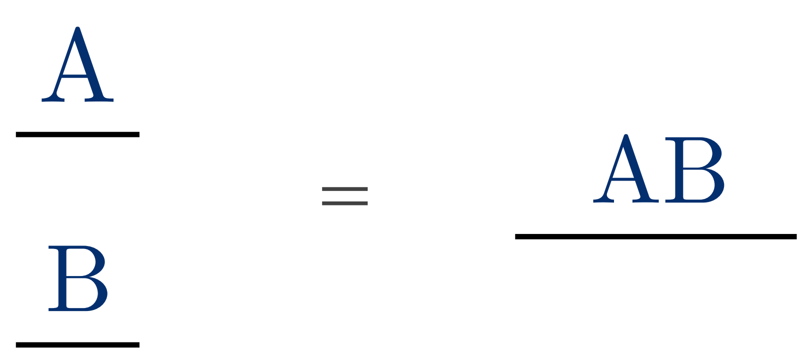
  - Associativity 

  - Unit 

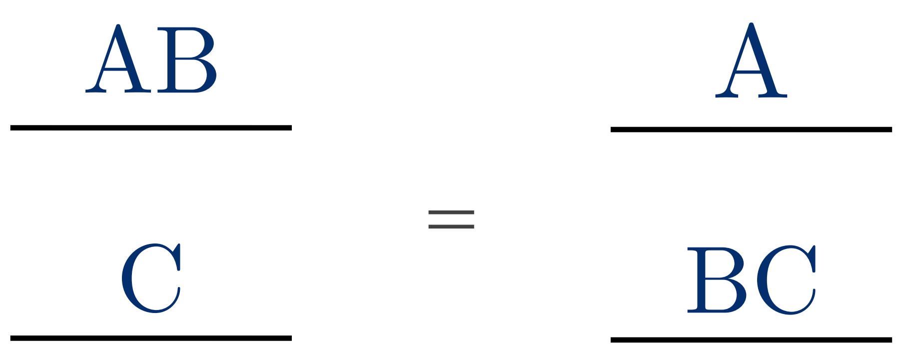
Reversible event: 

Parallel composition

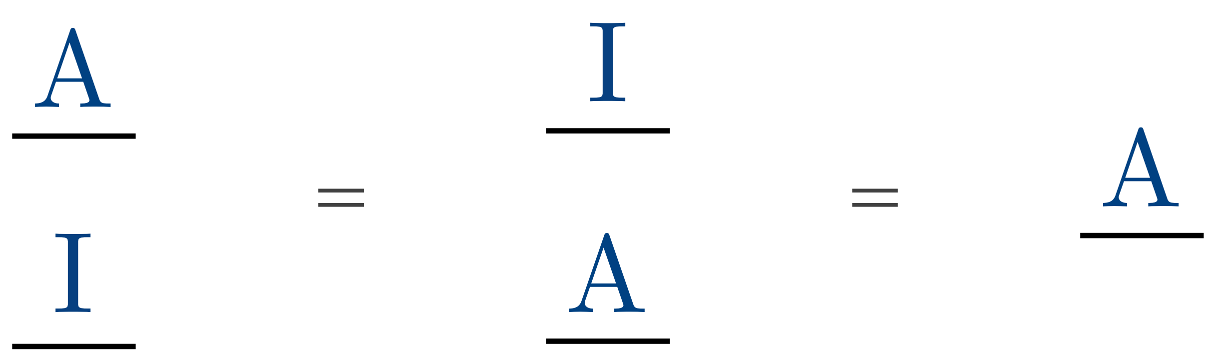
- Systems:



- Associativity

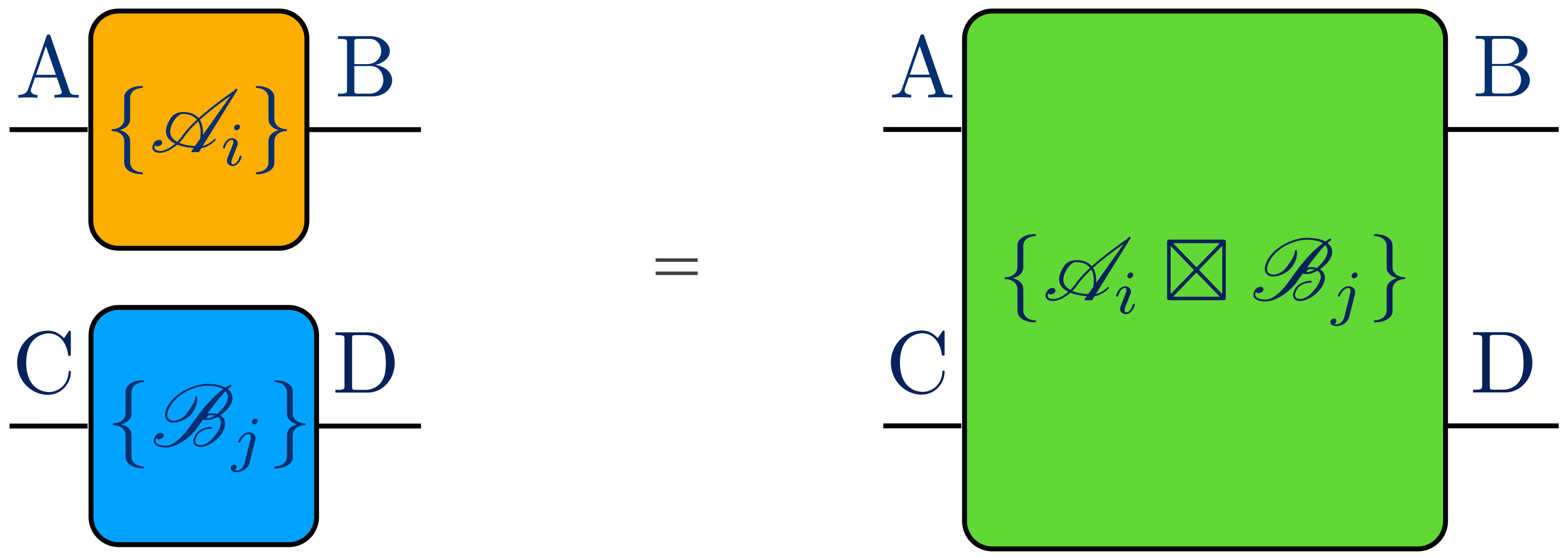


- Unit

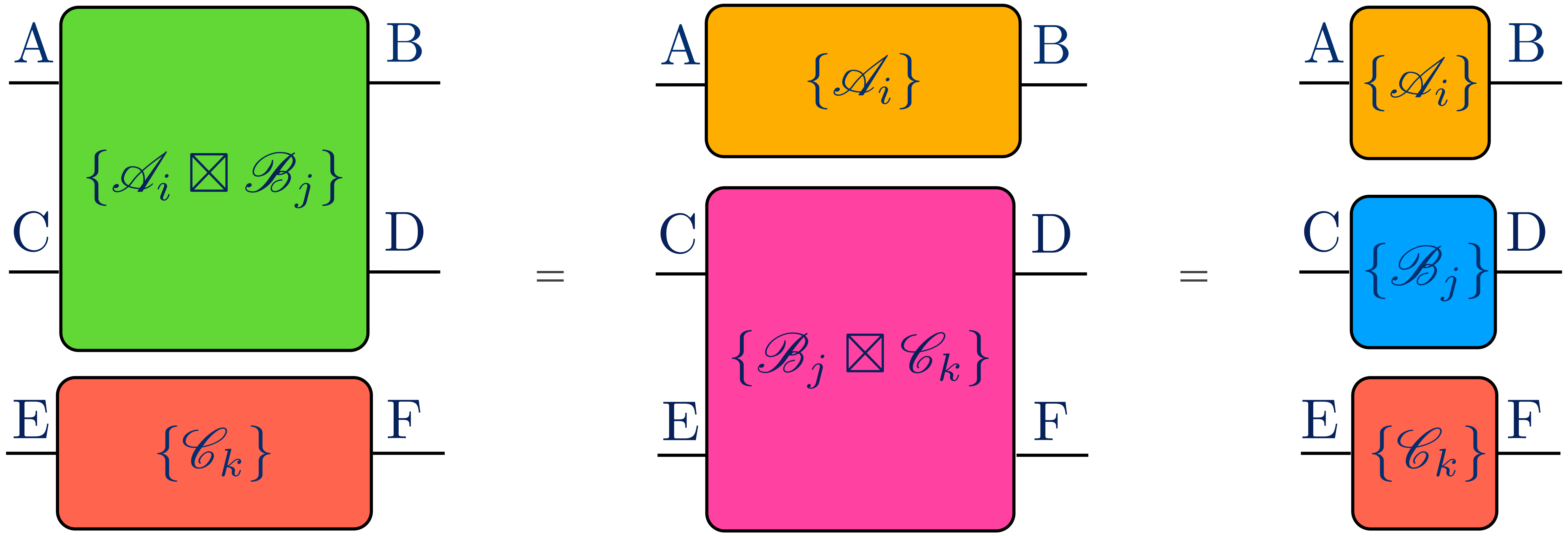


Parallel composition

- Tests:

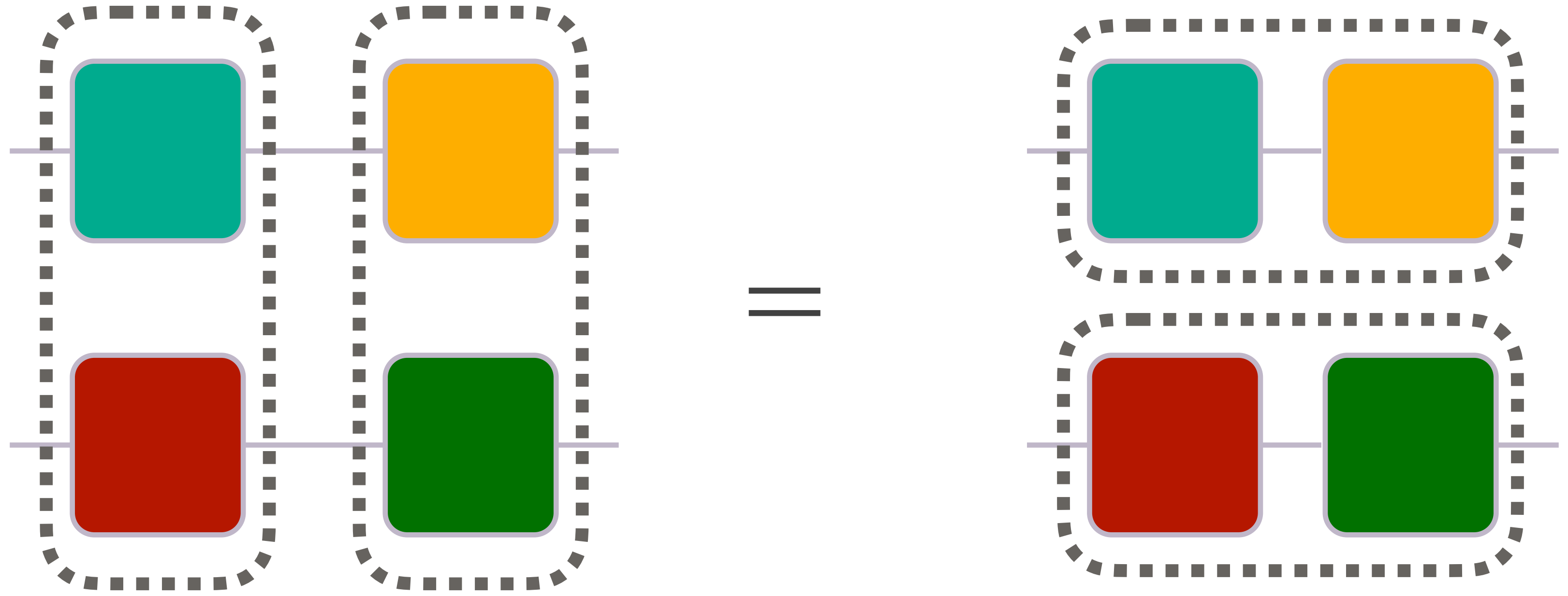


- Associativity



Monoidal structure

- Most important rule:



$$(\mathscr{A} \otimes \mathscr{B})(\mathscr{C} \otimes \mathscr{D}) = (\mathscr{A}\mathscr{C}) \otimes (\mathscr{B}\mathscr{D})$$

B. Coecke, AIP Conf. Proc. **810**, 81 (2006)

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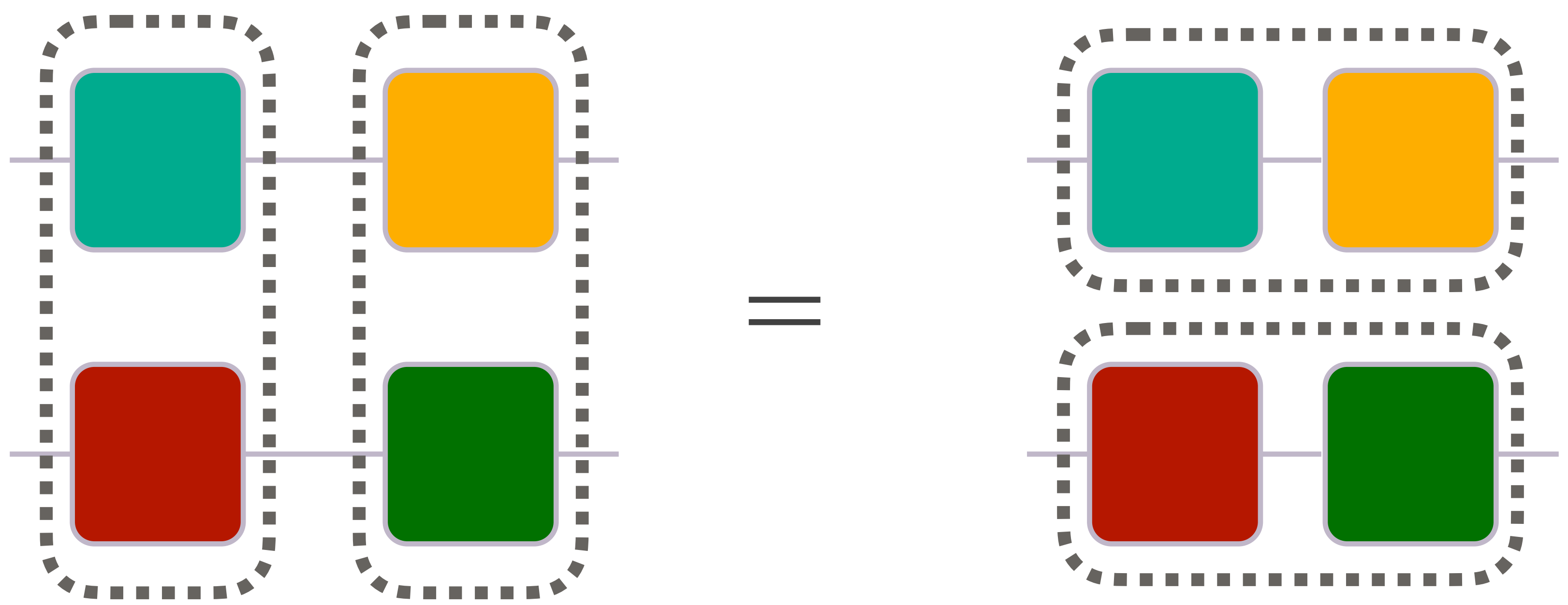

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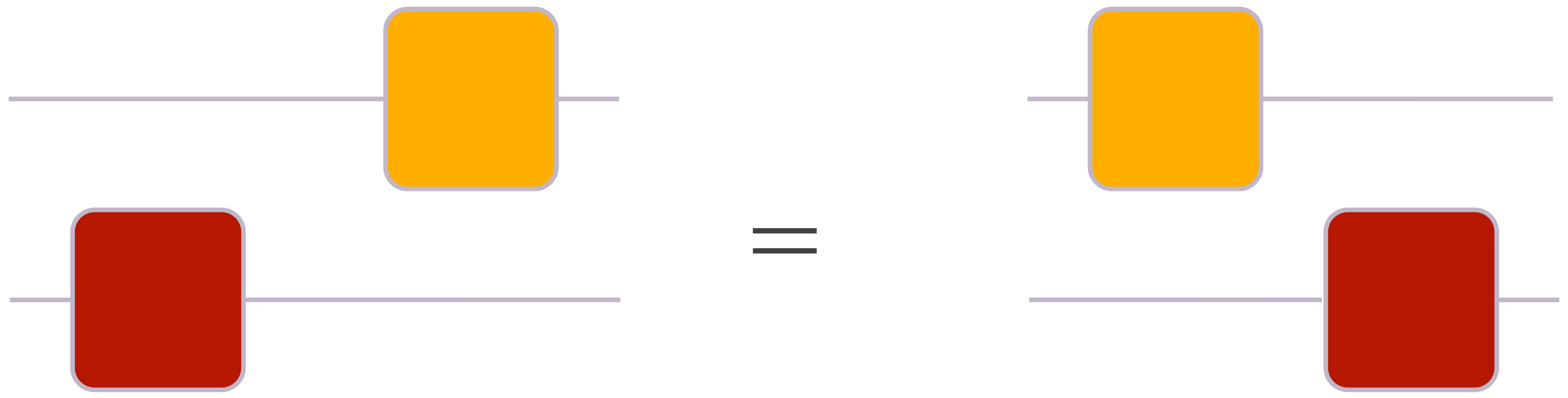
- Consequence:



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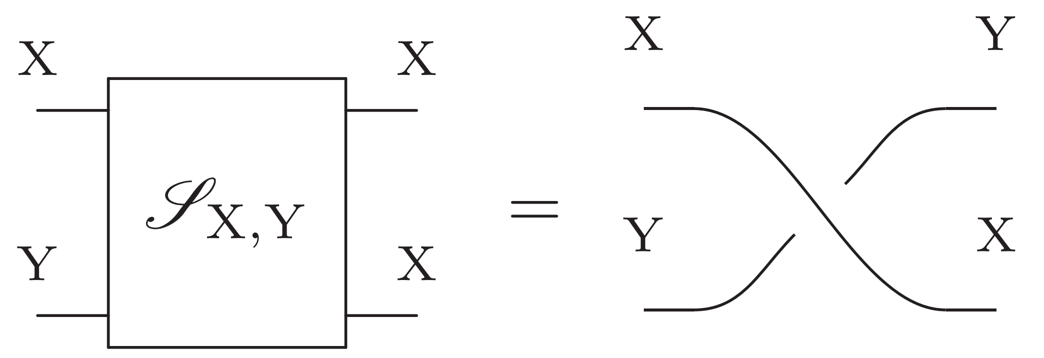
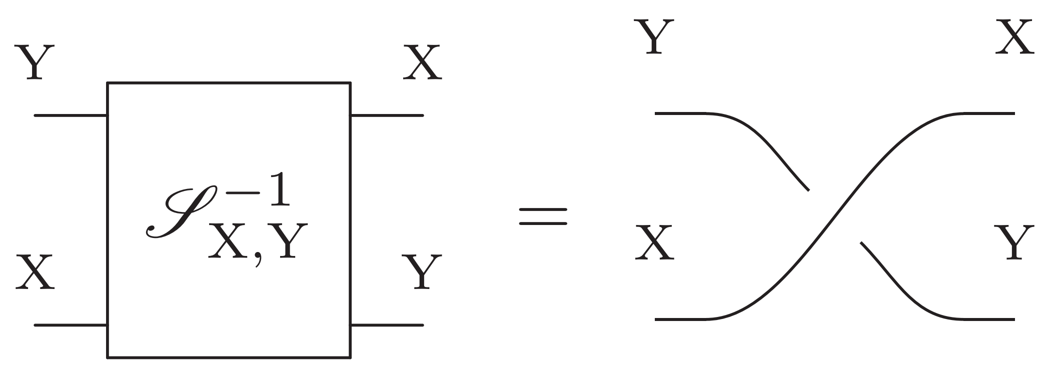
Braiding

- Every composite system AB is isomorphic to BA





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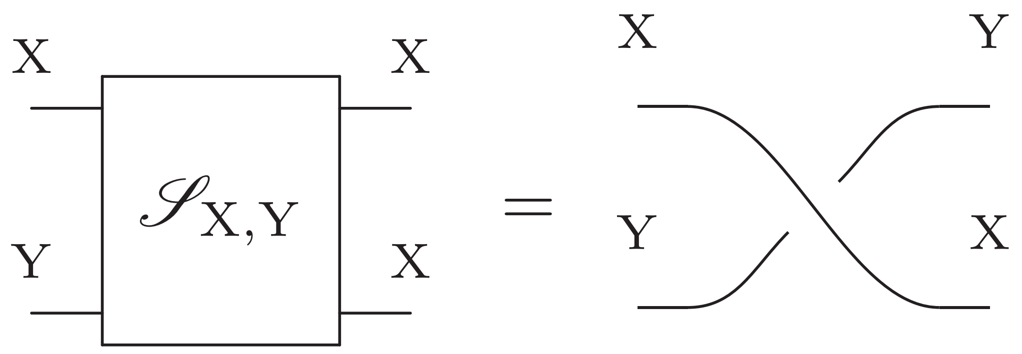
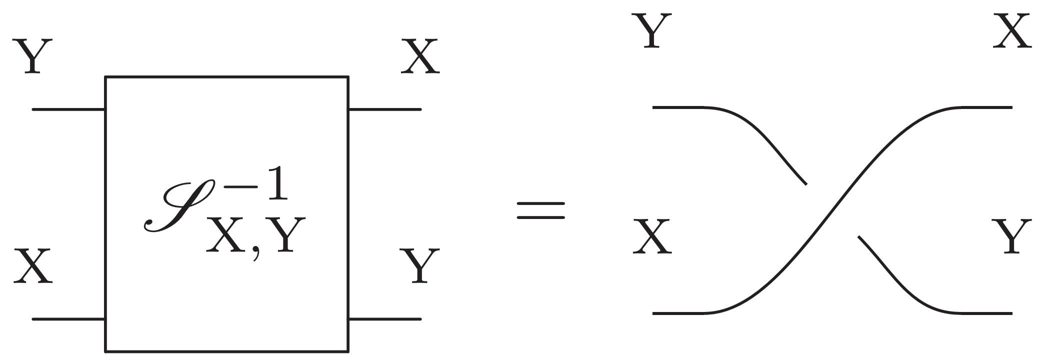
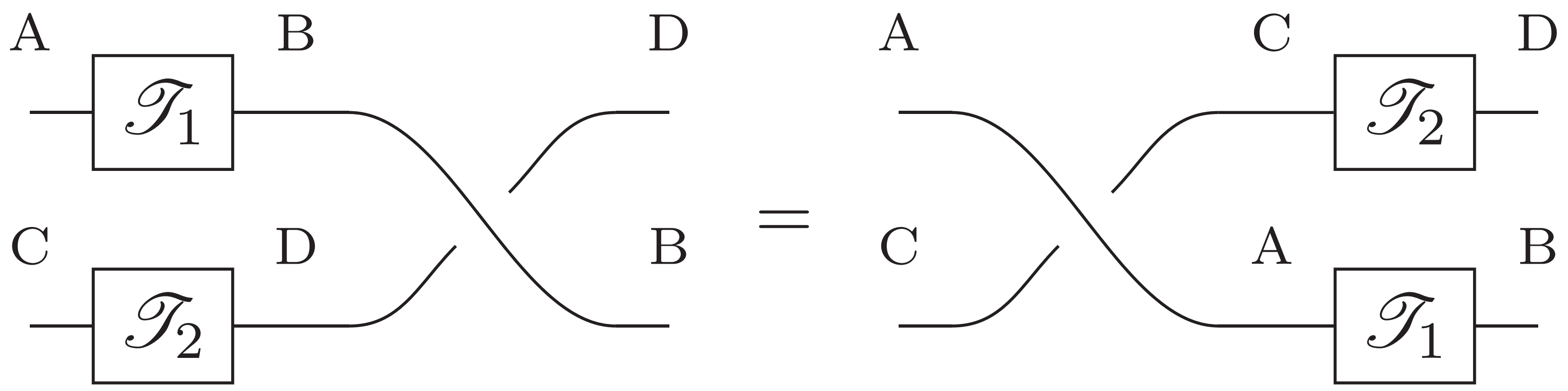




- Characteristic property of Swap



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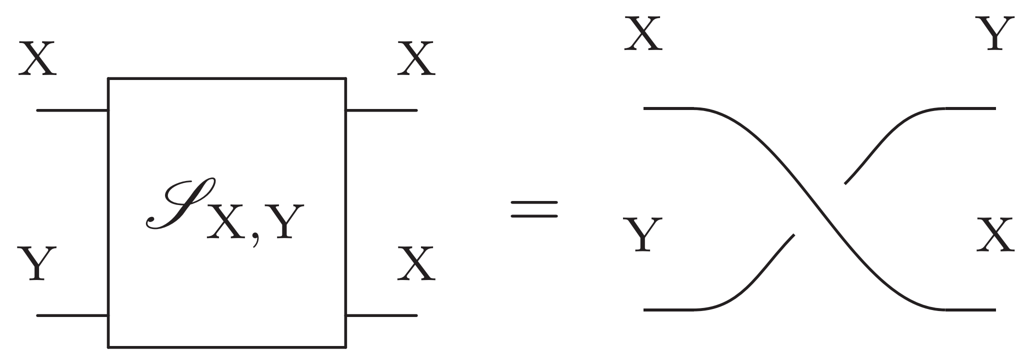

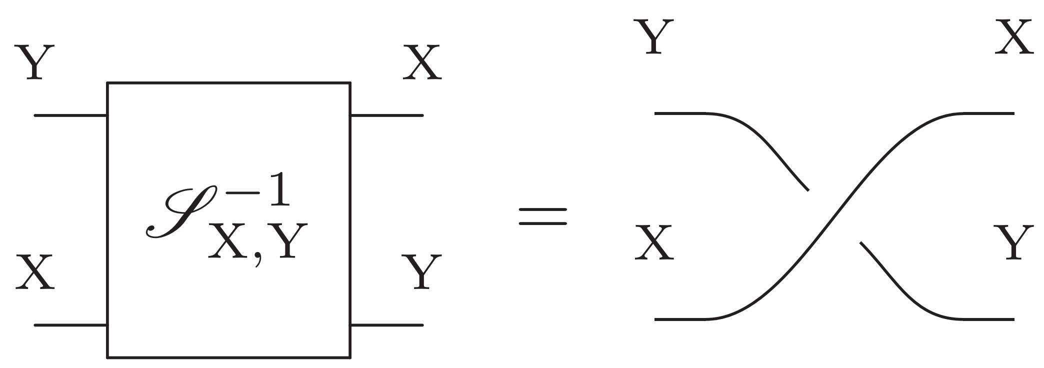

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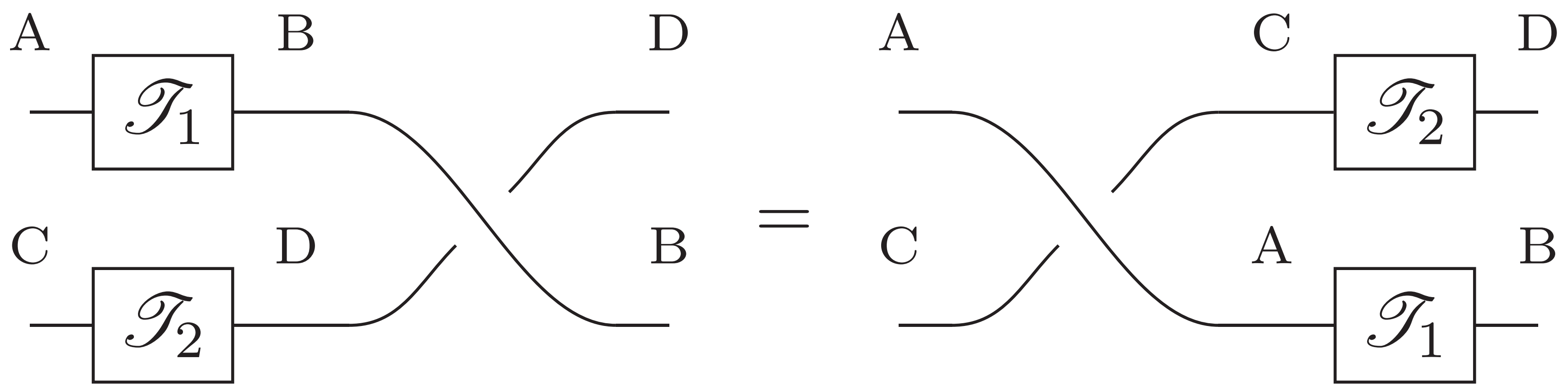
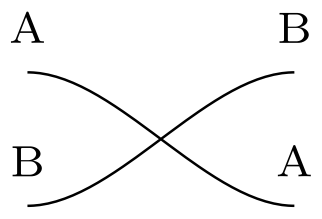


- Symmetric theory: $\mathscr{S}^{-1}_{AB} = \mathscr{S}_{BA}$

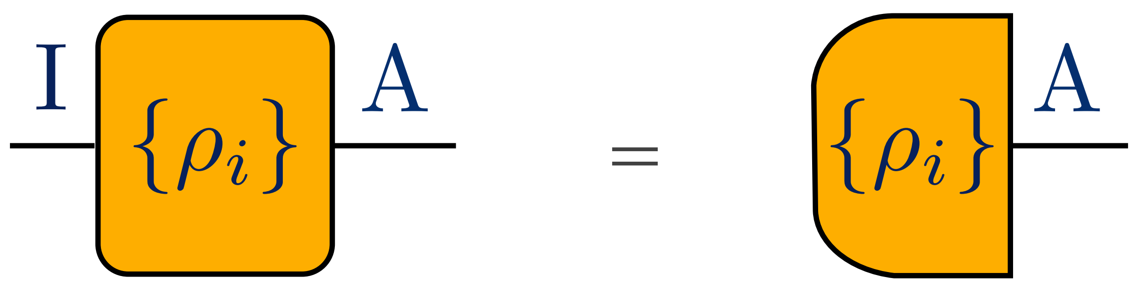


Preparation and observation

- Special tests

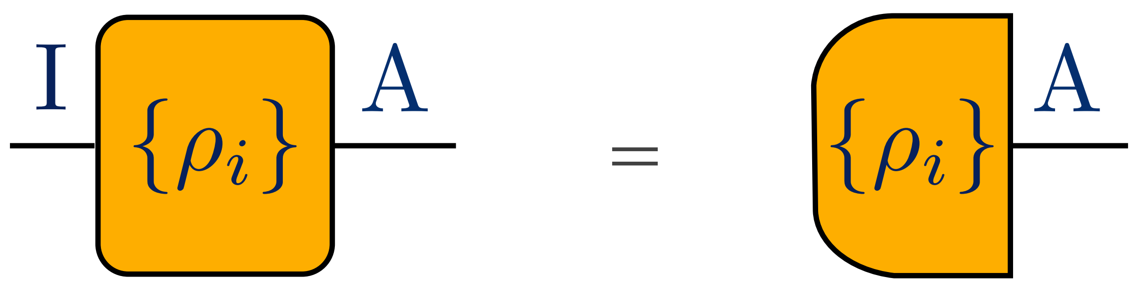
Preparation and observation

- Special tests
  - Trivial input: preparation test

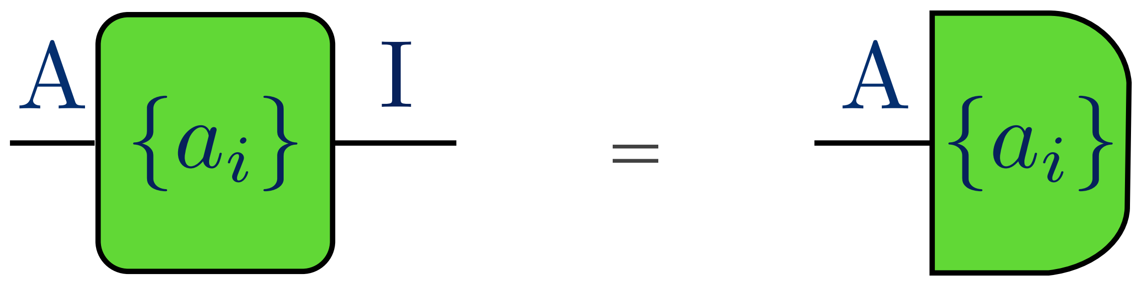


Preparation and observation

- Special tests
  - Trivial input: preparation test



  - Trivial output: observation test



Example I

Quantum theory

- Systems correspond to complex Hilbert spaces

$$\mathrm{A} \quad \Longleftrightarrow \quad \mathsf{H}_{\mathrm{A}} \simeq \mathbb{C}^{d_{\mathrm{A}}}$$

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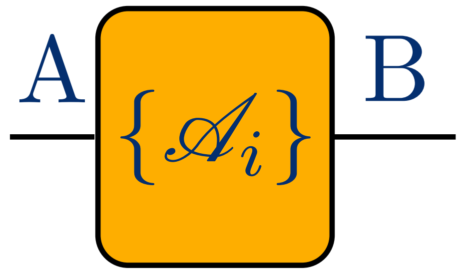

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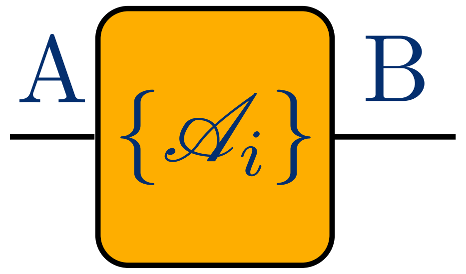

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- Preparations: ensembles $\{\rho_i\}; \ \forall i \ \rho_i \geq 0, \quad \mathrm{Tr}[\sum_i \rho_i] = 1$

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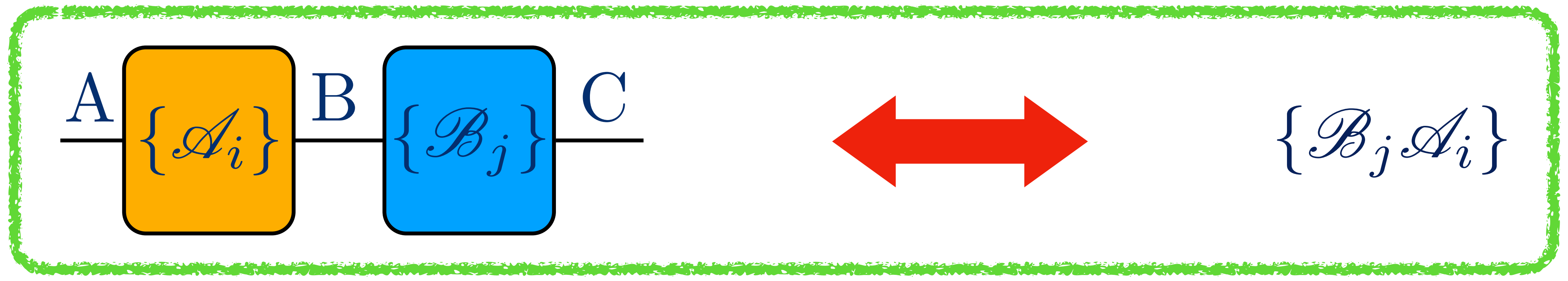
Quantum theory

- Sequential composition: composition of CP maps



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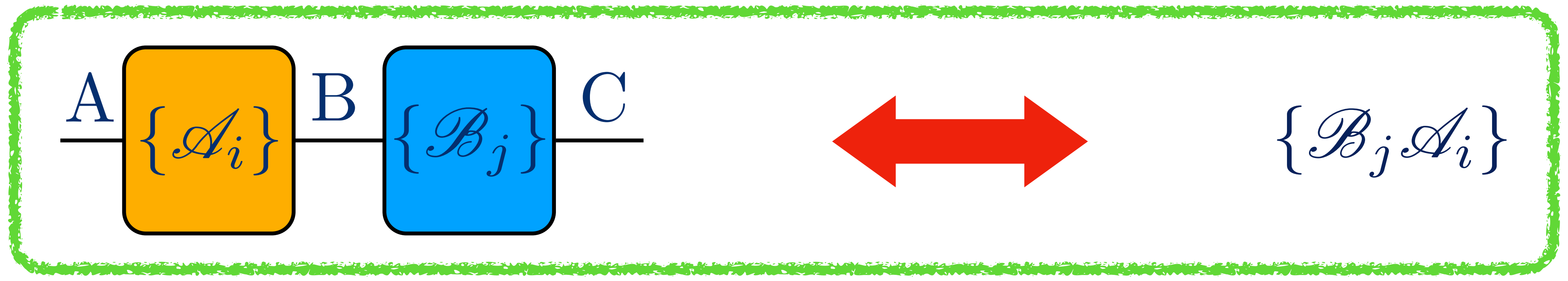

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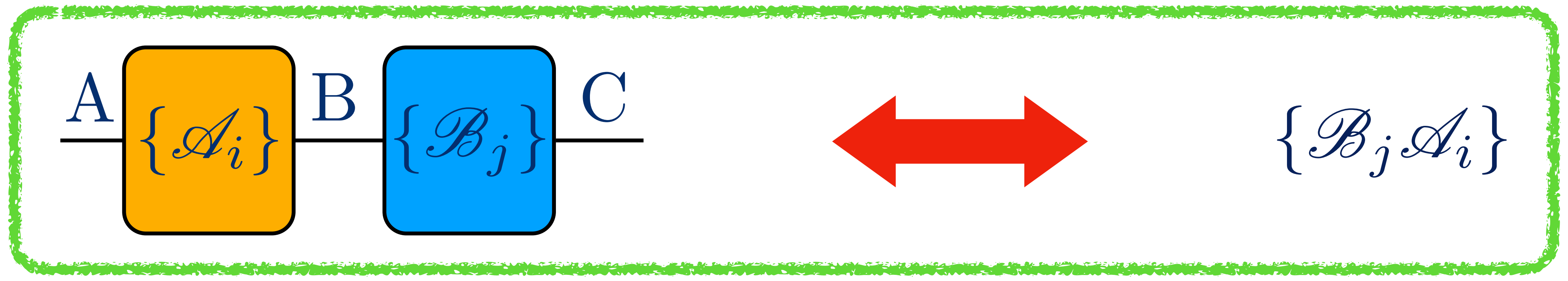

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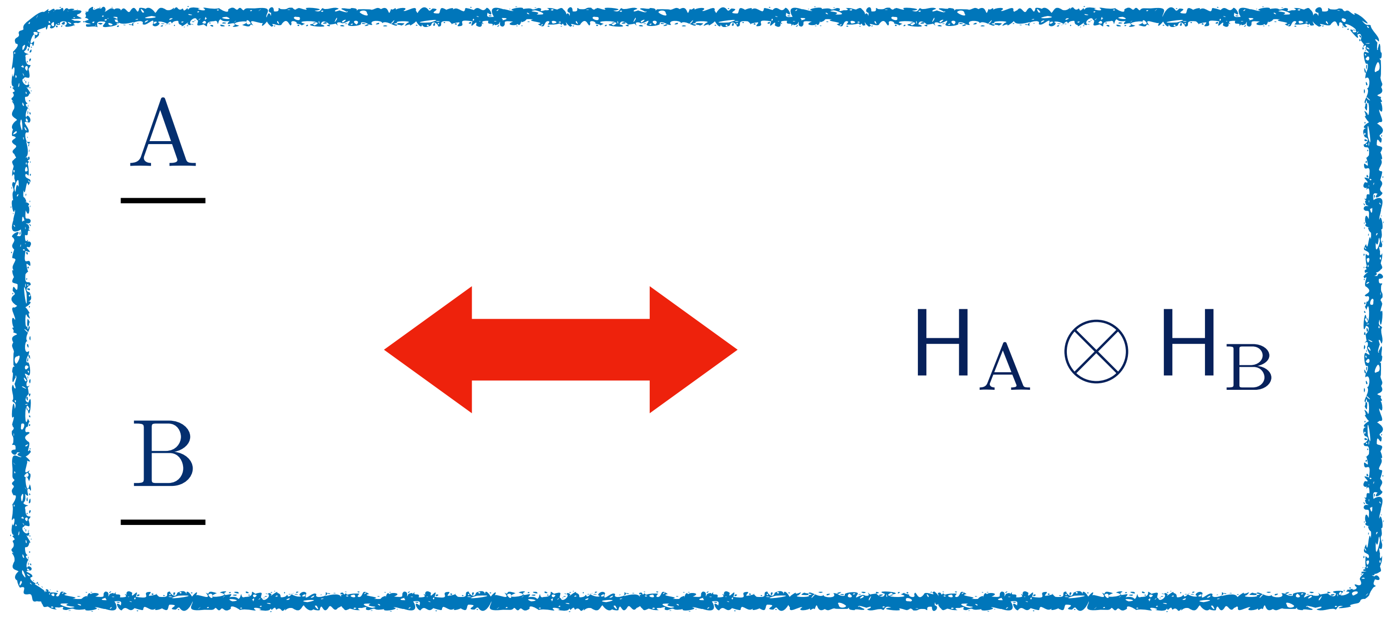

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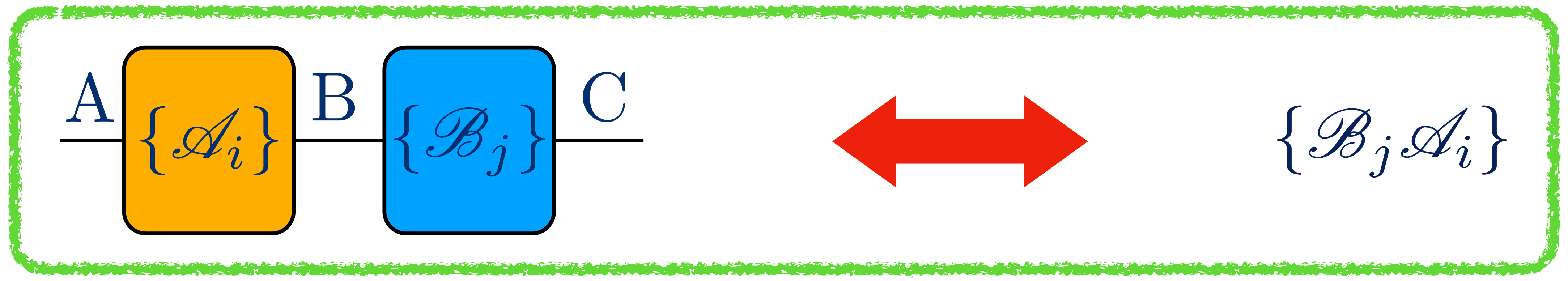

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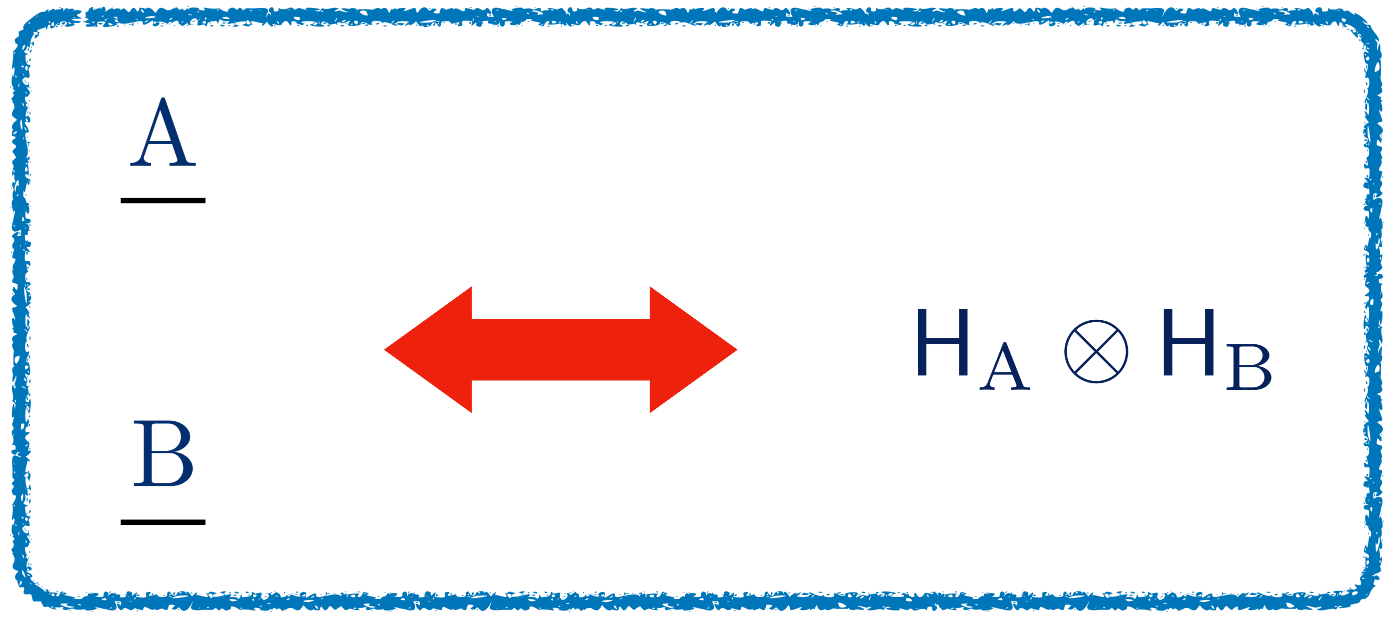

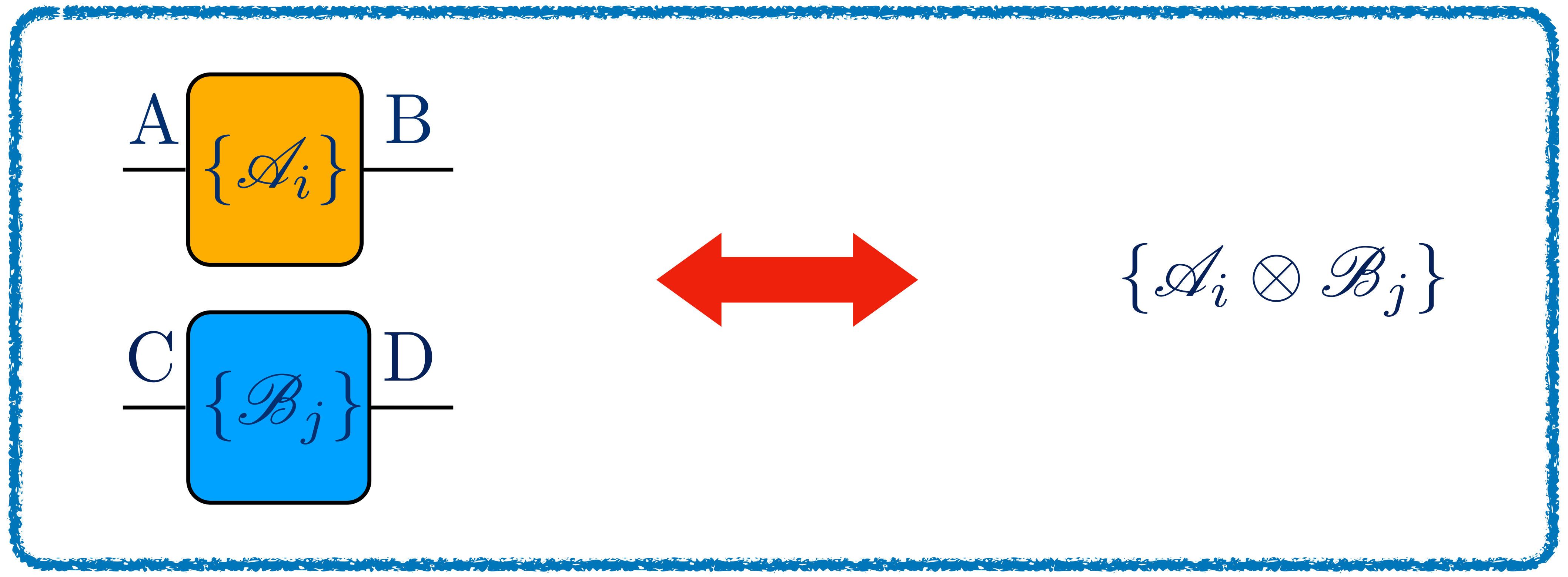

Example II

Classical theory

- Systems correspond to Real vector spaces

$$\mathrm{A} \quad \Longleftrightarrow \quad \mathbb{R}^{d_\mathrm{A}}$$

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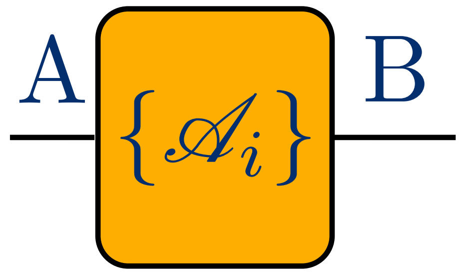

$$\longleftrightarrow \quad \{M^{(i)}\}, \quad M^{(i)}_{xy} \geq 0, \quad \sum_i M^{(i)} = M, \quad \forall y \sum_x M_{xy} = 1$$

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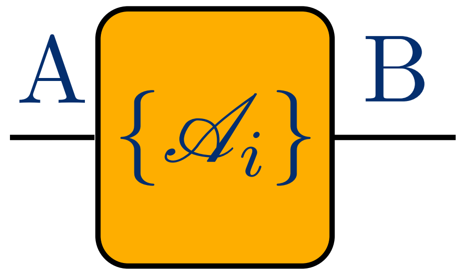

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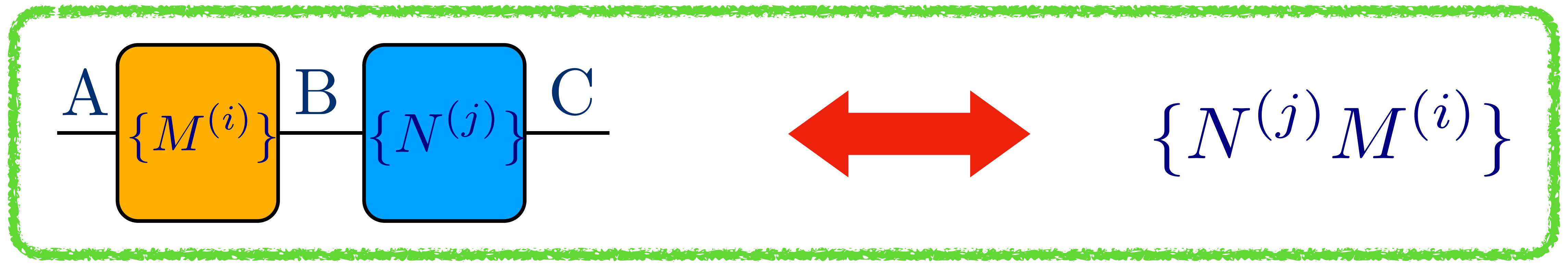
- States: probability vectors ($y \in \{*\}$)

Example II

Classical theory

- Sequential composition: matrix product



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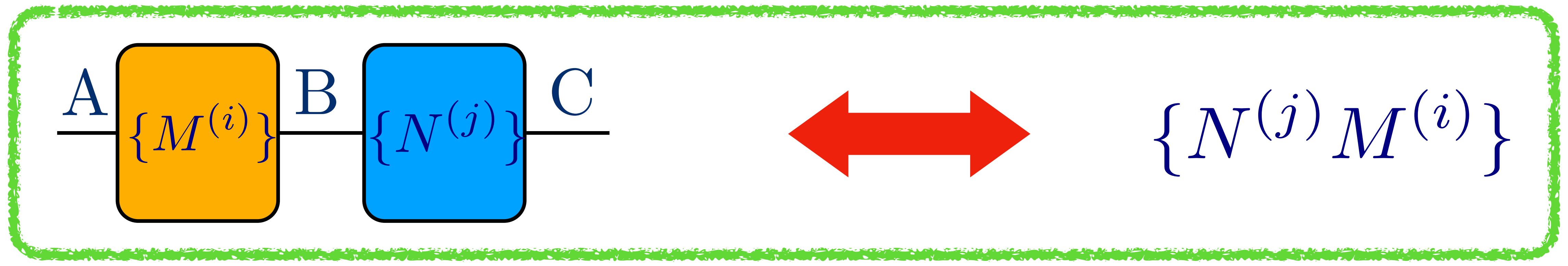

- Parallel composition: tensor product

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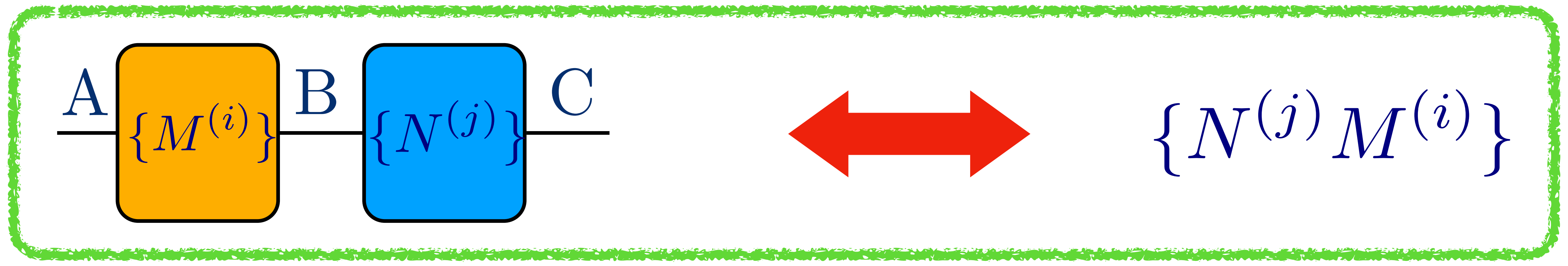

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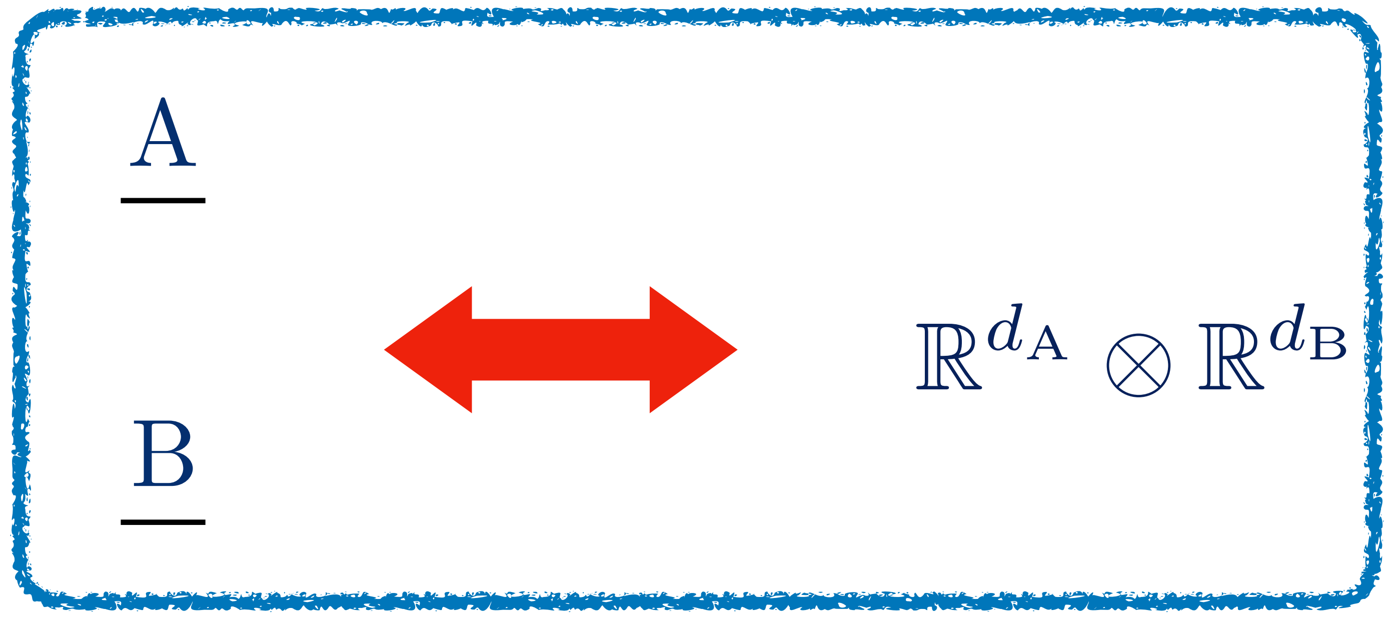

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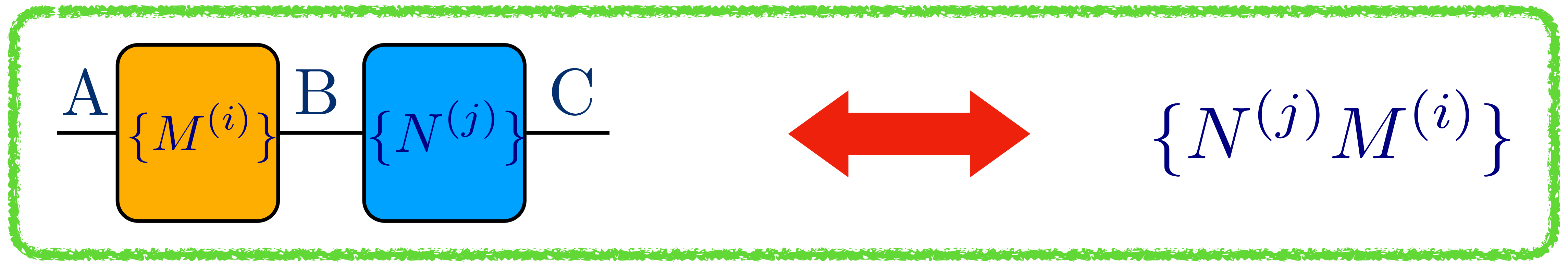

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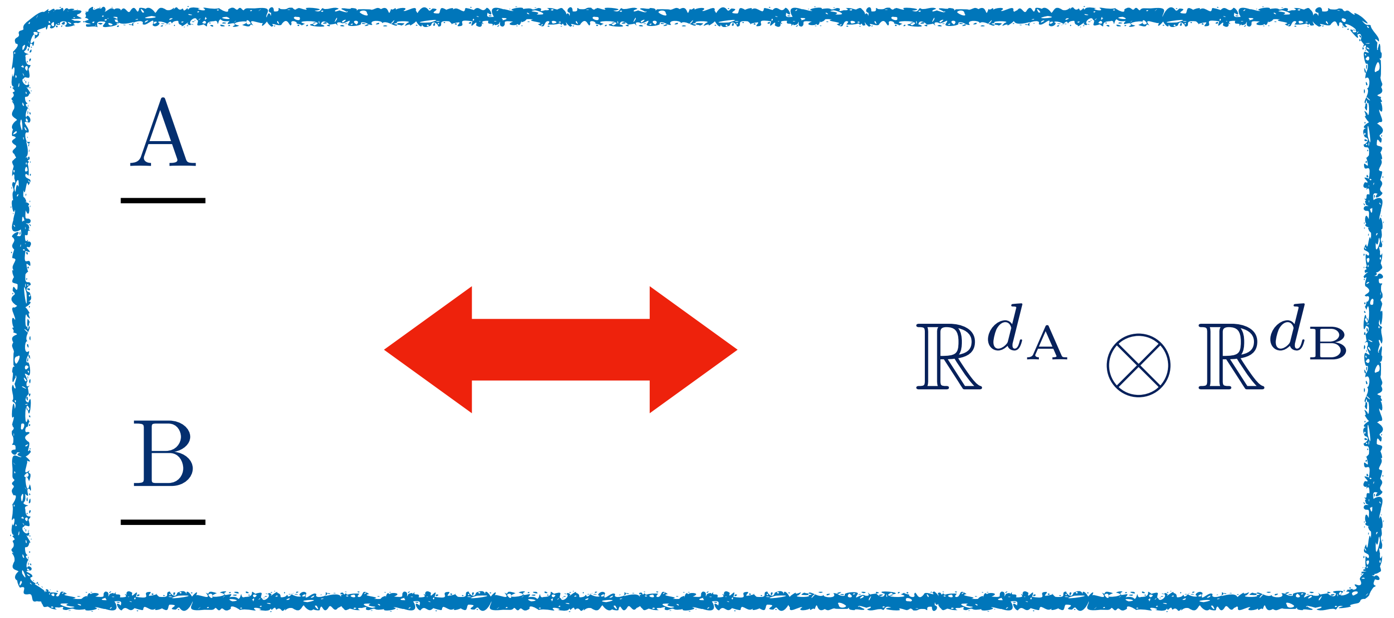
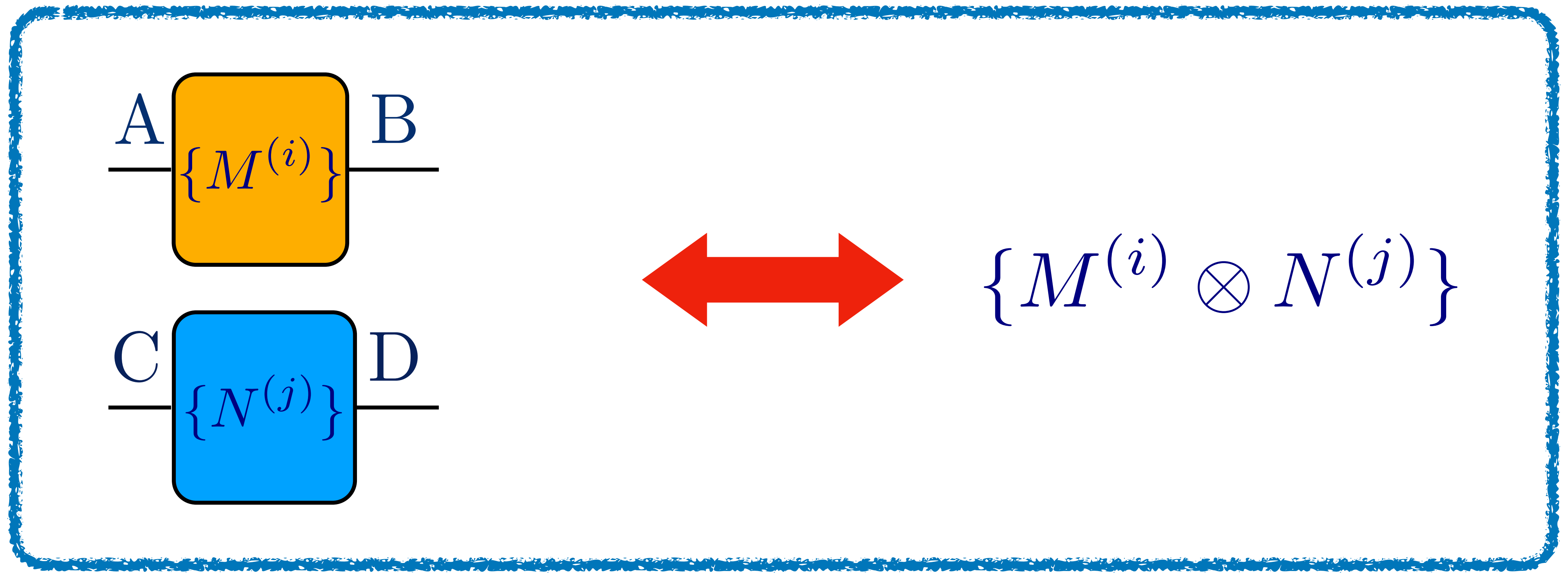
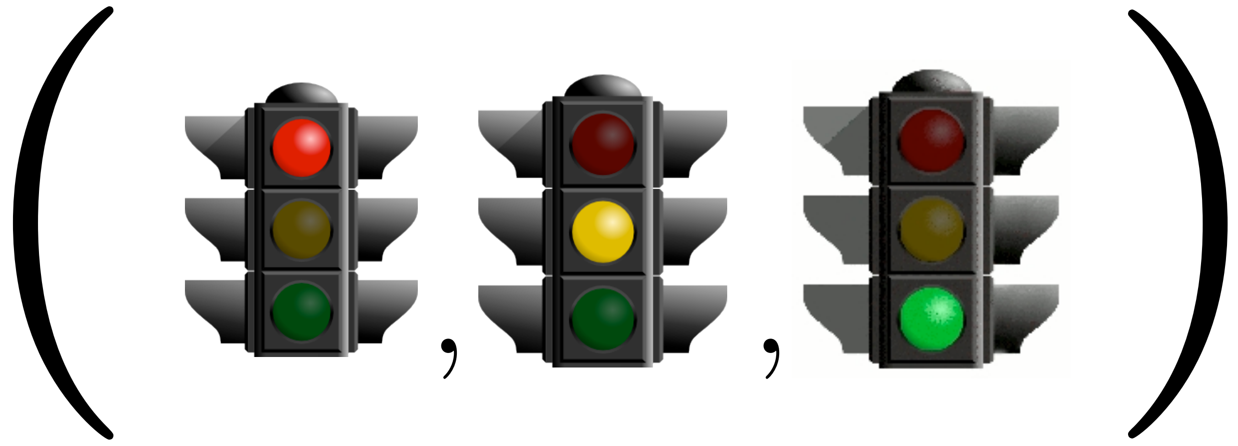




Coarse-Graining

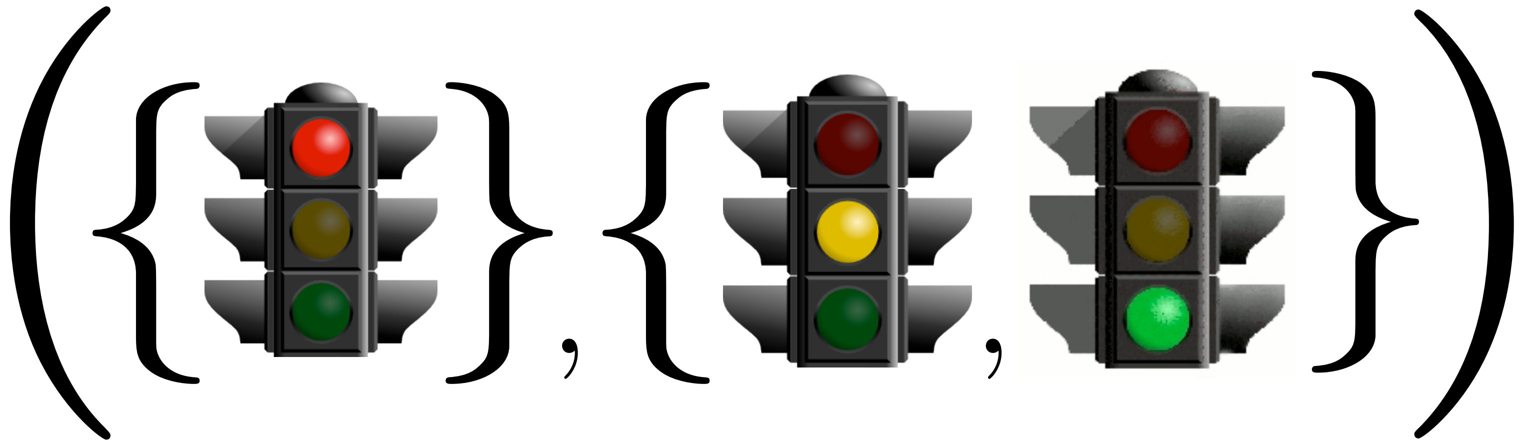
Example

$(\mathscr{C}_i)_{i \in X} =$



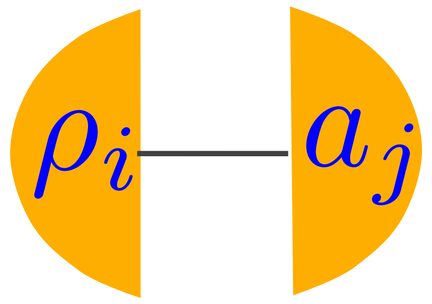
$X = (r, y, g)$

$(\mathscr{D}_j)_{j \in Y} =$



$Y = (r, \bar{r}) = (\{r\}, \{y, g\})$

Probabilistic theories

Every test of type I→I is a probability distribution 

$= \Pr(a_j, \rho_i)$

States are functionals on effects and vice-versa $[\![\mathrm{A}]\!]$, $[\![\bar{\mathrm{A}}]\!]$

G. Chiribella, G. M. D'Ariano, and PP, Phys. Rev. A **81**, 062348 (2010)

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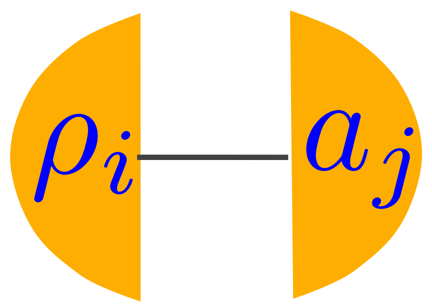
Every test of type I→I is a probability distribution 

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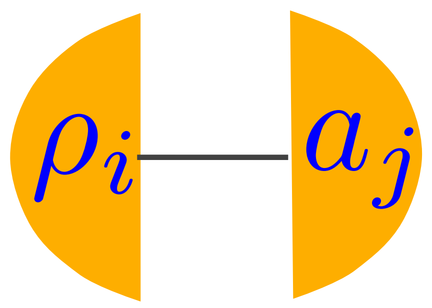
Every test of type I→I is a probability distribution 

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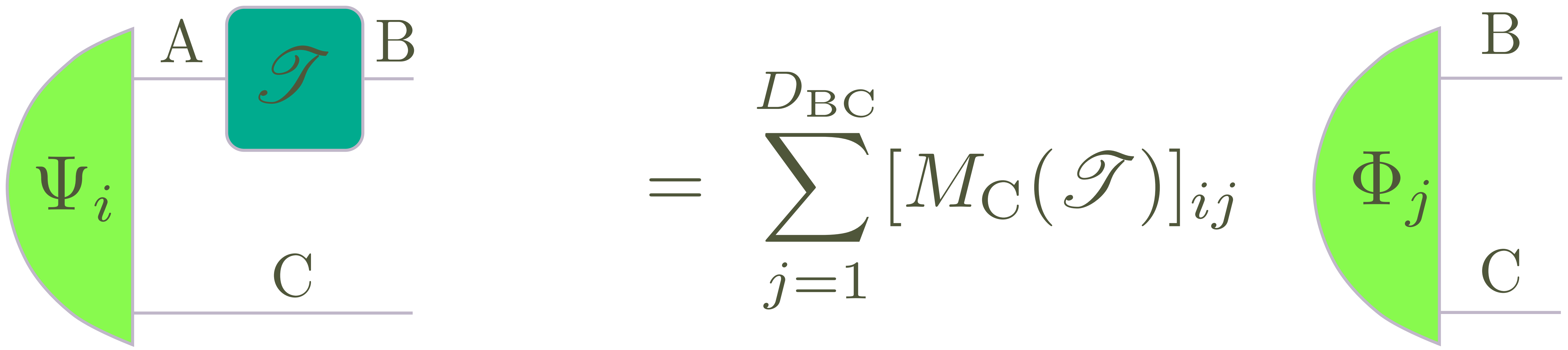
Coarse graining is represented by the sum

G. Chiribella, G. M. D'Ariano, and PP, Phys. Rev. A **81**, 062348 (2010)

Transformations

A transformation $\mathscr{T} \in [\![\mathrm{A} \to \mathrm{B}]\!]$ induces a family of linear maps:

$\{M_{\mathrm{C}}(\mathscr{T})\}_{\mathrm{C}}$ representing $\mathscr{T} \otimes \mathscr{I}_{\mathrm{C}}$ on $[\![\mathrm{AC}]\!]_{\mathbb{R}}$



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Transformations

Indeed, it is not sufficient to know the linear map induced by $\mathscr{T}$ on $[\![\mathrm{A}]\!]_{\mathbb{R}}$

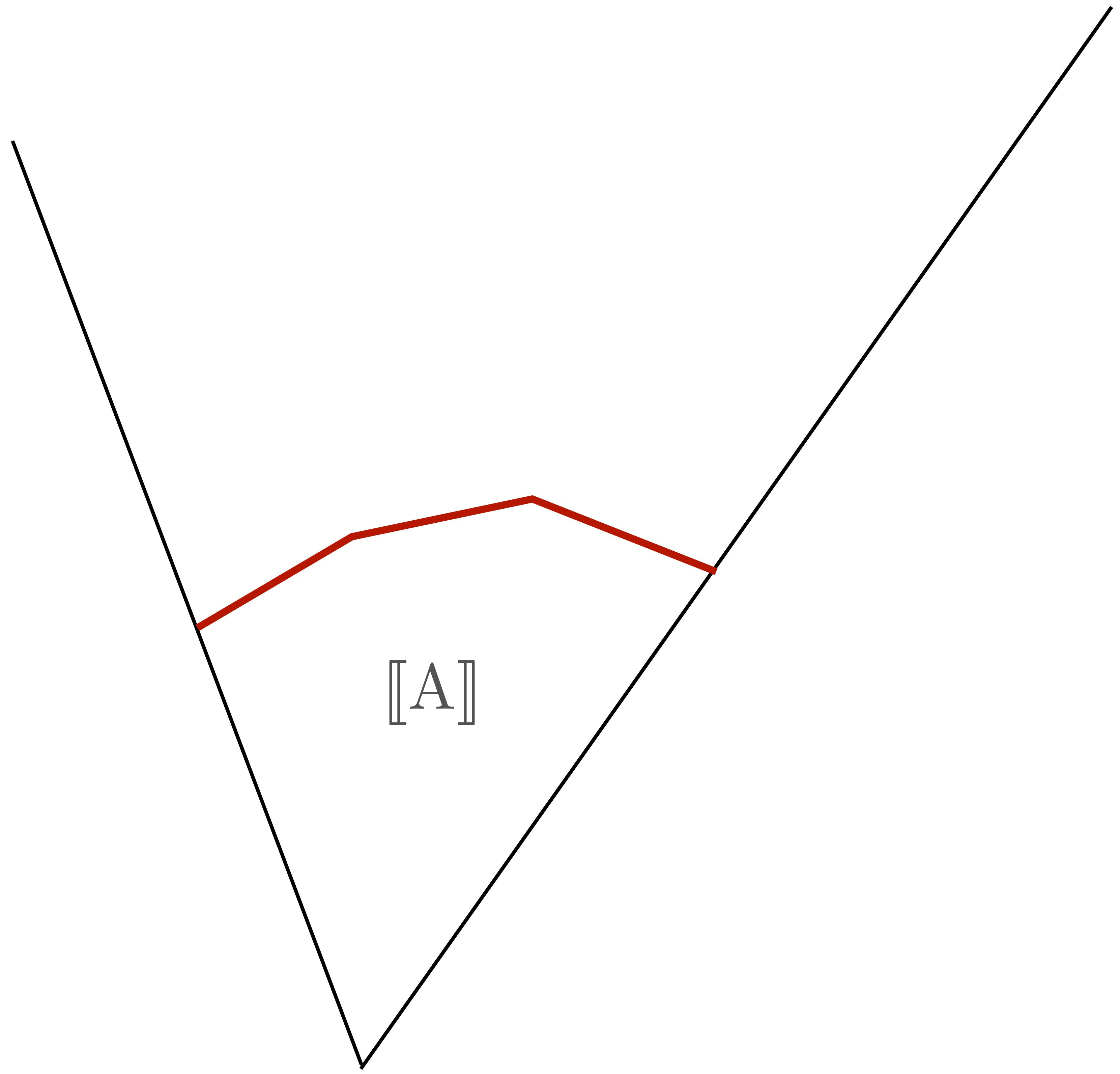
E.g.: transpose map in real quantum theory

- $\rho^T = \rho$ → $\mathscr{T}(\rho) = \mathscr{I}(\rho) \quad \forall \rho$

- $\sigma_y \otimes \sigma_y \in [\![\mathrm{AB}]\!]_{\mathbb{R}}$ $\quad (\mathscr{T} \otimes \mathscr{I}_{\mathrm{C}})(\sigma_y \otimes \sigma_y) = -\sigma_y \otimes \sigma_y$

States and effects

- States: convex set of $[\![\mathrm{A}]\!]_{\mathbb{R}}$ $[\![\mathrm{A}]\!]$



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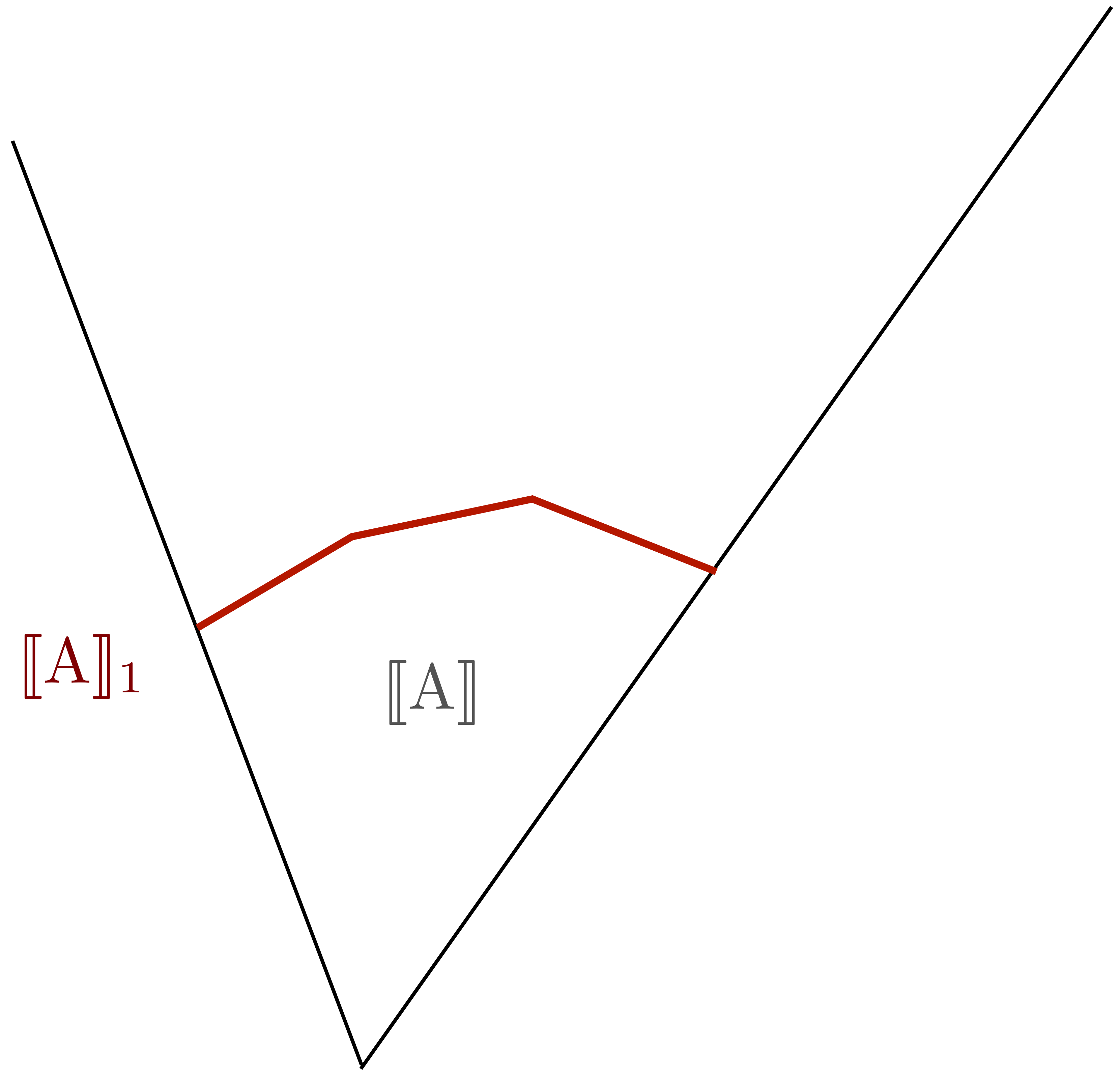

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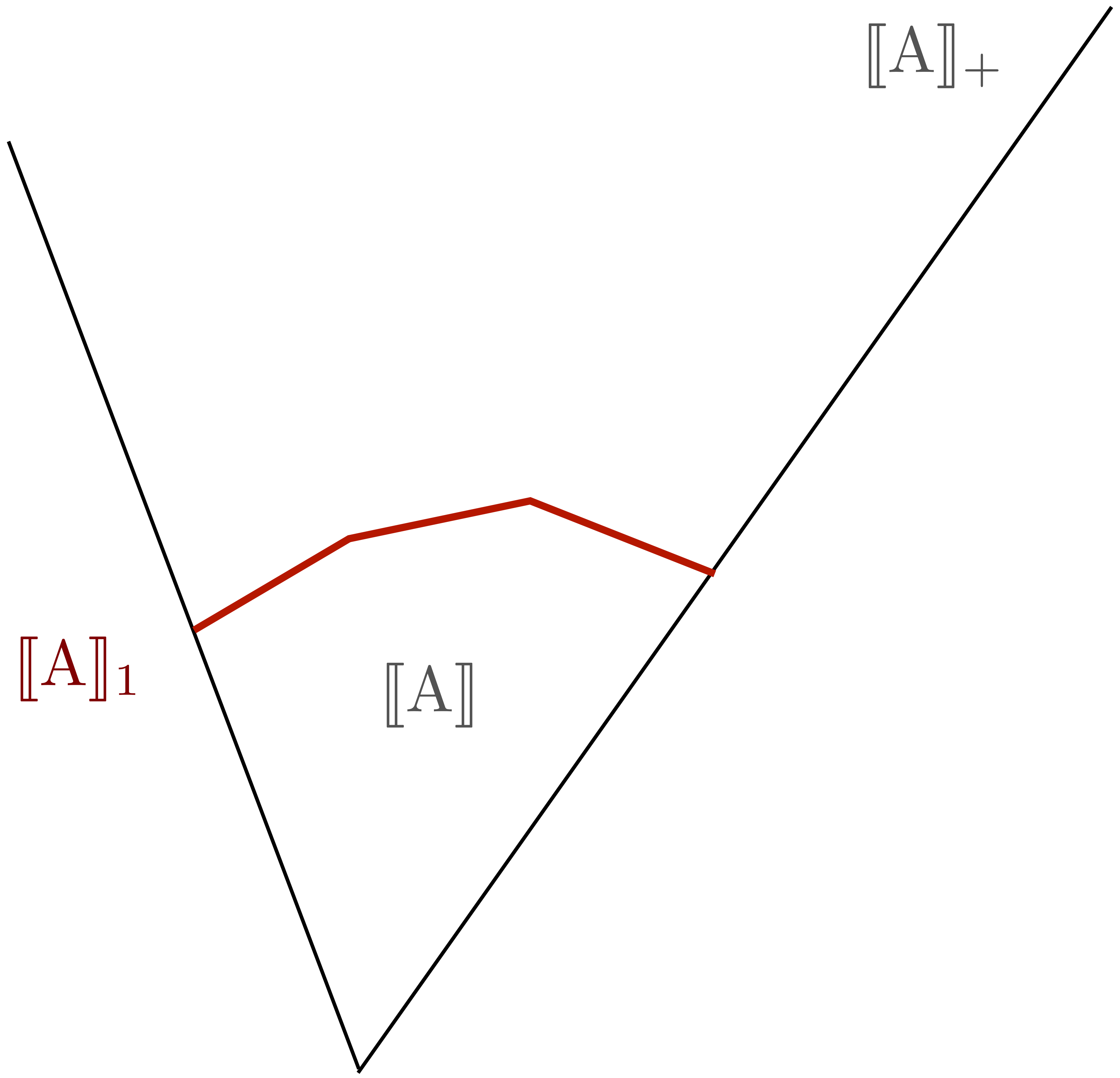

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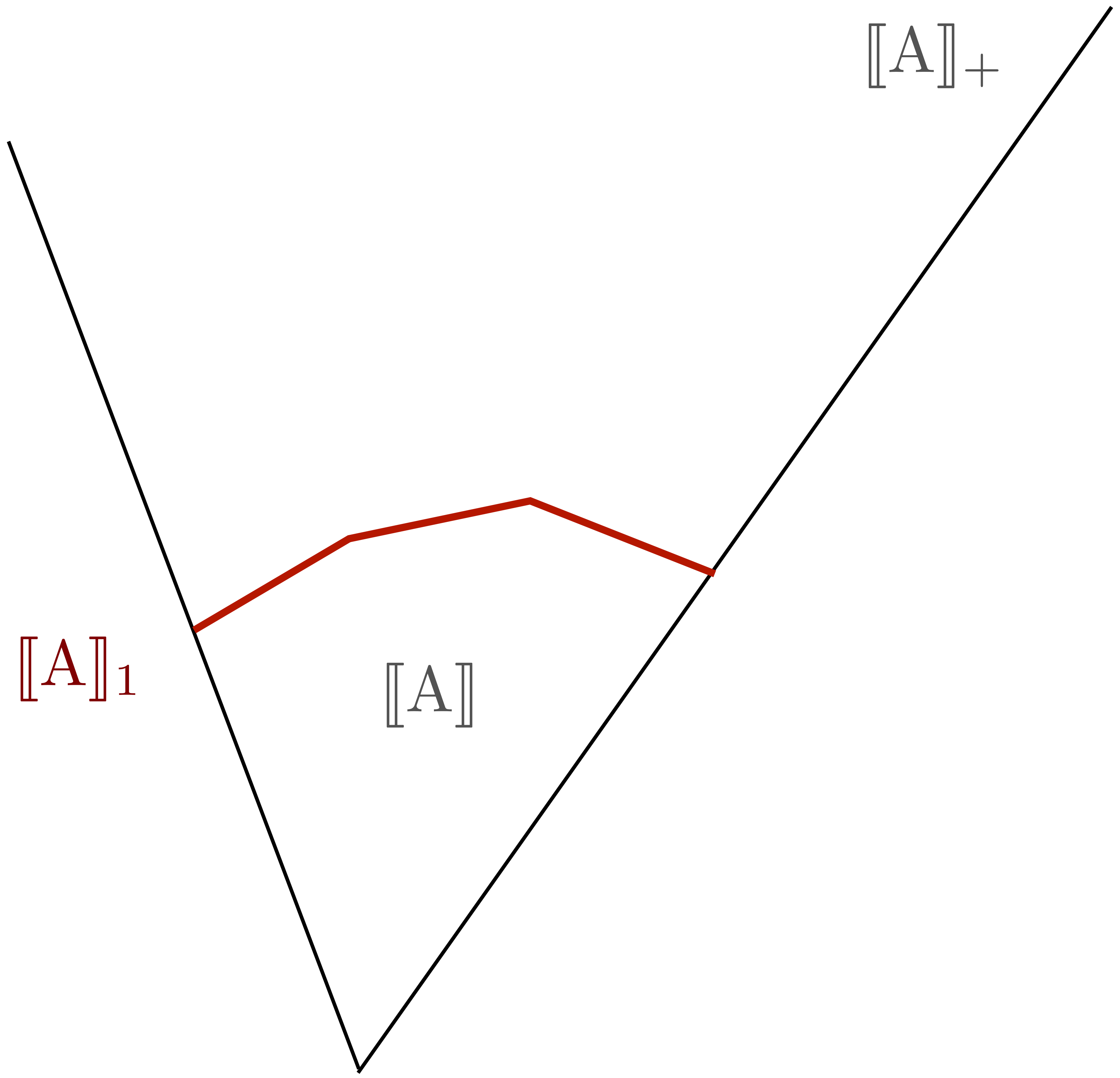
- Deterministic states belong to singleton preparation tests $[\![\mathrm{A}]\!]_1$
- Cone with base $[\![\mathrm{A}]\!]$ $[\![\mathrm{A}]\!]_+$
- Similar structures for effects

$$[\![\bar{\mathrm{A}}]\!]_1 \subseteq [\![\bar{\mathrm{A}}]\!] \subseteq [\![\bar{\mathrm{A}}]\!]_+ \subseteq [\![\bar{\mathrm{A}}]\!]_{\mathbb{R}}$$

and transformations

$$[\![\mathrm{A} \to \mathrm{B}]\!]_1 \subseteq [\![\mathrm{A} \to \mathrm{B}]\!] \subseteq [\![\mathrm{A} \to \mathrm{B}]\!]_+ \subseteq [\![\mathrm{A} \to \mathrm{B}]\!]_{\mathbb{R}}$$



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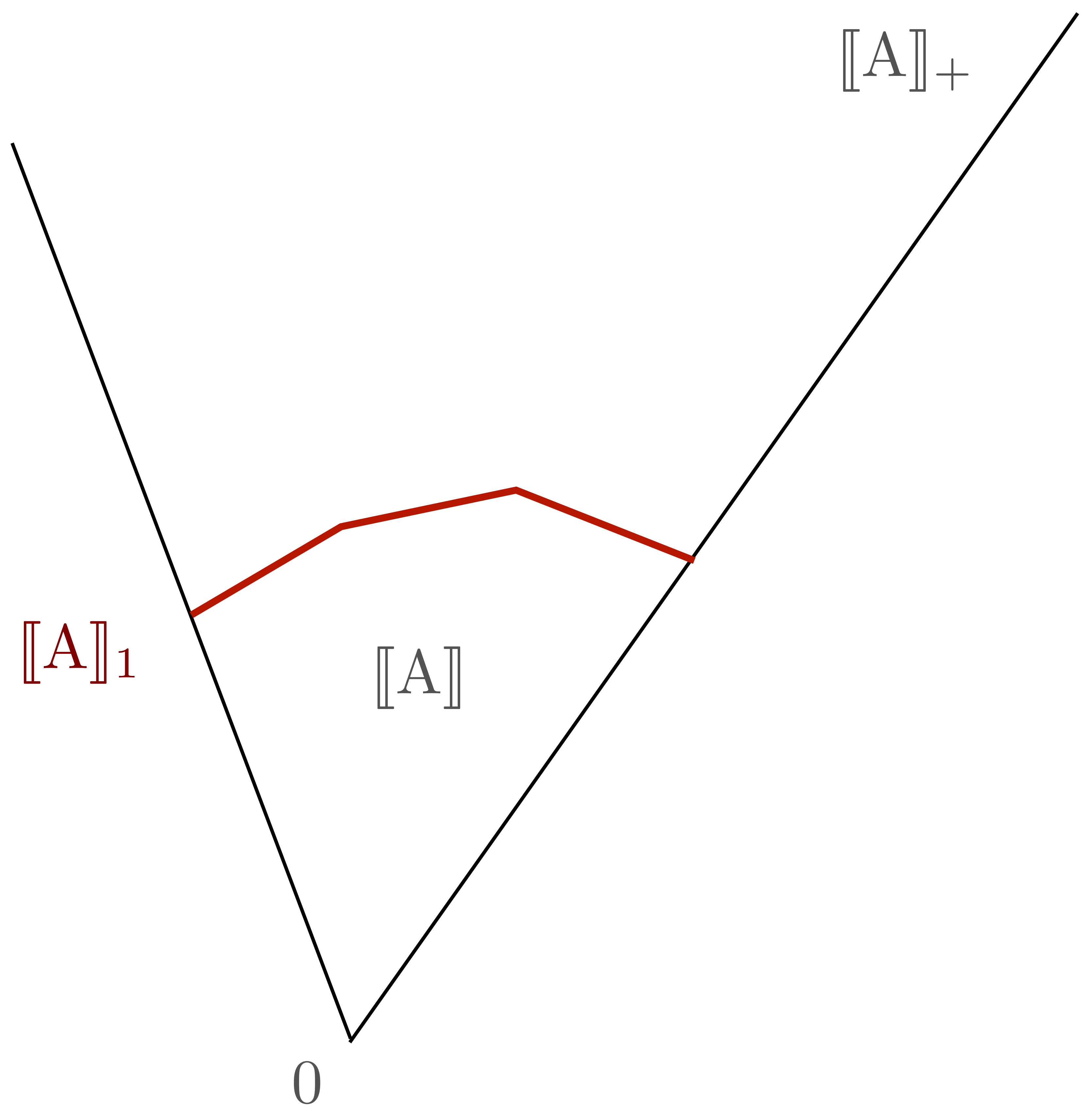
Cones

- The cone
  $[\![A]\!]_+ := \{\sigma \in [\![A]\!]_{\mathbb{R}} \mid \sigma = \lambda\rho,\ \lambda \geq 0,\ \rho \in [\![A]\!]\}$
  introduces an order

$$\tau \geq \nu \;\Leftrightarrow\; \tau - \nu \in [\![A]\!]_+$$



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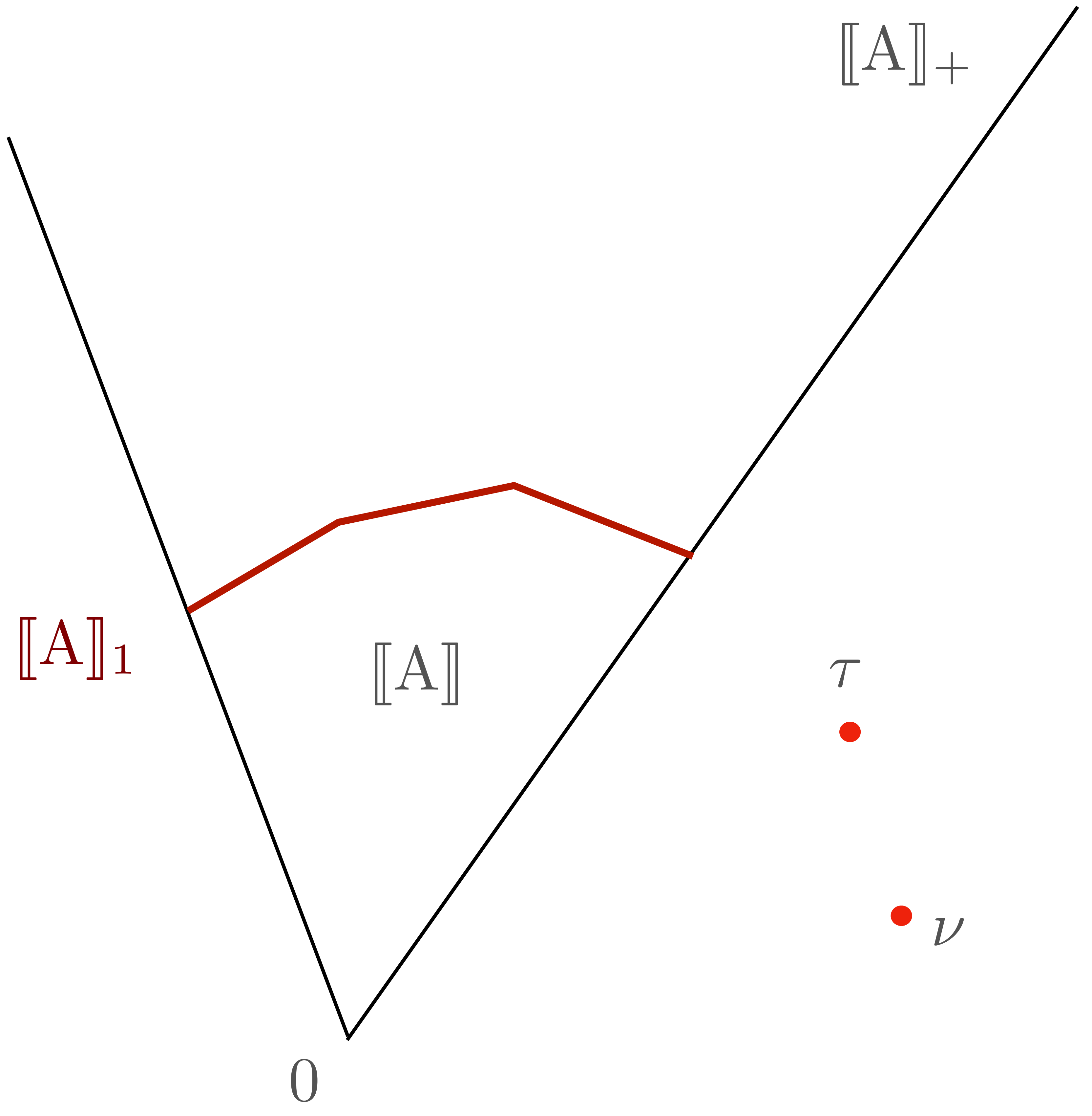

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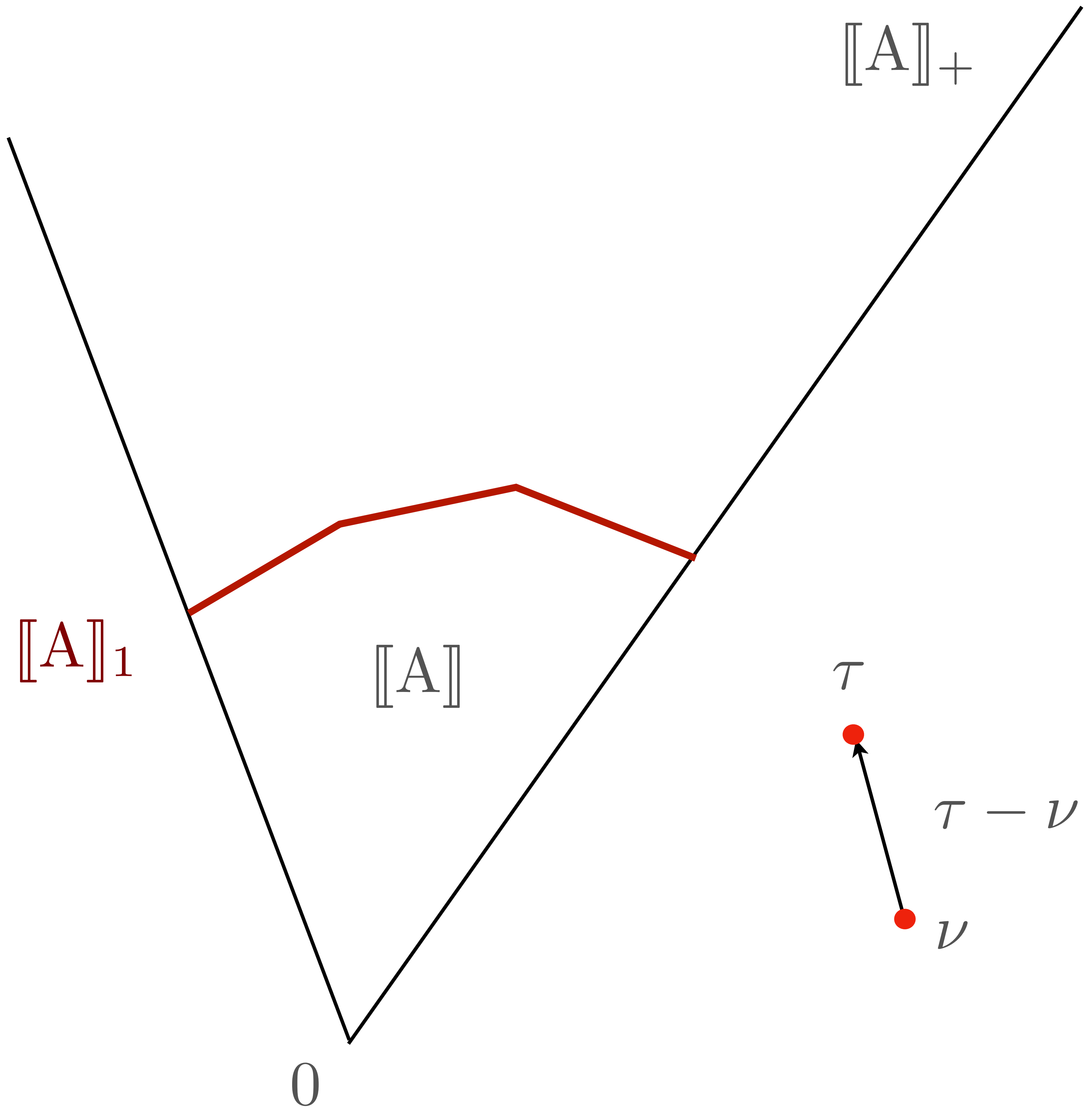

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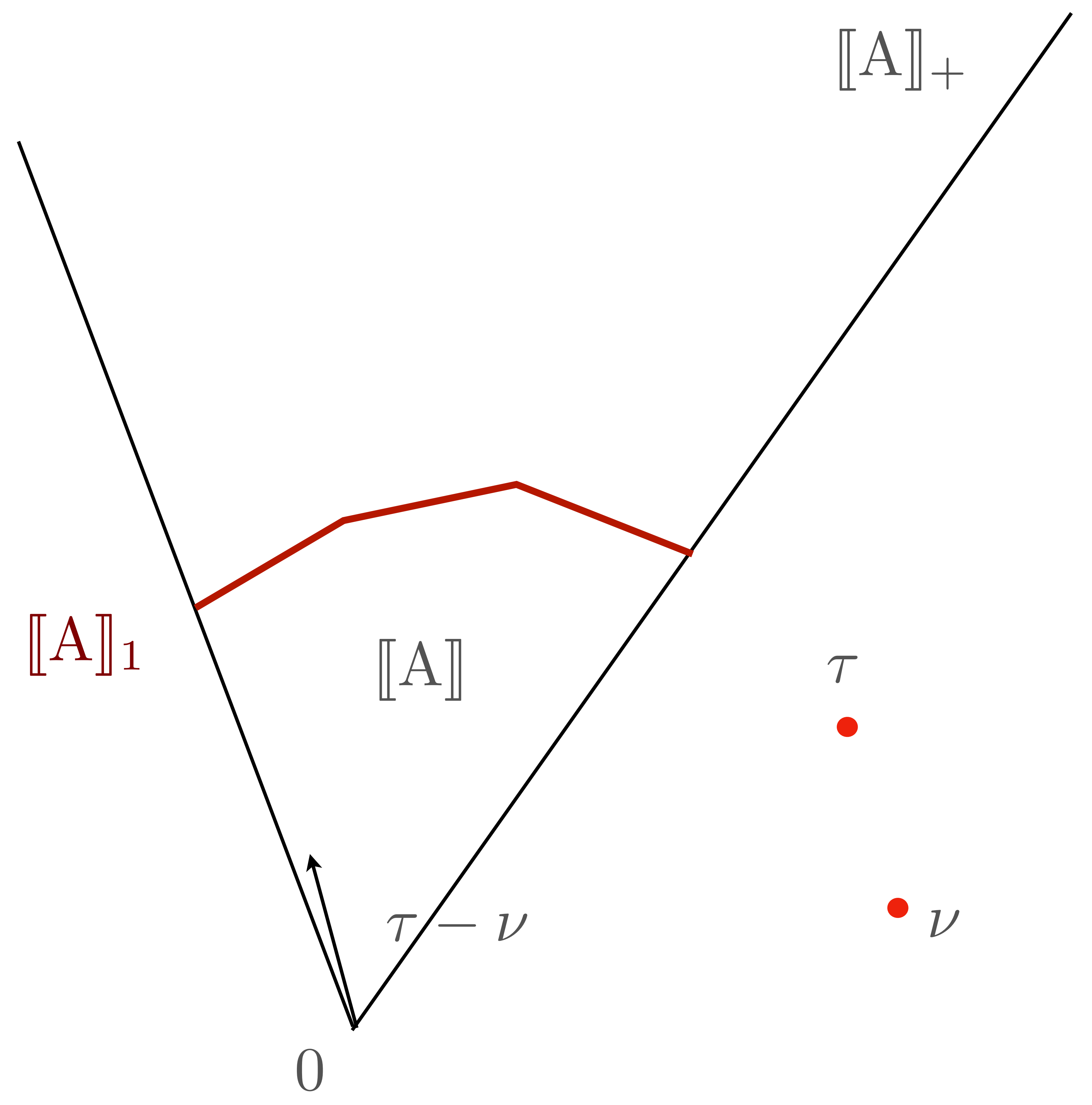

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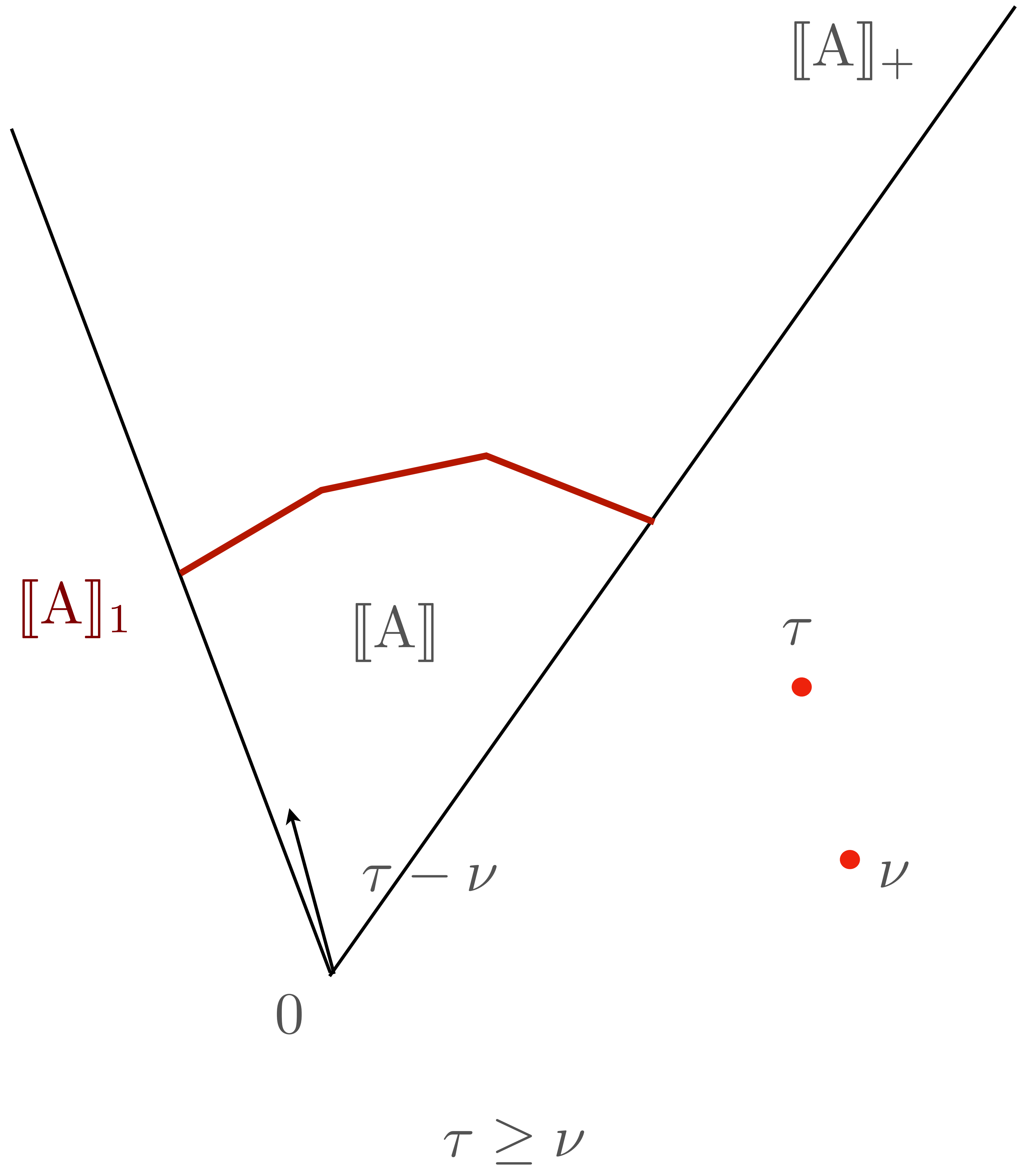

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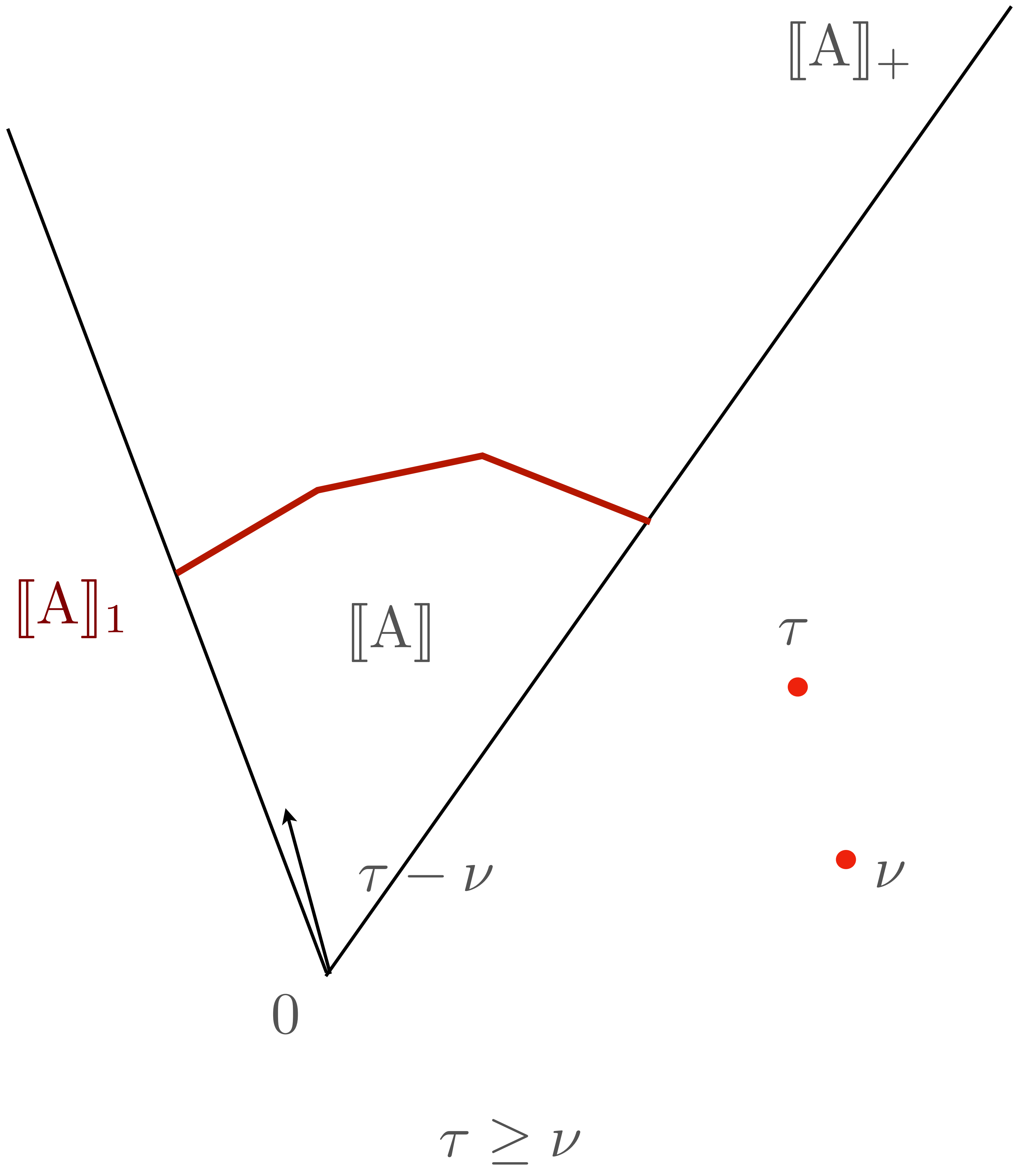
- In the same way define

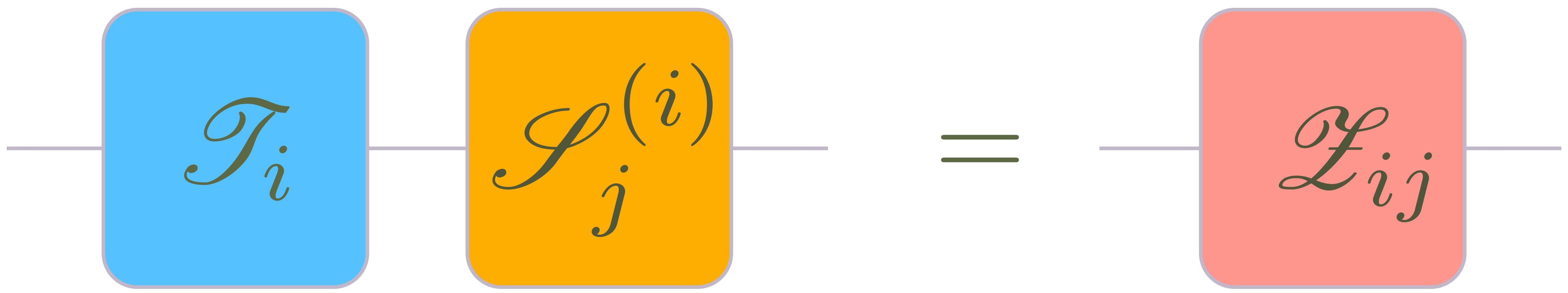
  $[\![\bar{A}]\!]_+, \quad [\![A \to B]\!]_+$

  and corresponding orderings

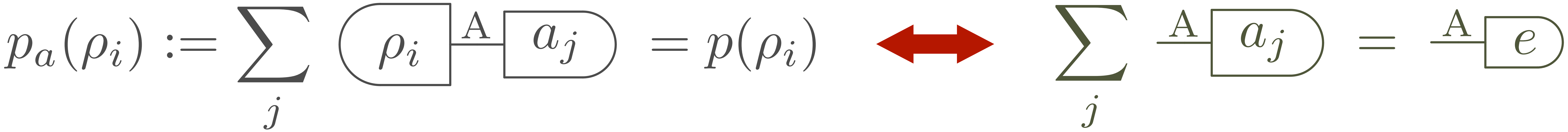


Causal theories

- Possibility of arbitrary conditional tests



- Causality implies no “backward” signalling

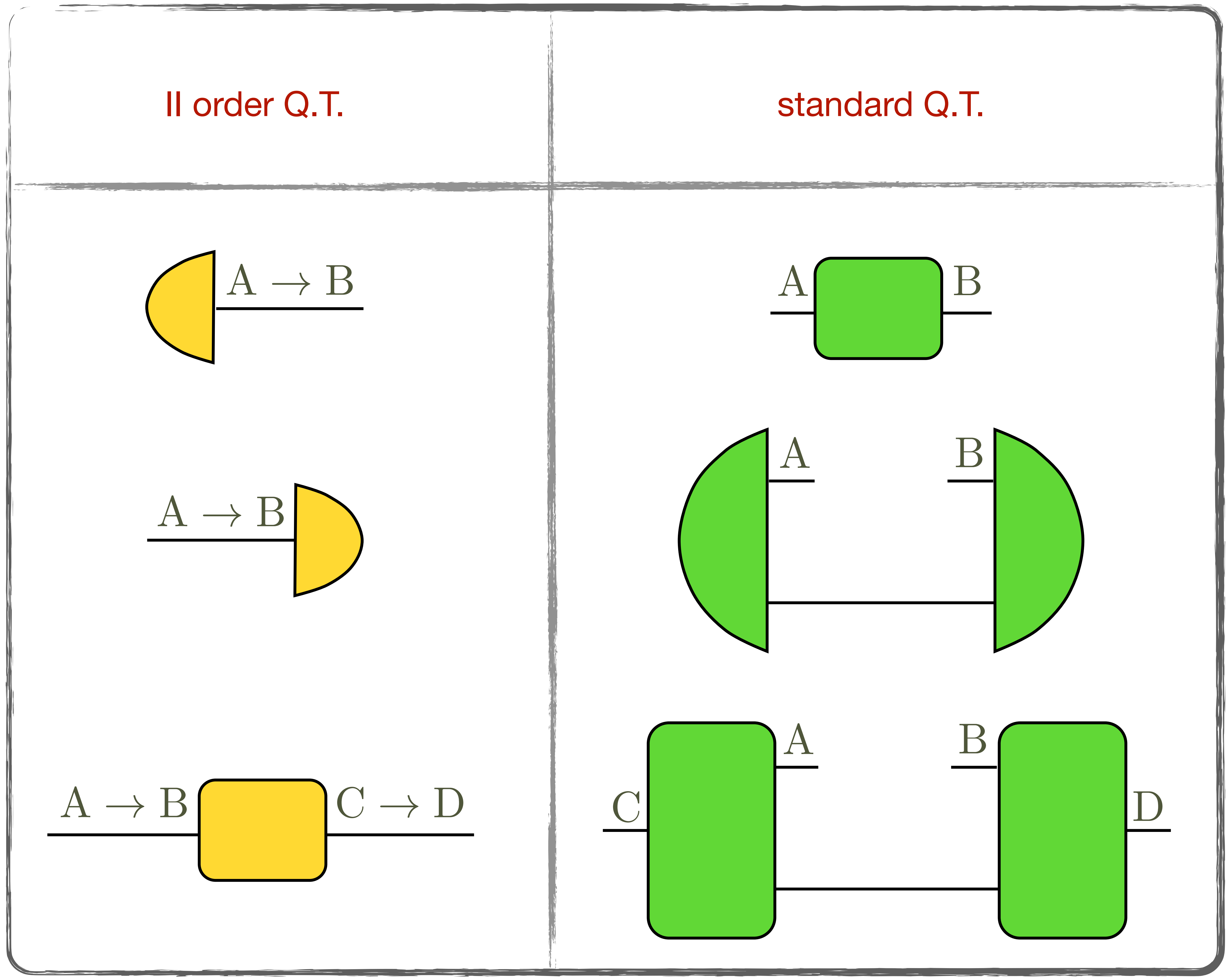


G. Chiribella, G. M. D’Ariano, and PP, Phys. Rev. A **81**, 062348 (2010)
PP, Quantum **4**, 294 (2020)

Non causal theories

Example

- Second order quantum theory
  - States: quantum operations (deterministic states: channels)
  - Effects: “quantum testers”
  - Transformations: “quantum supermaps”



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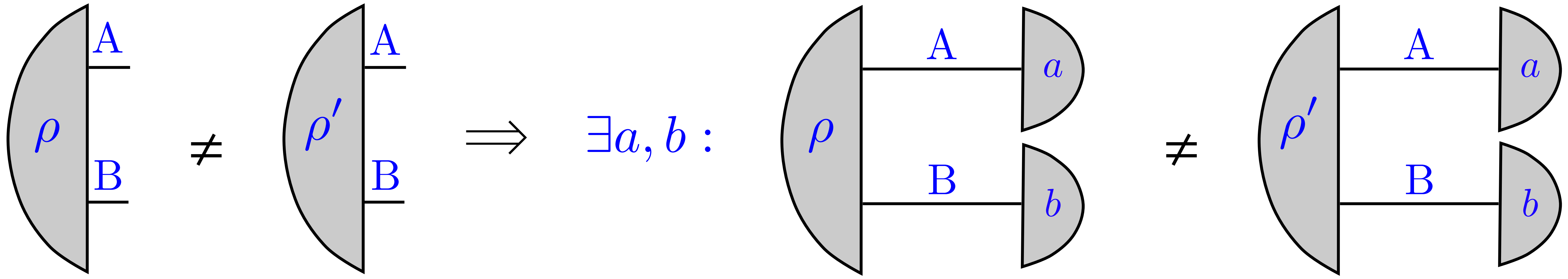
Local discriminability

AKA Local tomography/tomographic locality

- Every pair of bipartite states can be discriminated by local measurements



- Consequence 1:

- Consequence 2: $D_{\mathrm{AB}} = D_{\mathrm{A}} D_{\mathrm{B}}$ $\quad [\![\mathrm{AB}]\!]_{\mathbb{R}} = [\![\mathrm{A}]\!]_{\mathbb{R}} \otimes [\![\mathrm{B}]\!]_{\mathbb{R}}$

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Theories without local discriminability

Example 1: Real quantum theory

- Quantum theory with real Hilbert spaces $\mathrm{H_A} = \mathbb{C}^{d_\mathrm{A}}$
  - States: real density matrices
  - Effects: real positive operators
    bounded by the identity matrix
  - Transformations:
    CP TNI maps with real Kraus operators

Theories without local discriminability

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$$D_{\mathrm{A}} = \frac{d_{\mathrm{A}}(d_{\mathrm{A}}+1)}{2}$$

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  - Transformations: CP TNI maps with real Kraus operators
- Composite system: $d_{\mathrm{AB}} = d_{\mathrm{A}} d_{\mathrm{B}}$

$$D_{\mathrm{AB}} = \frac{d_{\mathrm{A}} d_{\mathrm{B}}(d_{\mathrm{A}} d_{\mathrm{B}}+1)}{2}$$

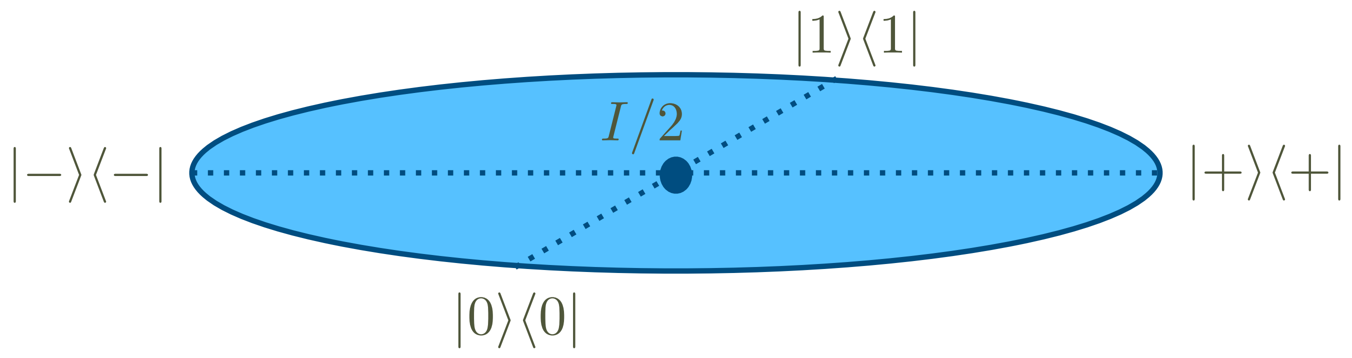
$$D_{\mathrm{A}} D_{\mathrm{B}} = \frac{d_{\mathrm{A}} d_{\mathrm{B}}(d_{\mathrm{A}} d_{\mathrm{B}} + d_{\mathrm{A}} + d_{\mathrm{B}} + 1)}{4} \leq D_{\mathrm{AB}}$$

Theories without local discriminability

Example 1: Real quantum theory

- The real qubit (rebit)

$d_R = 2 \;\Rightarrow\; D_R = 3$



$|\pm\rangle := 1/\sqrt{2}(|0\rangle \pm |1\rangle)$

$[\![\mathrm{R}]\!]_{\mathbb{R}}$

$[\![\mathrm{R}^2]\!]_{\mathbb{R}}$

Theories without local discriminability

Example 1: Real quantum theory

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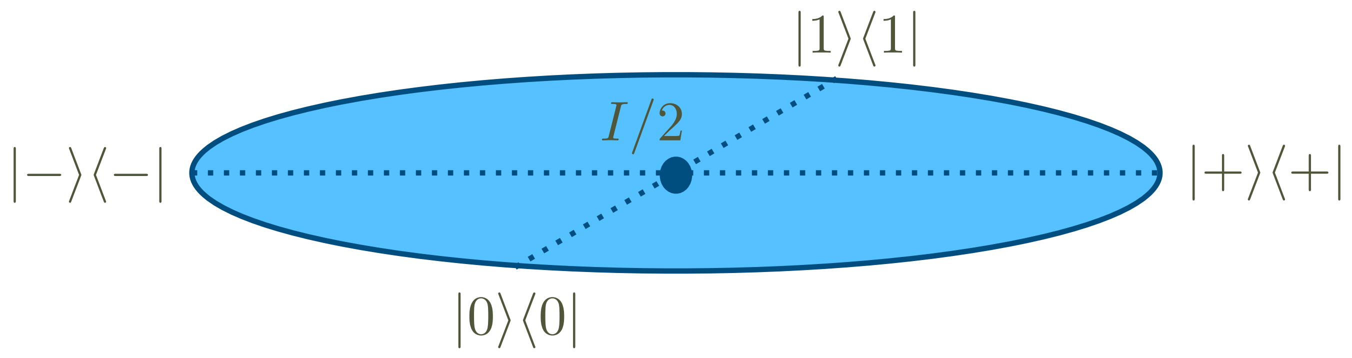

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- Two rebits

$d_{\mathrm{R}^2} = 4 \;\Rightarrow\; D_{\mathrm{R}^2} = 10 = D_{\mathrm{R}}^2 + 1$

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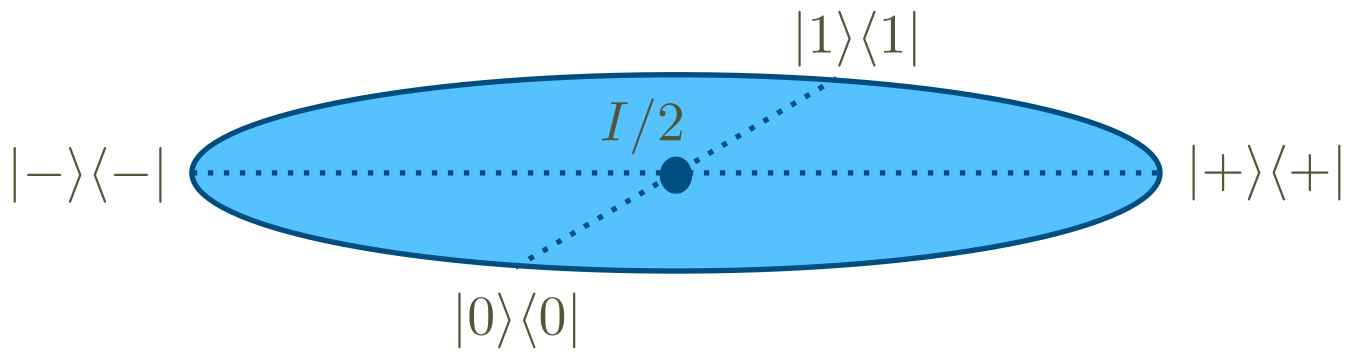

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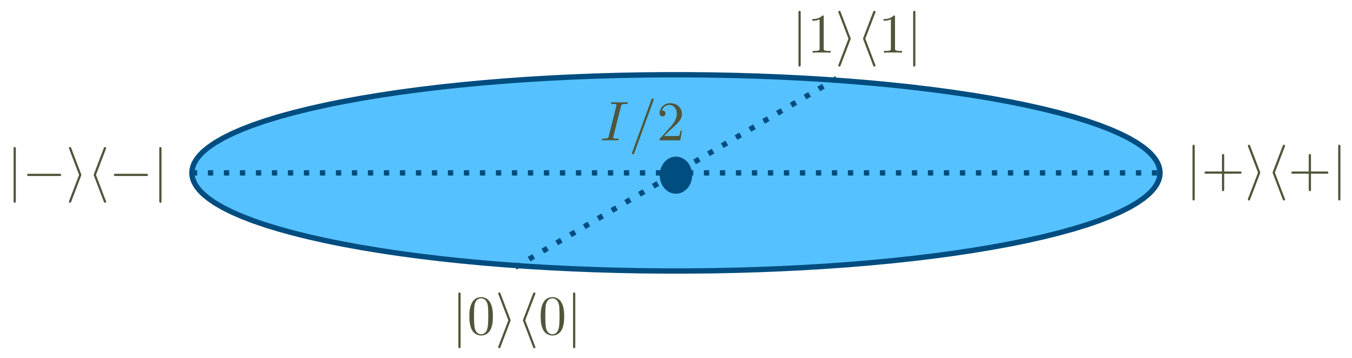

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Theories without local discriminability

Example 2: Fermionic quantum theory

- The theory is meant to provide a realisation of the fermion algebra

$$\{\varphi_i^\dagger, \varphi_j\} = \delta_{ij} I, \quad \{\varphi_i, \varphi_j\} = 0$$

for an arbitrary finite number N of Local Fermionic Modes (LFM) $1 \leq i, j \leq N$

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- Example of basis state: $|0010110\rangle = \varphi_3^\dagger \varphi_5^\dagger \varphi_6^\dagger |0000000\rangle$

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Theories without local discriminability

Example 2: Fermionic quantum theory

- The representation depends on the chosen ordering of the LFMs

$$J(\varphi_i) := I_1 \otimes \cdots \otimes I_{i-1} \otimes \sigma_i^- \otimes {\sigma^z}_{i+1} \cdots \otimes {\sigma^z}_N$$

$$J(XY) := J(X)J(Y) \qquad J(X^\dagger) := J(X)^\dagger$$

$$J(aX + bY) := aJ(X) + bJ(Y)$$

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- Jordan-Wigner representation
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- Used to prove computational equivalence of Fermionic and standard quantum computation*

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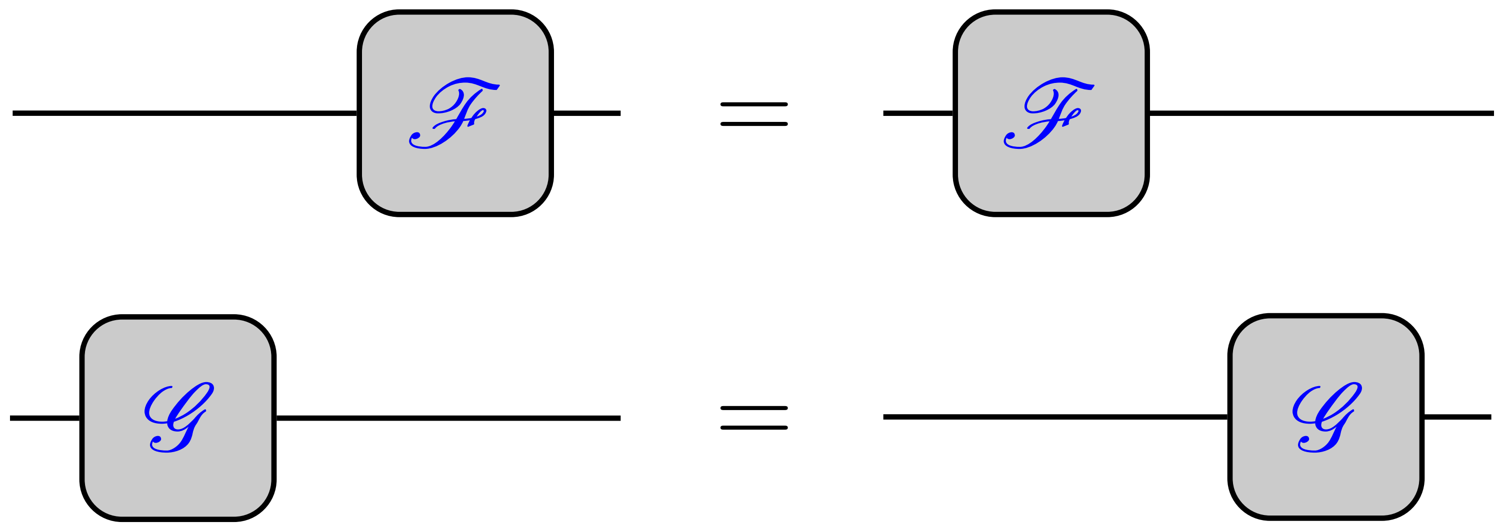

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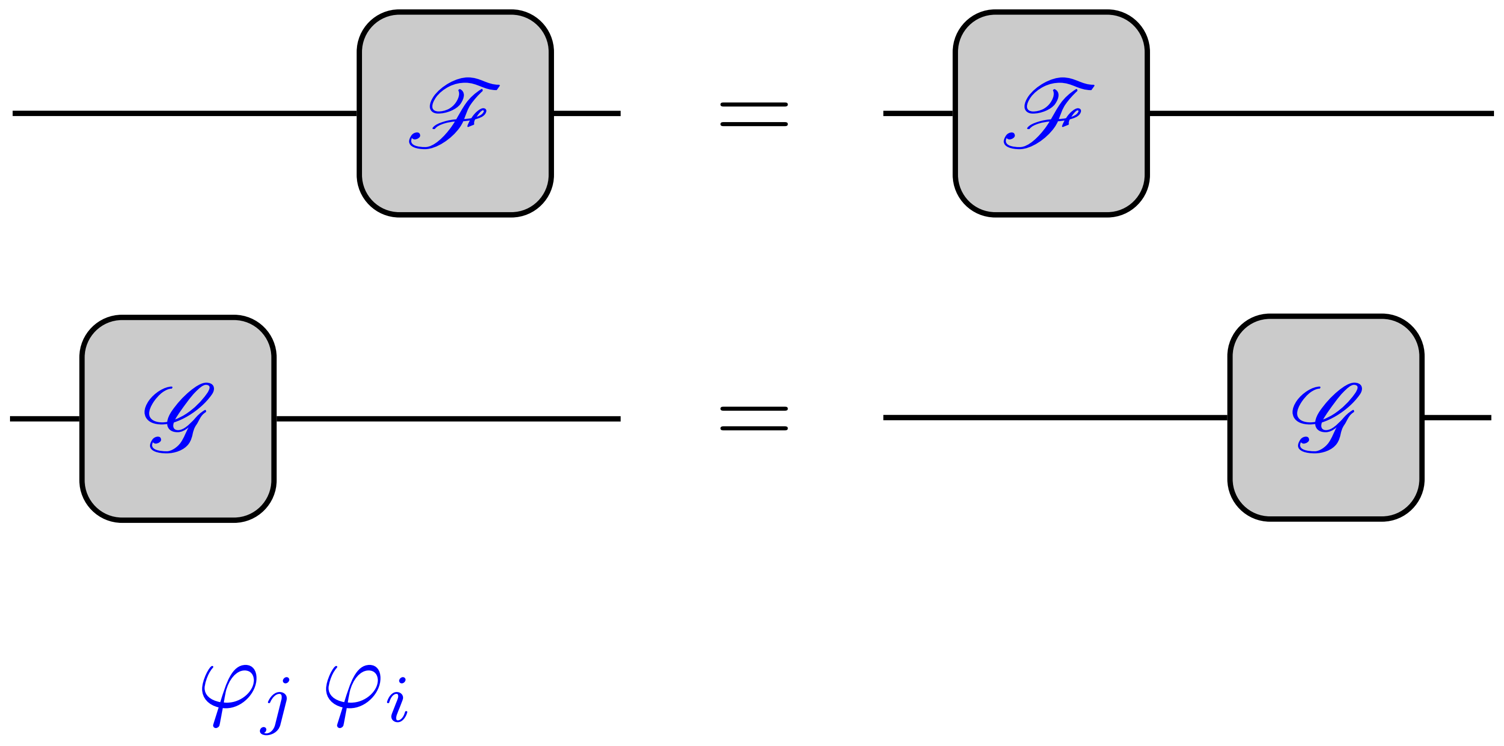

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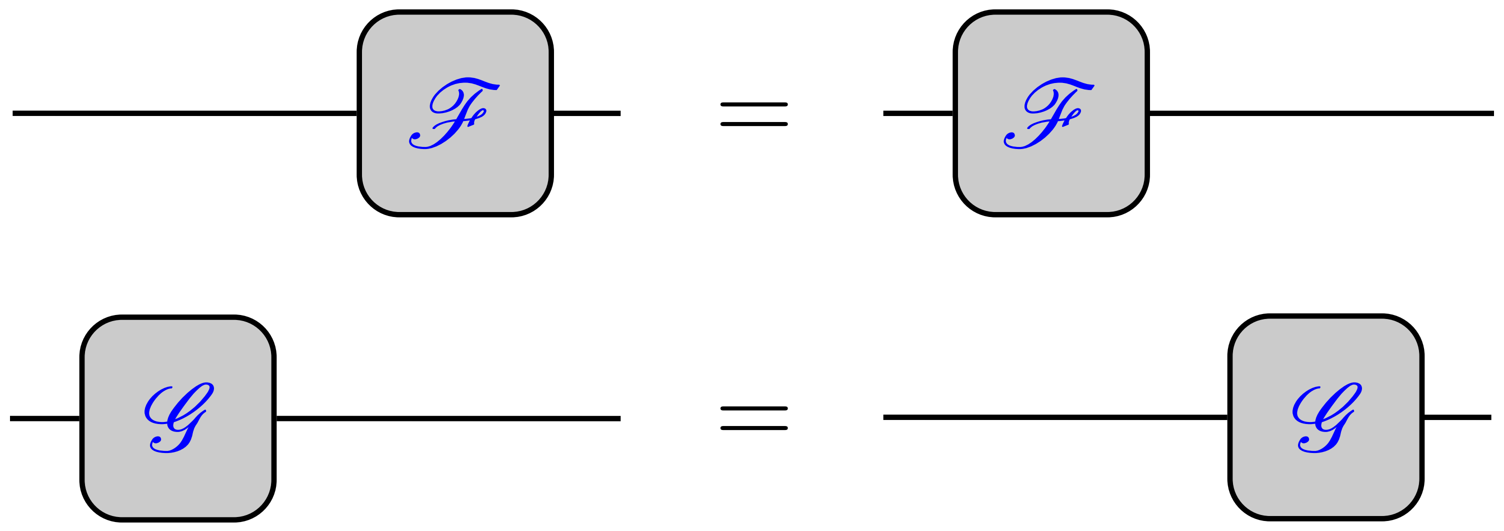

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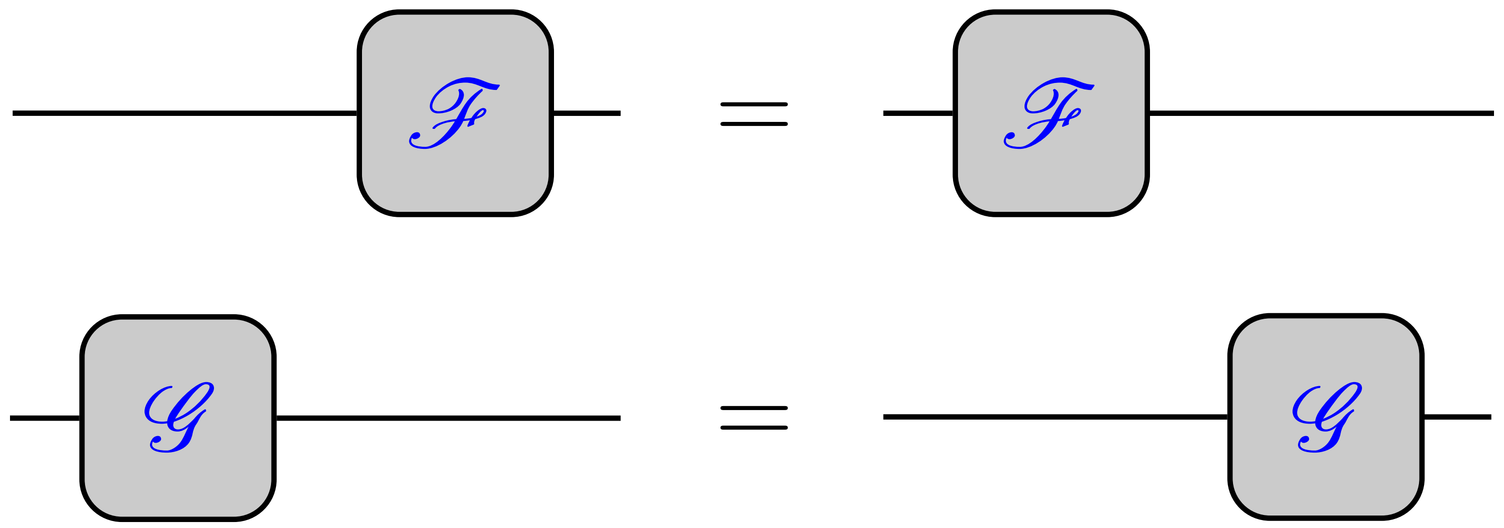

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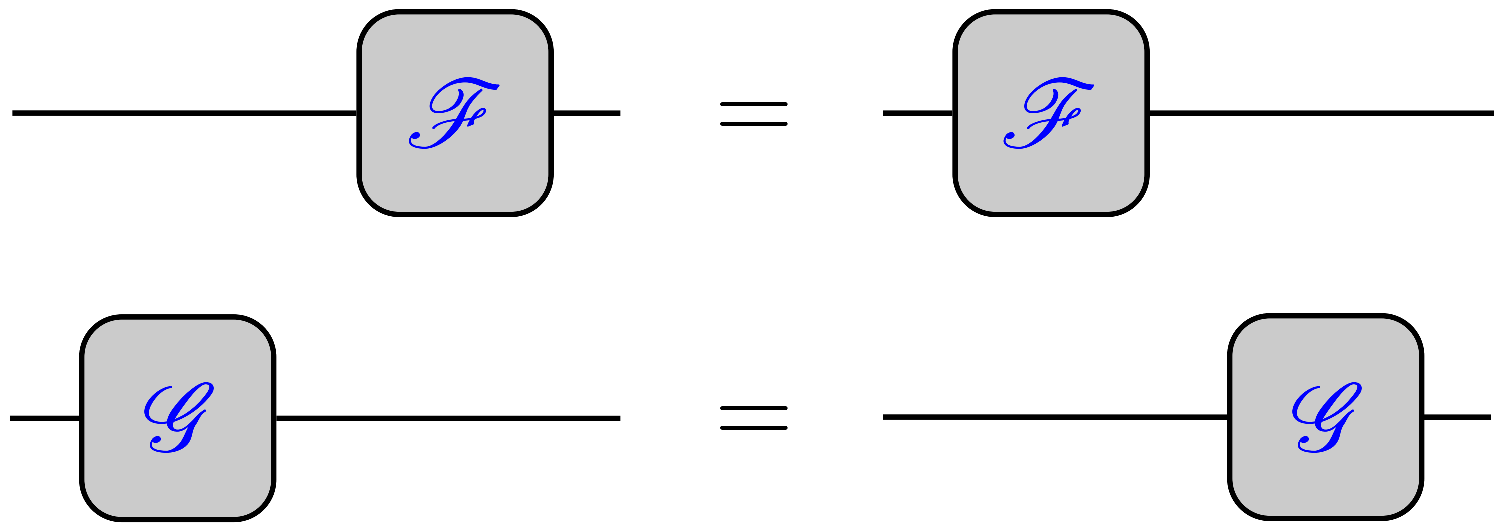

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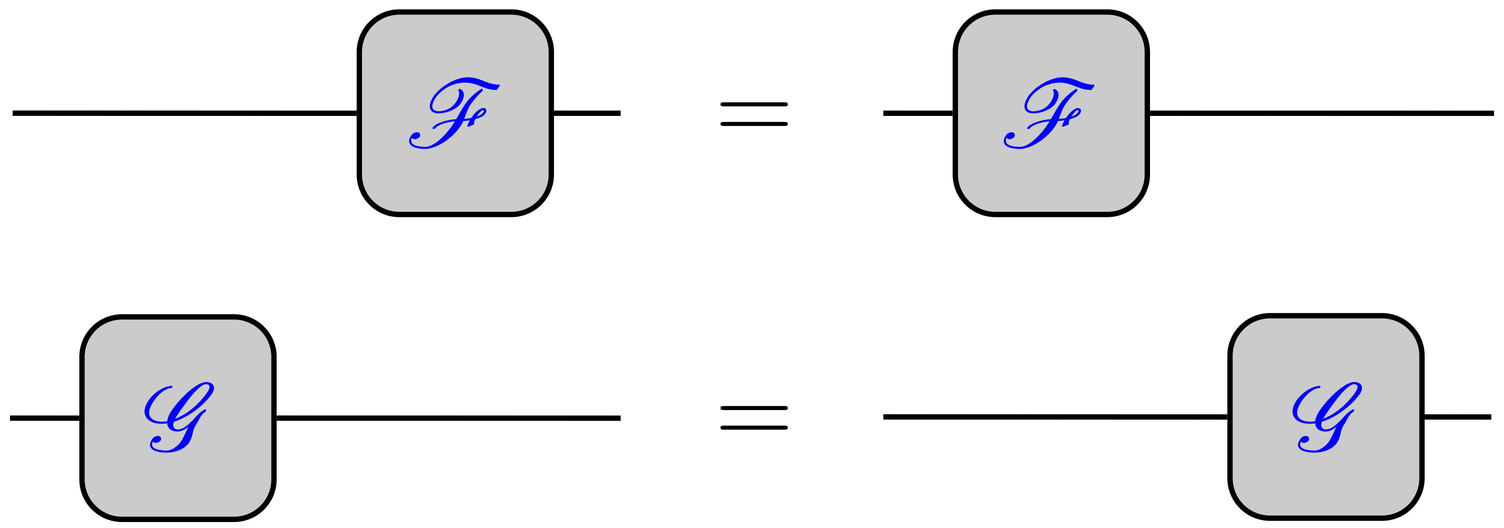

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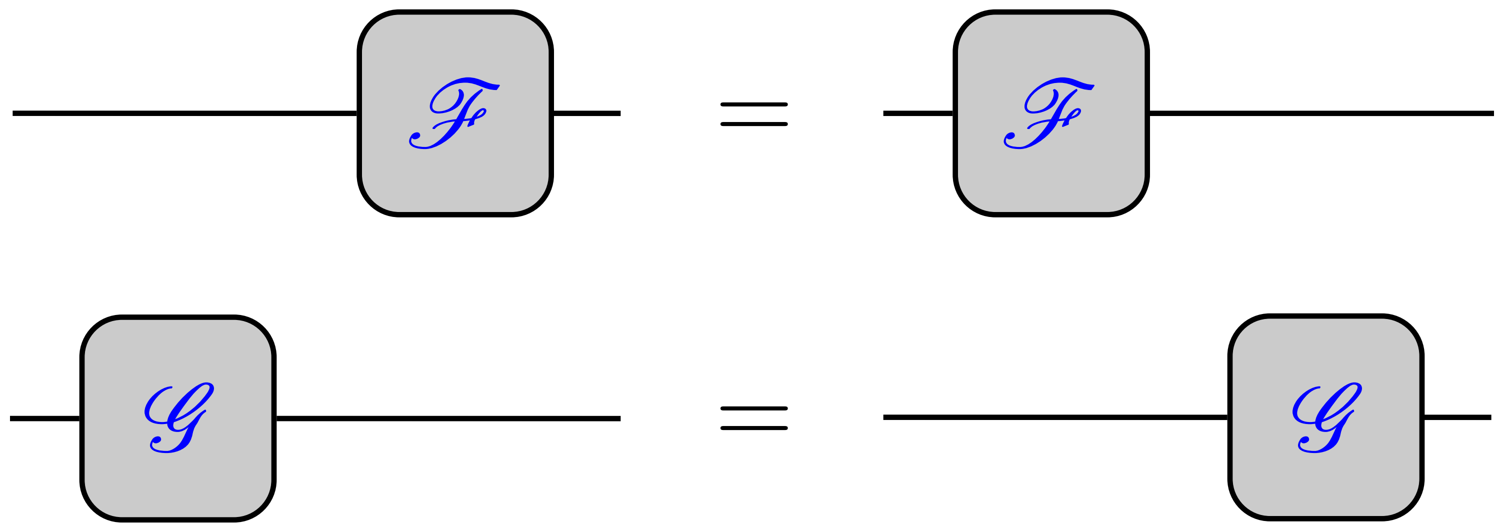

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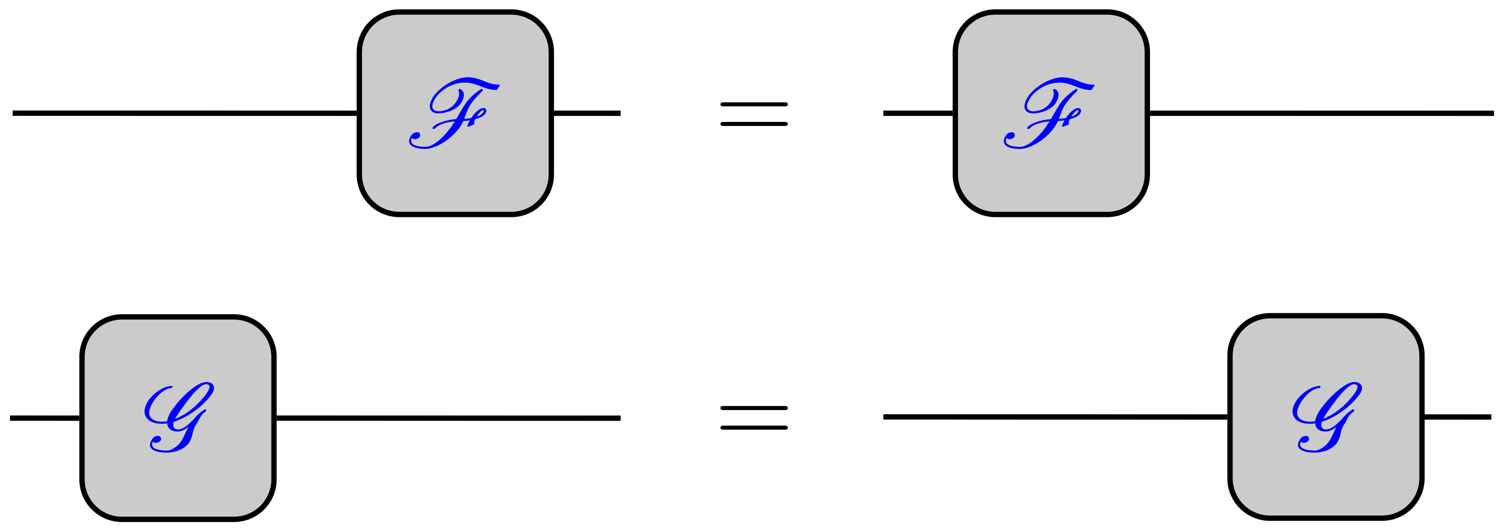

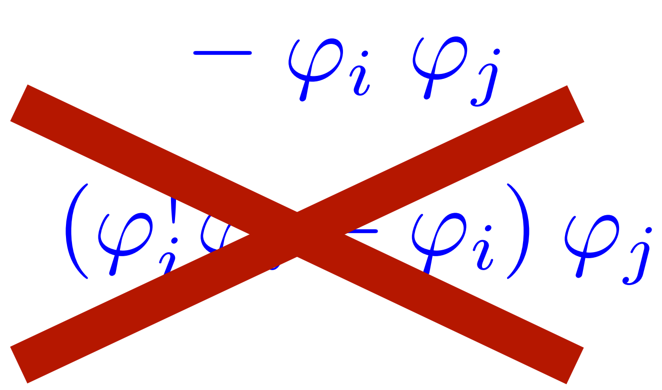

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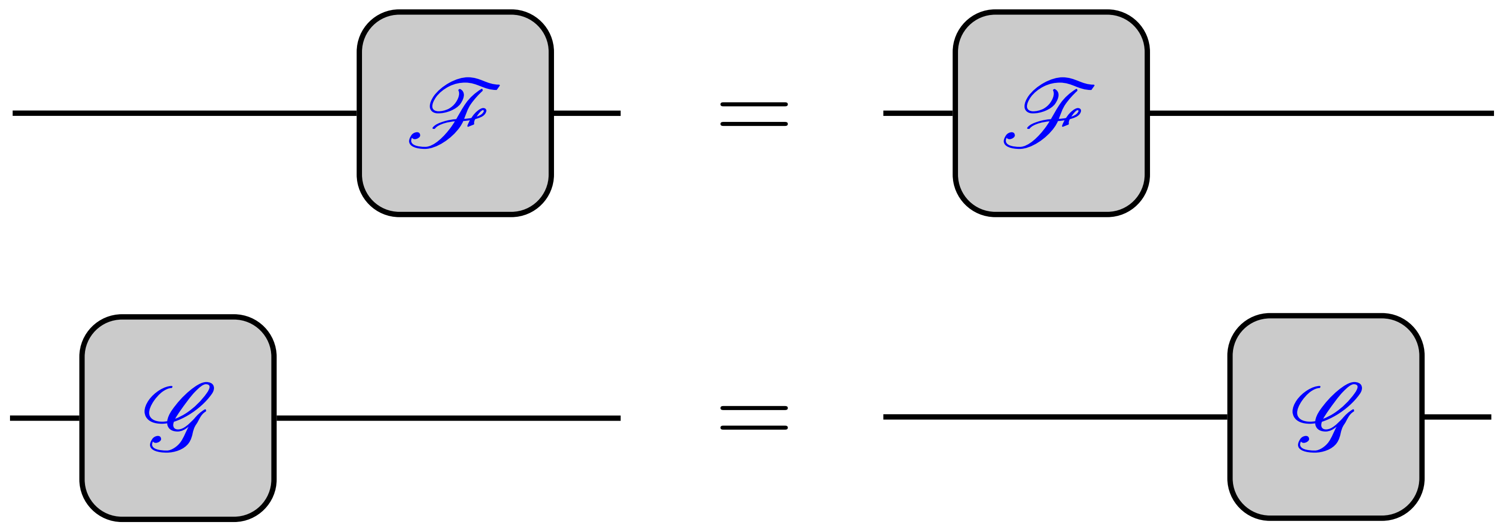

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- States and effects are combinations even products of field operators

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- This corresponds to a parity superselection rule

$$|\psi\rangle = |00\rangle,\ |10\rangle,\ a|10\rangle + b|01\rangle, \ldots$$

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- Parity superselection → block-diagonal structure for states

$$J(\rho) = \left(\begin{array}{c|c} p\rho_O & 0 \\ \hline 0 & (1-p)\rho_E \end{array}\right)$$

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Purification

- Two axioms in one

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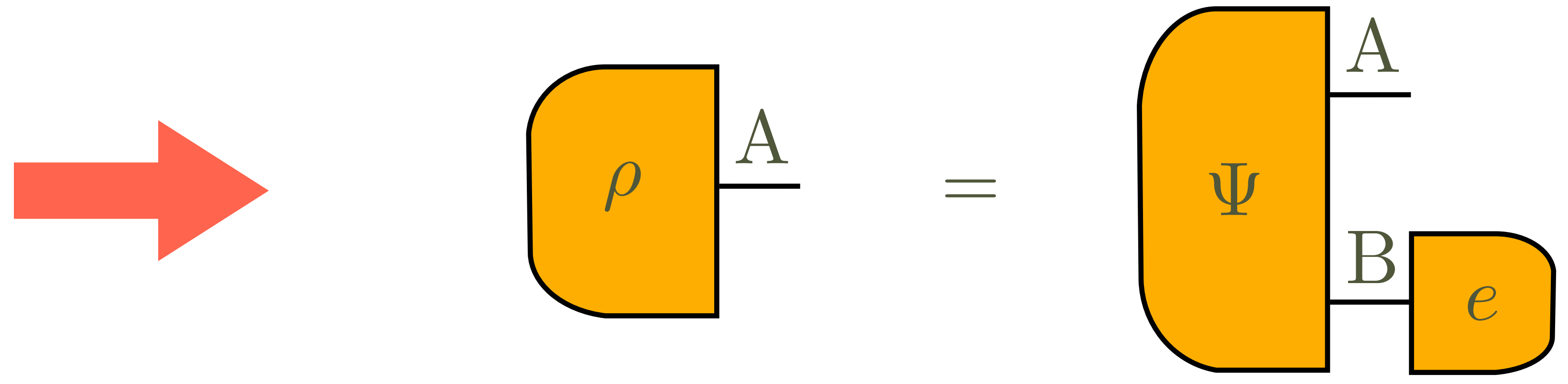
$$\forall \rho \in [\![\mathrm{A}]\!] \quad \exists \mathrm{B},\ \Psi \in [\![\mathrm{AB}]\!]$$



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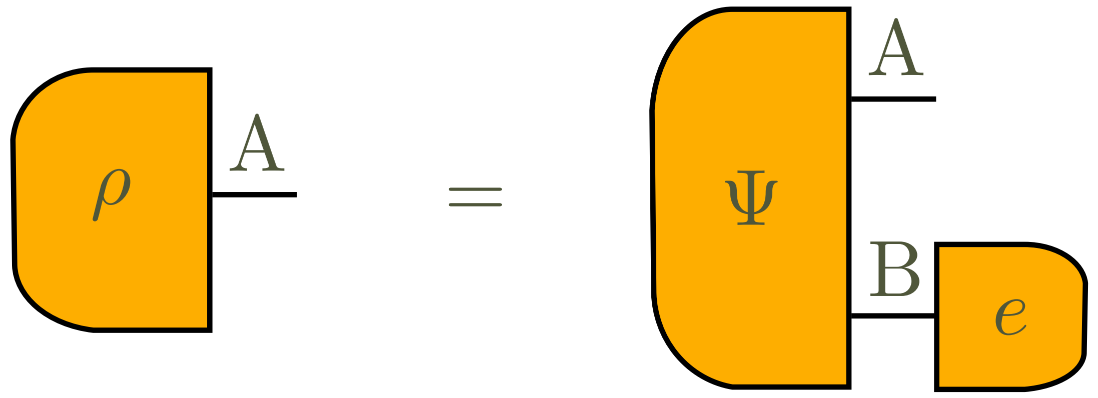
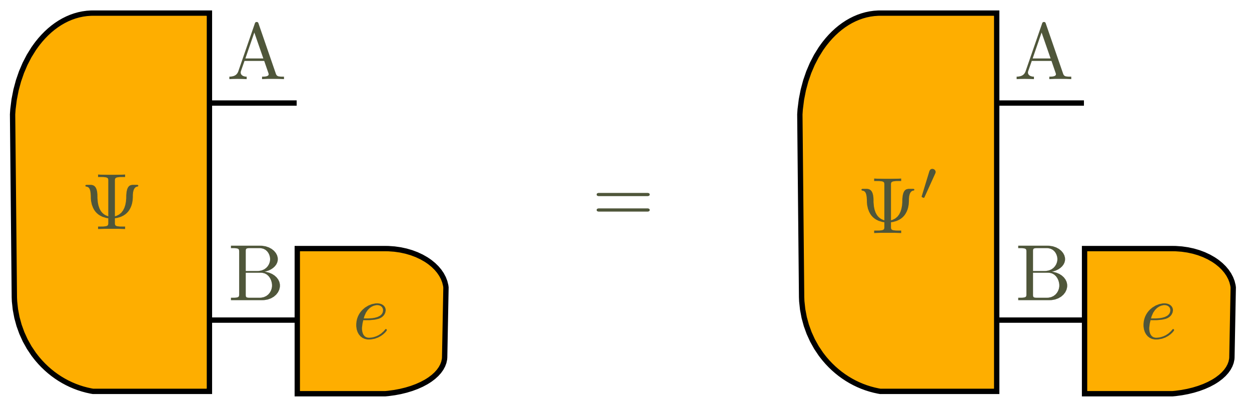
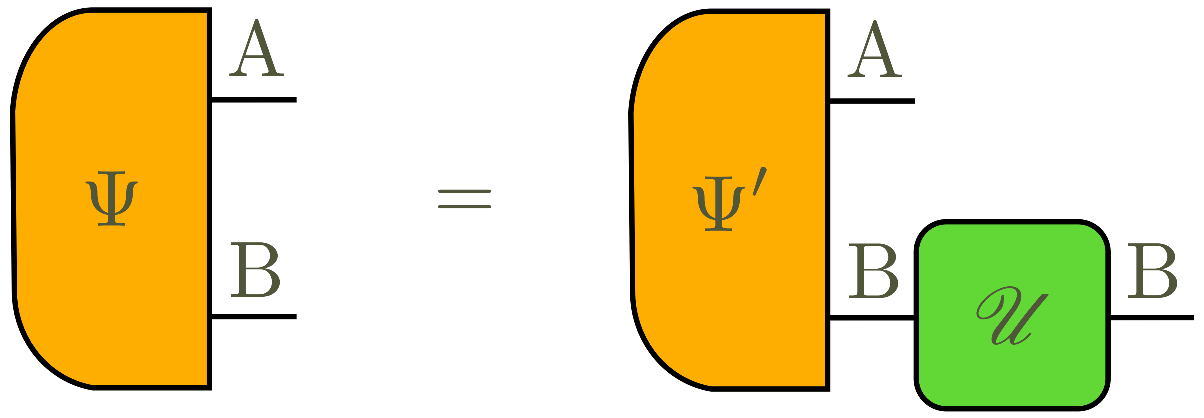


  - Essential uniqueness



$\exists \mathscr{U}$:



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Theories with purification

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- System types: natural numbers $m \in \mathbb{N}$

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Example 2: Quantum Theory

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Example 3: Fermionic quantum theory

- Every state in FQT can be purified

$$\rho = p\rho_O + (1-p)\rho_E \qquad \rho_X = \sum_i q_i^X |\psi_i^X\rangle\langle\psi_i^X|$$

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- We can always purify ρ with an even-parity pure state

$$|\Psi^E\rangle := \sum_i \left\{ \sqrt{pq_i^E}|\psi_i^E\rangle|\psi_i^E\rangle + \sqrt{(1-p)q_i^O}|\psi_i^O\rangle|\psi_i^O\rangle \right\}$$

or with with an odd-parity pure state

$$|\Psi^O\rangle := \sum_i \left\{ \sqrt{pq_i^E}|\psi_i^E\rangle|\psi_i^O\rangle + \sqrt{(1-p)q_i^O}|\psi_i^O\rangle|\psi_i^E\rangle \right\}$$

G. M. D'Ariano, F. Manessi, PP, and A. Tosini, Int. J. Mod. Phys. A **29**, 1430025 (2014)

Theories with purification

Example 3: Fermionic quantum theory

- Every state in FQT can be purified

$$\rho = p\rho_O + (1-p)\rho_E \qquad \rho_X = \sum_i q_i^X |\psi_i^X\rangle\langle\psi_i^X|$$

- We can always purify ρ with an even-parity pure state

$$|\Psi^E\rangle := \sum_i \left\{ \sqrt{pq_i^E}|\psi_i^E\rangle|\psi_i^E\rangle + \sqrt{(1-p)q_i^O}|\psi_i^O\rangle|\psi_i^O\rangle \right\}$$

or with with an odd-parity pure state

$$|\Psi^O\rangle := \sum_i \left\{ \sqrt{pq_i^E}|\psi_i^E\rangle|\psi_i^O\rangle + \sqrt{(1-p)q_i^O}|\psi_i^O\rangle|\psi_i^E\rangle \right\}$$

- Purification is unique up to reversible transformations

G. M. D'Ariano, F. Manessi, PP, and A. Tosini, Int. J. Mod. Phys. A **29**, 1430025 (2014)

Operational norm

Task: discrimination

- We assume causality
- States $p_{\mathrm{err}} := p_0 \left(\rho_0\right]\overset{A}{—}\left[a_1\right) + p_1 \left(\rho_1\right]\overset{A}{—}\left[a_0\right)$

$$= \frac{1}{2}\Big(1 + \left(p_0\rho_0 - p_1\rho_1\right]\overset{A}{—}\left[a_1 - a_0\right)\Big)$$

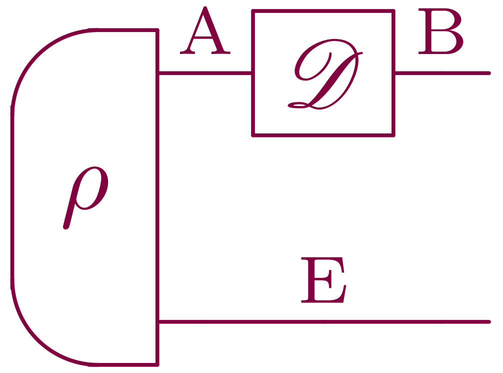
$$= \frac{1}{2}\Big(1 + p_0 - p_1 - 2\left(p_0\rho_0 - p_1\rho_1\right]\overset{A}{—}\left[a_0\right)\Big)$$

$$\|\eta\|_{\mathrm{op}} := \max_{a \in \mathsf{Eff}(\mathrm{A})} \left(\eta\right]\overset{A}{—}\left[a\right)$$

$$p_{\mathrm{opt}} = \frac{1}{2}\left(1 + p_0 - p_1 - \|p_0\rho_0 - p_1\rho_1\|_{\mathrm{op}}\right)$$

- Transformations

$$\|\mathscr{D}\|_{\mathrm{op}} := \max_{\mathrm{E}, \rho \in \mathsf{St}(\mathrm{AE})} \left\| \quad \right\|_{\mathrm{op}}$$

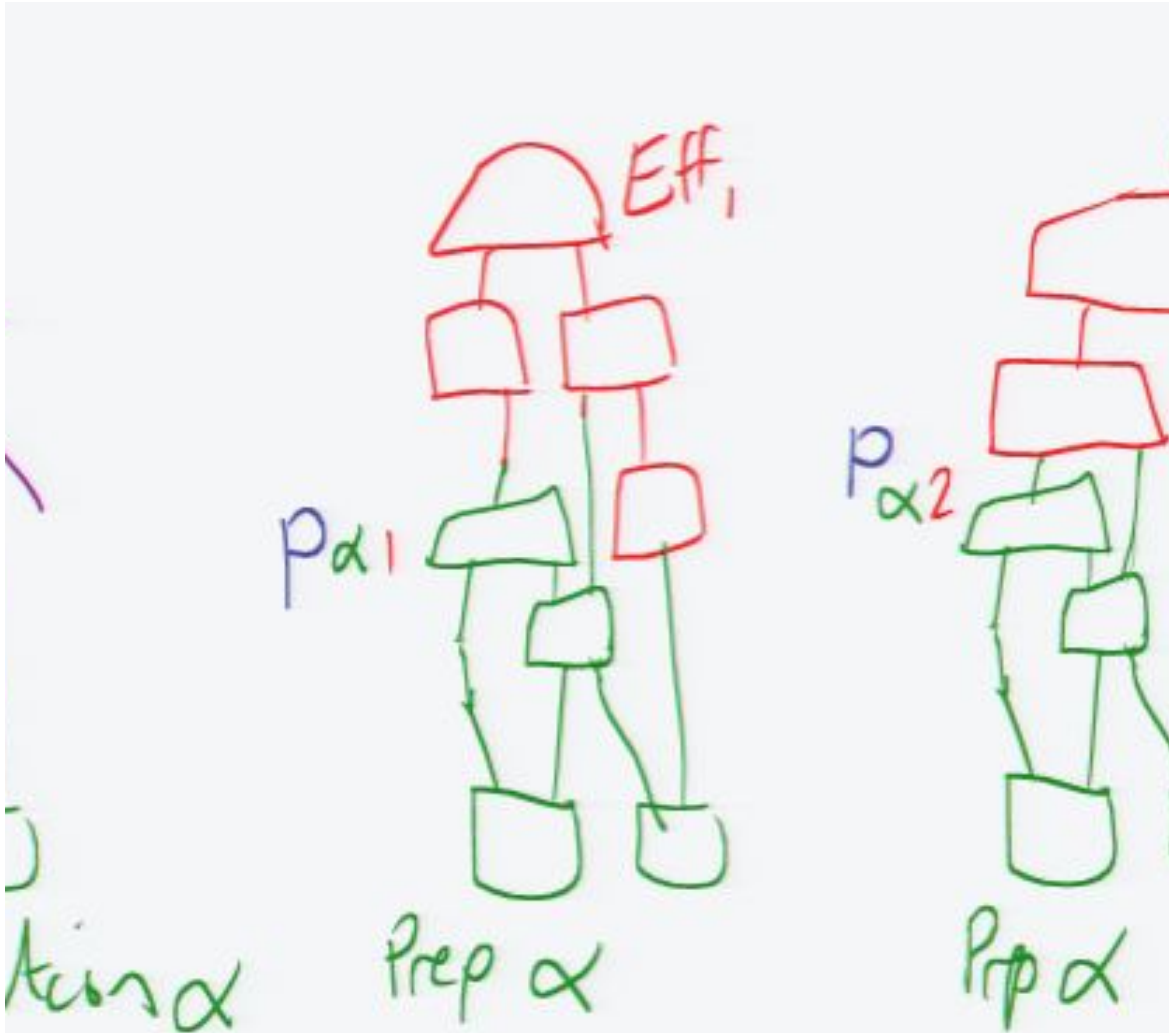


  - Effects

$$\|d\|_{\mathrm{op}} = \max_{\rho \in \mathsf{St}_1(\mathrm{A})} \left| \left(\rho\right]\overset{A}{—}\left[a\right) \right|$$

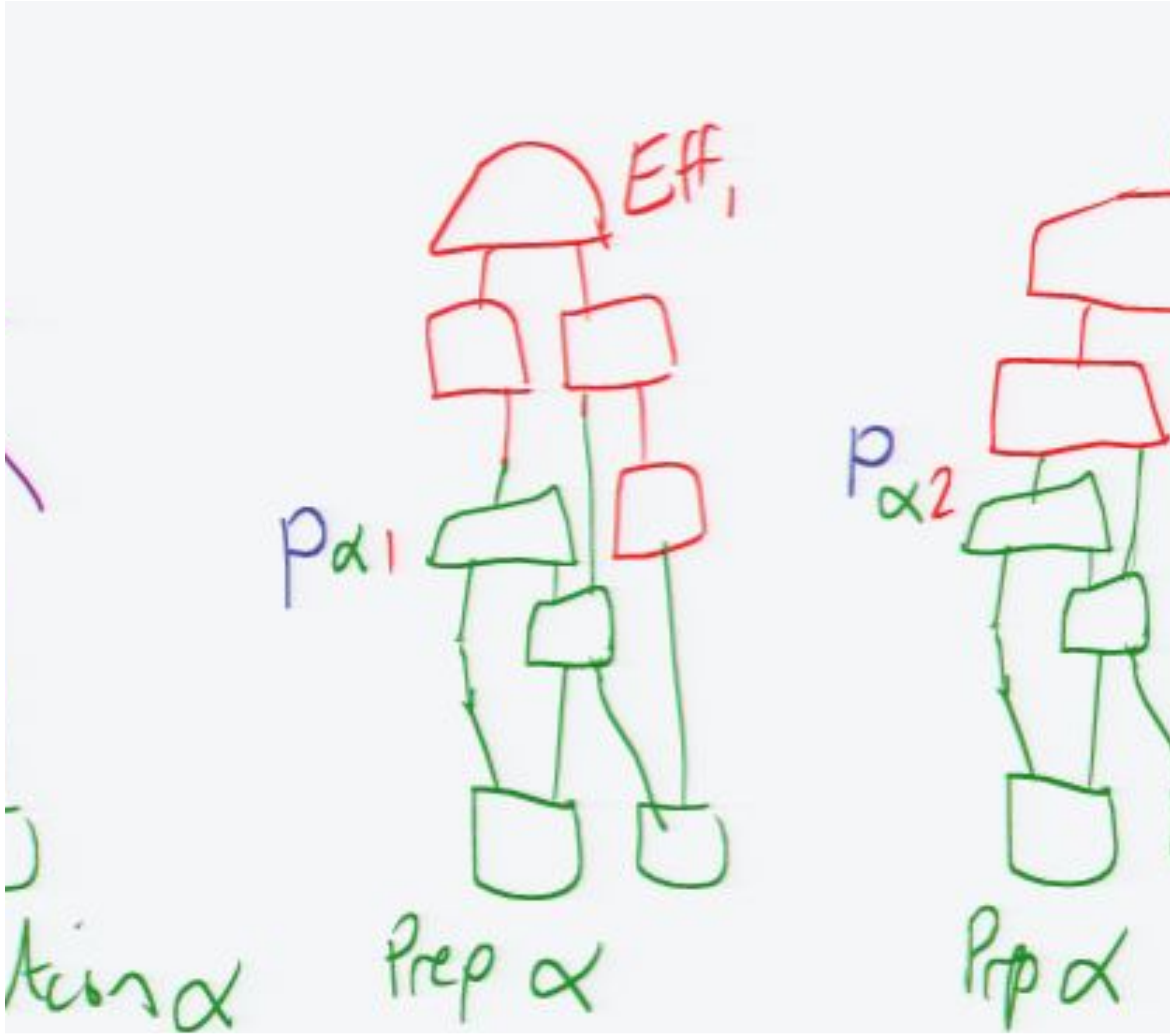
Summary

- Compositional structure



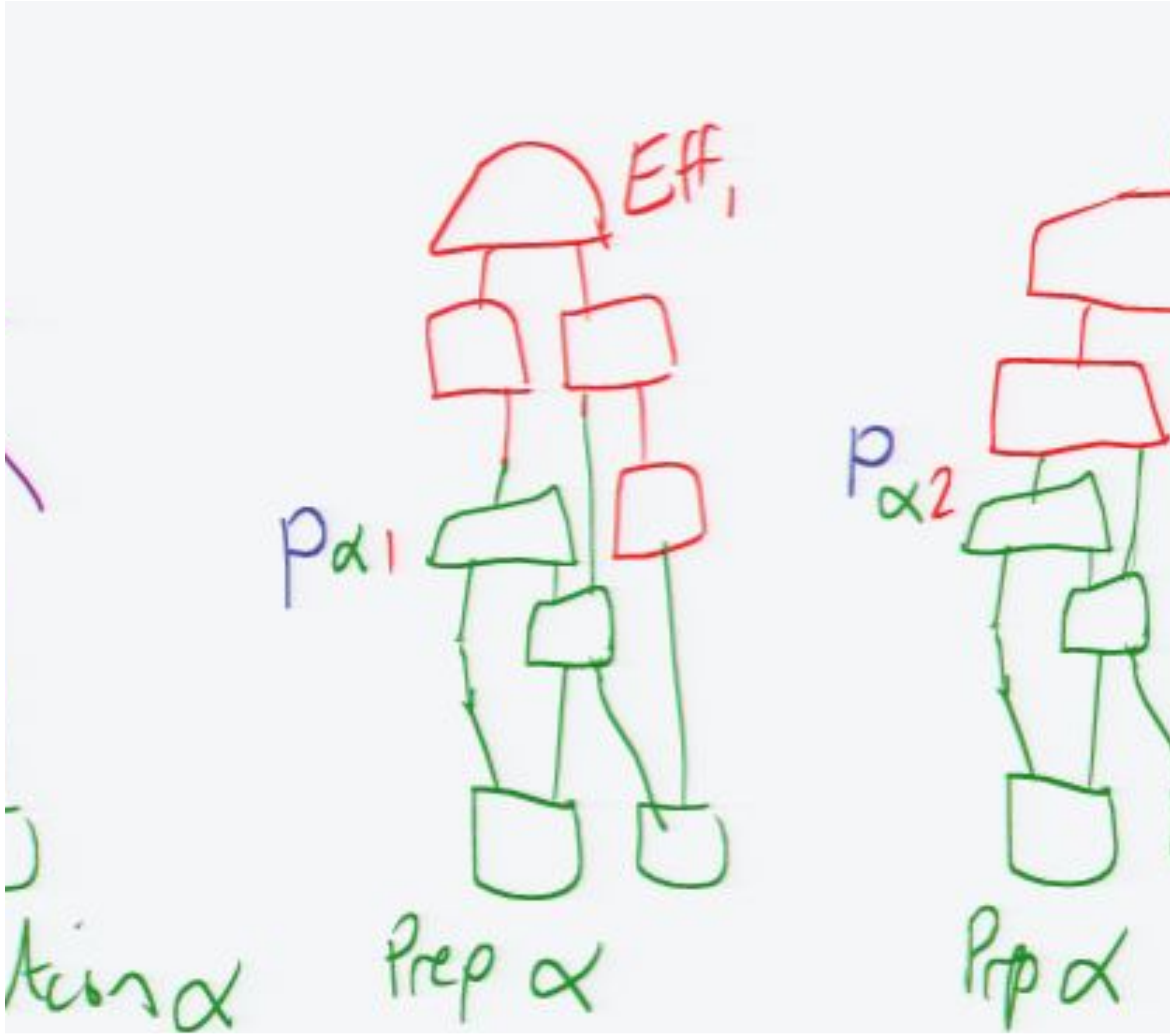
Summary

- Compositional structure
- Probabilistic structure



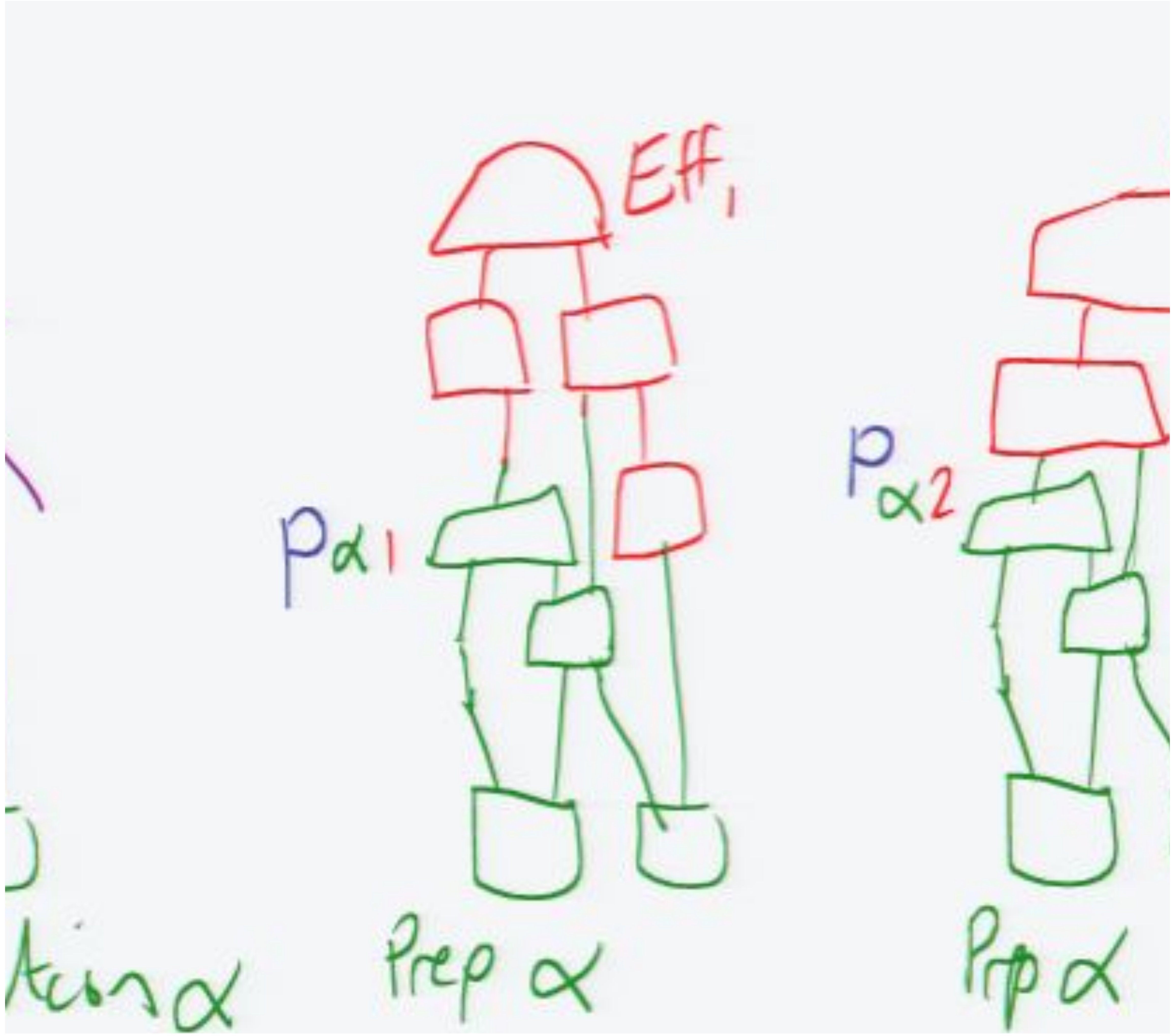
Summary

- Compositional structure
- Probabilistic structure
- Properties:



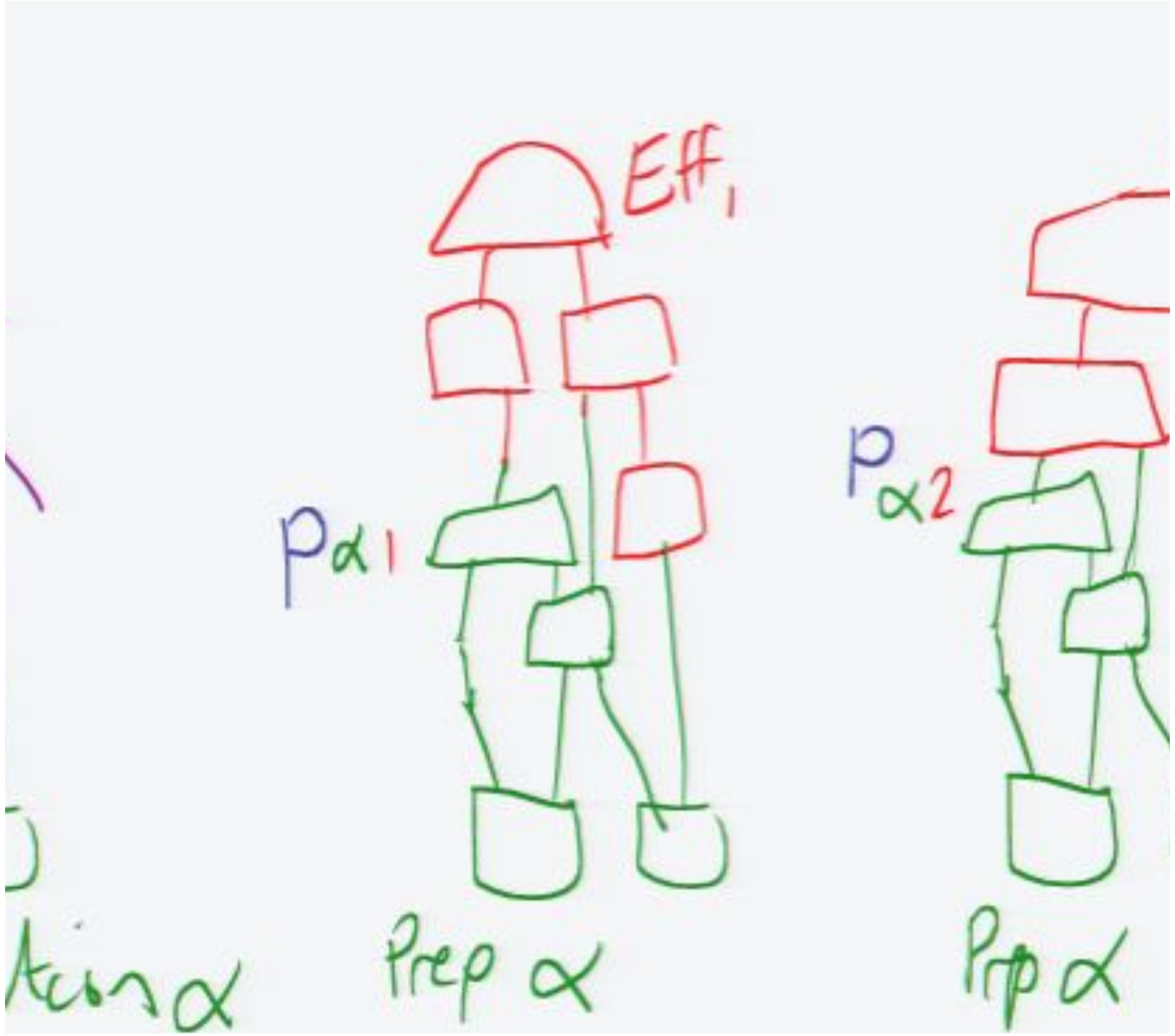
Summary

- Compositional structure
- Probabilistic structure
- Properties:
    - Causality



Summary

- Compositional structure
- Probabilistic structure
- Properties:
  - Causality
  - Local discriminability



Summary

- Compositional structure
- Probabilistic structure
- Properties:
    - Causality
    - Local discriminability
    - Purification