Operational probabilistic theories and cellular automata: how I learned to stop worrying and love C* algebras

School on Advanced Topics in Quantum Information and Foundations
Quantum Information Unit and the Yukawa Institute for Theoretical Physics, Kyoto University

UNIVERSITÀ DI PAVIA

INFN Istituto Nazionale di Fisica Nucleare

QUit quantum information theory group

Paolo Perinotti - February 8-12 2021

Lecture 2
Quantum Cellular automata

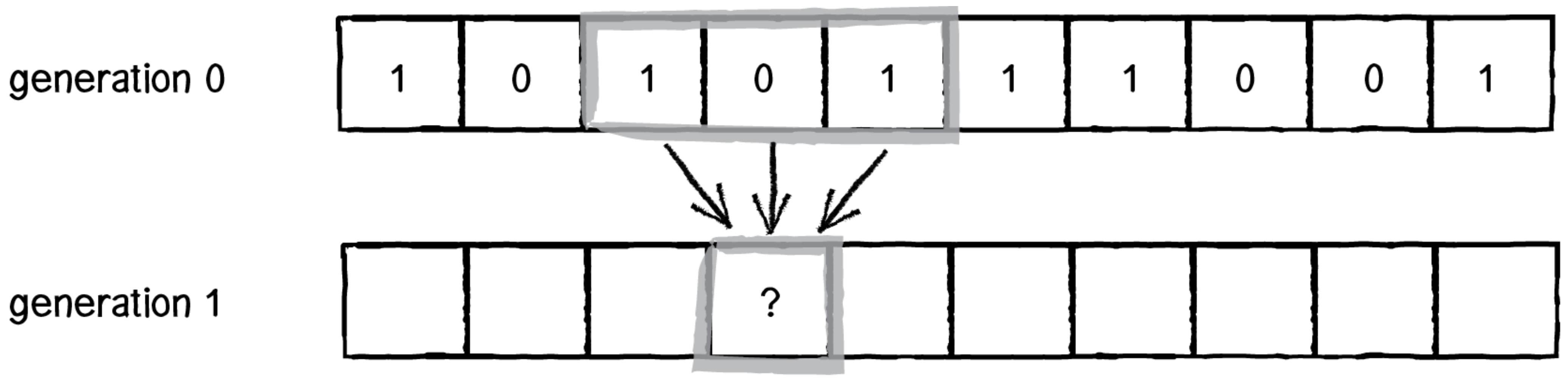
Summary

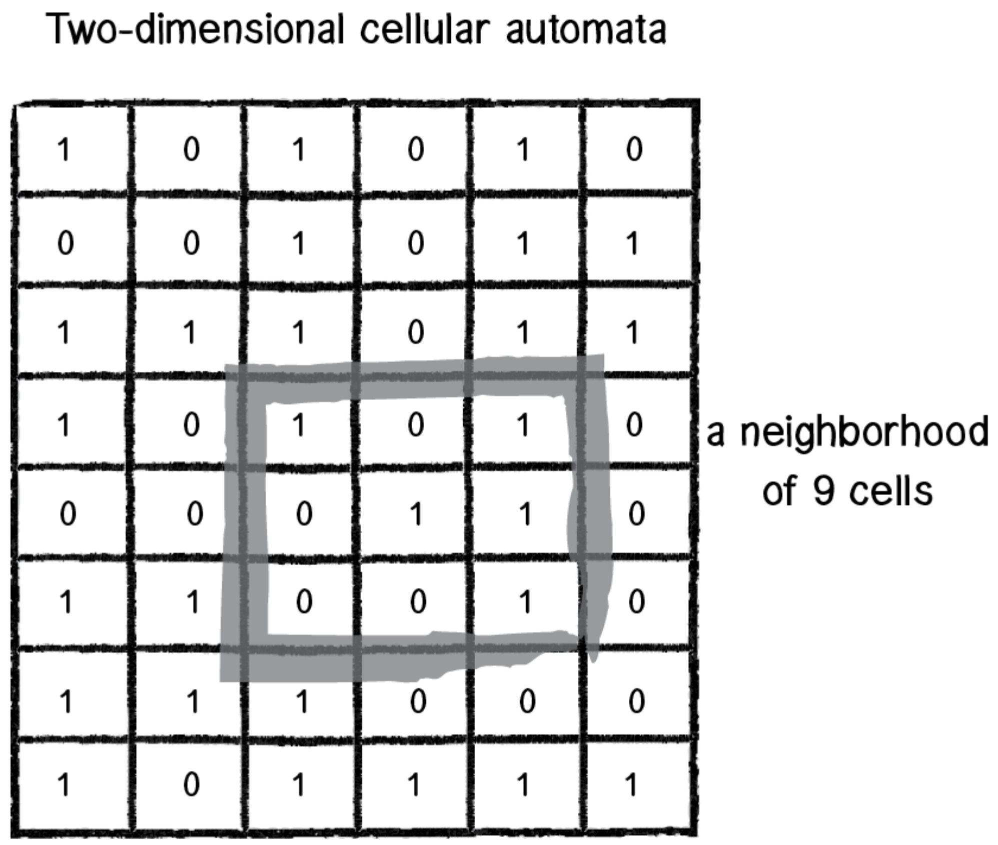
- Cellular Automata
- QCA
  - finite quantum systems
  - Infinite systems - inductive limit
  - Infinite systems - topological closure
- Neighbourhood
- The problem of quantisation
- Margolus decompositions
- Fermionic CA and quantum walks

Cellular Automata

J. Von Neumann and A. W. Burks, “Theory of self-reproducing automata” 1966

Conway’s game of life





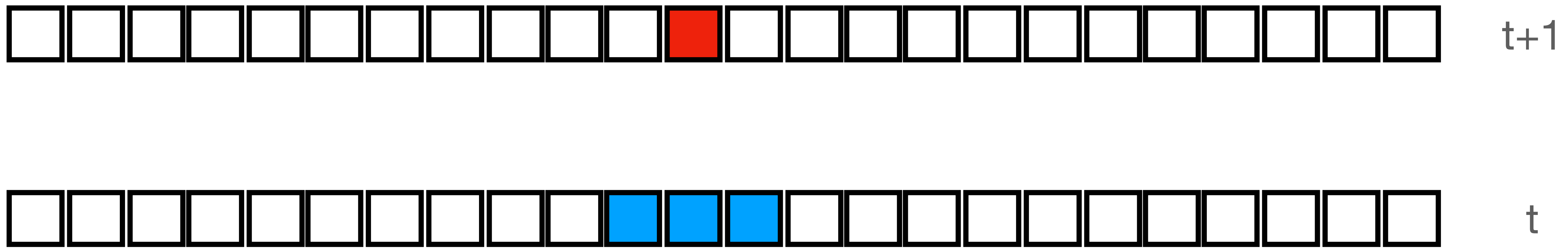
Definition (Cellular Automata)**.** *A* cellular automaton *(CA) is a 4-tuple* $(\mathrm{L}, \Sigma, \mathcal{N}, f)$ *consisting of (1) a d-dimensional lattice of cells* L *indexed* $i \in \mathbb{Z}^d$*, (2) a finite set of states* Σ*, (3) a finite neighborhood scheme* $\mathcal{N} \subset \mathbb{Z}^d$*, and (4) a local transition function* $f : \Sigma^{\mathcal{N}} \to \Sigma$*.*

K. Wiesner, Quantum Cellular Automata. In: Meyers R. (eds.) Encyclopedia of Complexity and Systems Science.

Quantum cellular automata

Problems

- What is a local update rule?



B. Schumacher and R.F. Werner, arXiv:quant-ph/040517

Quantum cellular automata

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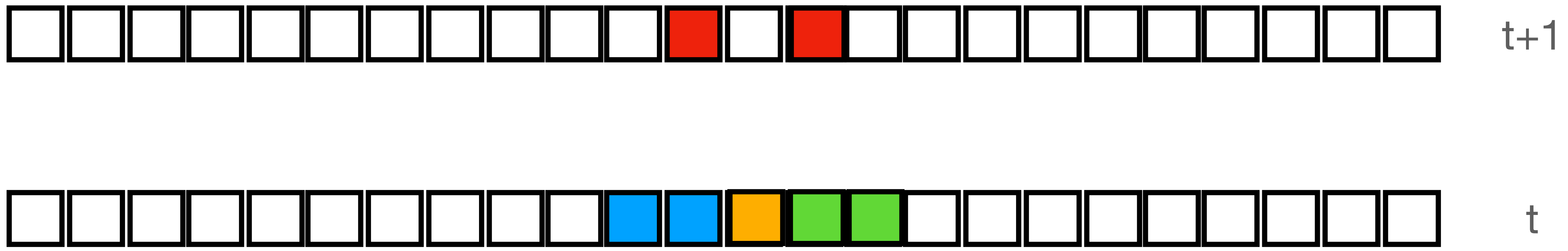

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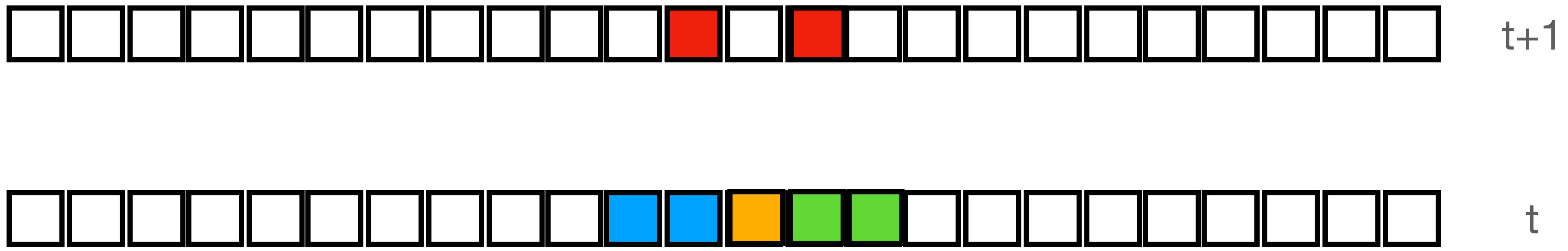

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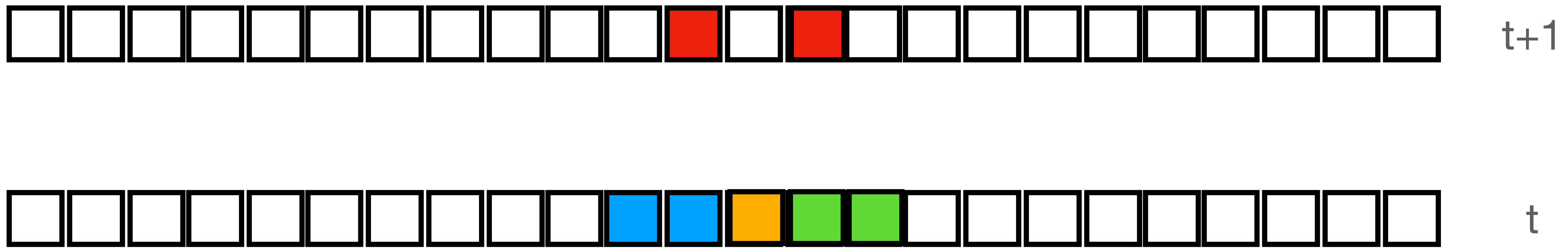
Quantum cellular automata

Problems

- What is a local update rule?
- No-cloning theorem
- Non commutativity of local operations
- Heart of the problem: definition through action on states



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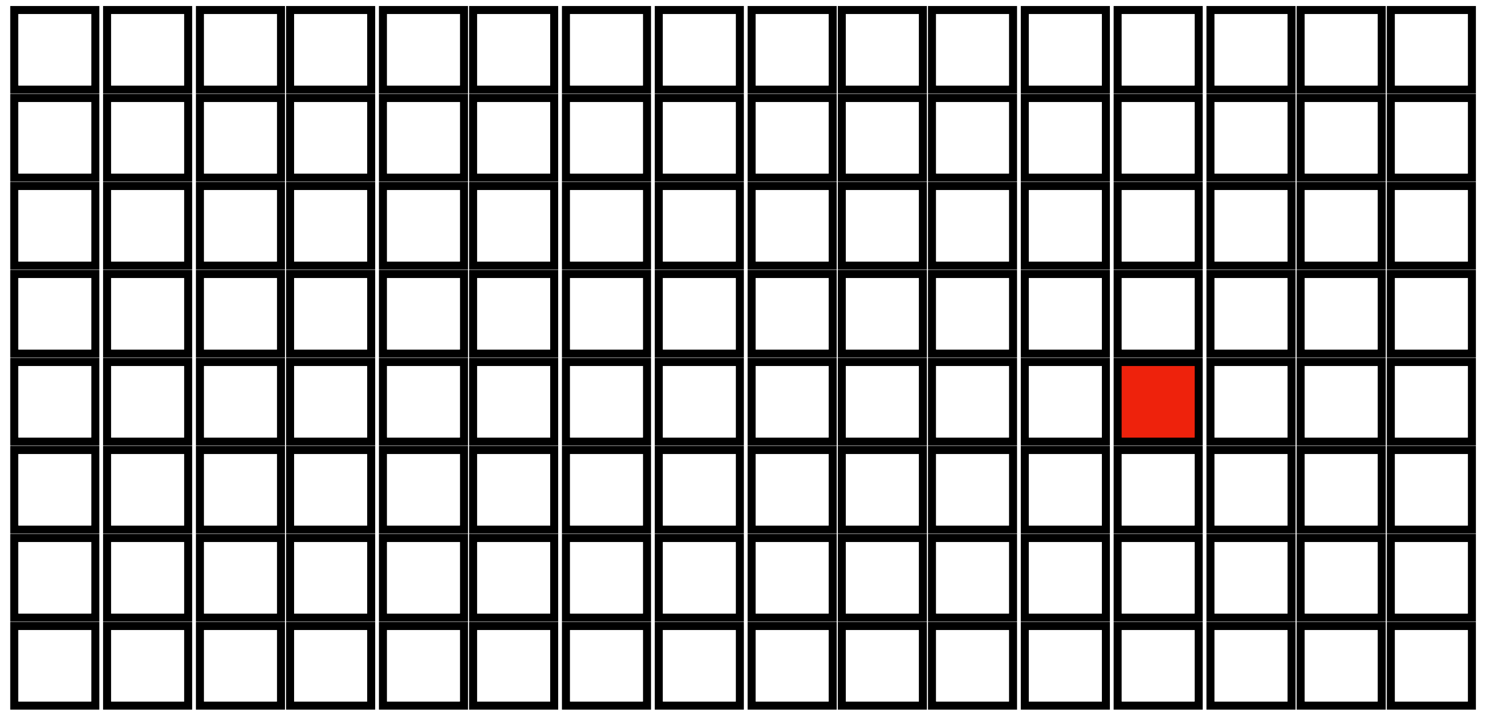
Quantum cellular automata

Finite case

 $= \mathcal{H}_x$

C.A.: $U : \bigotimes_x \mathcal{H}_x \to \bigotimes_x \mathcal{H}_x$



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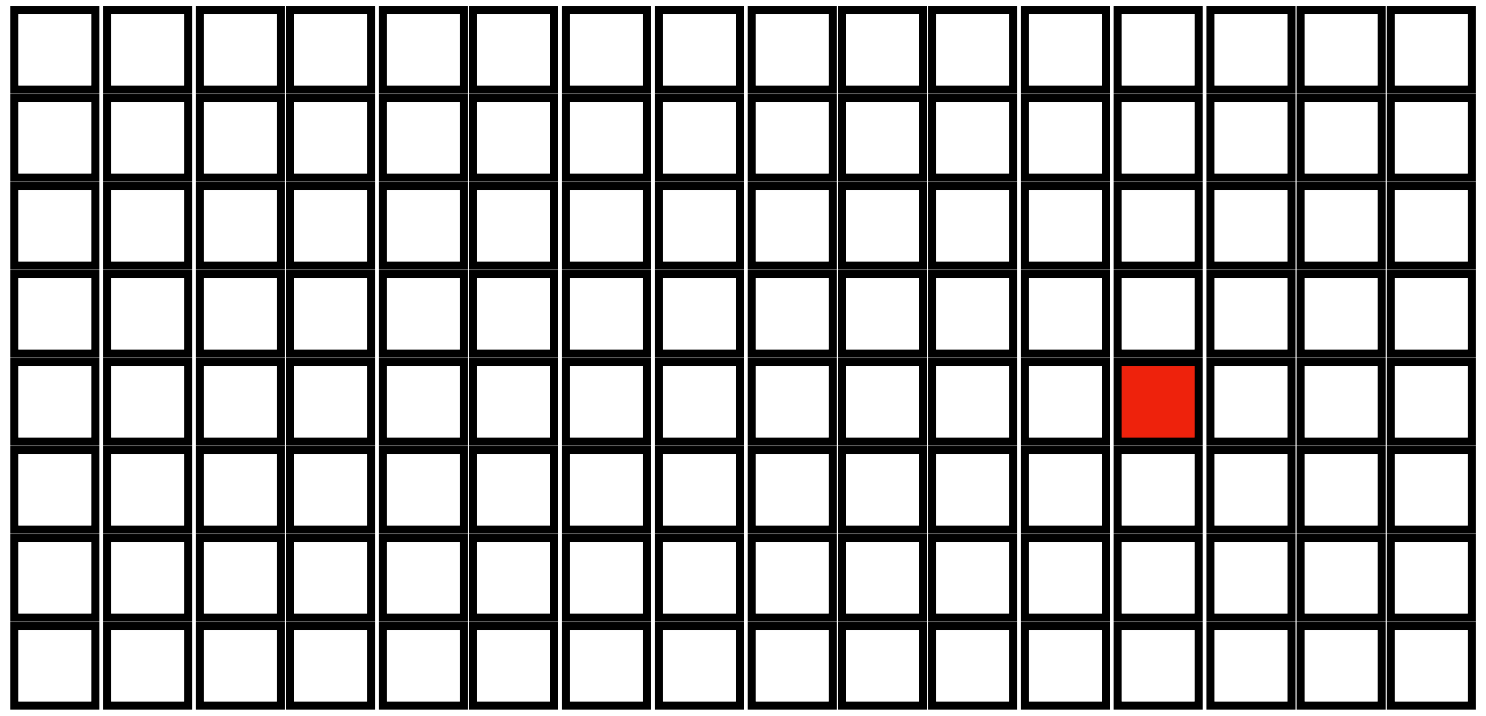
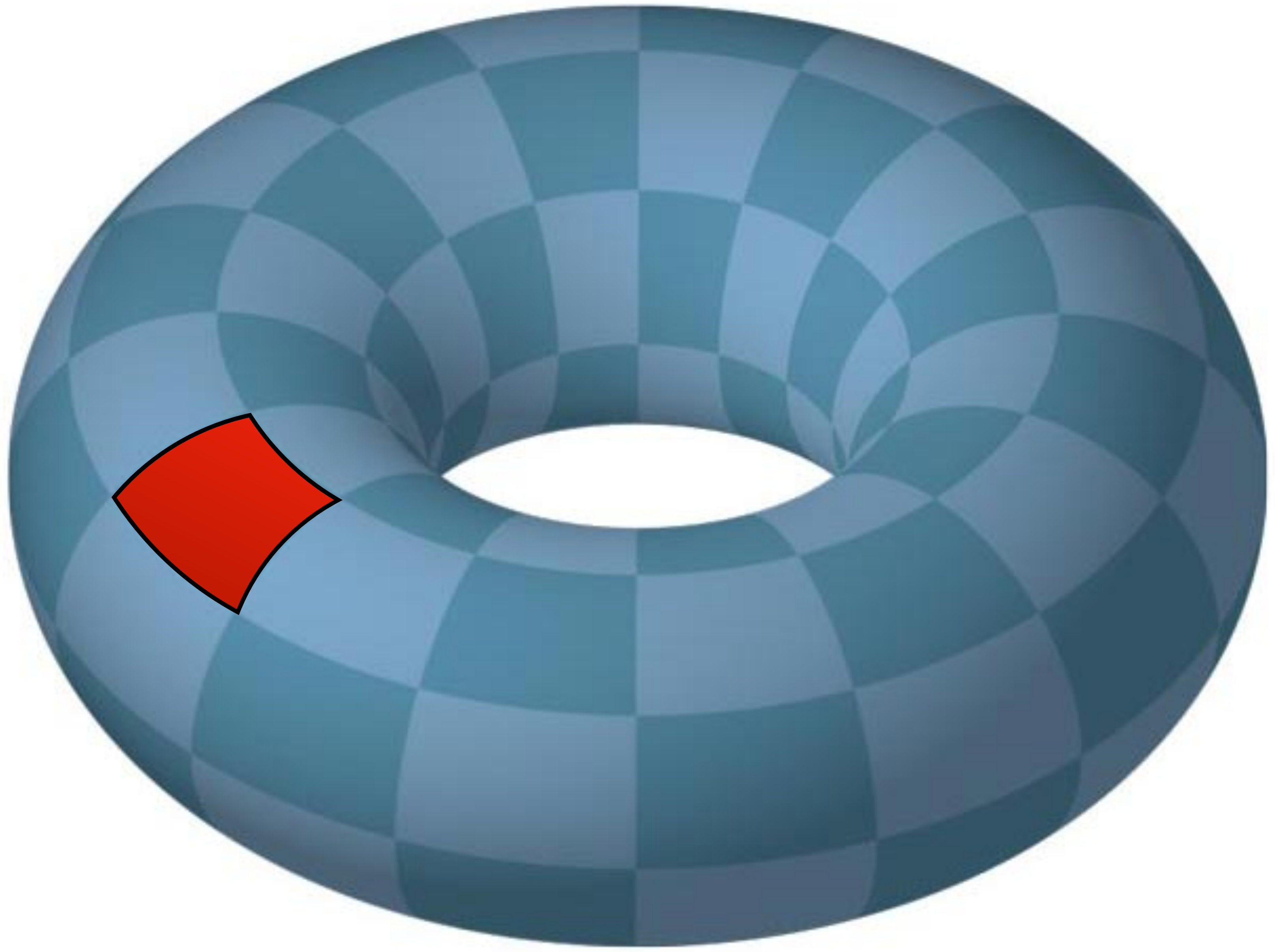
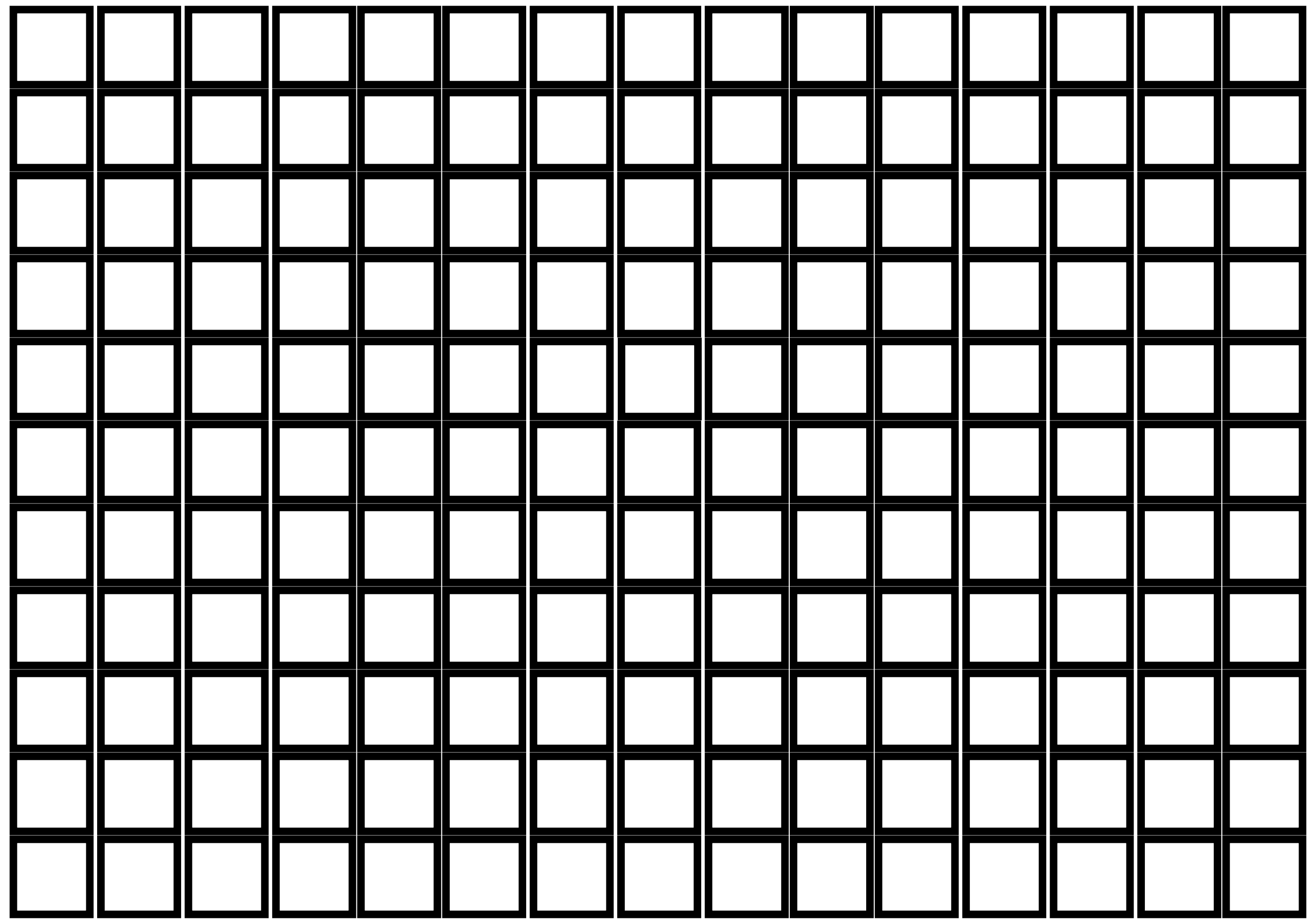


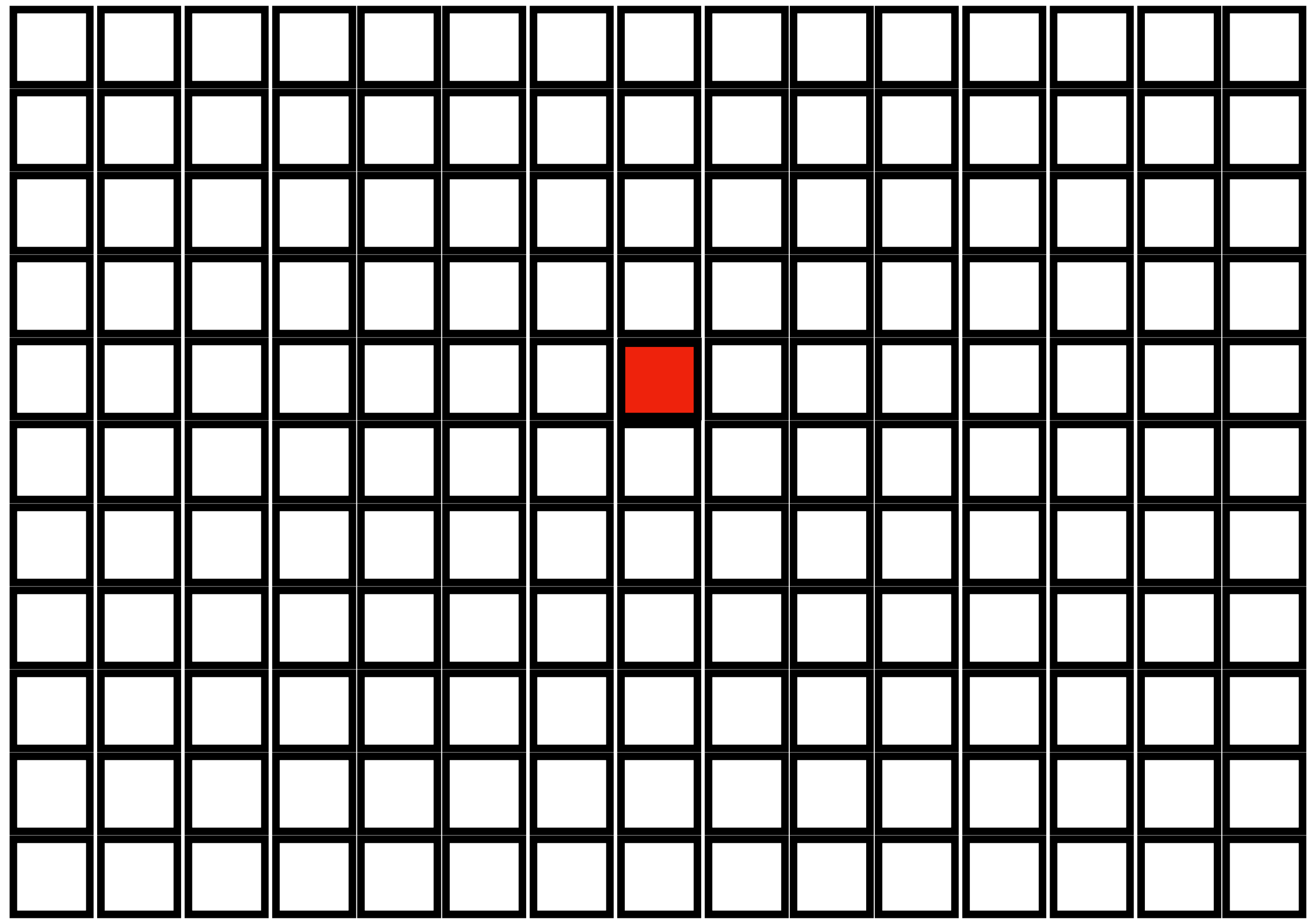


Local system and local rule



Local system and local rule

$$p(A_x = a \mid \rho) = \mathrm{Tr}[P(a)_x \rho] = \mathrm{Tr}[\{P(a) \otimes I_{\bar{x}}\}\rho]$$



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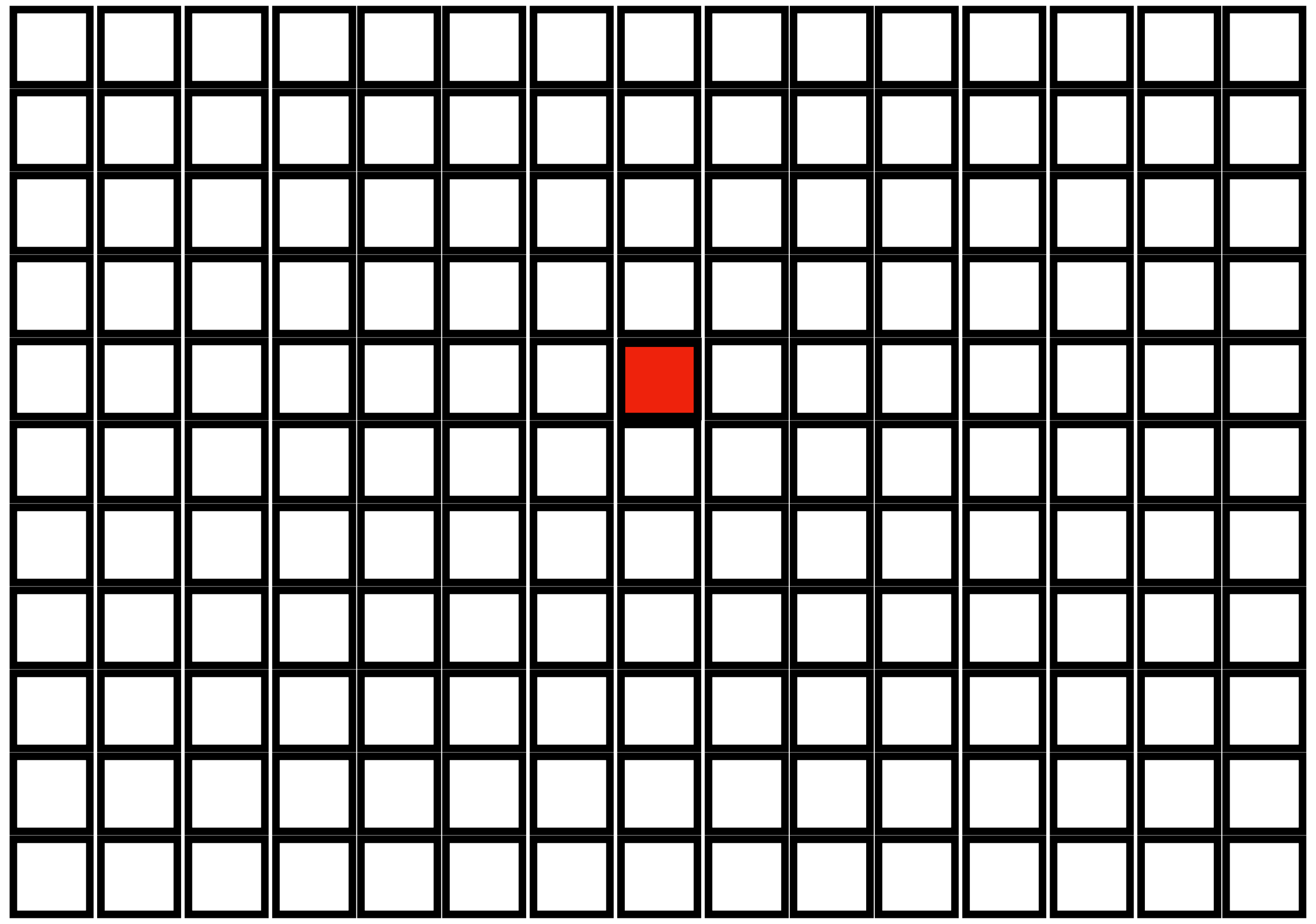

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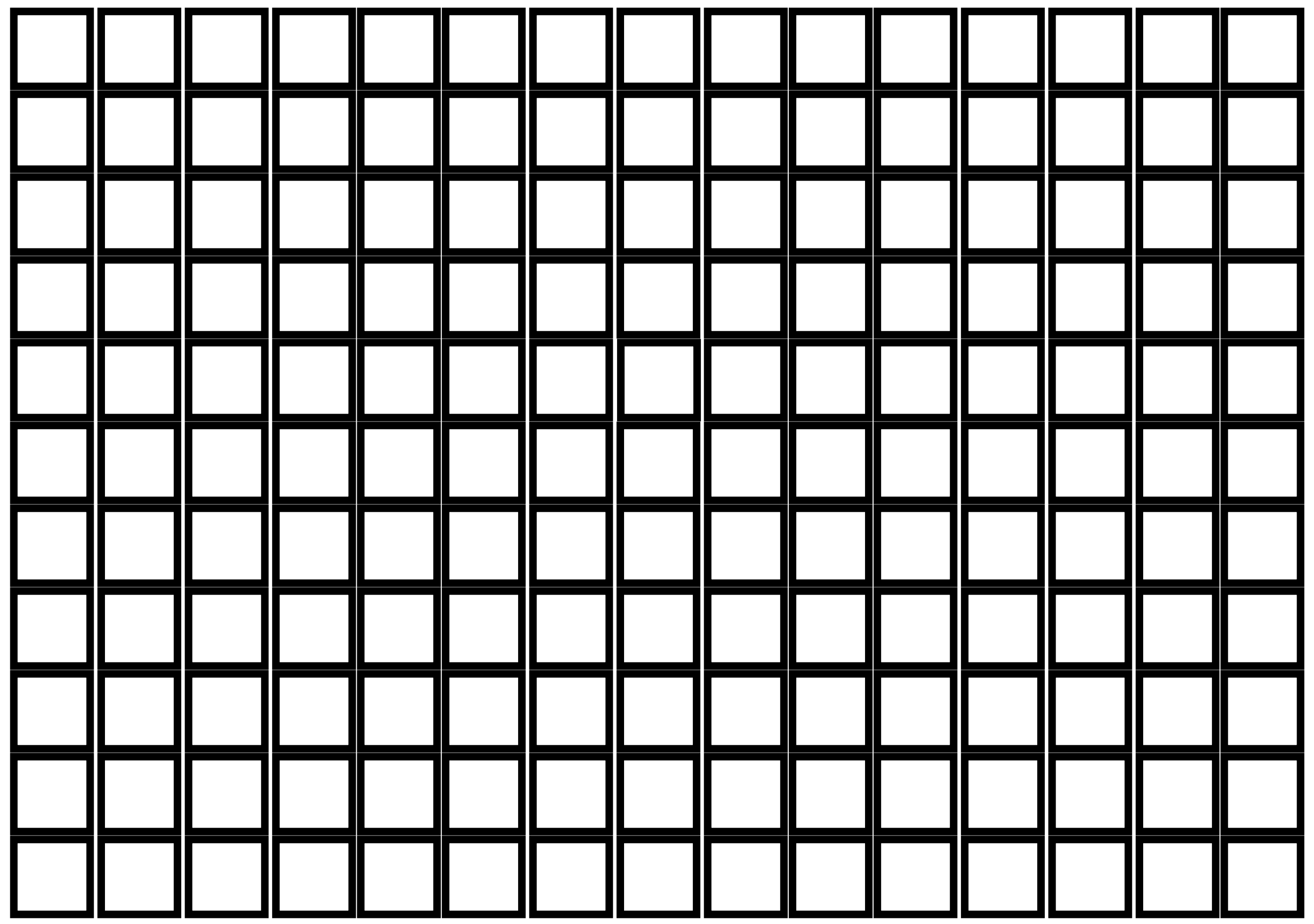

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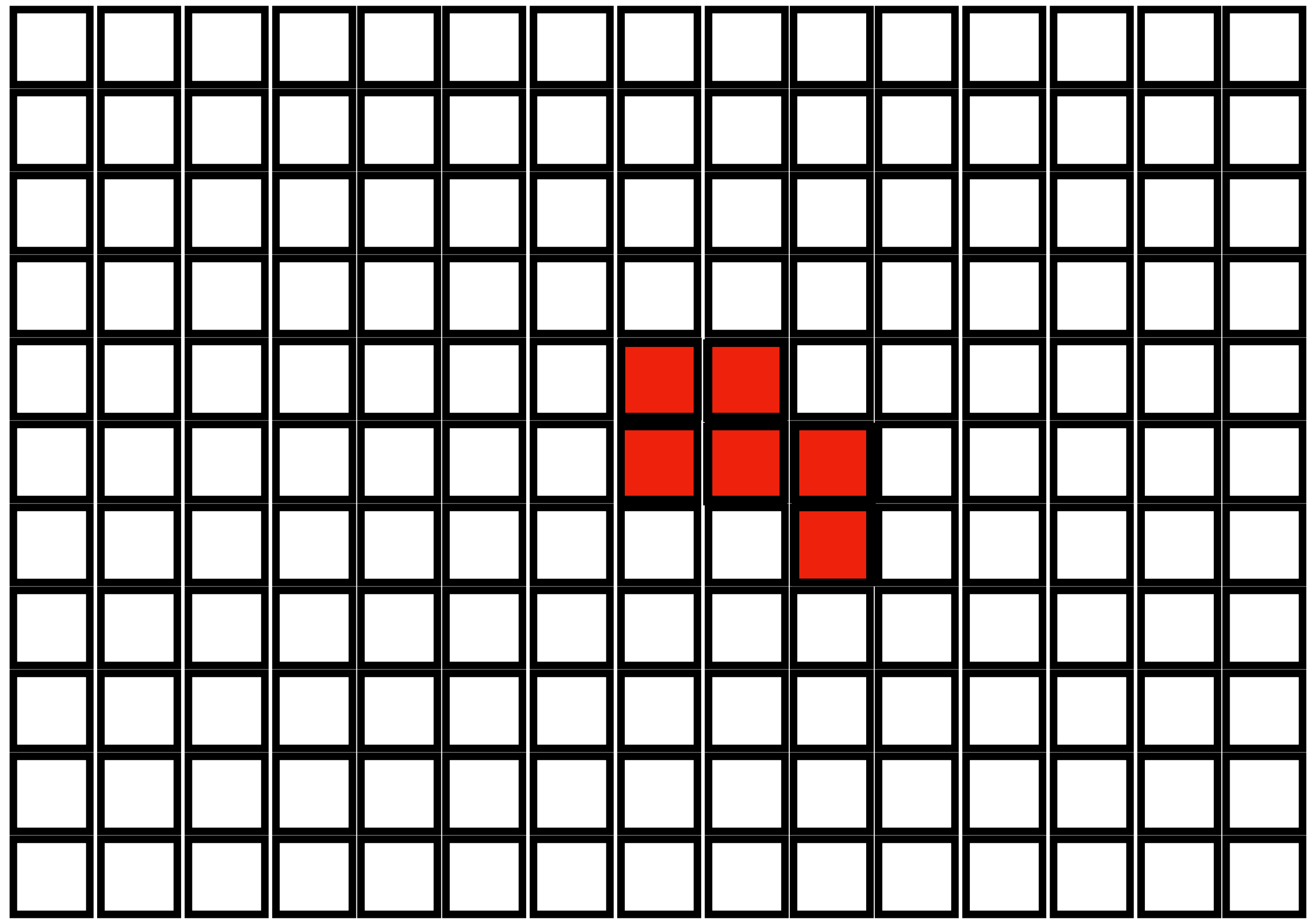

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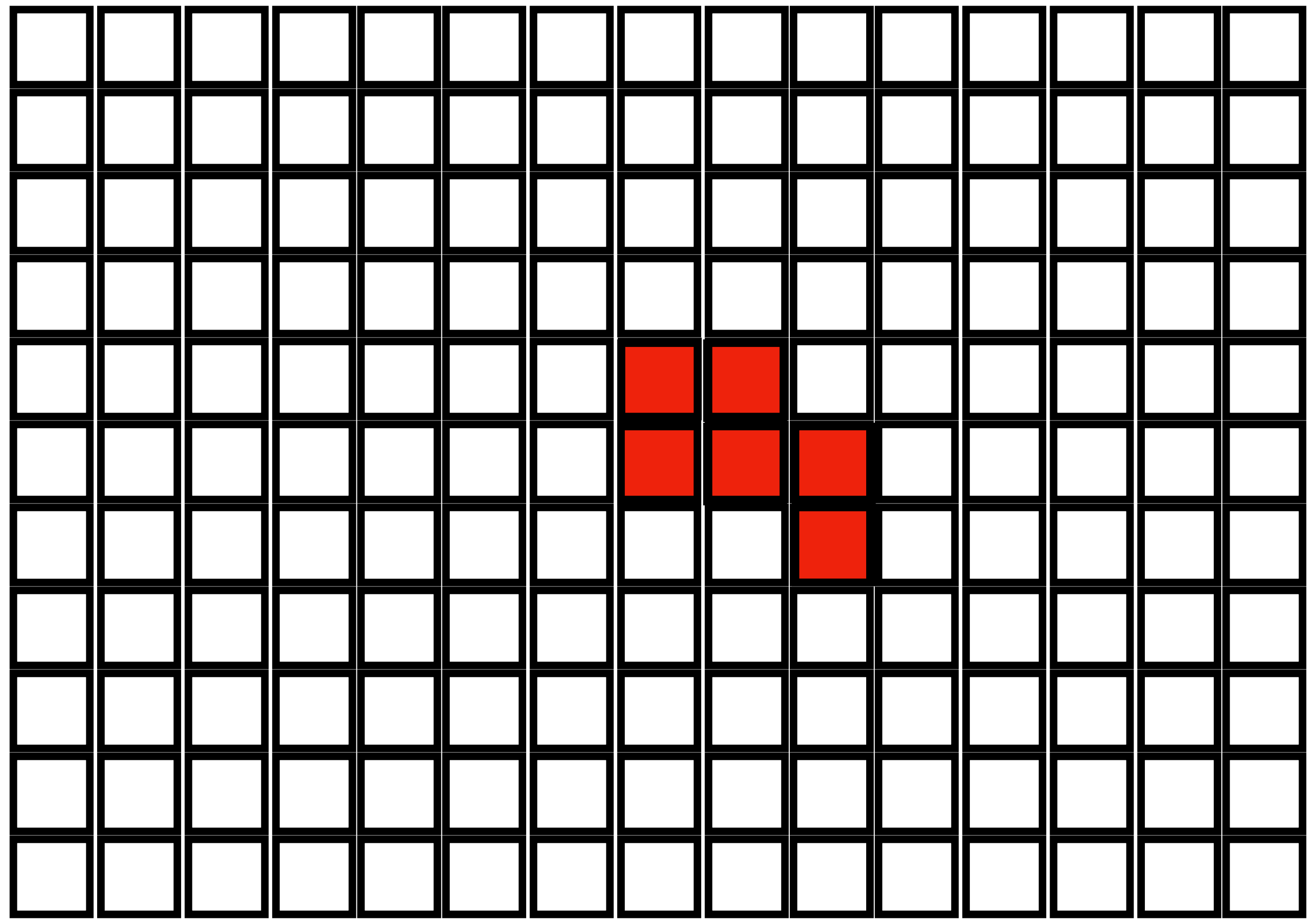

$$A_R \in \mathsf{Span}\left(\bigotimes_{x \in R} \mathsf{A}_x\right)$$

Heisenberg picture

$$p(A_x = a \mid U\rho U^\dagger) = \mathrm{Tr}[P(a)_x U\rho U^\dagger] = \mathrm{Tr}[U^\dagger P(a)_x U\rho]$$

Quantum cellular automata

Neighbourhood of a cell

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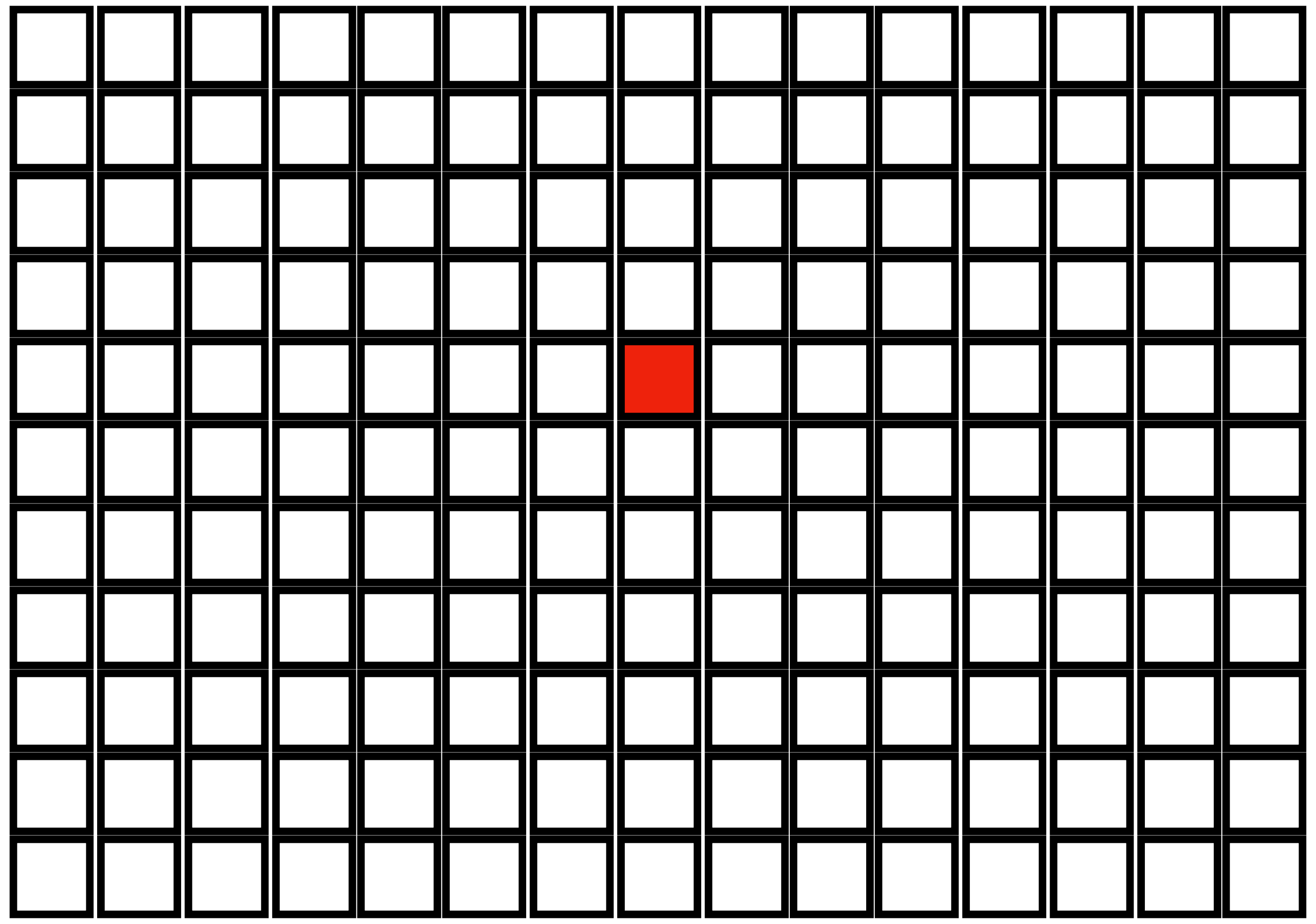

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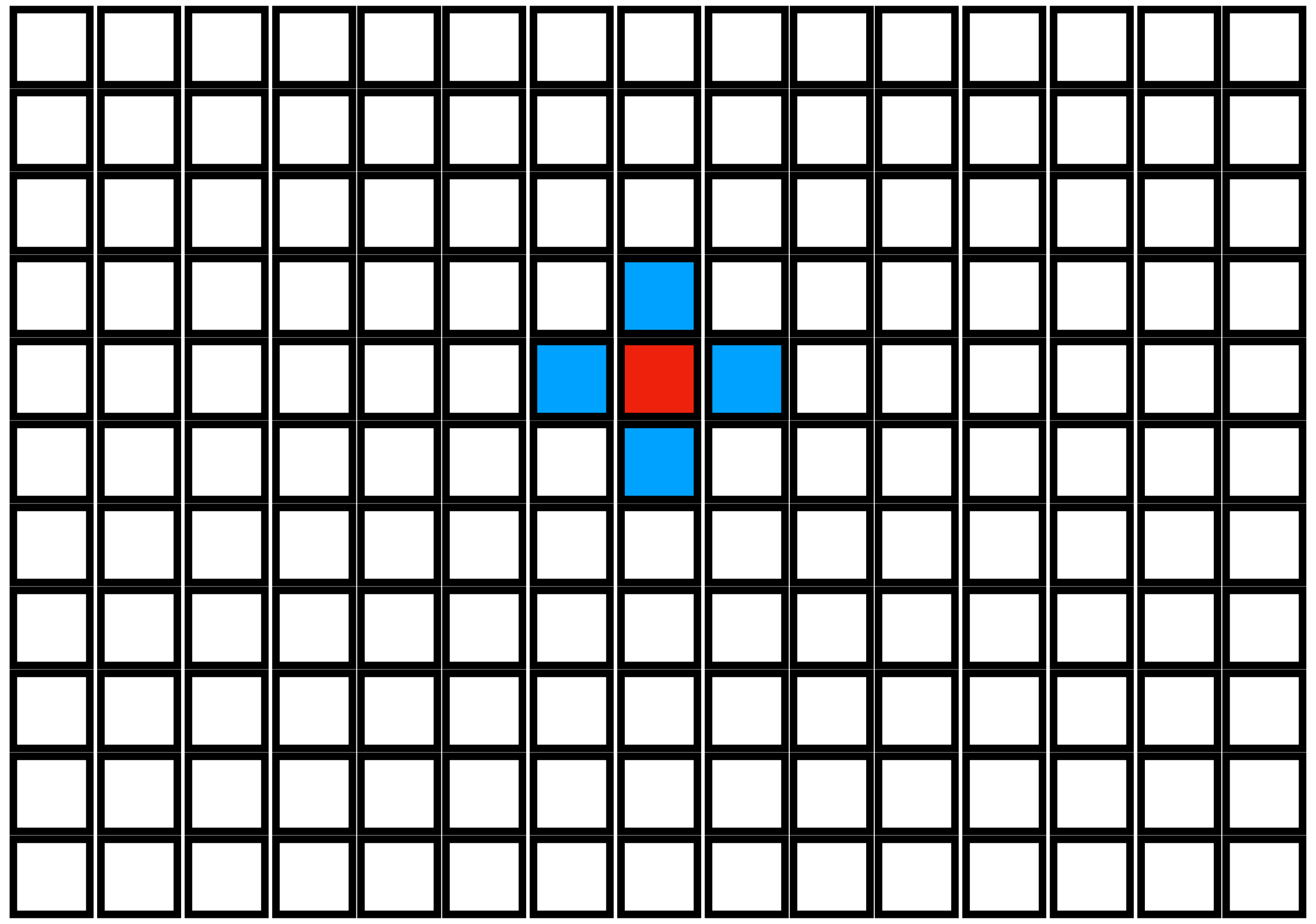

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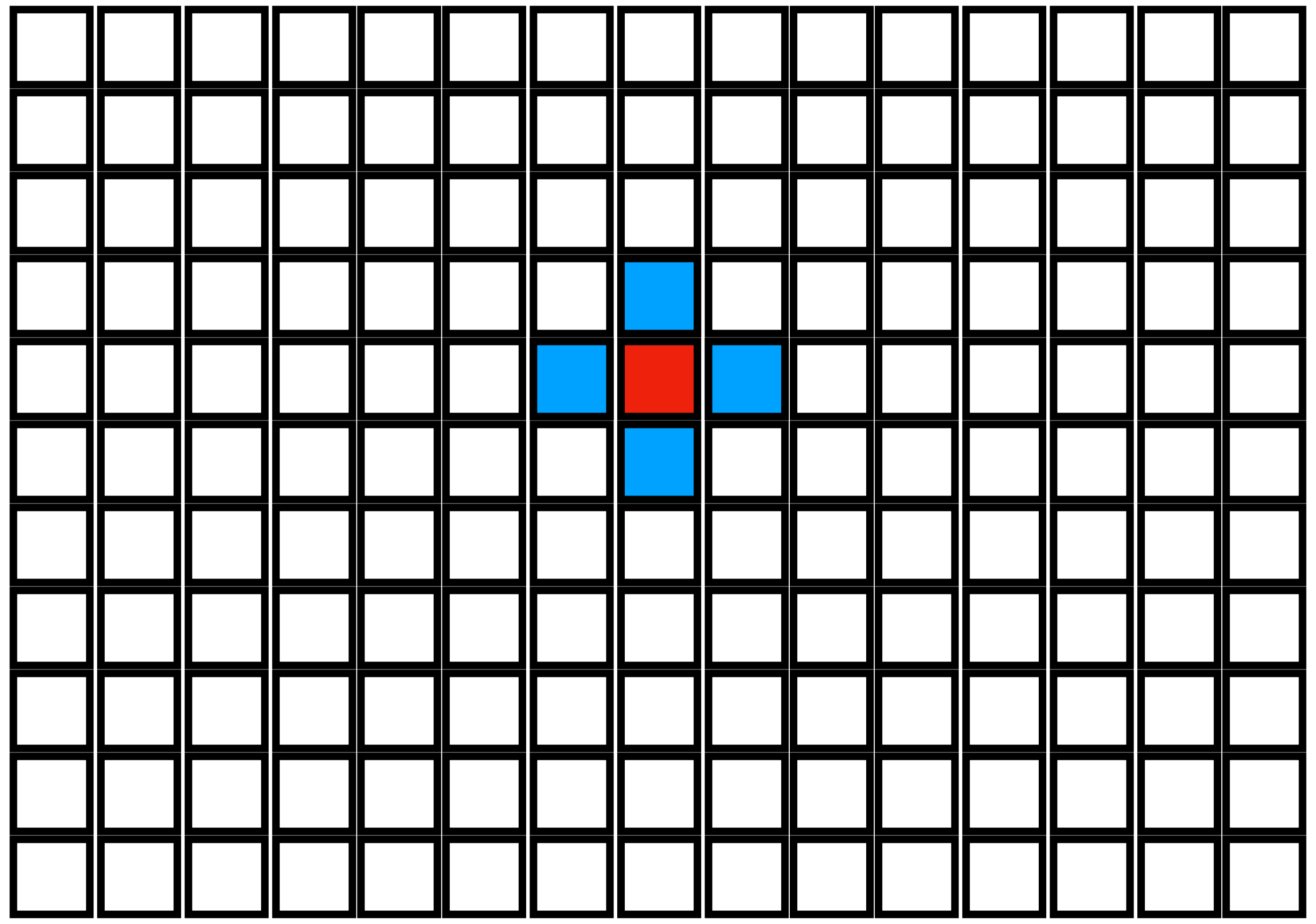

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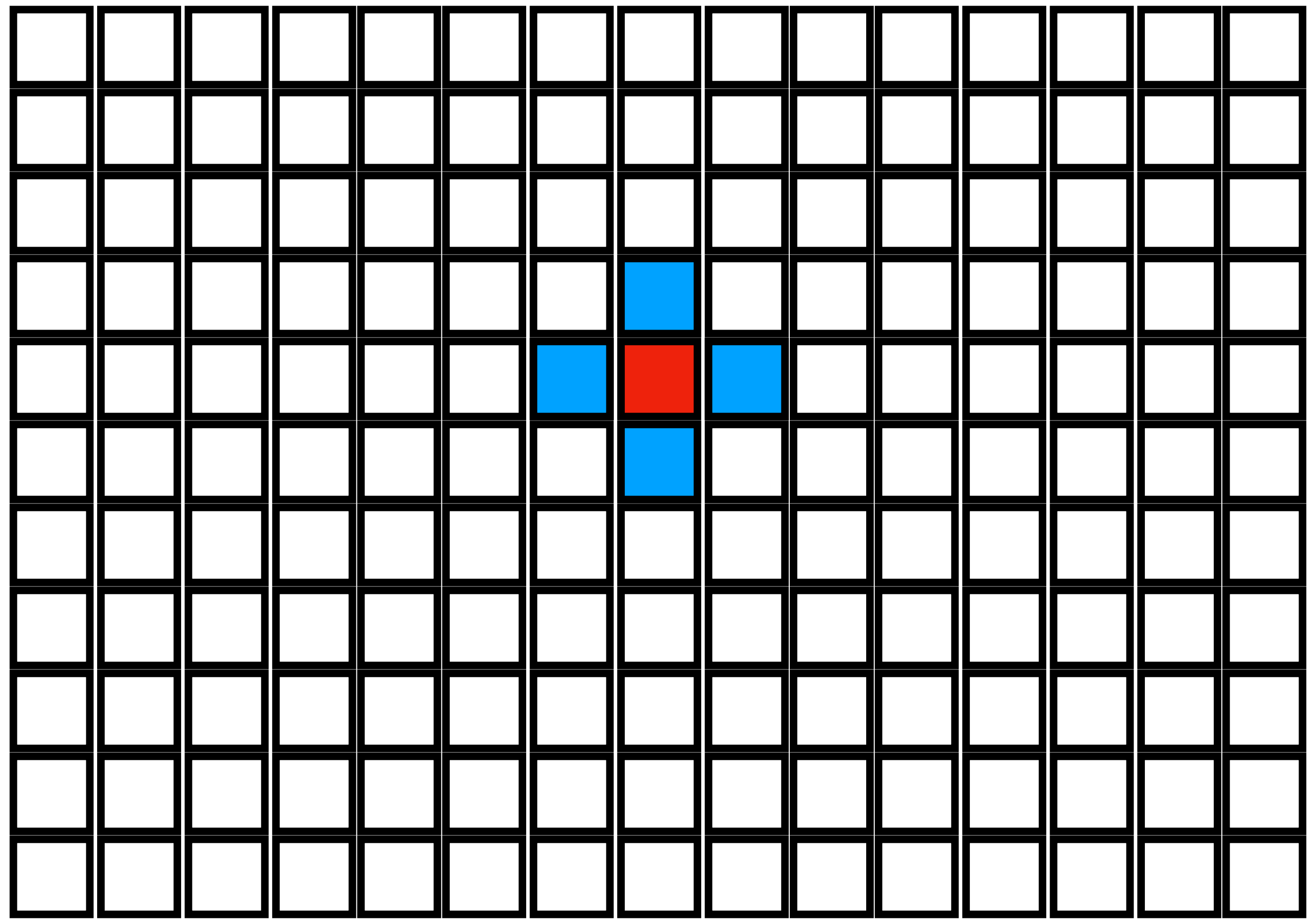

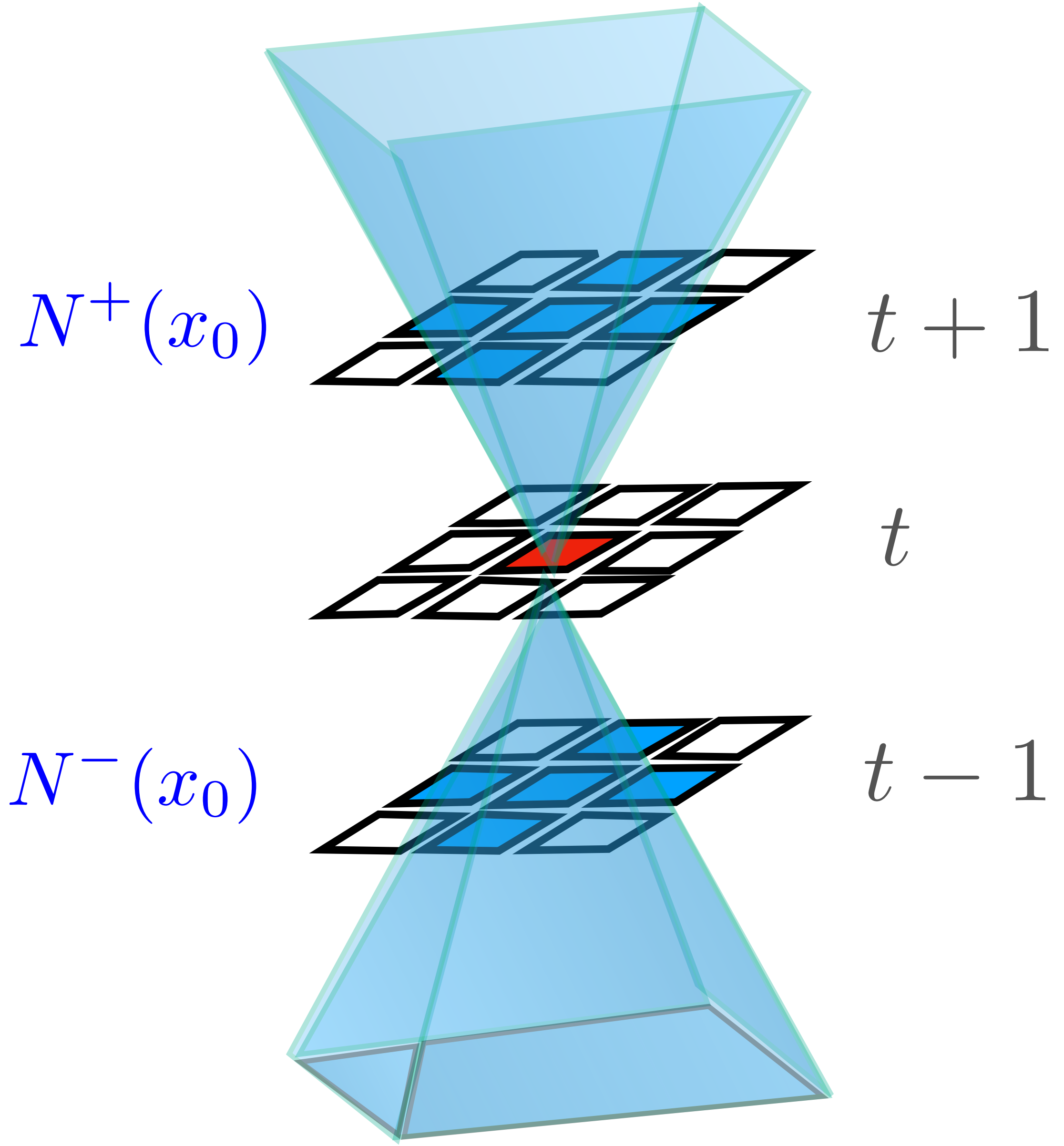

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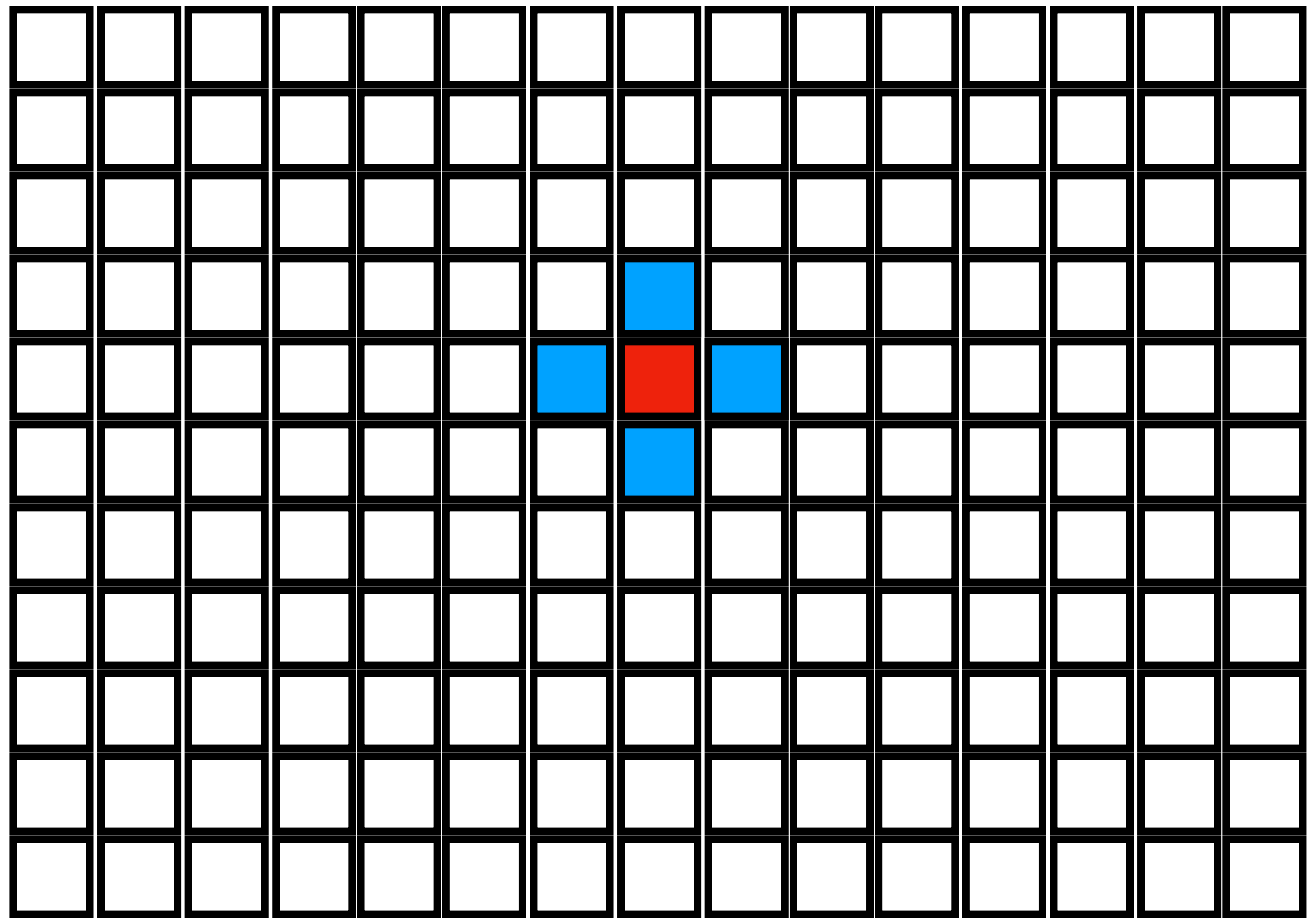

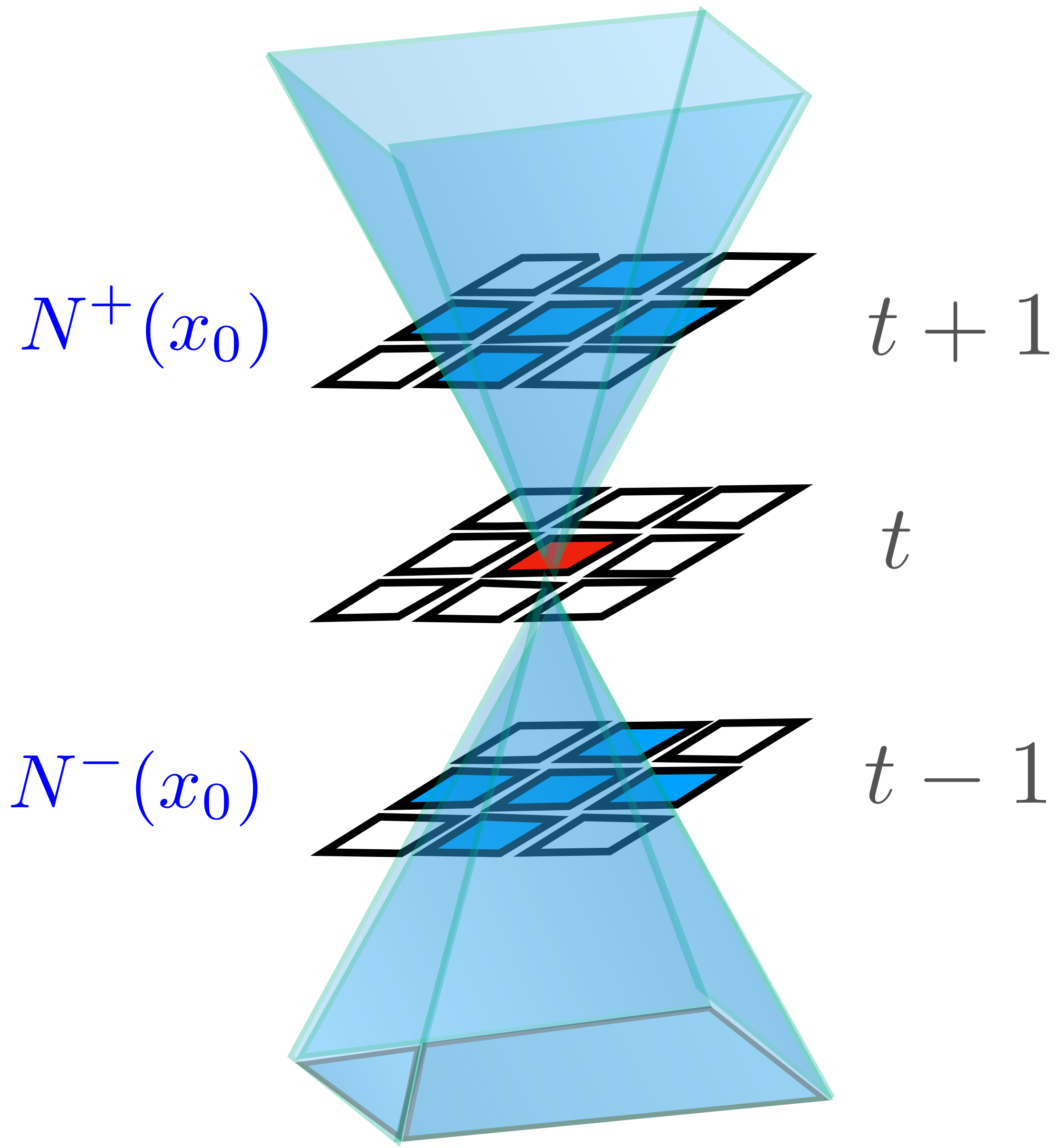

More precisely, $N^-(x_0)$ is the smallest set of cells such that $U^{-1} \mathsf{A}_{x_0} U = \mathsf{A}_{N^-(x_0)} \otimes I_{\bar{x}_0}$

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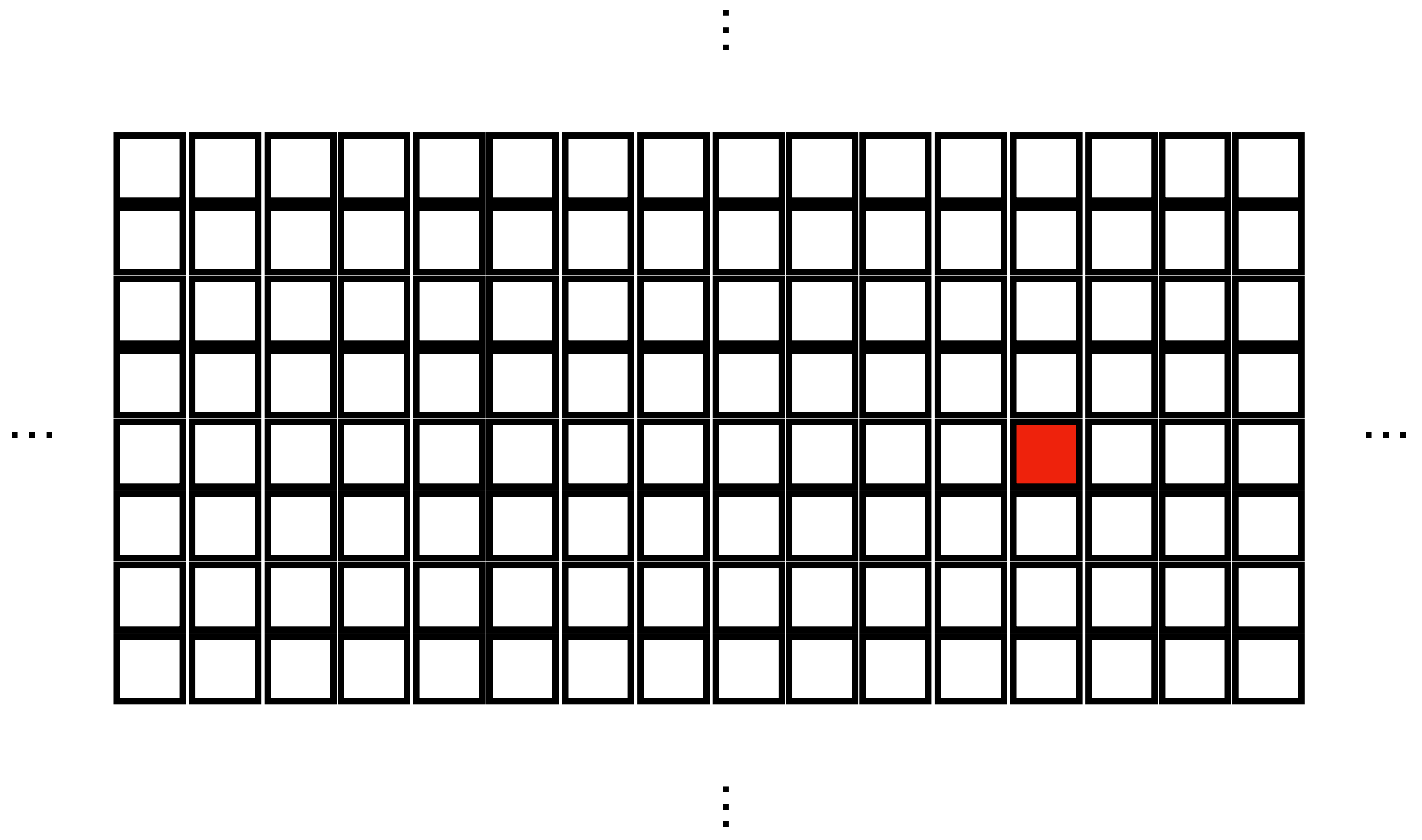

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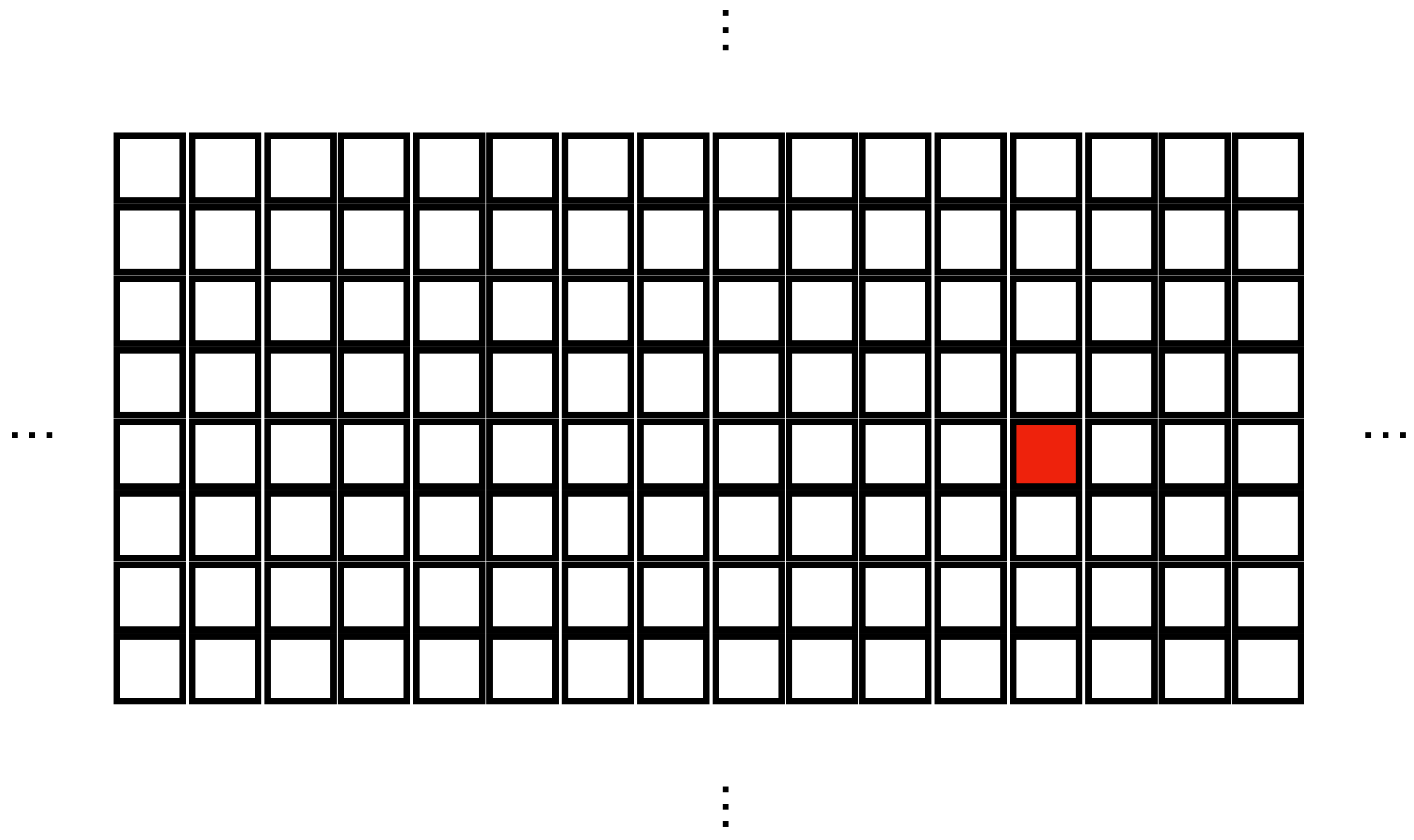

$$\mathsf{A} = \text{`}\bigotimes_x\text{'}\,\mathsf{A}_x$$

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Infinite system

Inductive limit

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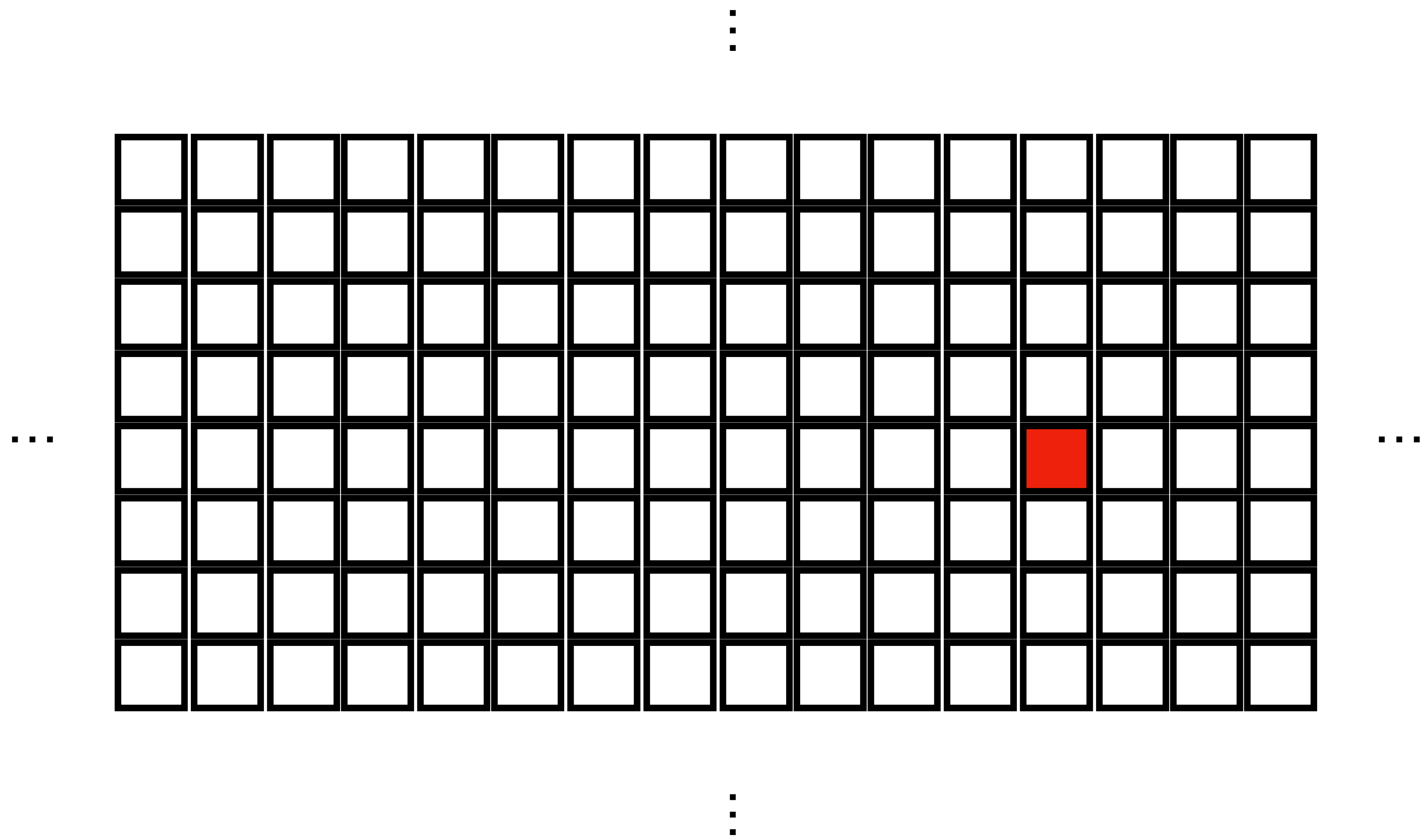

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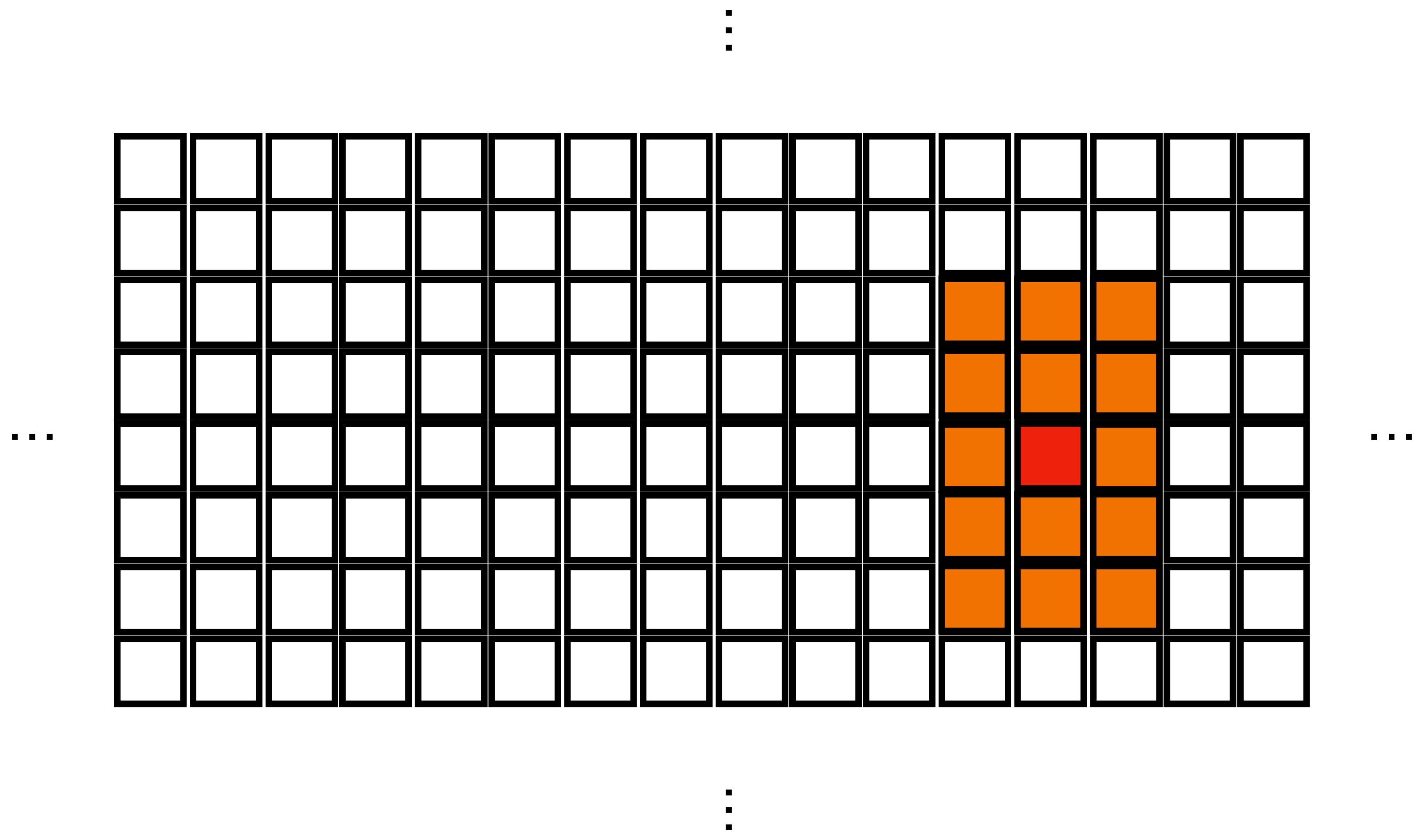

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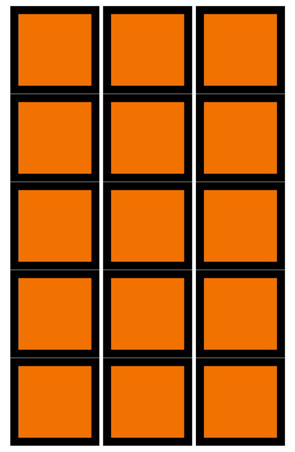

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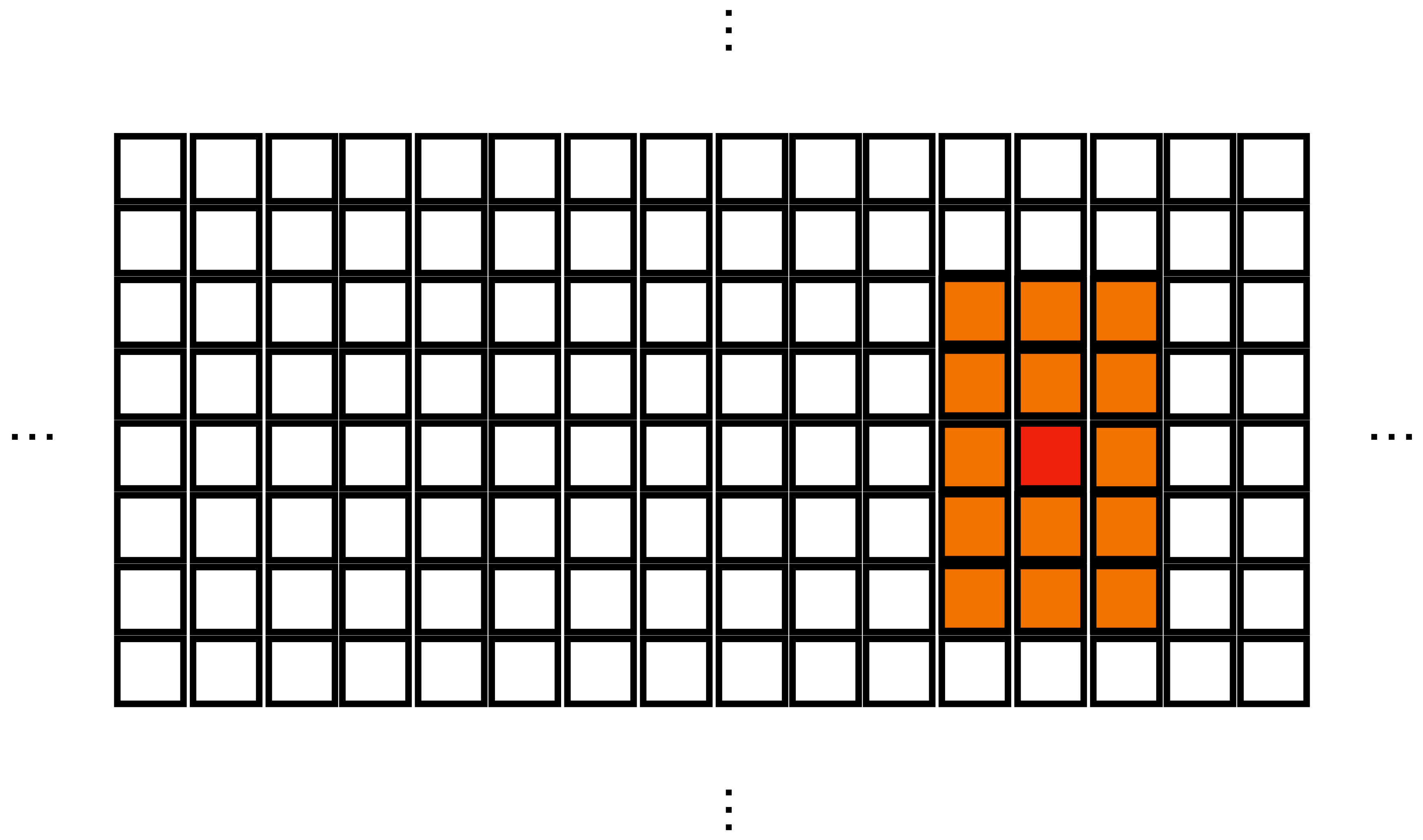

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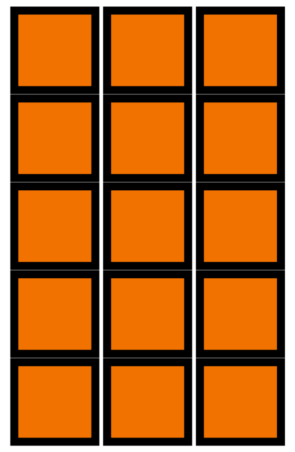

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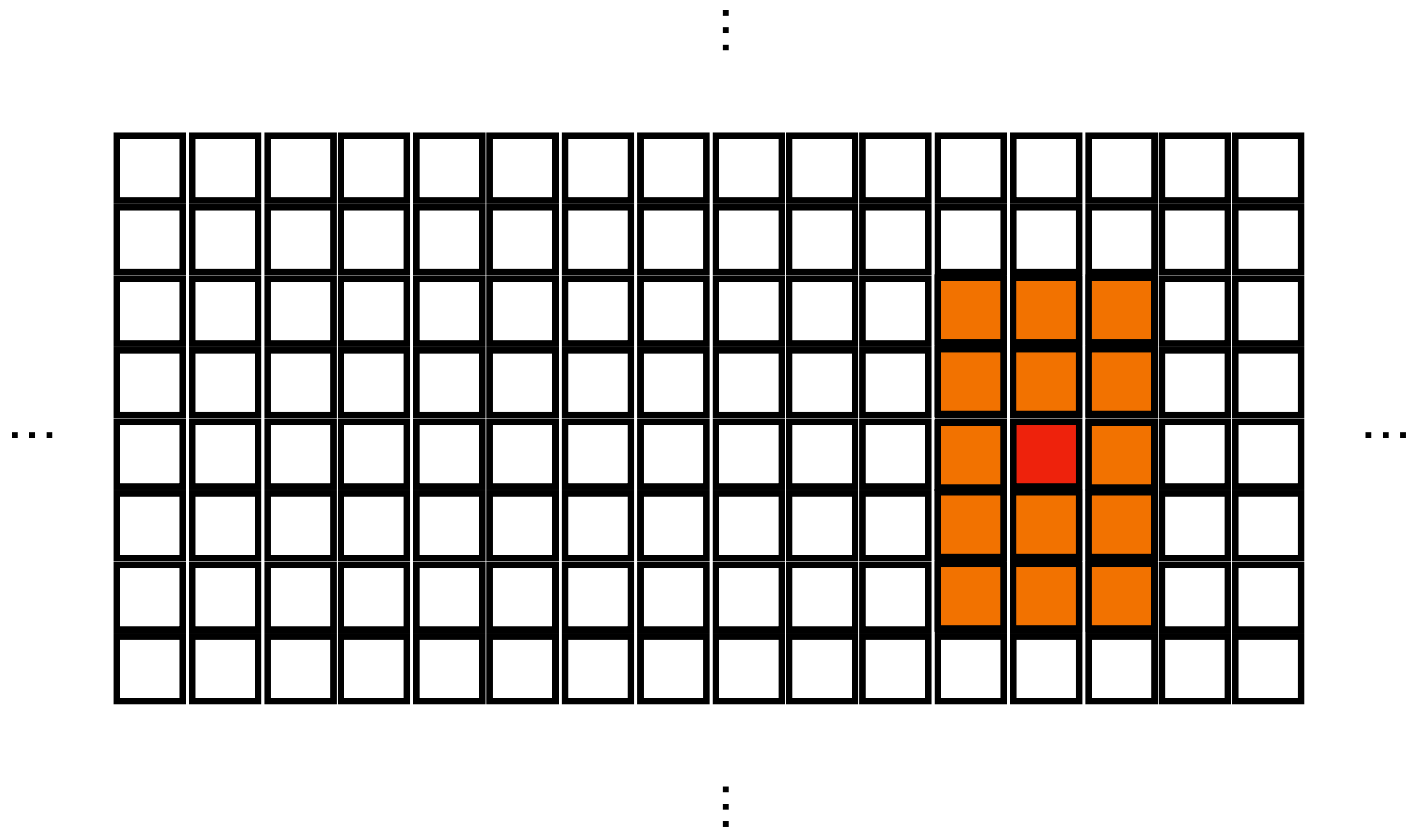

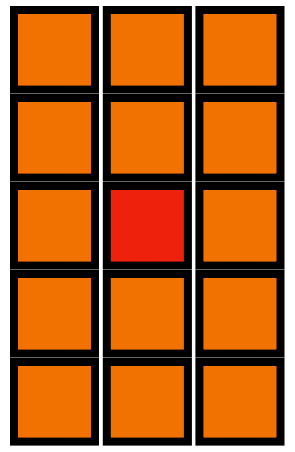

A_{x_0} ‘$\subseteq$’ A_R

Infinite system

Inductive limit

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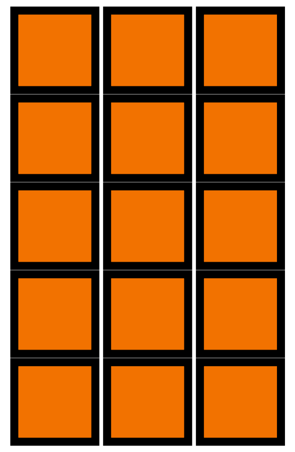

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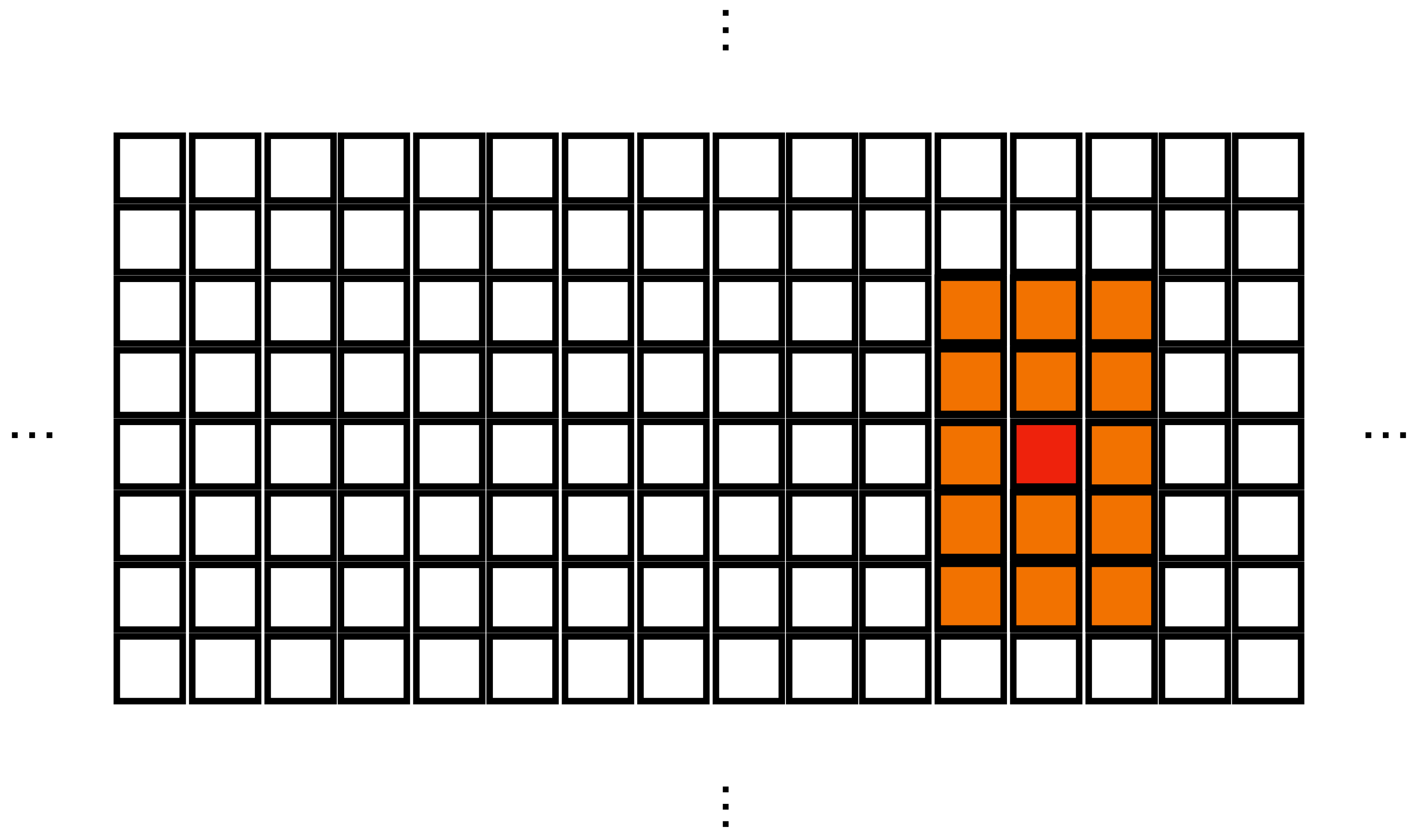
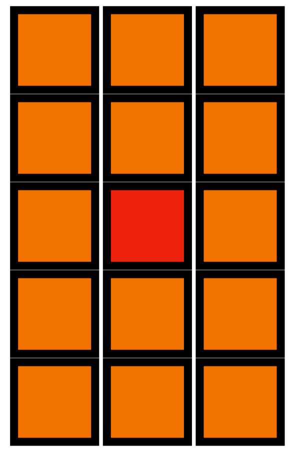




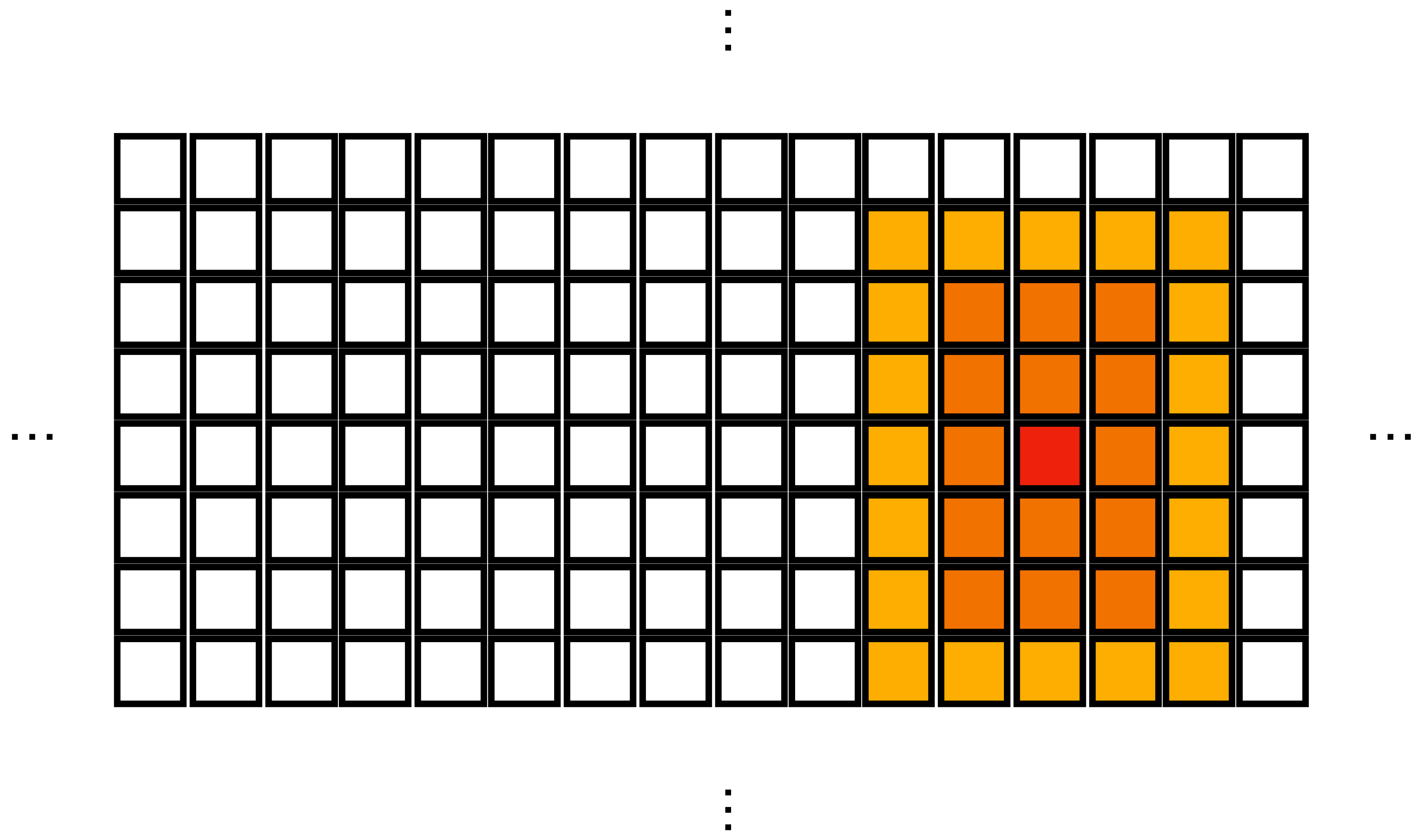
A_{x_0}‘ $\subseteq$ ’A_R

$$f_{x_0,R} : \mathsf{A}_{x_0} \to \mathsf{A}_R$$

$$f_{x_0,R}(B) = B \otimes I_{R \setminus \{x_0\}}$$

Infinite system

Inductive limit



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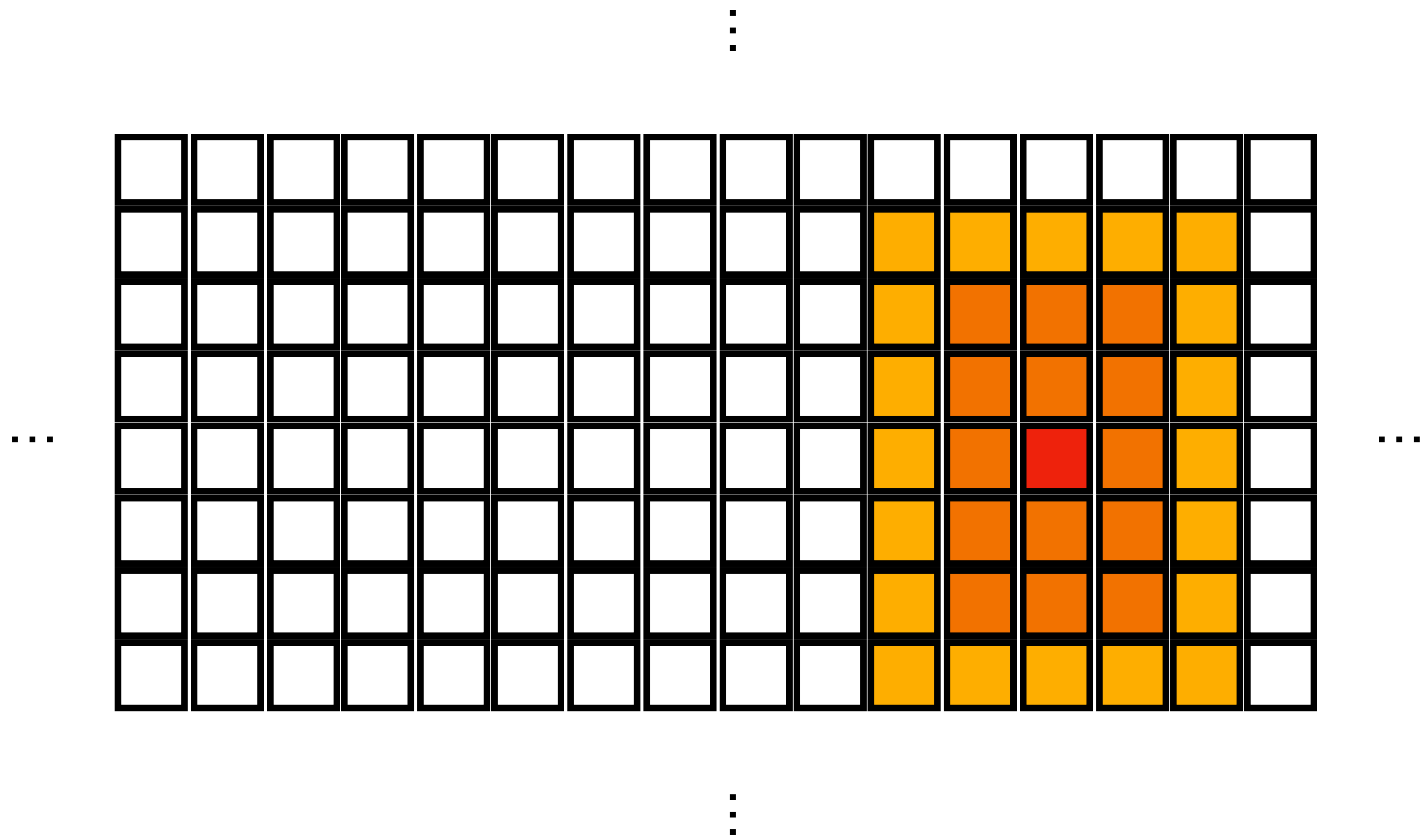

$R_0 \subseteq R_1 \subseteq R_2 \subseteq \dots$

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$f_{ij} : \mathsf{A}_{R_i} \to \mathsf{A}_{R_j}$

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Infinite system

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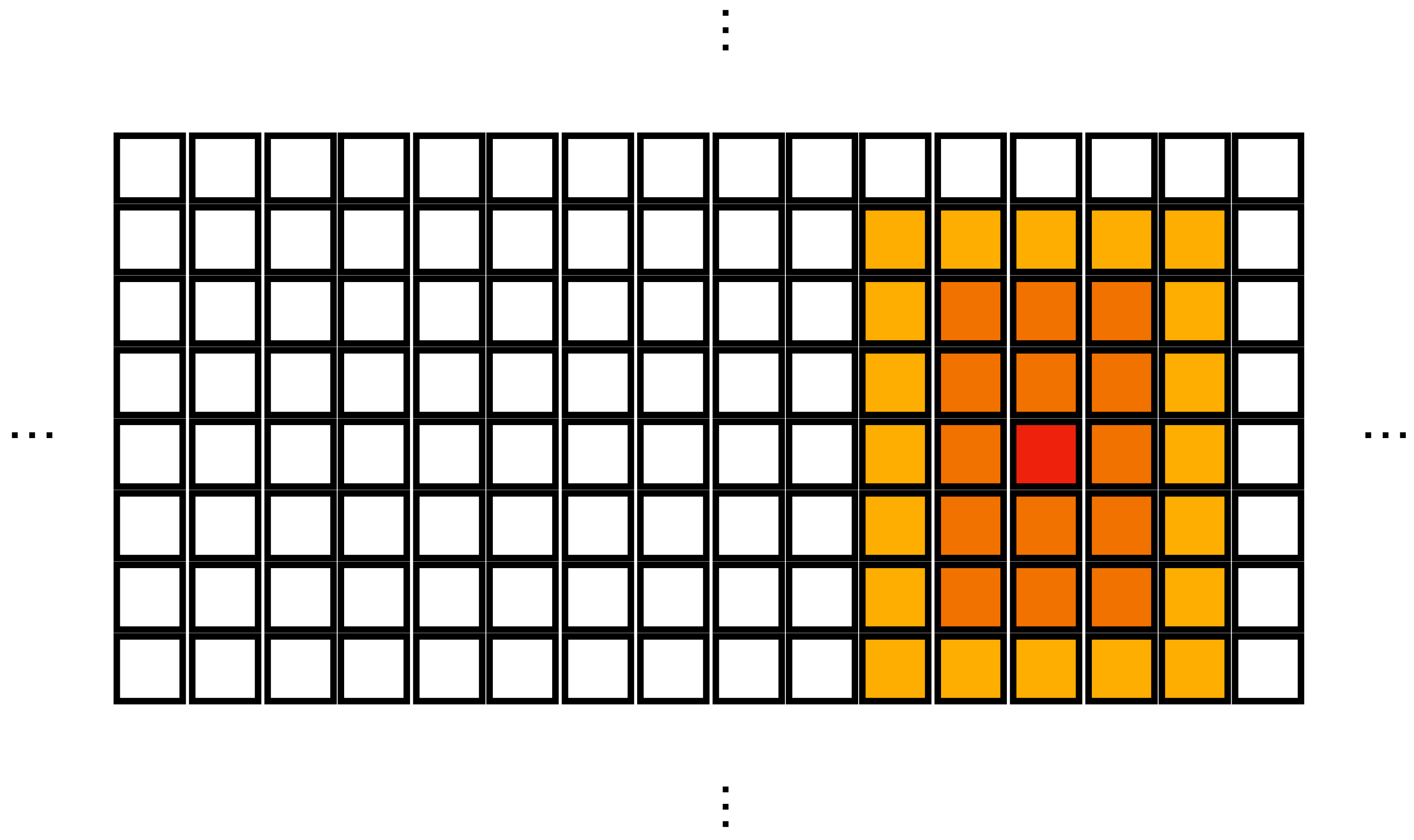

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$B_j \sim B_i \;\; \text{if} \;\; \exists k \geq i, j \;\; \text{s.t.} \;\; f_{ik}(B_i) = f_{jk}(B_j)$

$$\mathsf{A}_L := \bigsqcup_i \mathsf{A}_{R_i} / \sim$$

Infinite system

Topological limit

- Inductive limit: all local operators on arbitrarily large but finite regions

Infinite system

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- $B_{i_n} \sim C_{i_n}$ if $\forall \varepsilon > 0 \exists n_0$ s.t. $\forall n \geq n_0$ $\|B_{i_n} - C_{i_n}\|_{\text{op}} \leq \varepsilon$

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- A is the quasi-local algebra

Quasi-local algebra

Topological limit

- The algebra A contains the local algebra as a subalgebra

$$\mathsf{A}_L \subseteq \mathsf{A} \quad B_i \in \mathsf{A}_L \; \leftrightarrow \; B_{i_n} \in \mathsf{A}, \; \forall n \in \mathbb{N} \; B_{i_n} = B_i$$

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$$\|B\|_{\text{op}} := \lim_{n\to\infty} \|B_{i_n}\|_{\text{op}}$$

- Define the adjoint of $B = \lim_{n\to\infty} B_{i_n}$ as

$$B^\dagger := \lim_{n\to\infty} B_{i_n}^\dagger$$

Quasi-local algebra

C*-algebra structure

- It is easily proved that A is a c*-algebra, i.e. a Banach vector space with

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$$(AB)C = A(BC)$$

$$(A+B)C = AC + BC$$

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  - an involution $* : A \mapsto A^* \quad (A^*)^* = A$

$$(\lambda A + \mu B)^* = \bar{\lambda} A^* + \bar{\mu} B^*$$

$$(AB)^* = B^* A^*$$

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  - an associative and distributive product

$$A, B, C \in \mathsf{A} \quad AB \in \mathsf{A}$$

$$(AB)C = A(BC)$$

$$(A + B)C = AC + BC$$

$$A(B + C) = AC + AB$$

  - an involution $* : A \mapsto A^* \quad (A^*)^* = A$

$$(\lambda A + \mu B)^* = \bar{\lambda} A^* + \bar{\mu} B^*$$

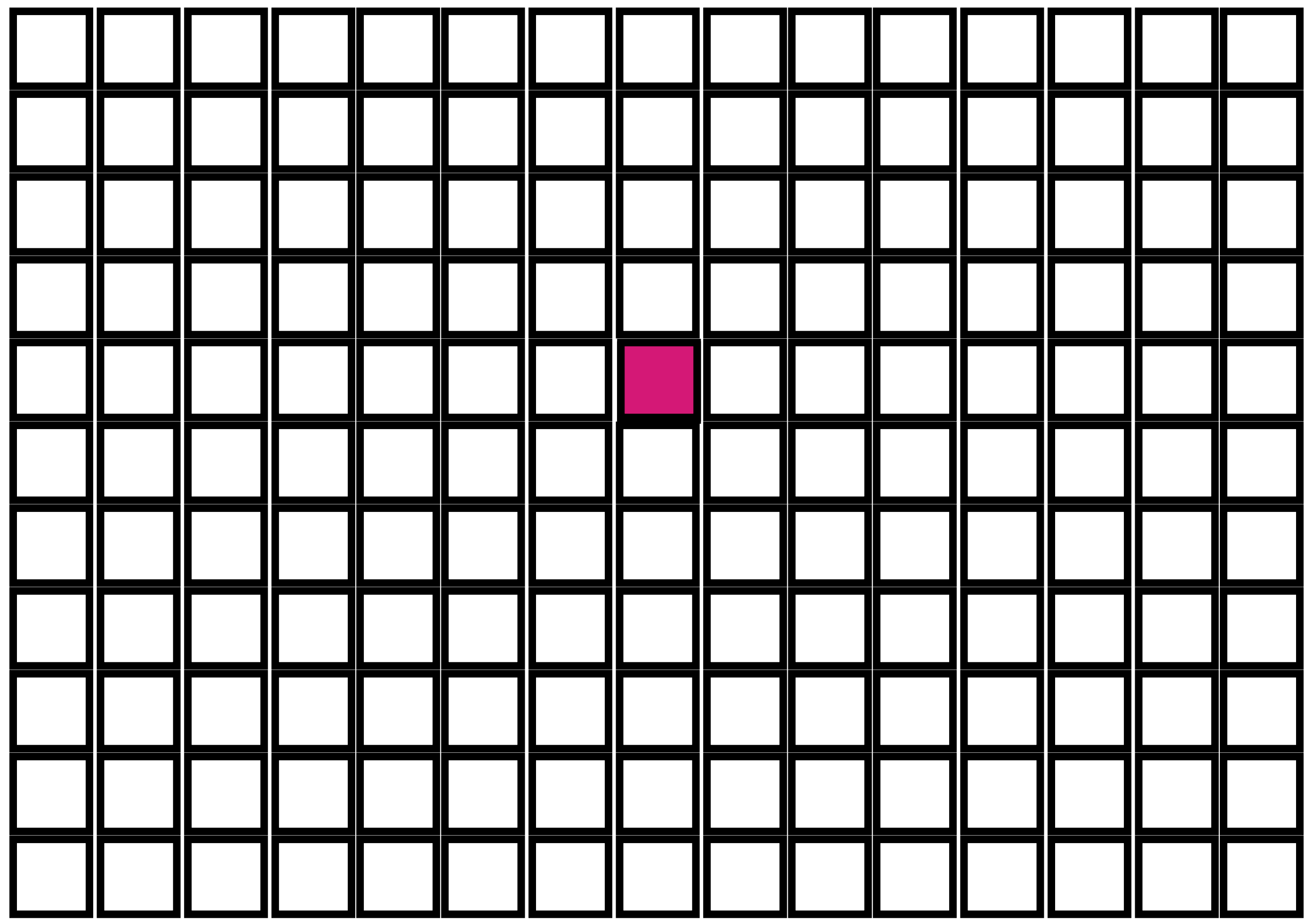
$$(AB)^* = B^* A^*$$

- and the norm satisfies $\|AB\| \leq \|A\| \, \|B\|, \quad \|A^* A\| = \|A\|^2$

Quantum Cellular Automaton

Infinite case

- The following definition was given for QCA on $\mathbb{Z}^d$



B. Schumacher and R.F. Werner, arXiv:quant-ph/040517

Quantum Cellular Automaton

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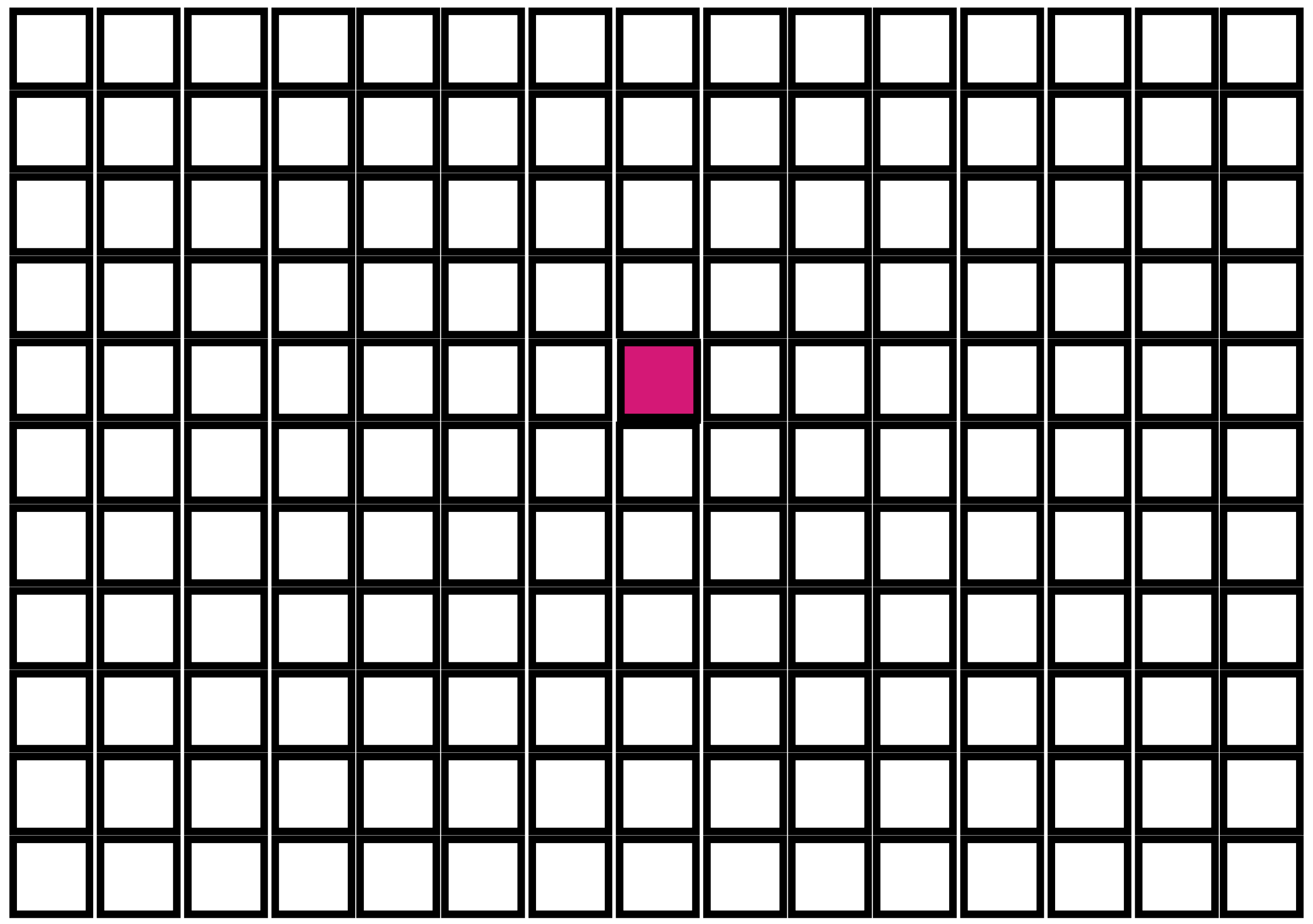

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Quantum Cellular Automaton

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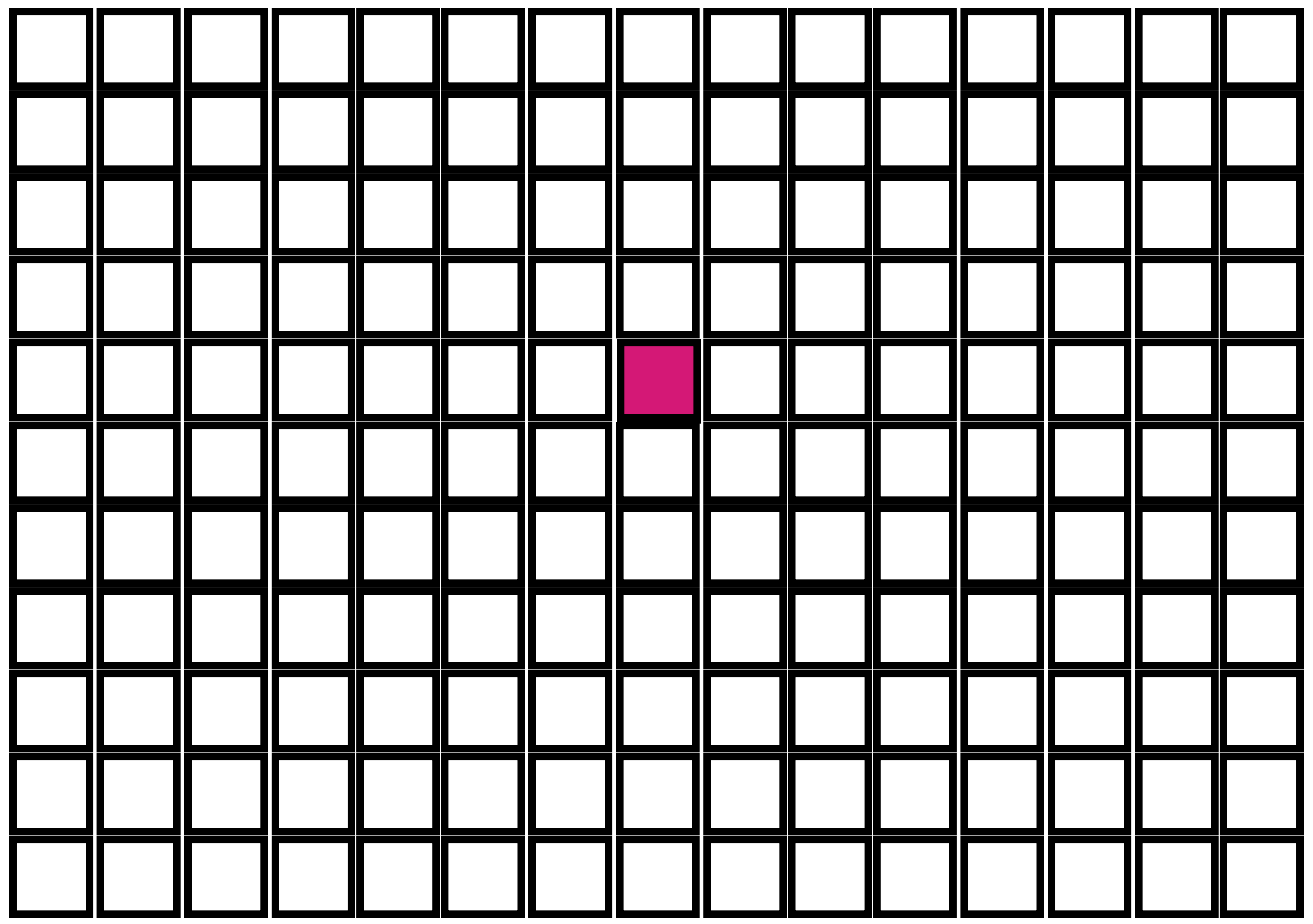

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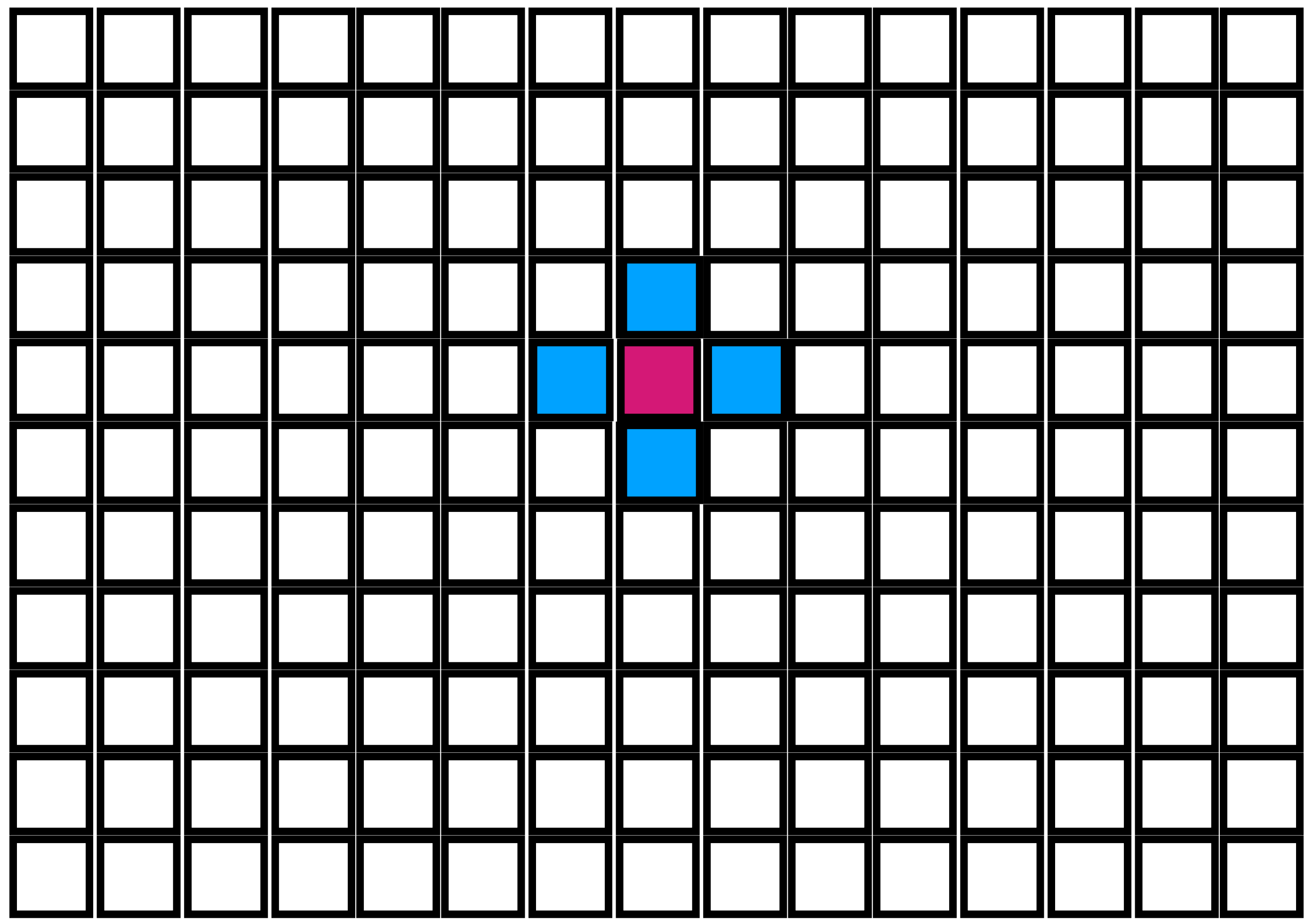

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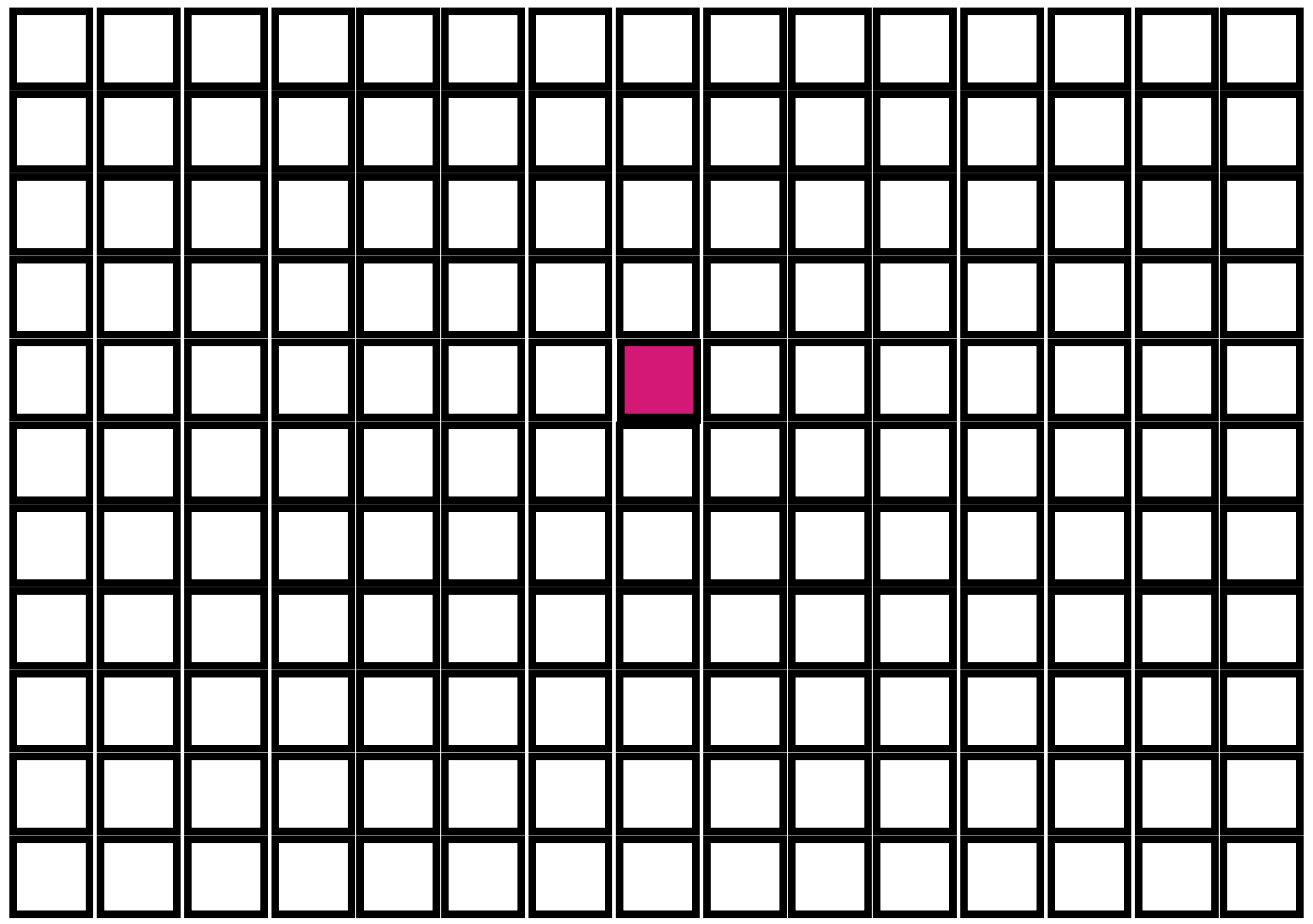

    - Translationally invariant

$$\mathscr{V}[\mathcal{T}_z(\mathsf{A}_x)] = \mathcal{T}_z[\mathscr{V}(\mathsf{A}_x)]$$

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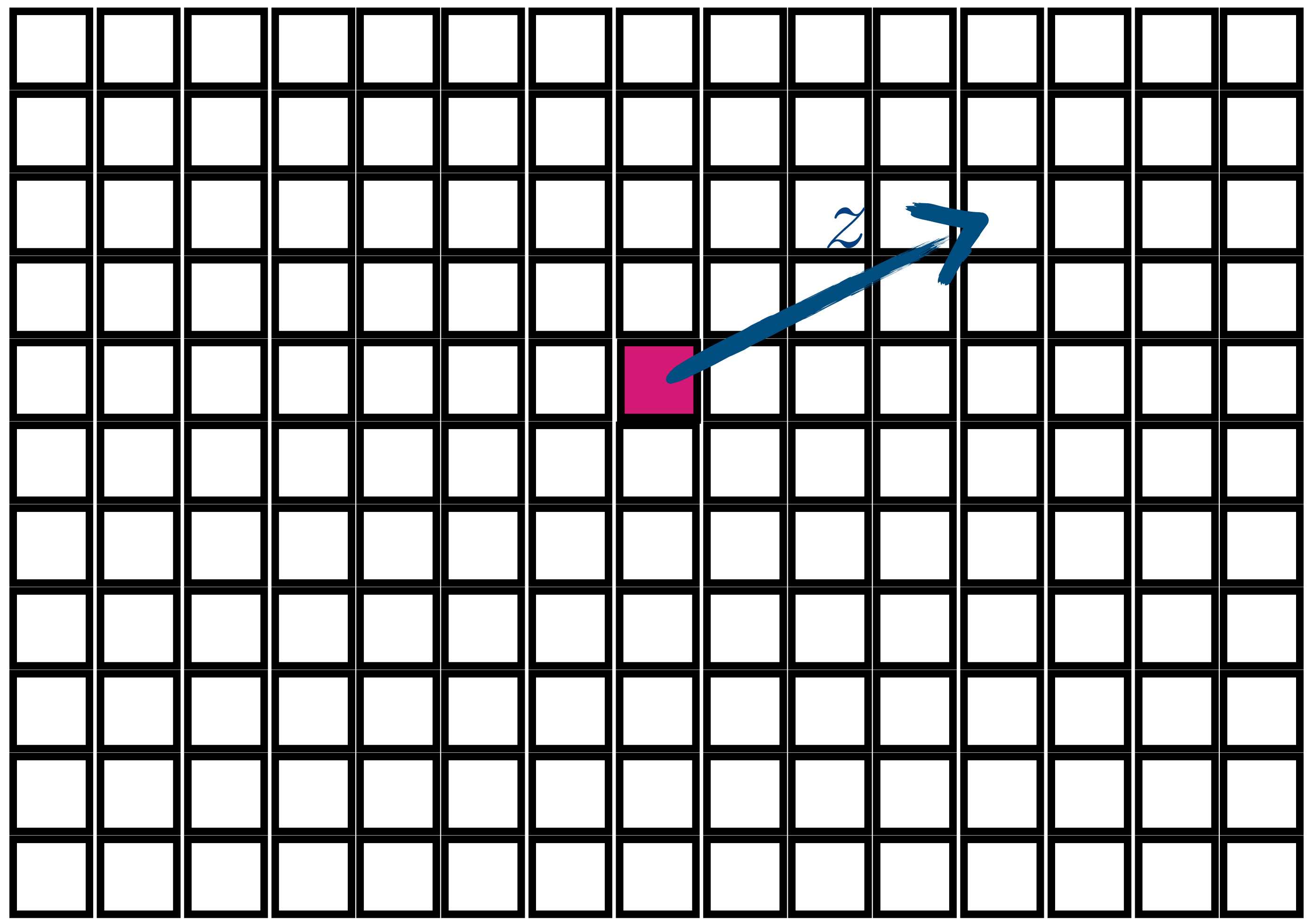

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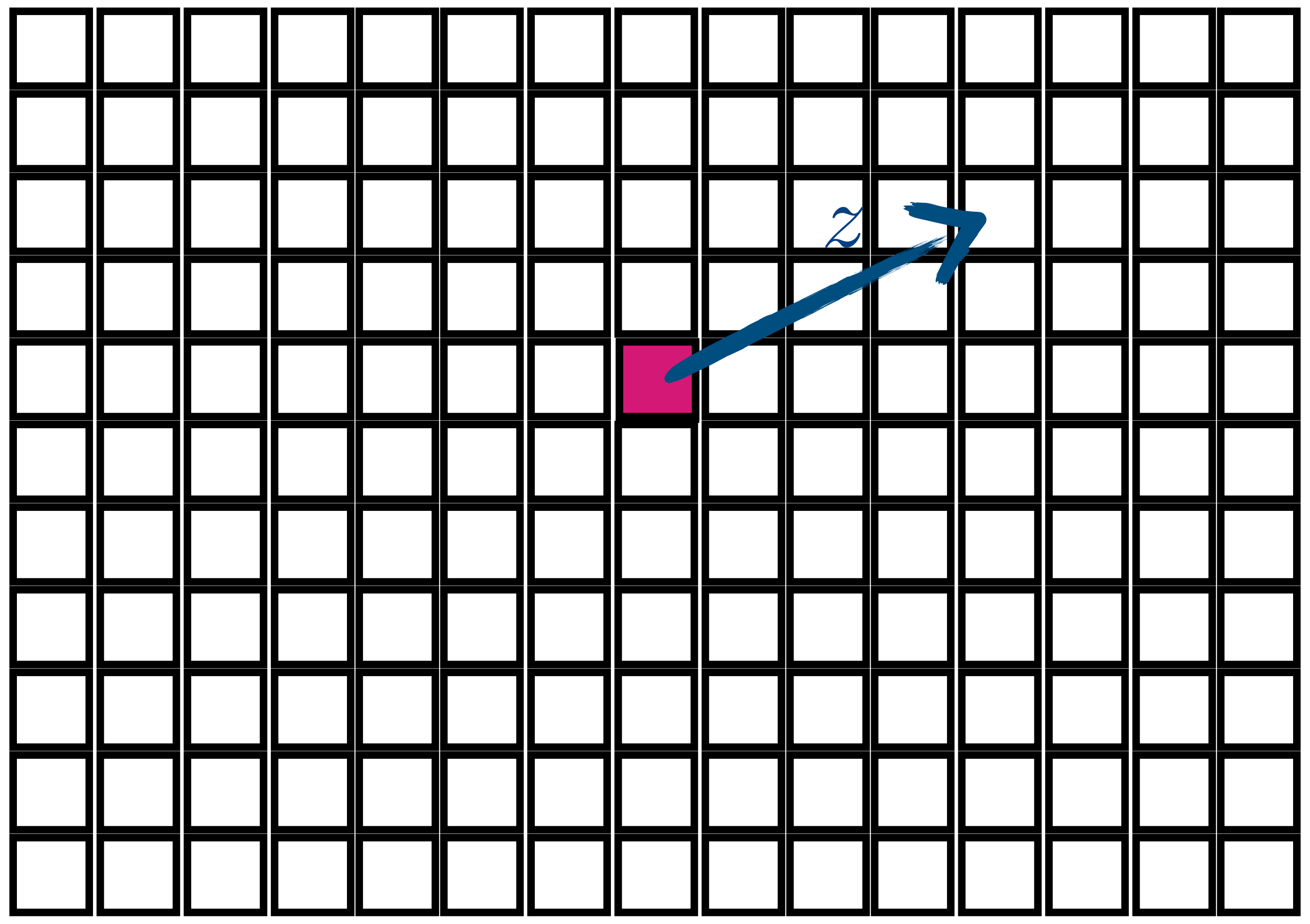

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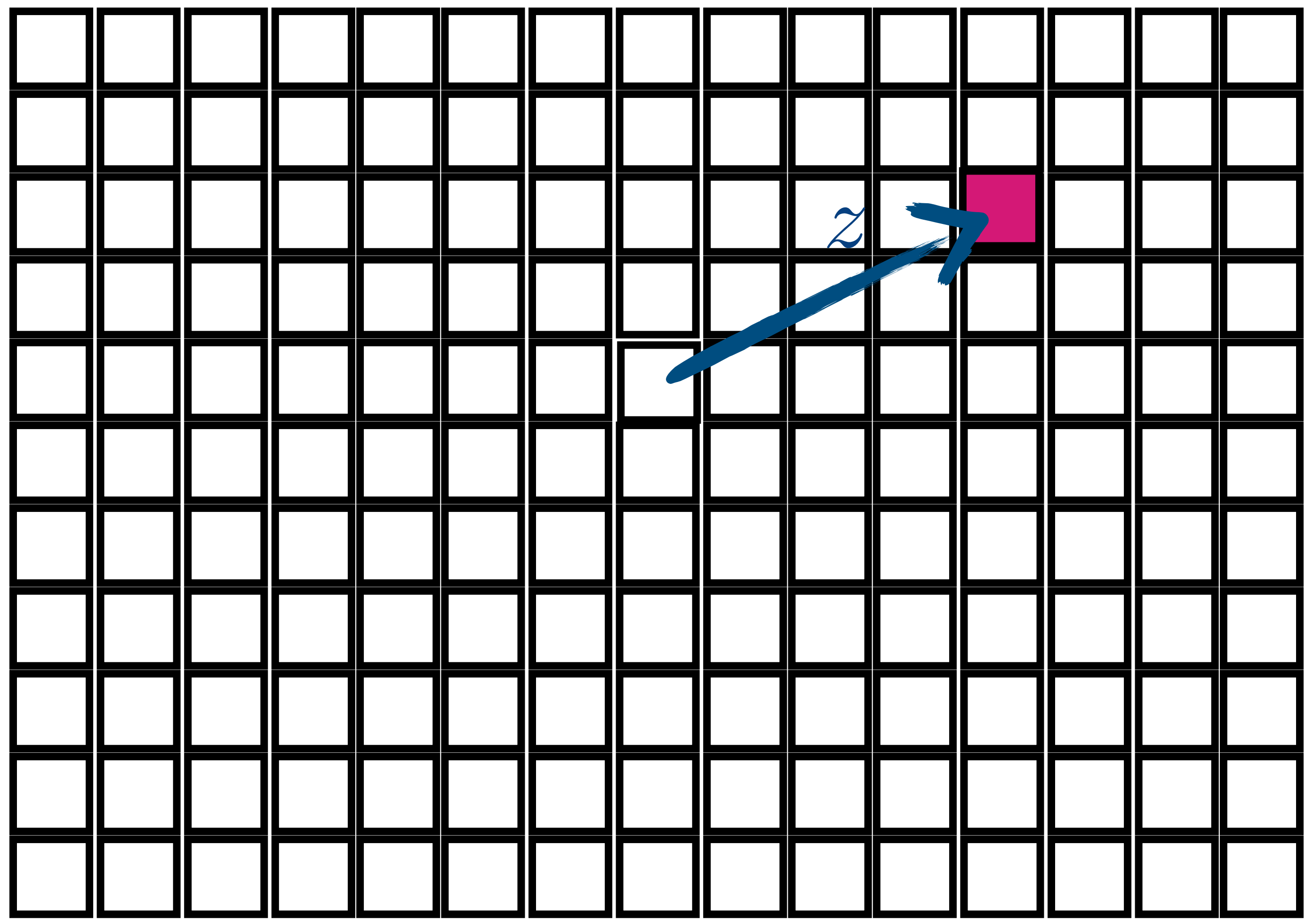

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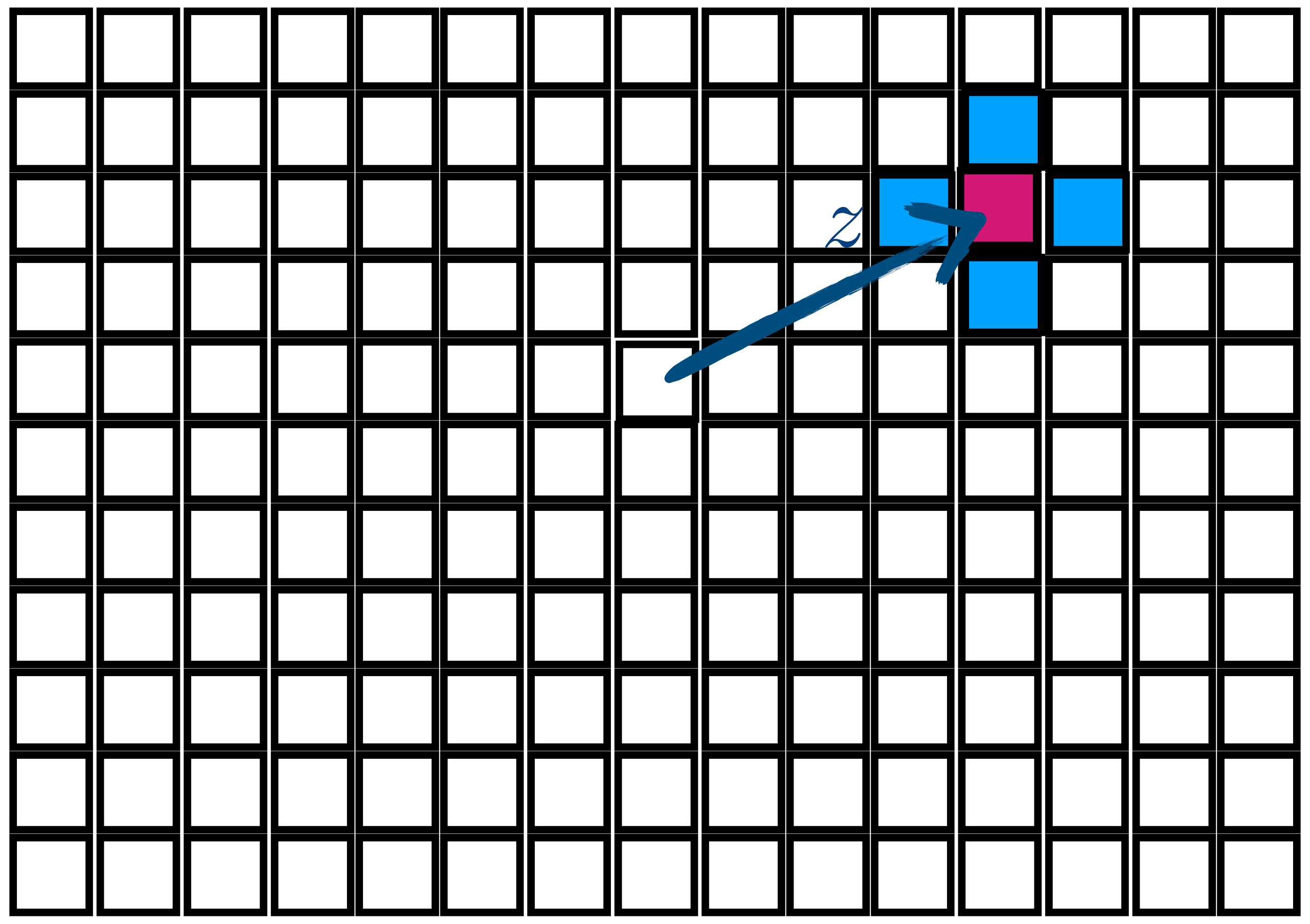

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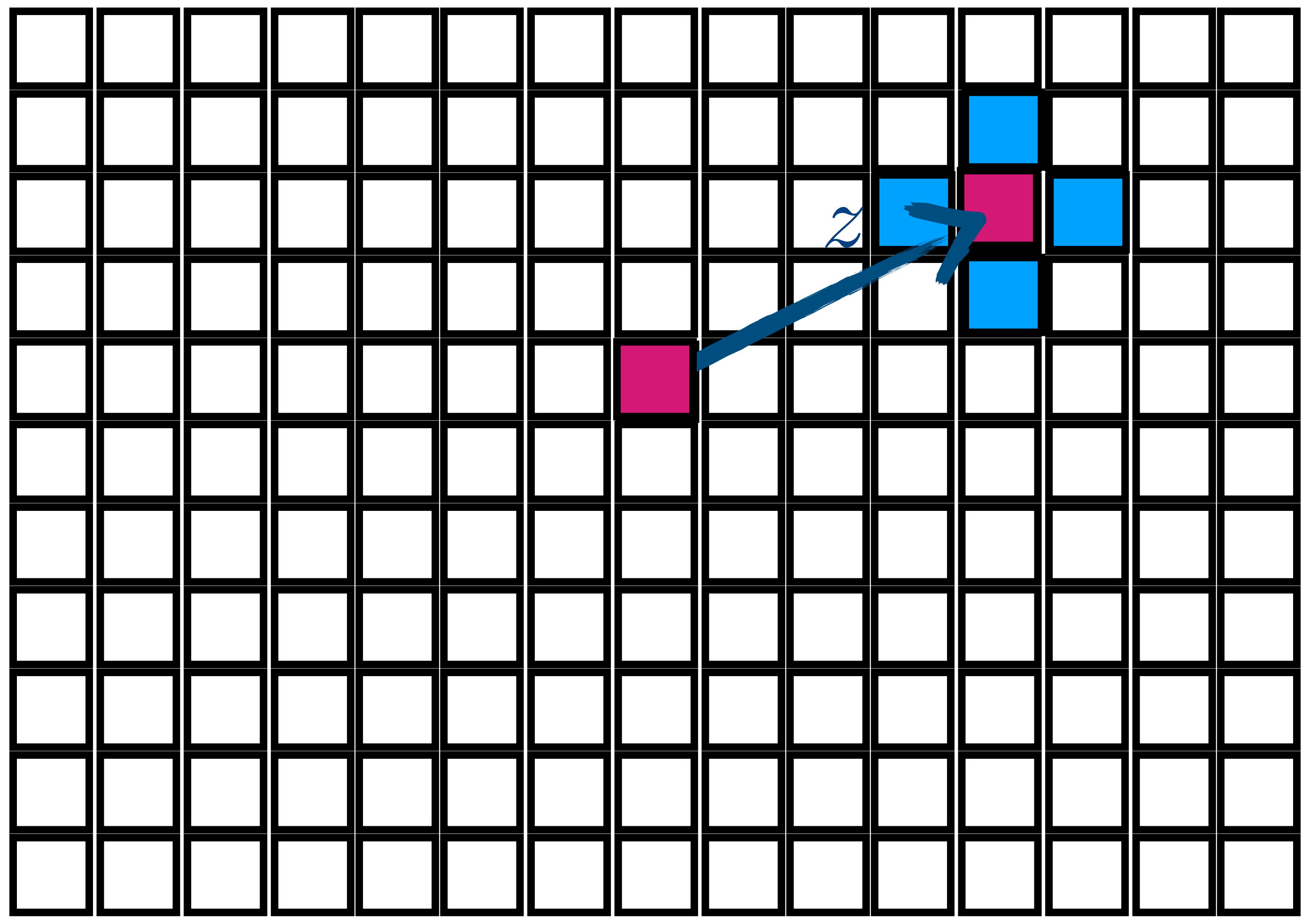

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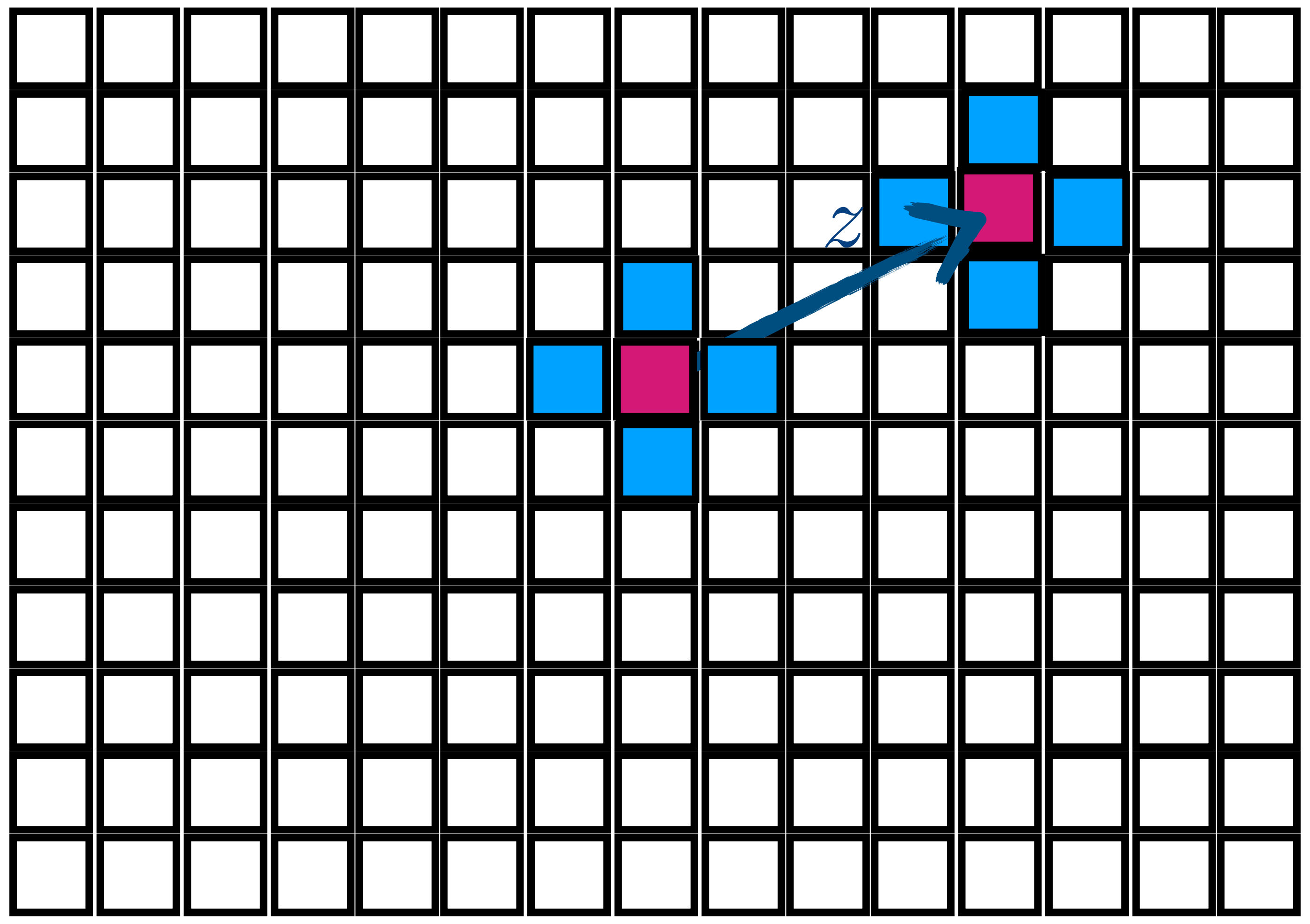

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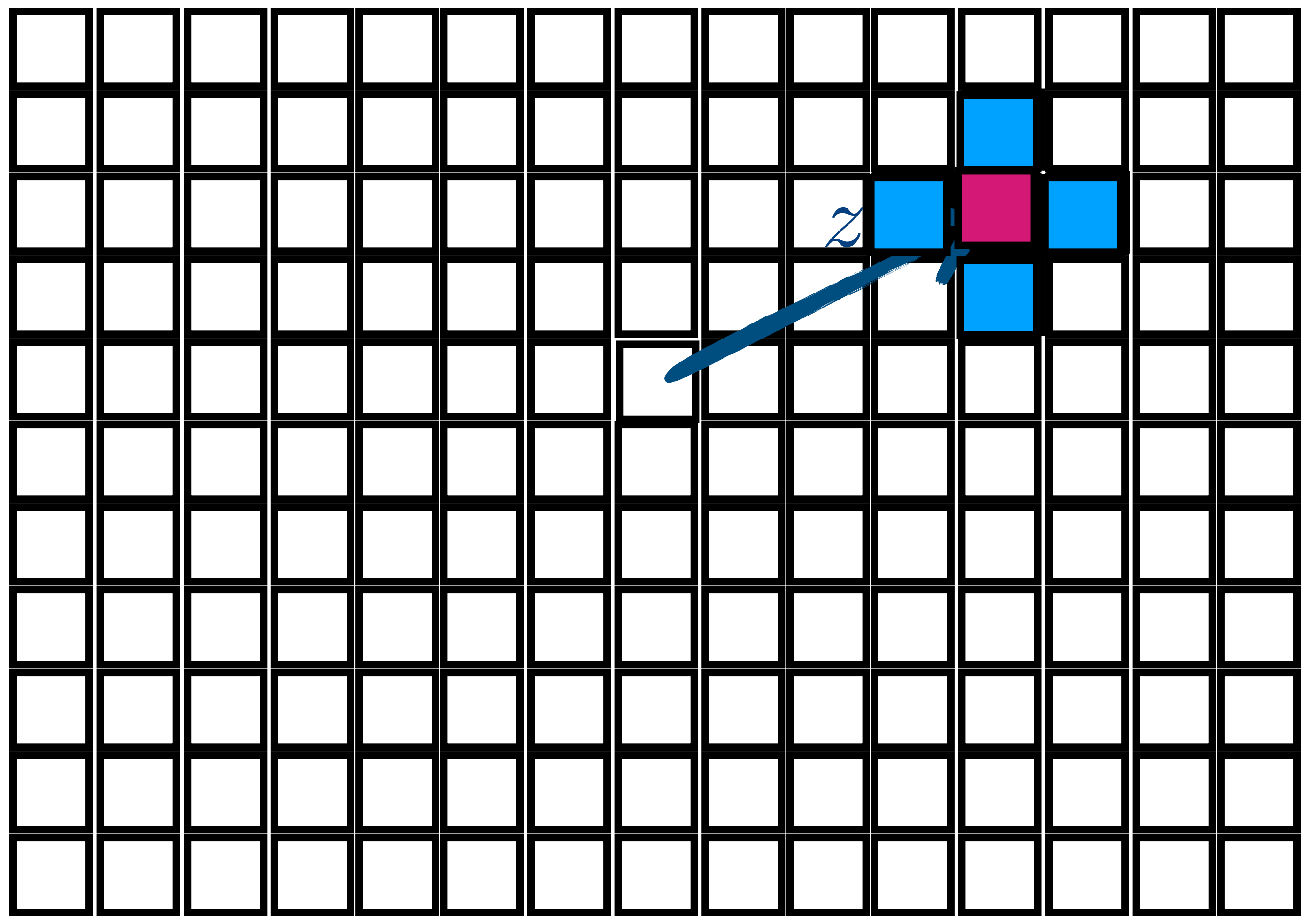

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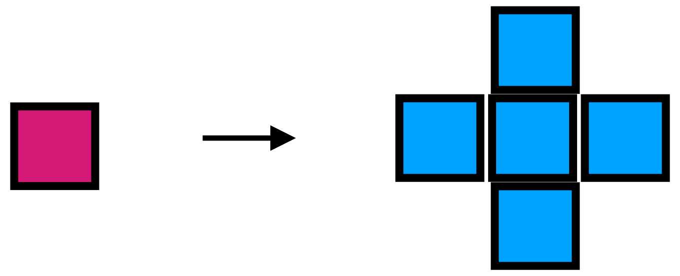
Local update rule

Consequence of the definition

- The QCA induces a local rule $\mathcal{V}_x : \mathsf{A}_x \to \mathsf{A}_{N^+(x)}$

$$\mathcal{V}_x(B_x) := \mathcal{V}(B_x)$$



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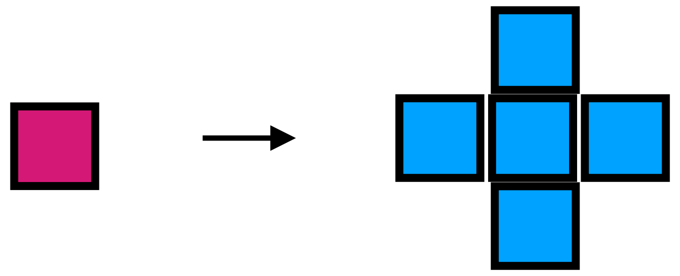

- The local rule at x=0 completely specifies the QCA

$$\begin{aligned}\mathcal{V}(B_x) &= \mathcal{T}_x\{\mathcal{V}[\mathcal{T}_x^{-1}(B_x)]\} \\ &= \mathcal{T}_x\{\mathcal{V}_0[\mathcal{T}_x^{-1}(B_x)]\}\end{aligned}$$

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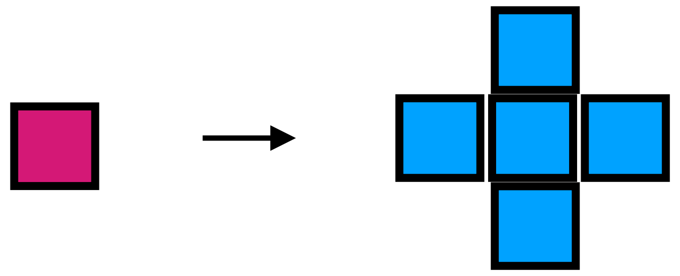

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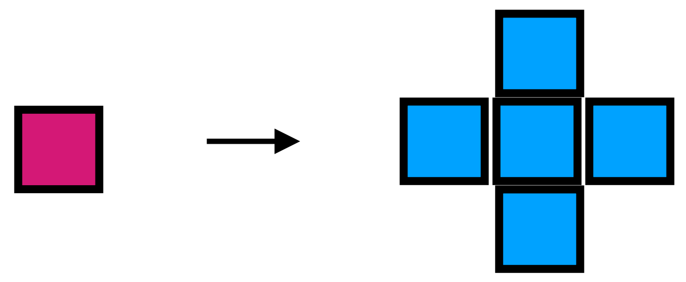

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- The image of every local effect can be obtained using $\mathcal{V}_0$ since $B_R = \sum_i \bigotimes_{x \in R} B_x^{(i)}$

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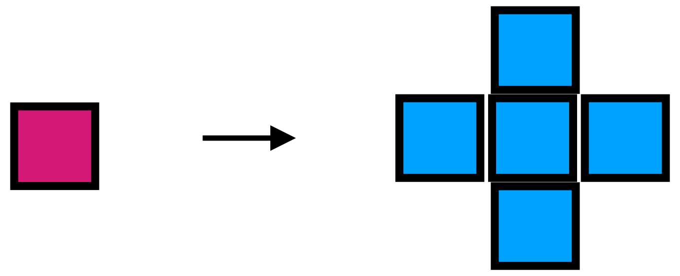

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  - The image of every local effect can be obtained using $\mathcal{V}_0$ since $B_R = \sum_i \bigotimes_{x \in R} B_x^{(i)}$
  - Boundedness: image of limits (of the quasi-local algebra) is determined

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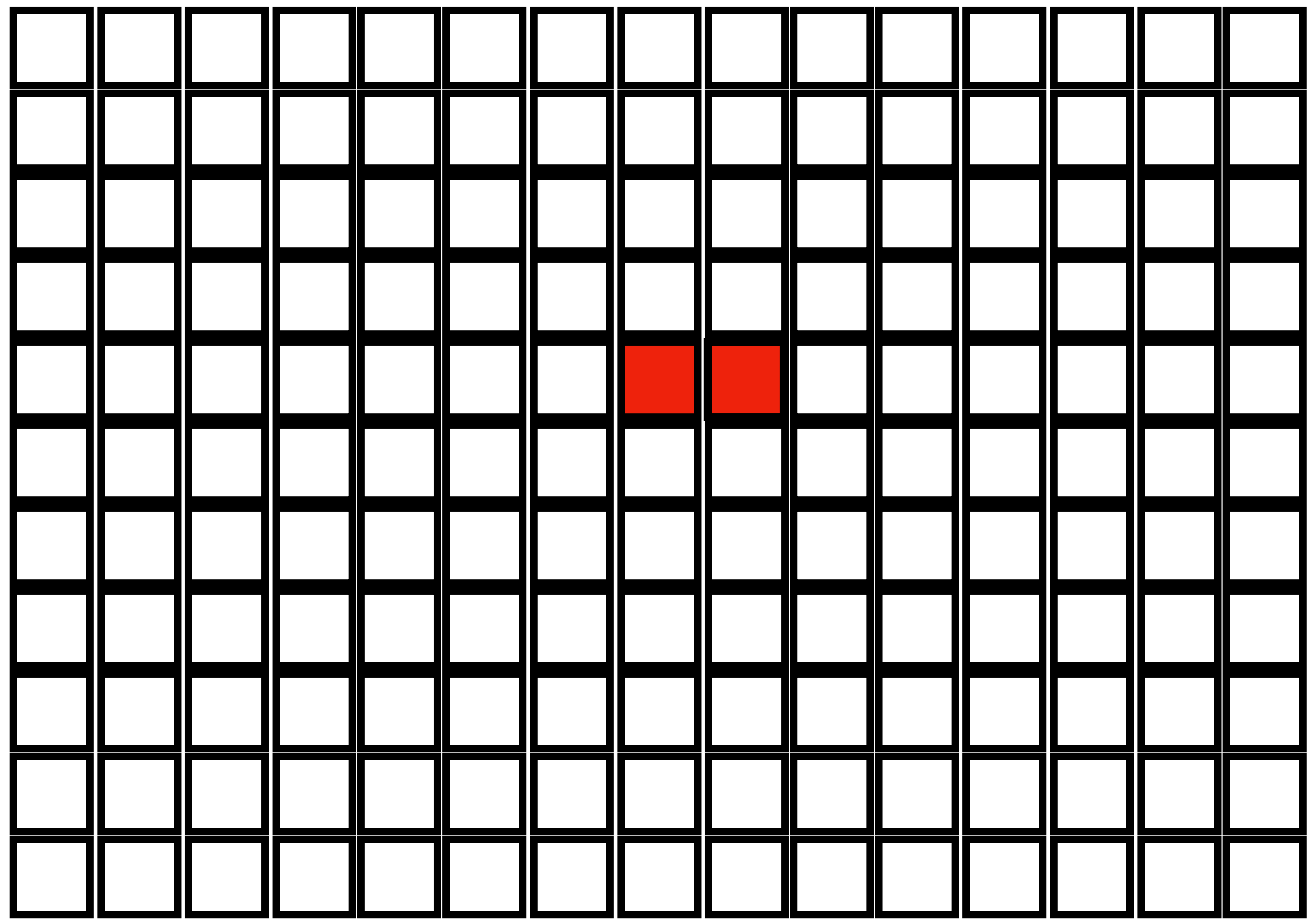
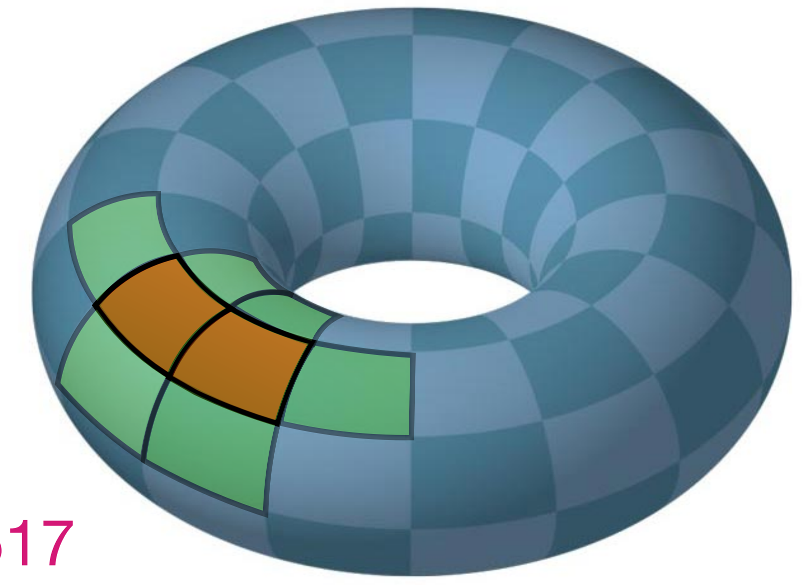
Wrapping lemma

Reduces the local rule to that of a finite QCA

- Local algebras commute and so must do their images under $\mathcal{V}$





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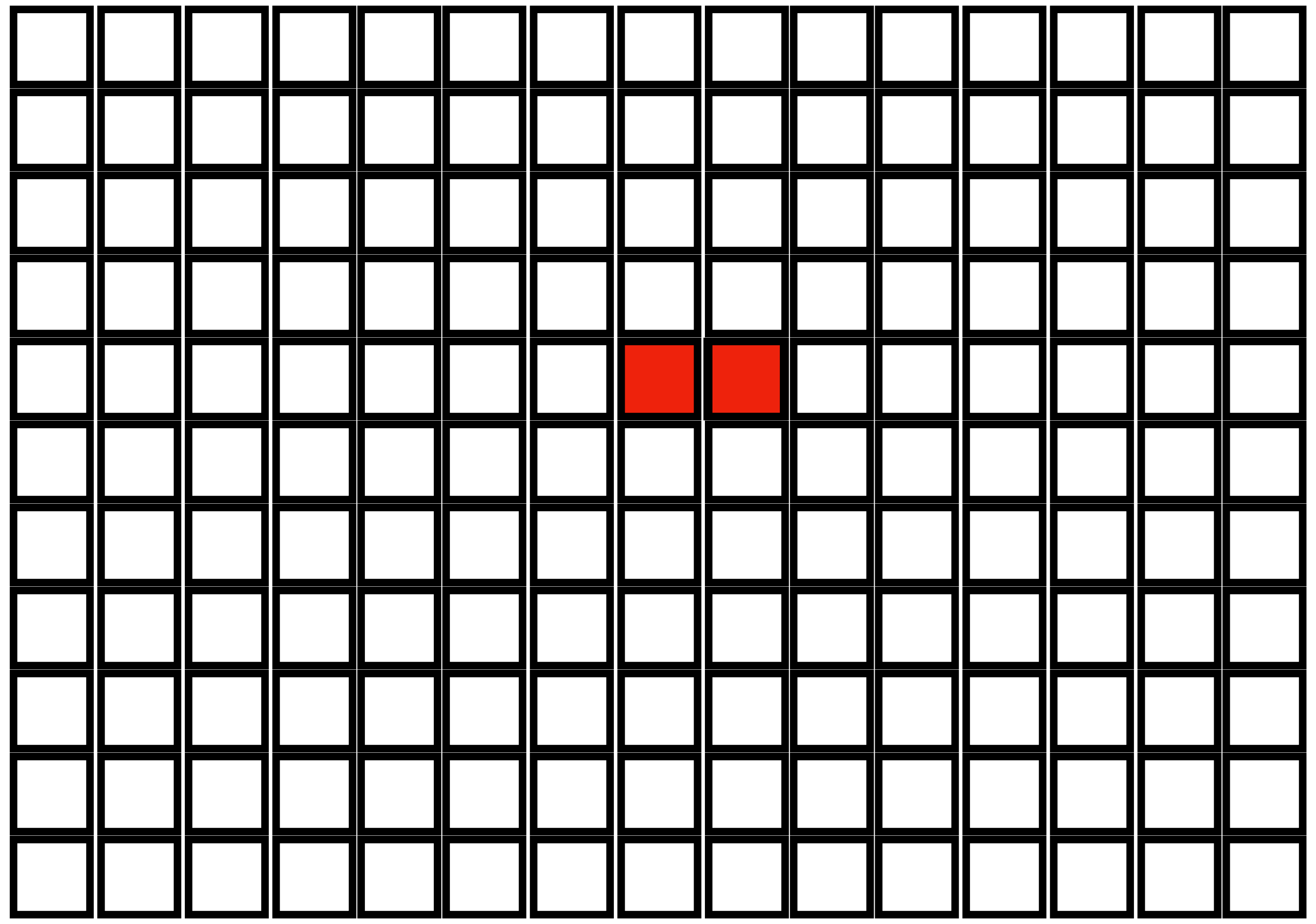

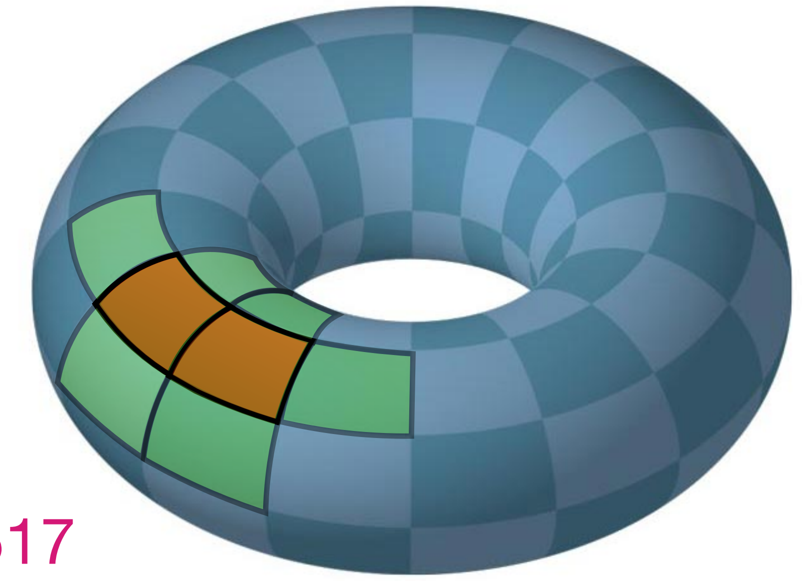

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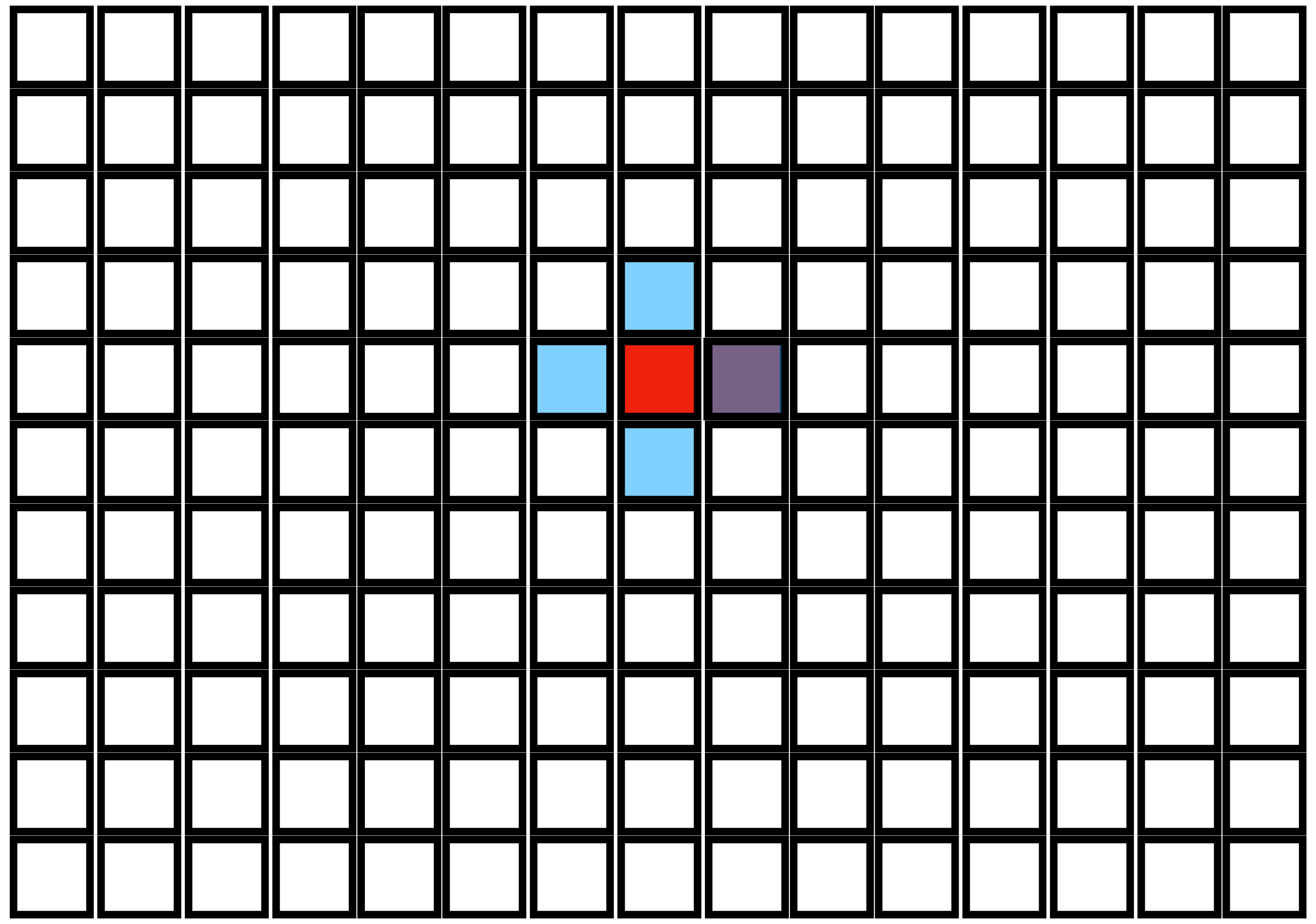

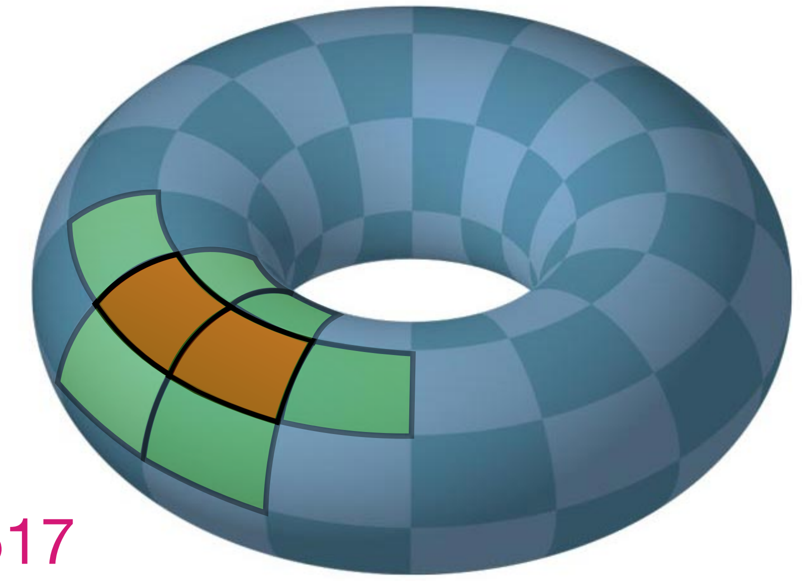

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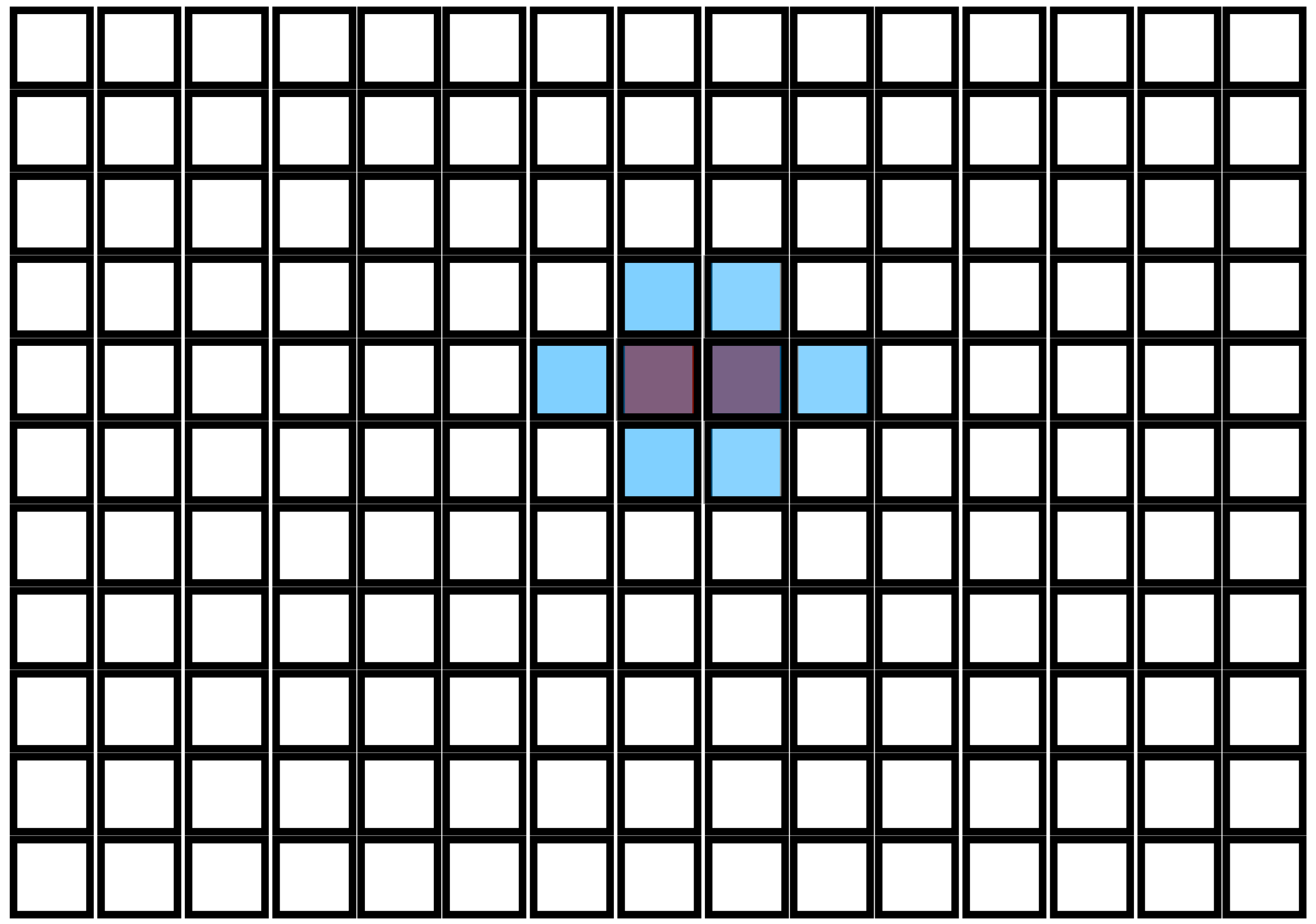

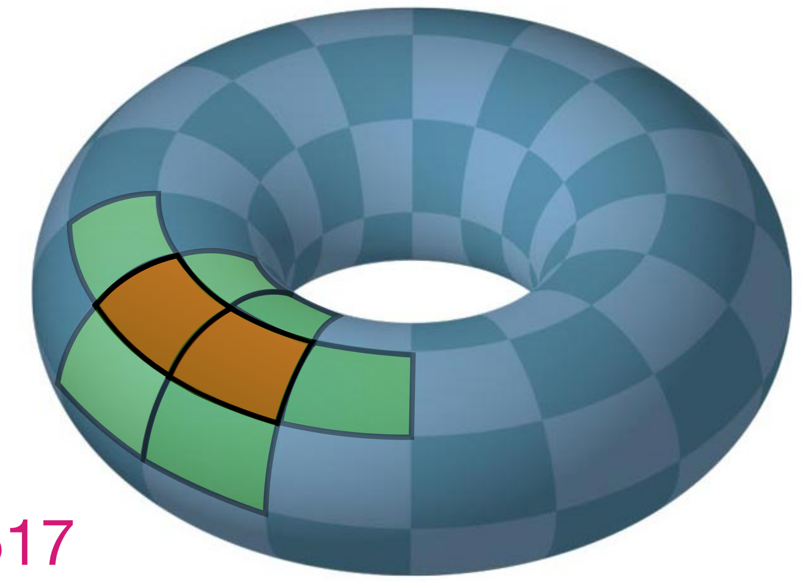

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- Also the converse is true

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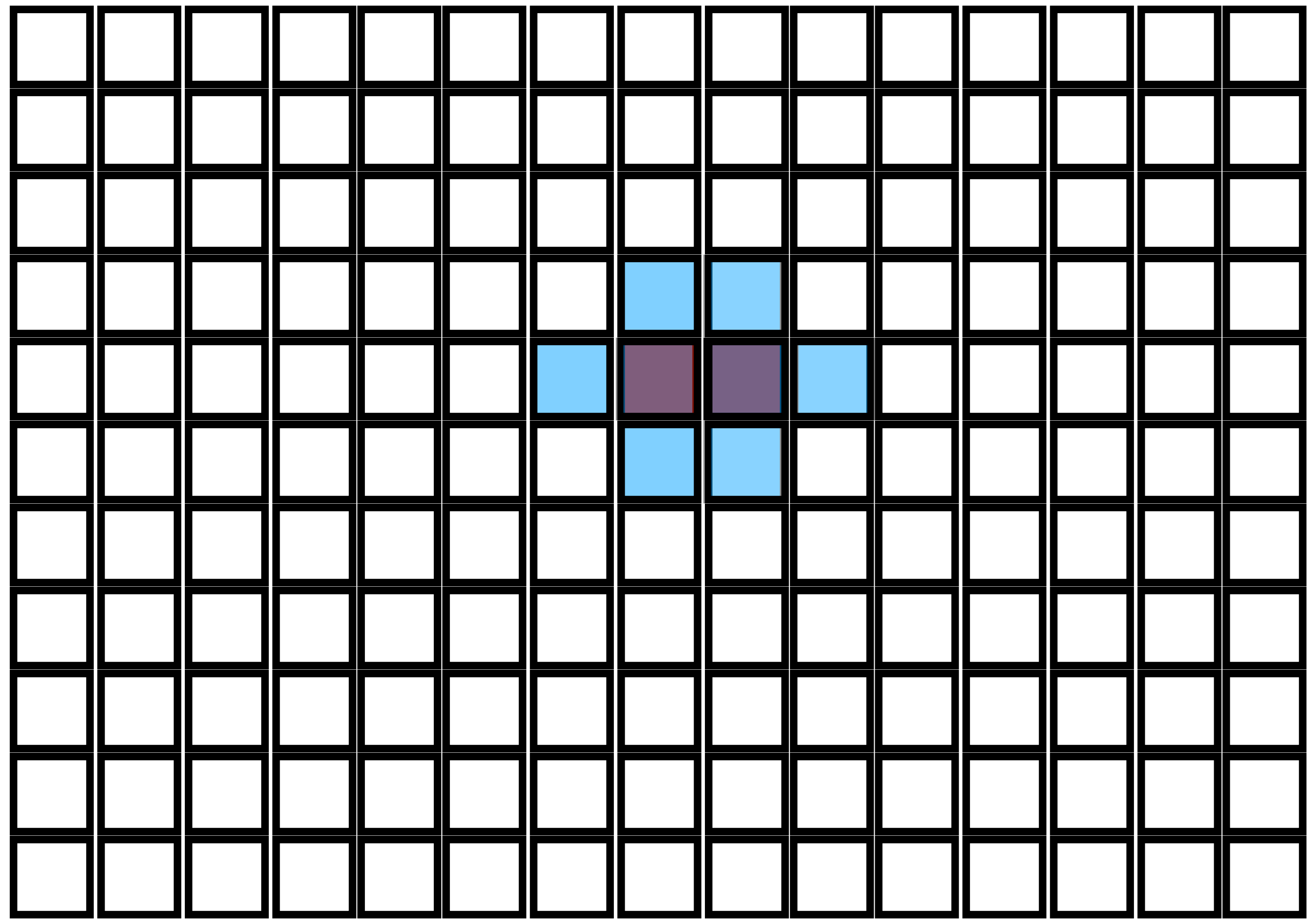

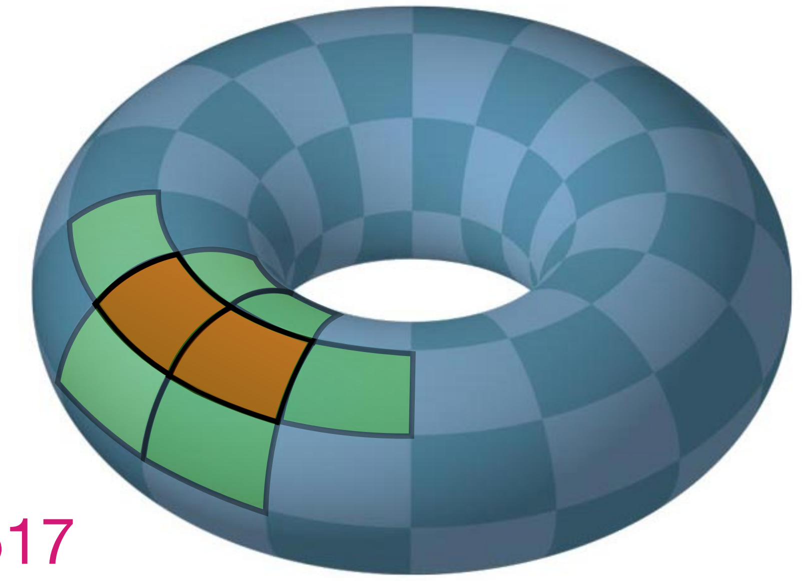

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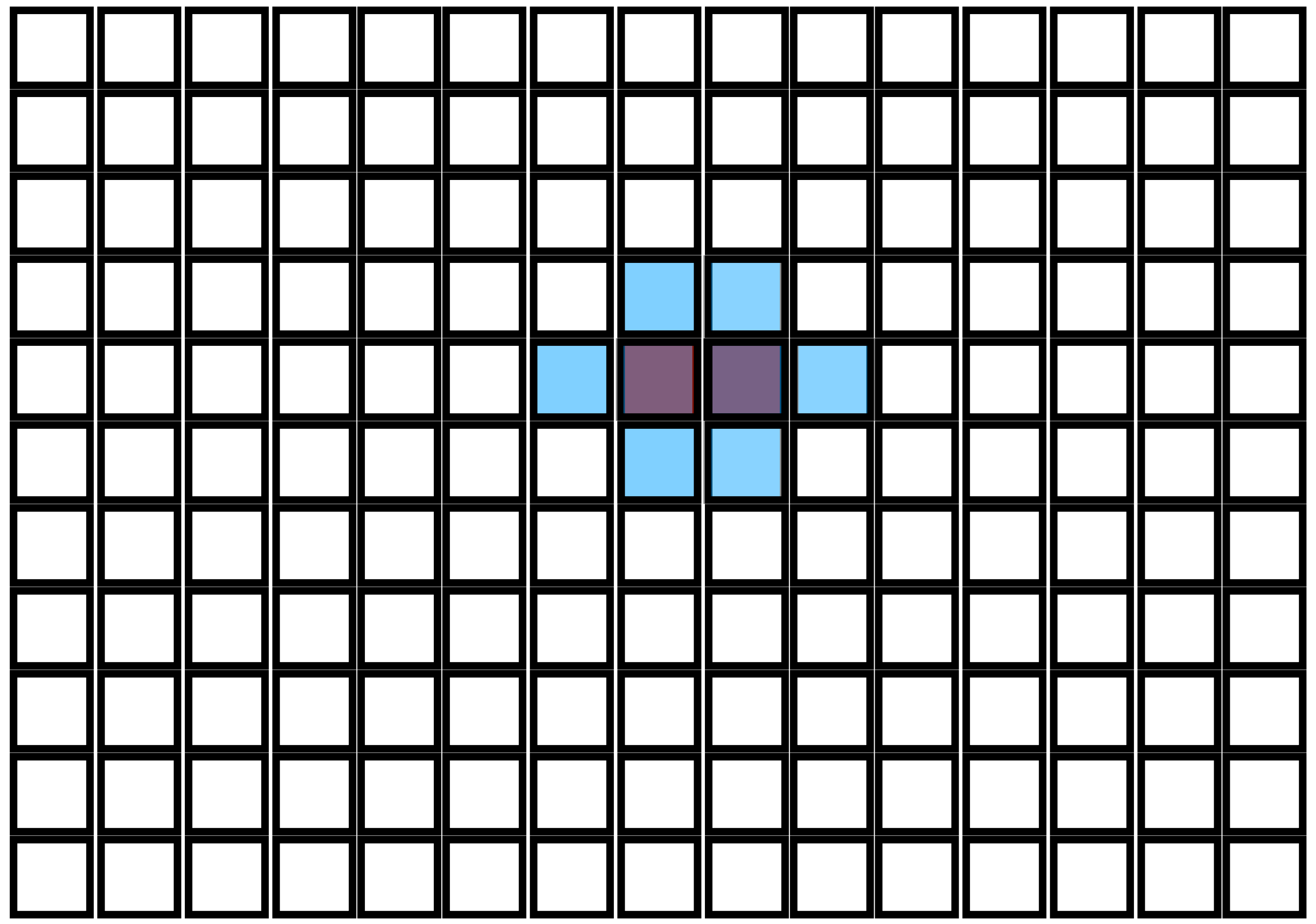
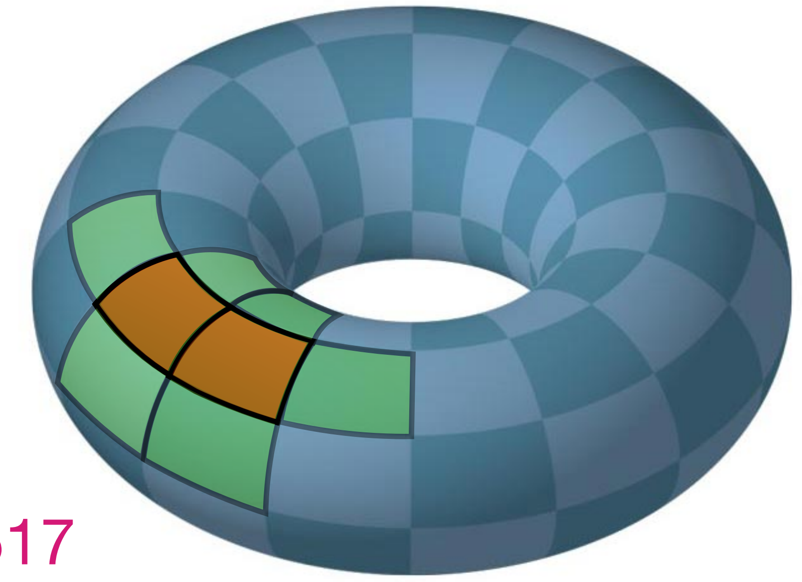


- The commutation condition is local



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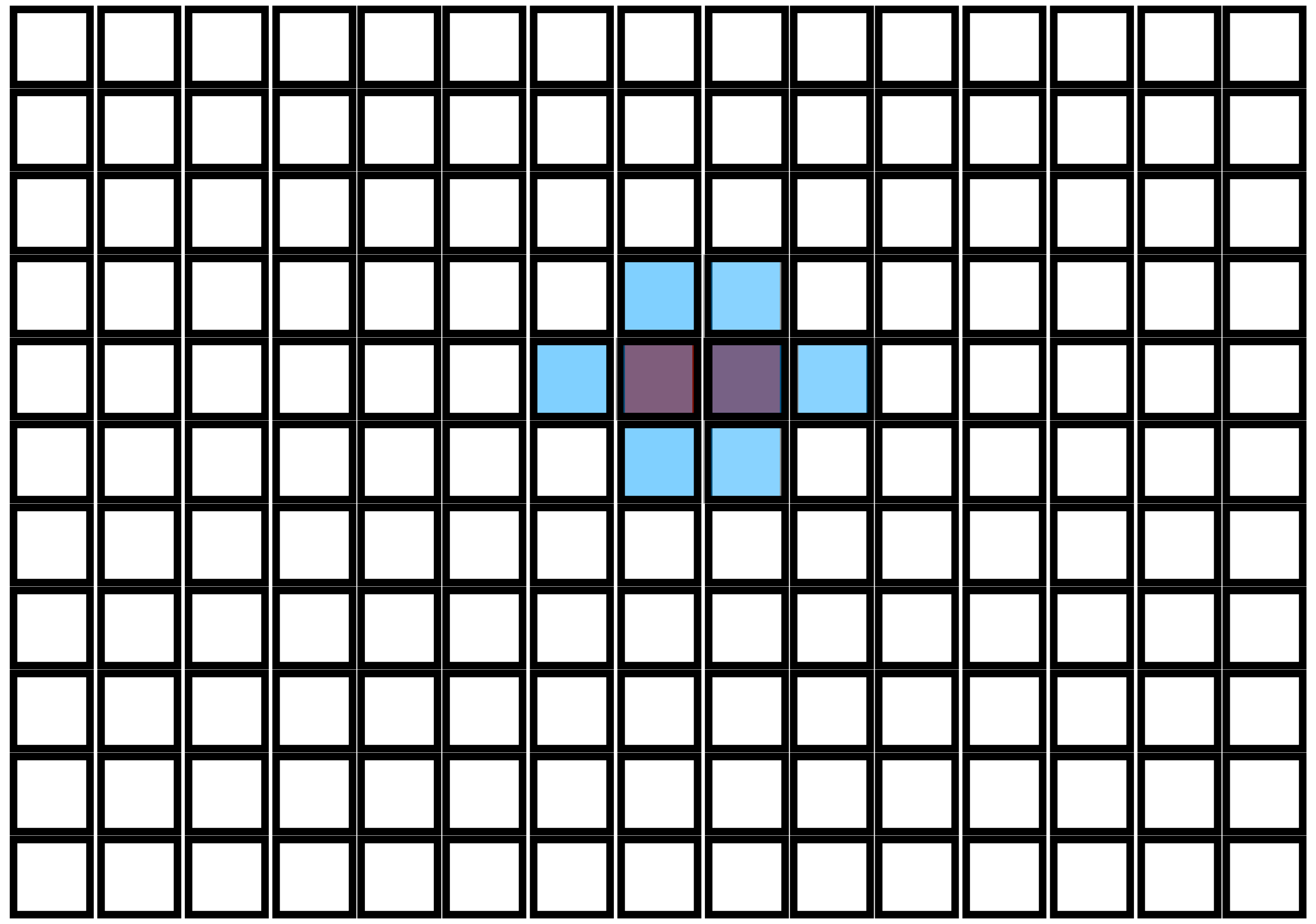
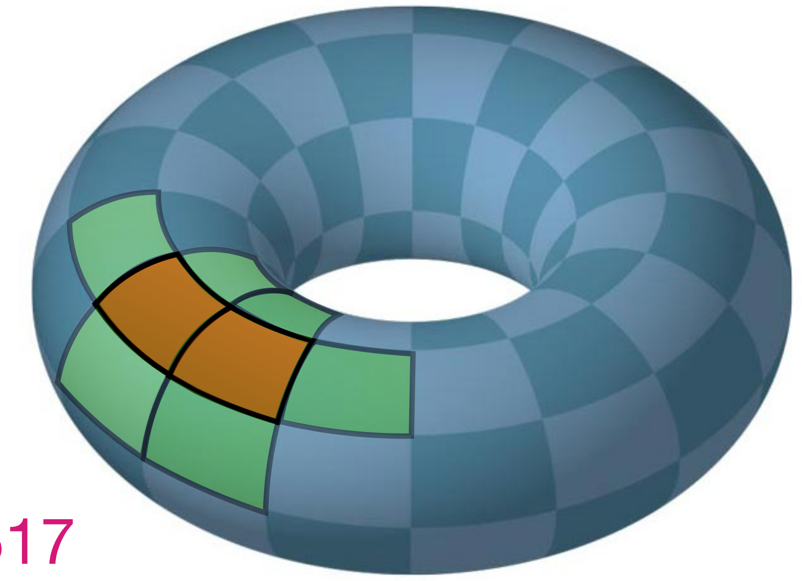


- The commutation condition is local
  - The same for infinite/finite QCA



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Wrapping lemma

Unitarity of QCAs

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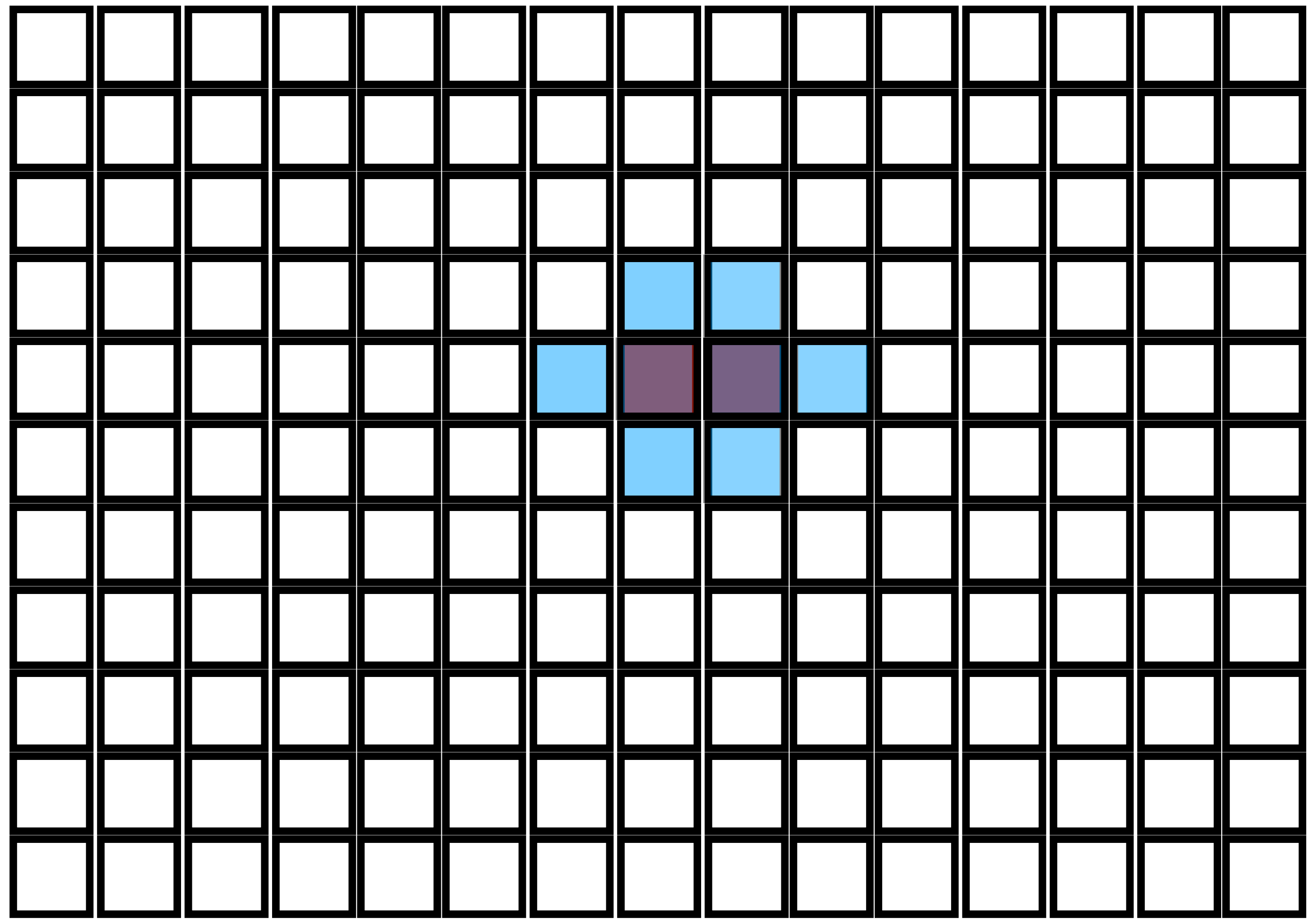

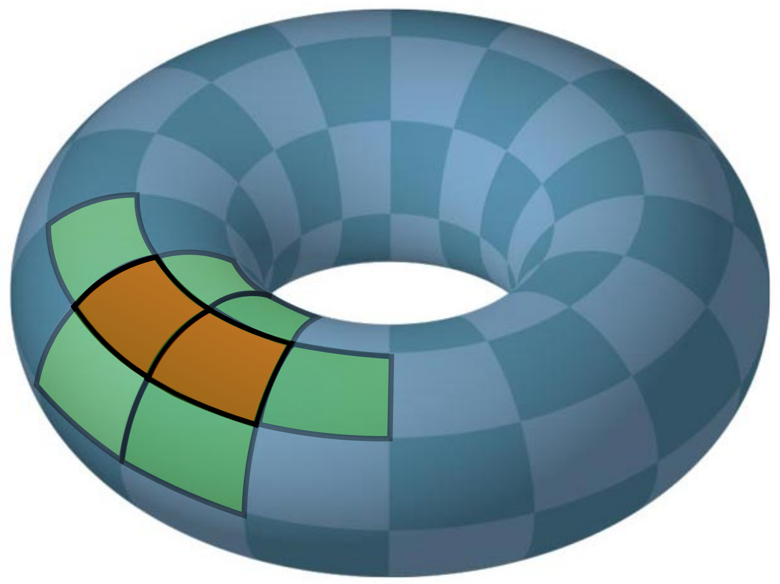

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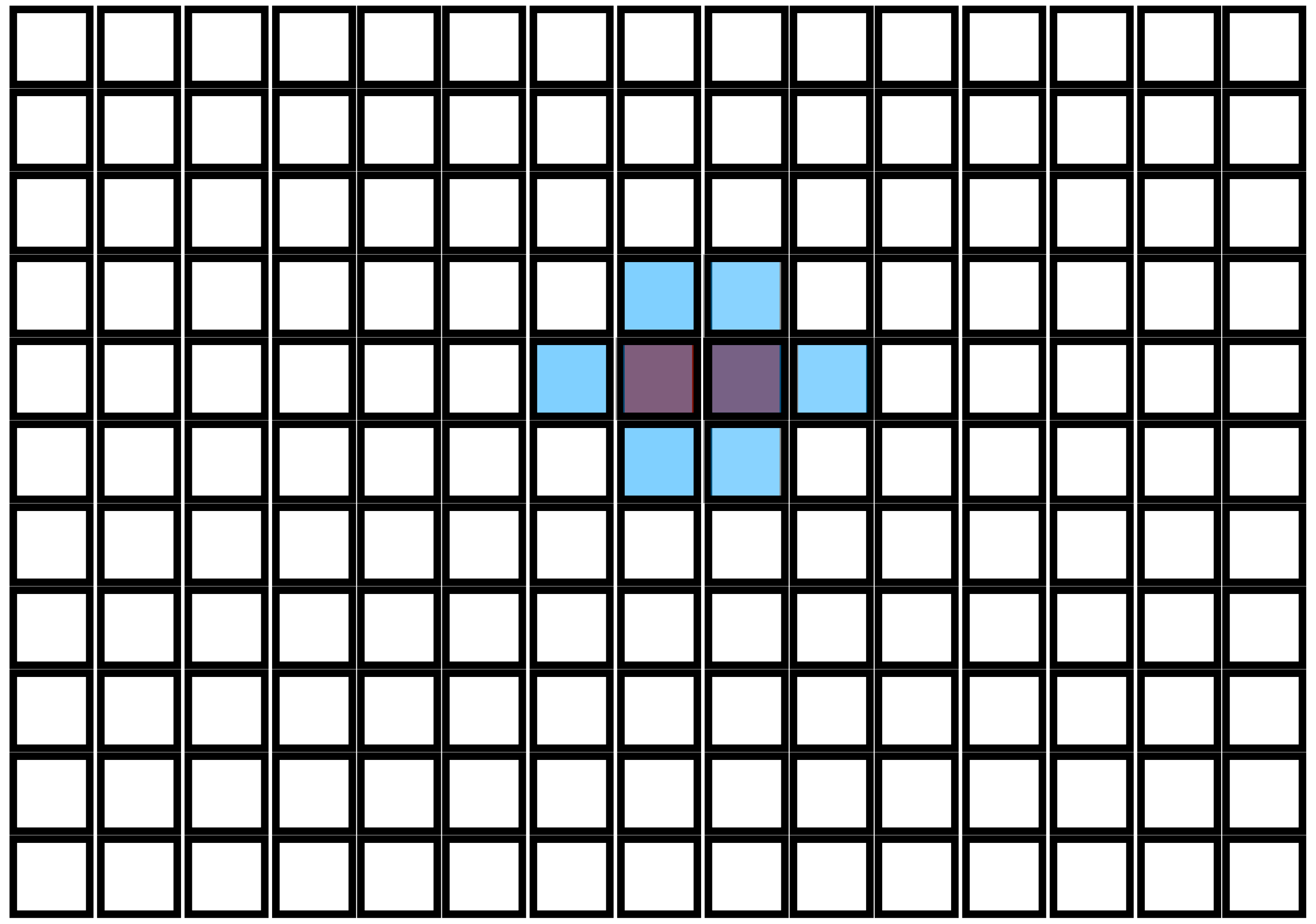
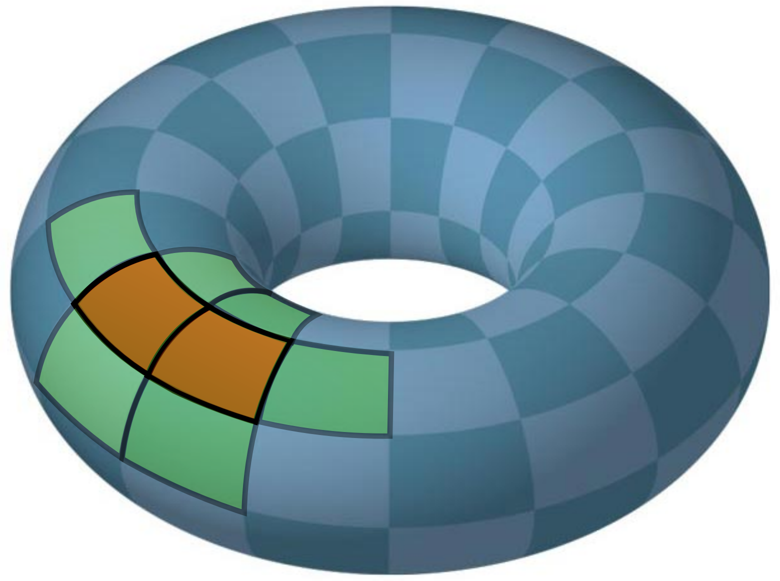
- By Stinespring theorem in the finite case

$$\mathcal{V}(B_R) = U(I \otimes B_R)U^\dagger$$
$$= \sum_i \prod_{x \in R} U_x(I \otimes B_x^{(i)})U_x^\dagger$$





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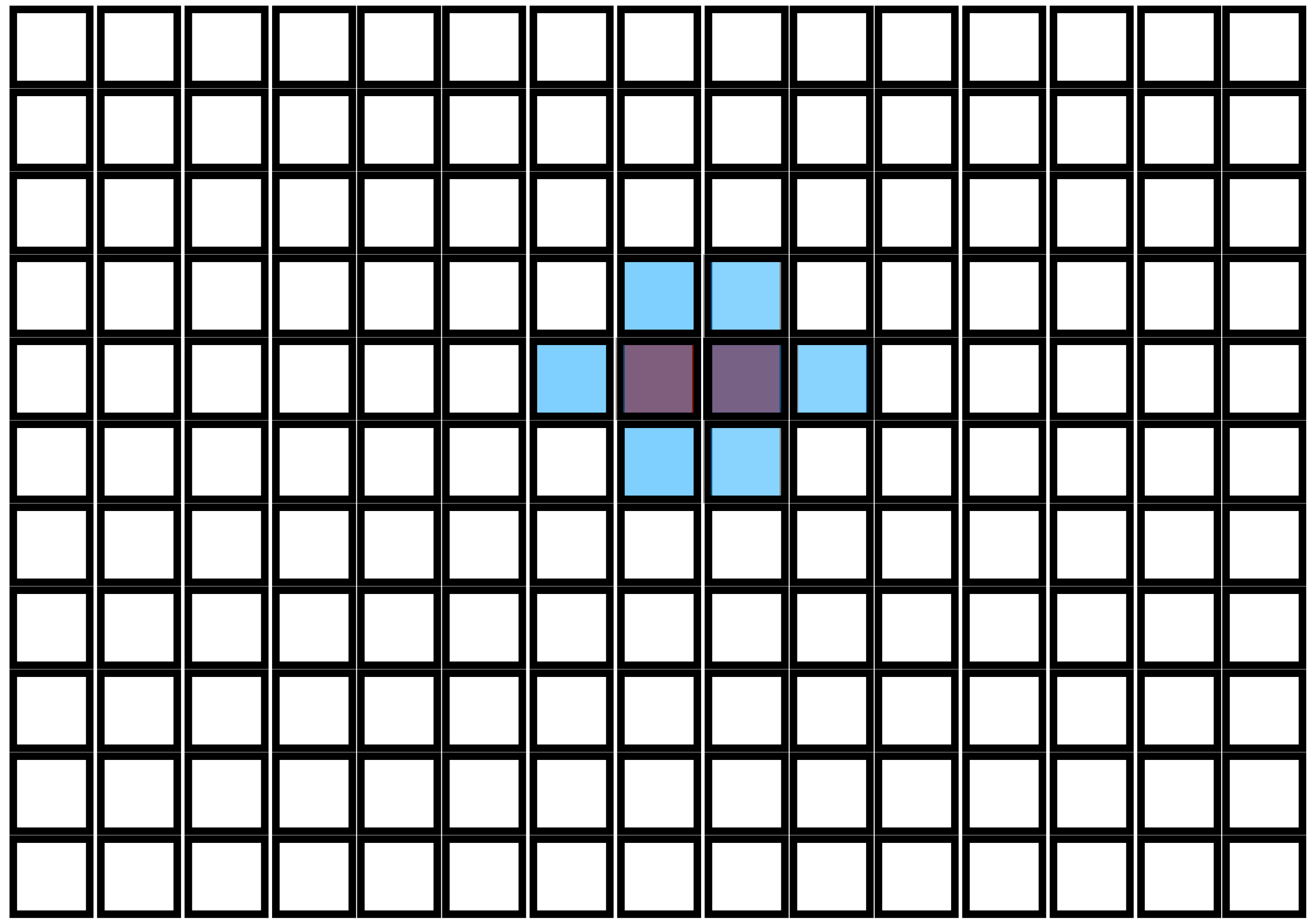

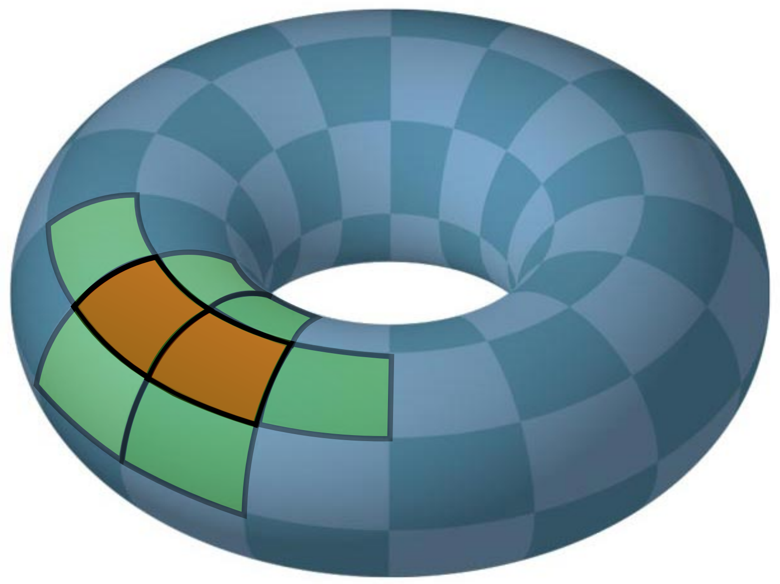

B. Schumacher and R.F. Werner, arXiv:quant-ph/040517

Wrapping lemma

Unitarity of QCAs

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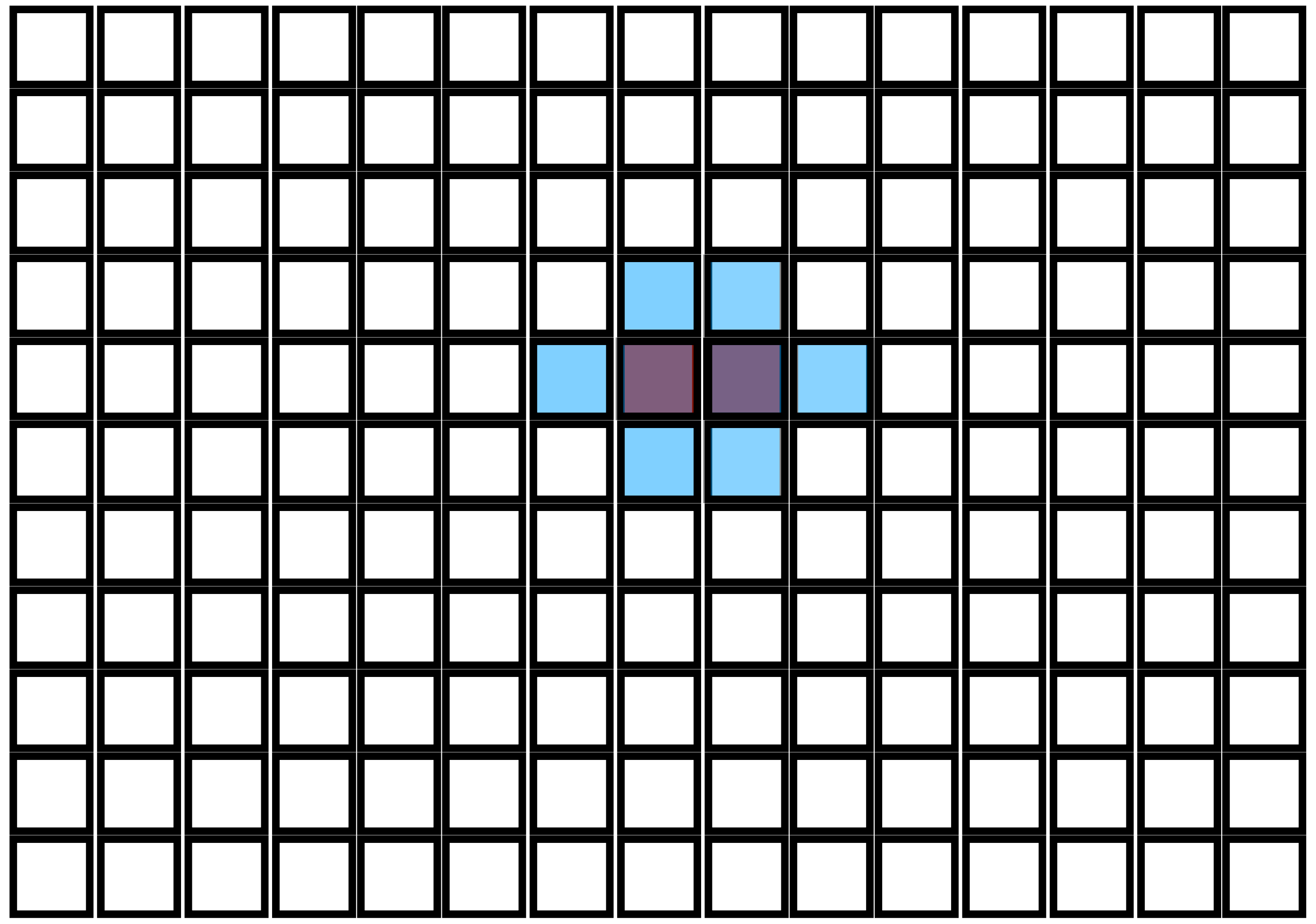

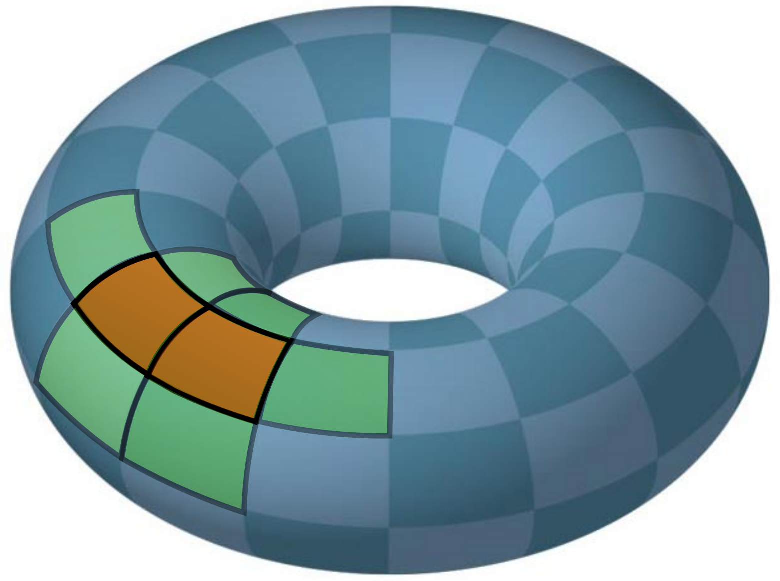

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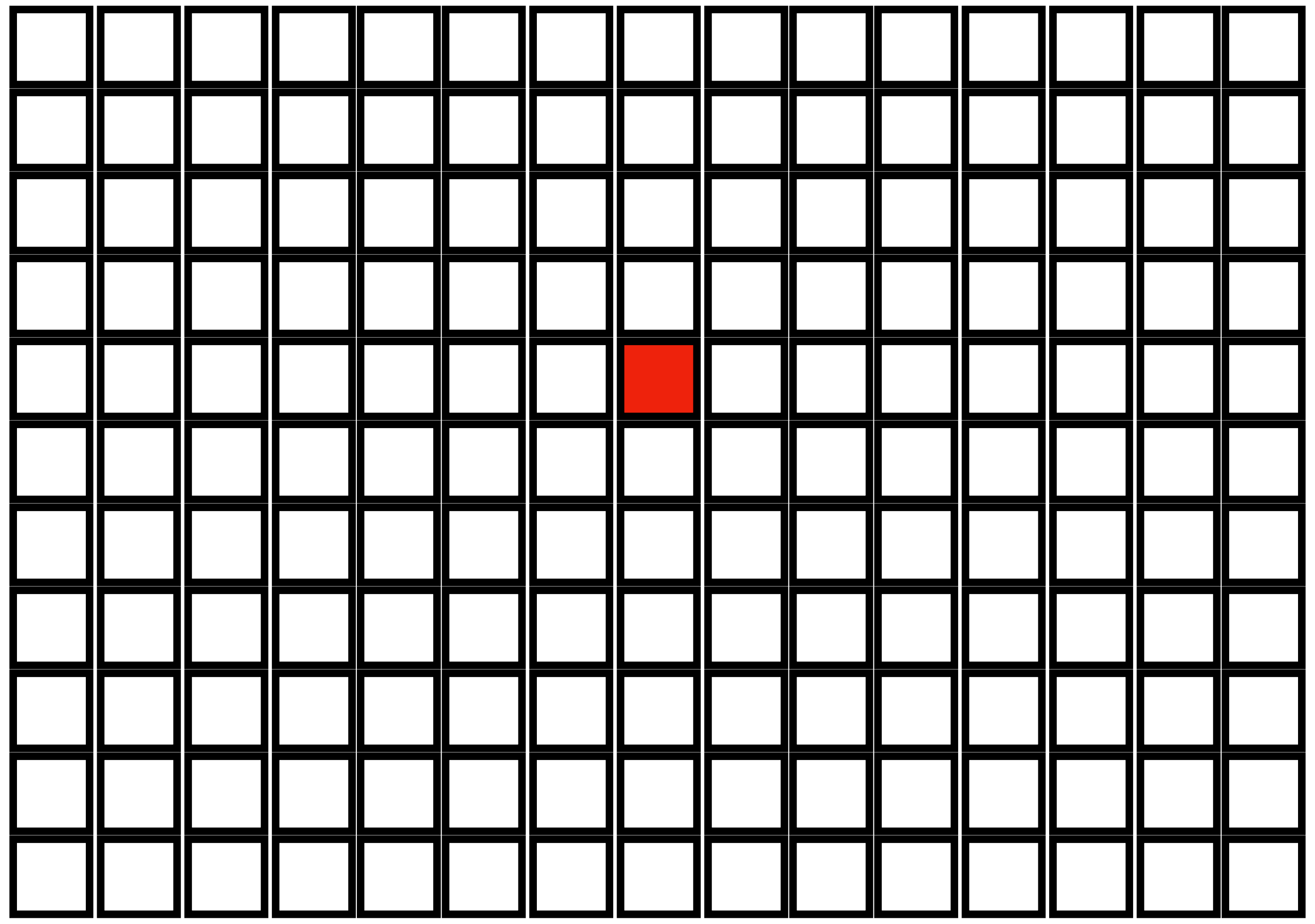
- The inverse of $\mathcal{V}$ on $\mathbb{Z}^d$ is a QCA itself



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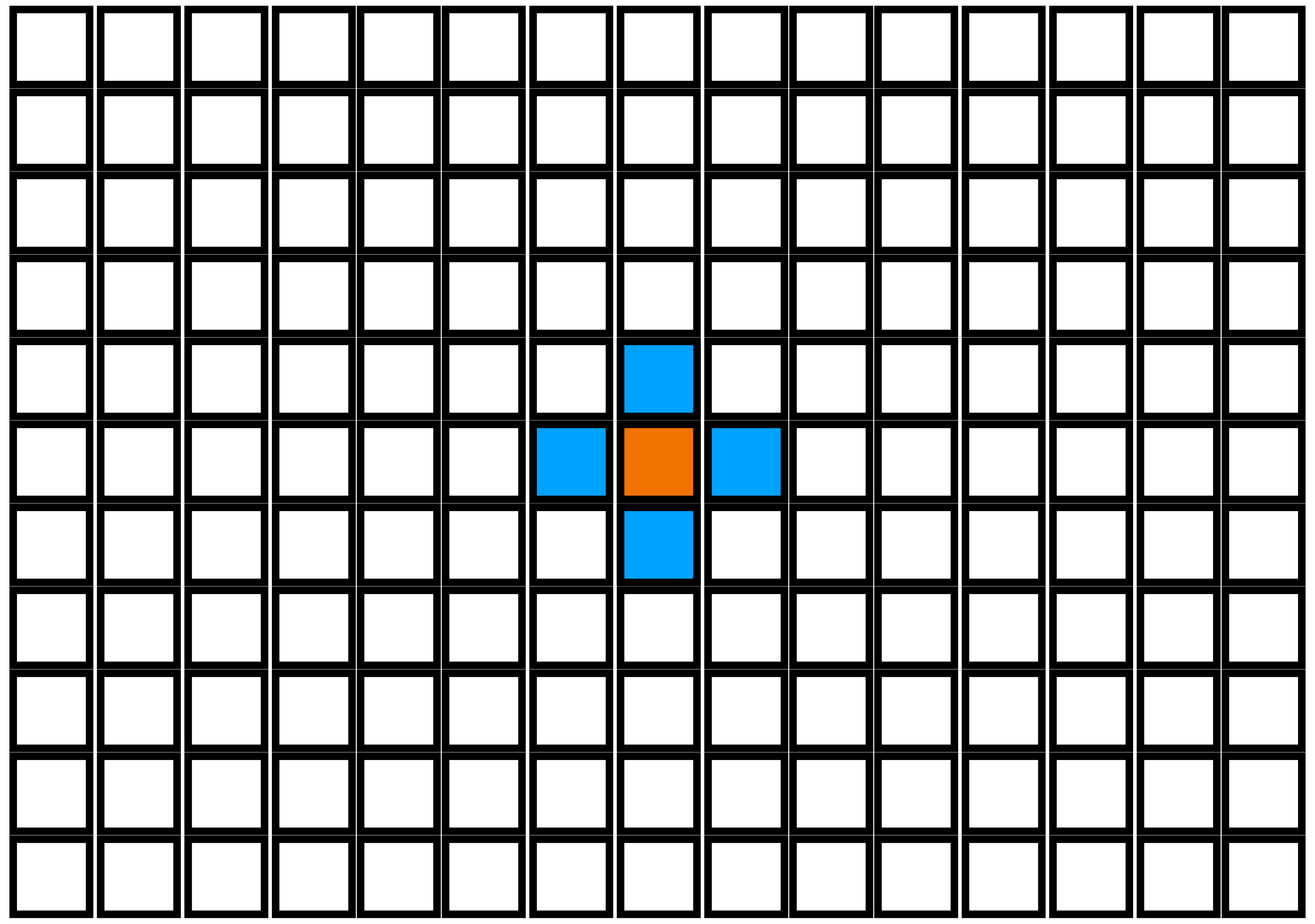

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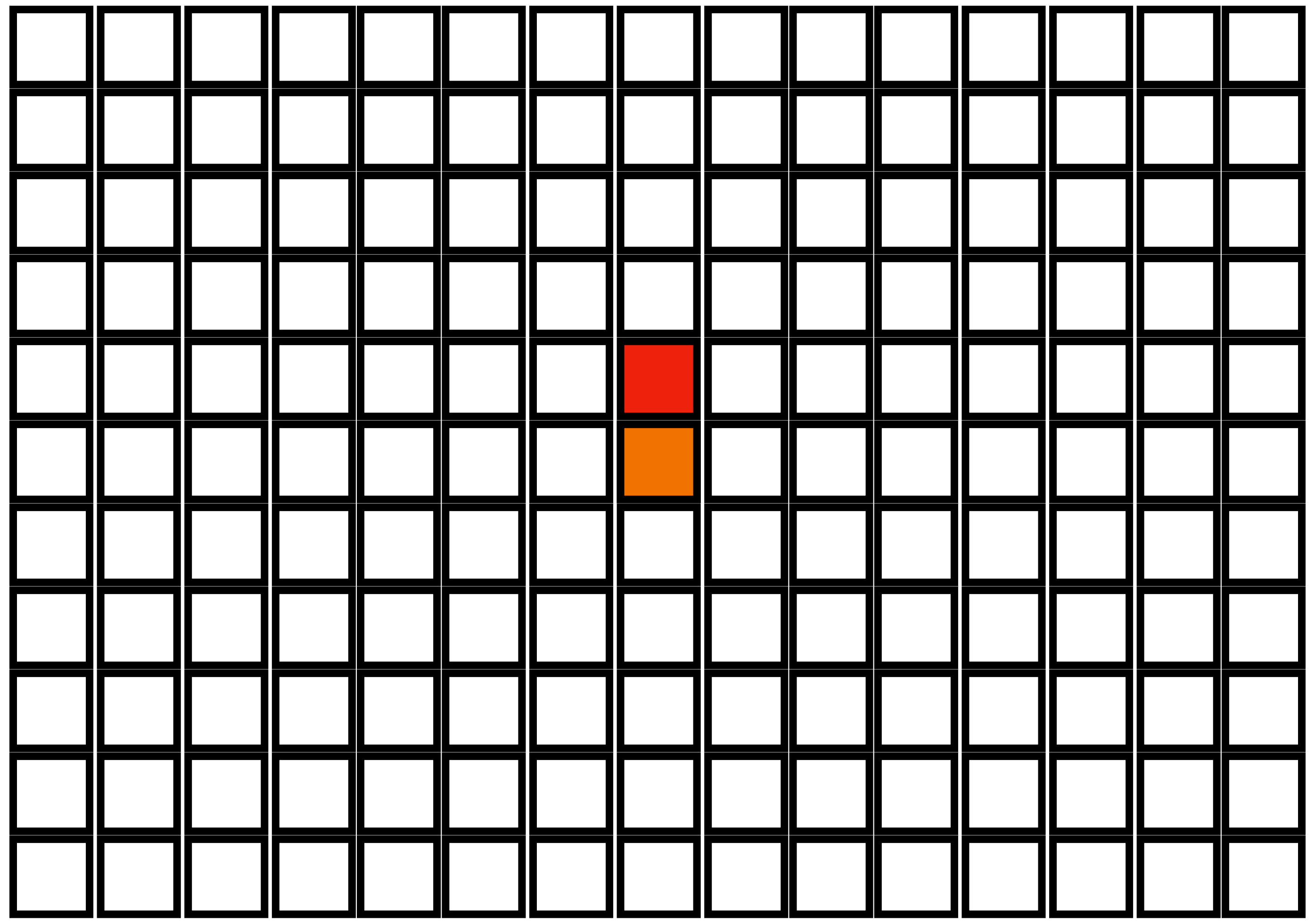

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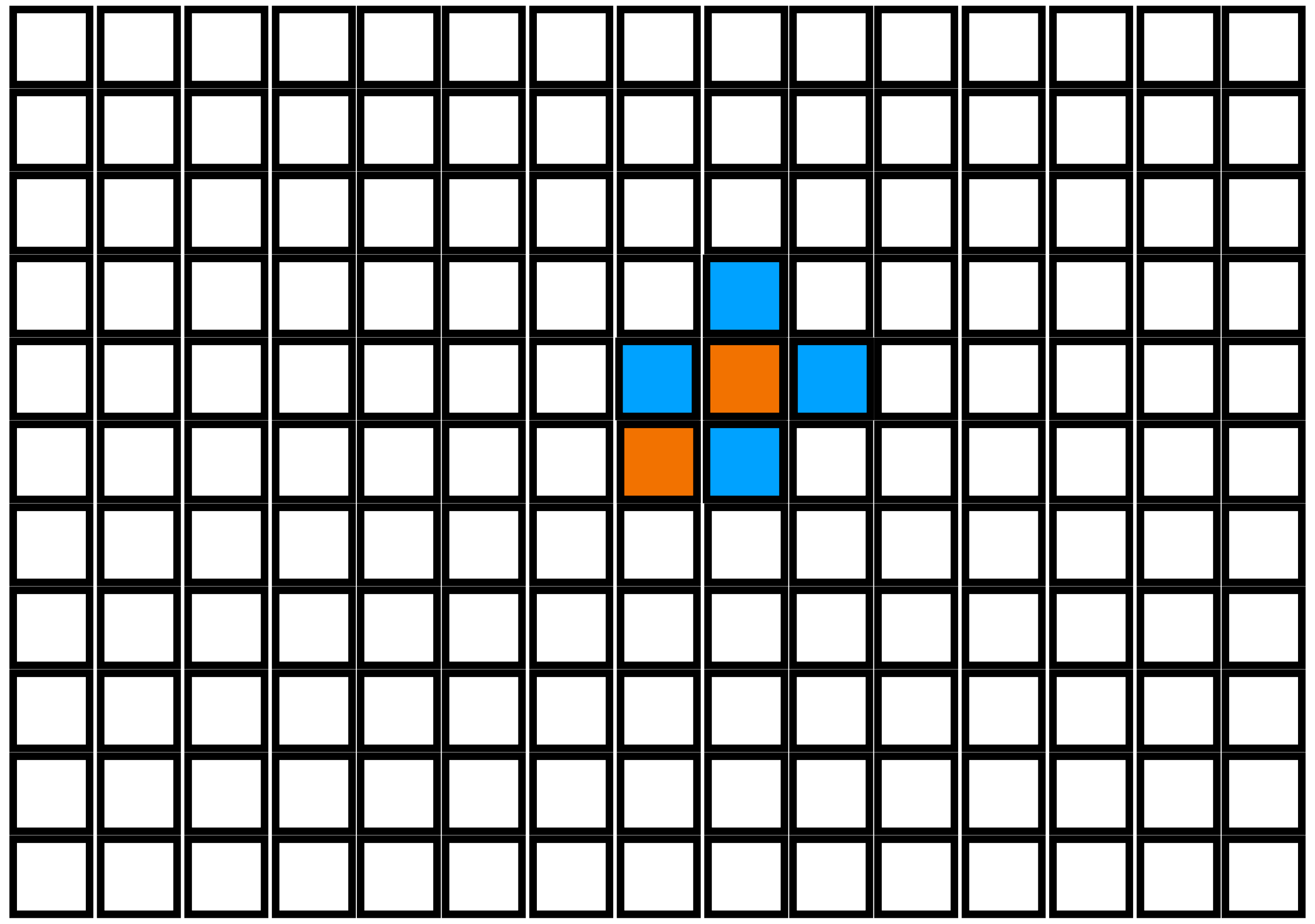

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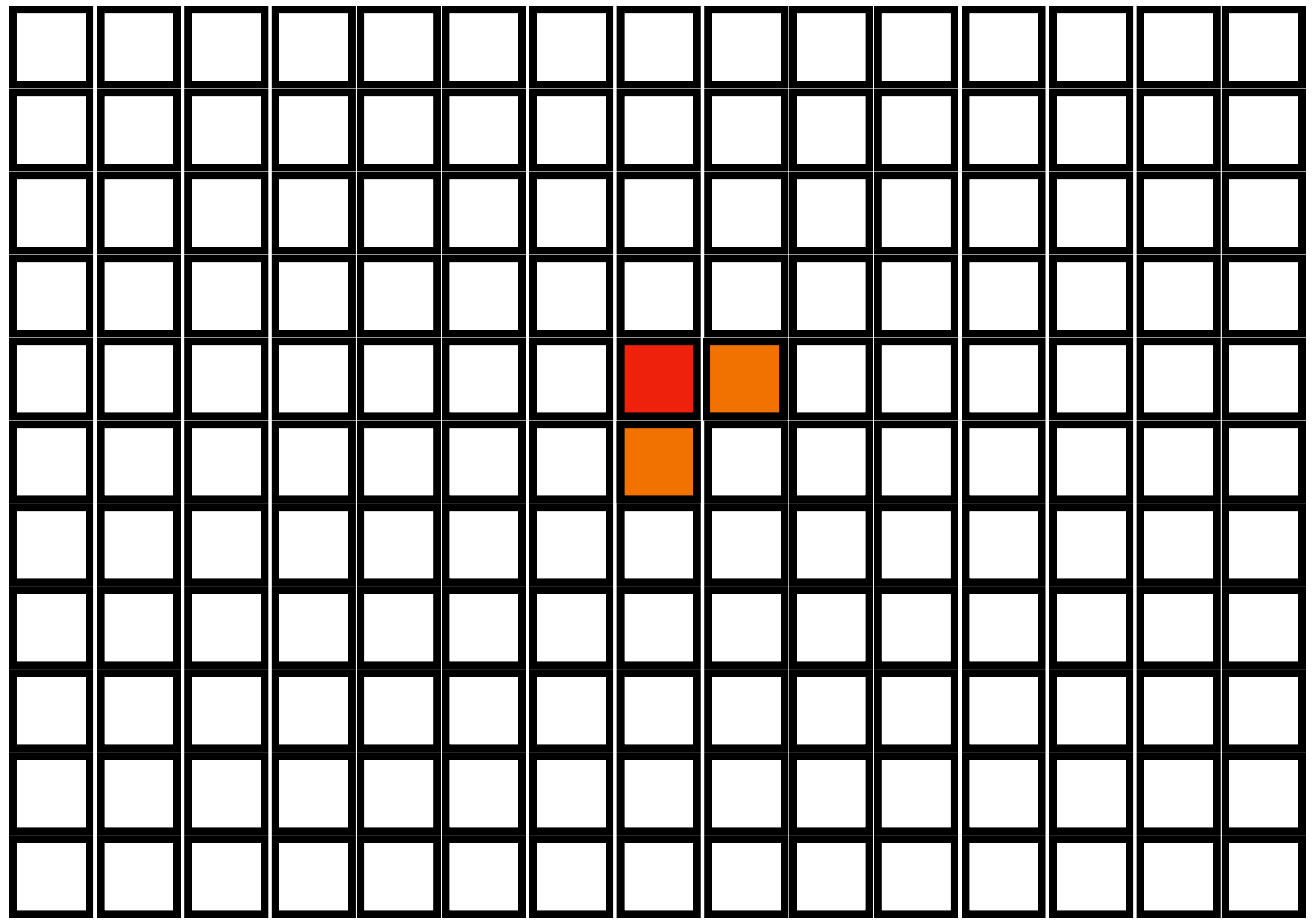

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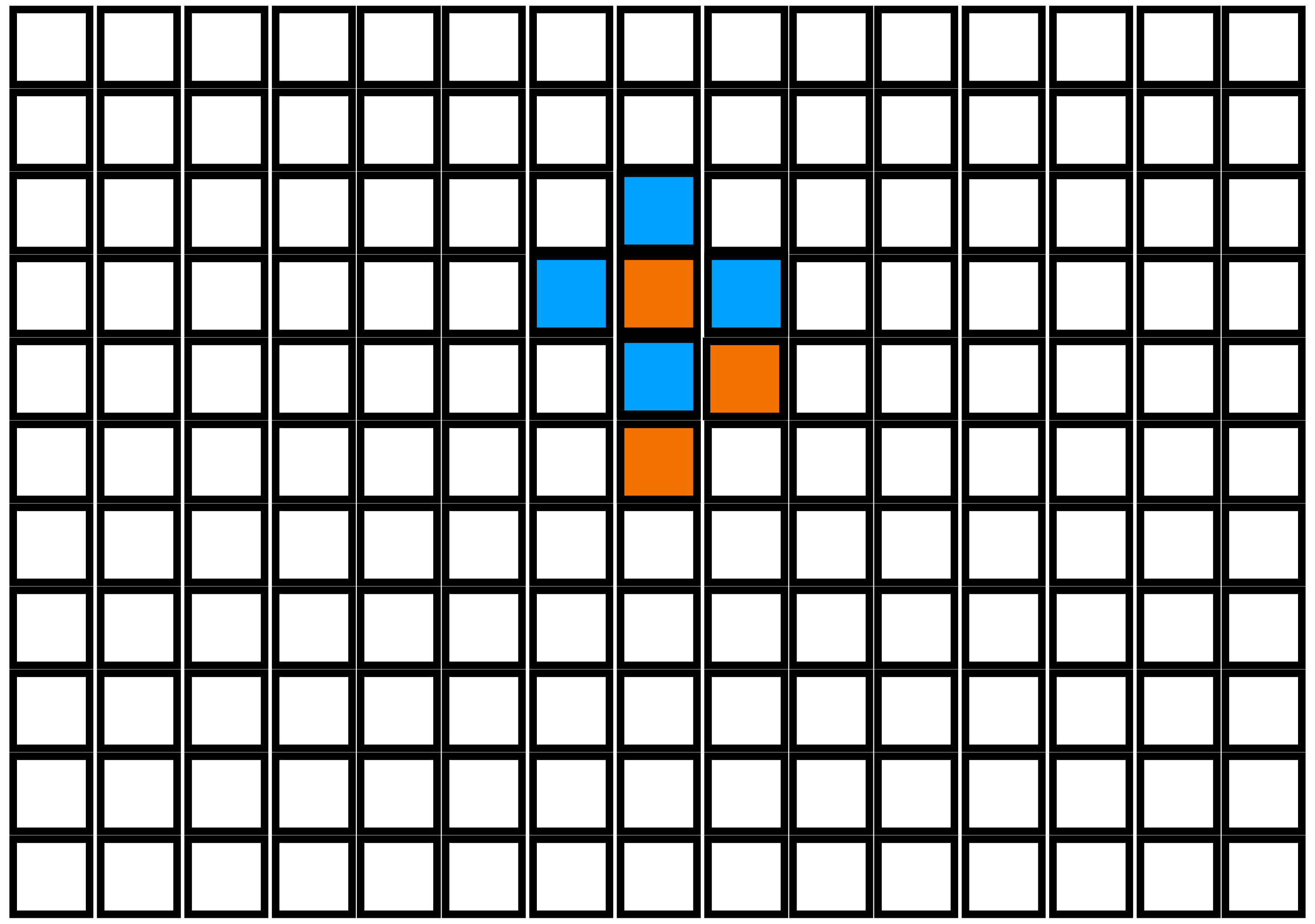

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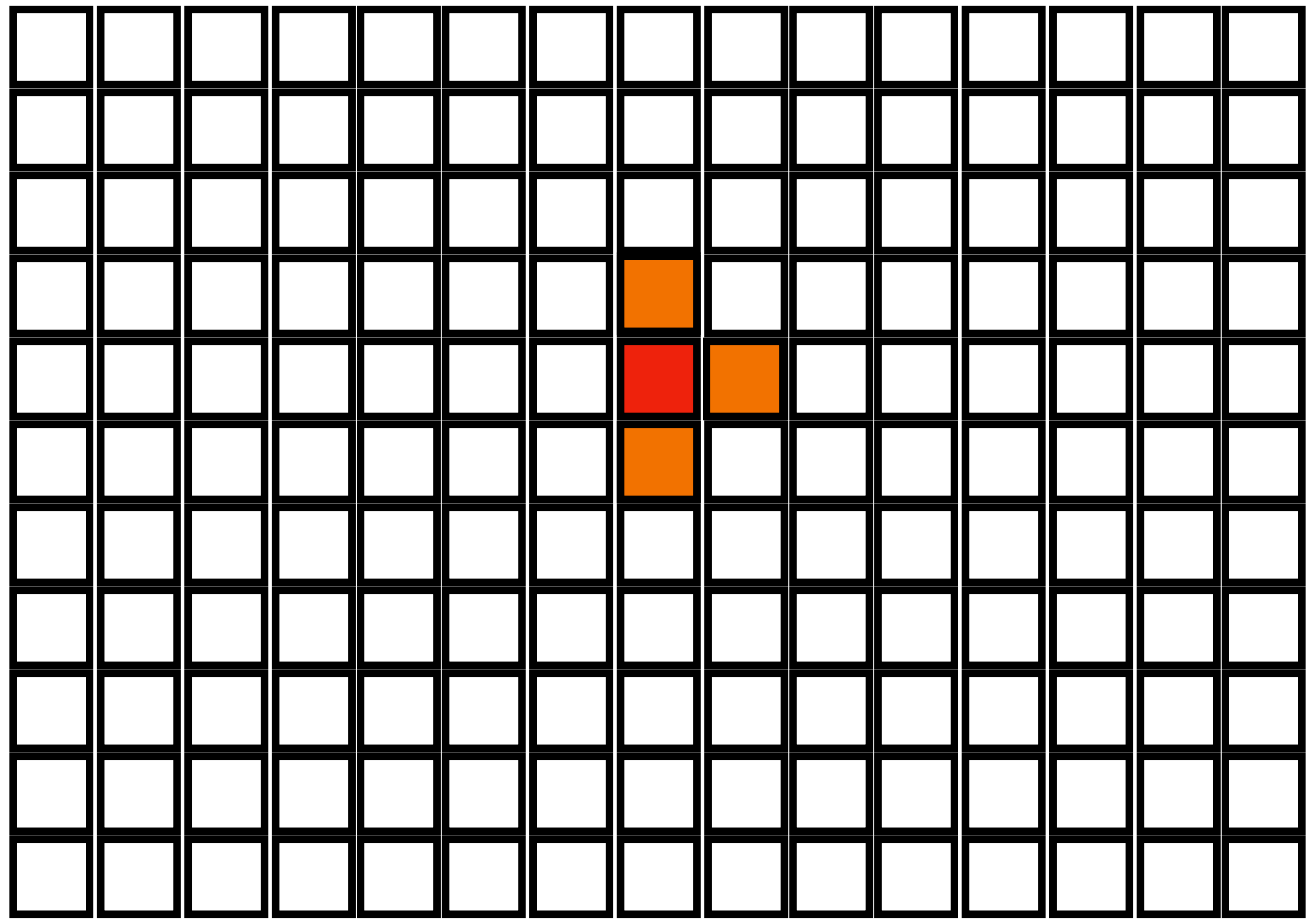

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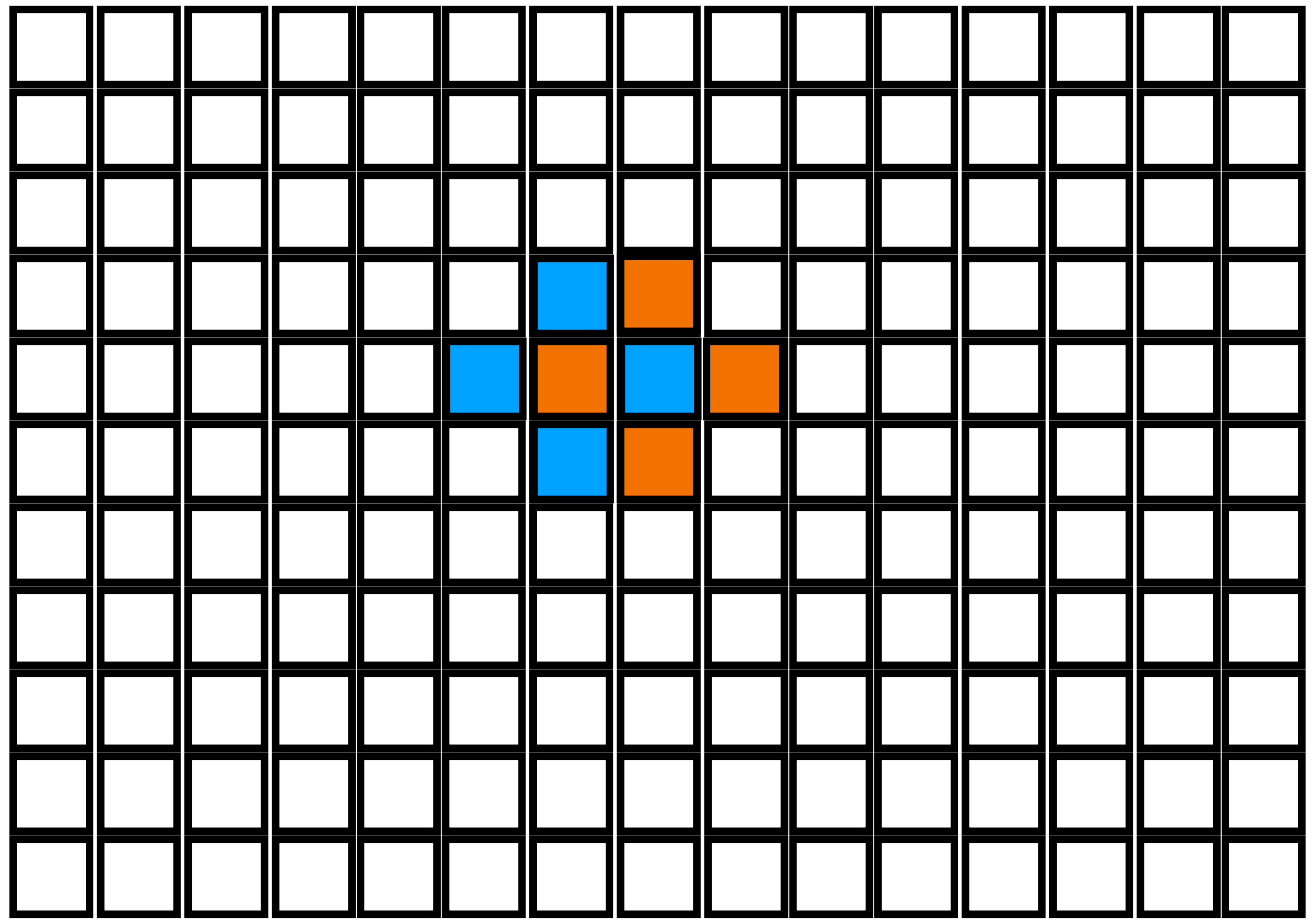

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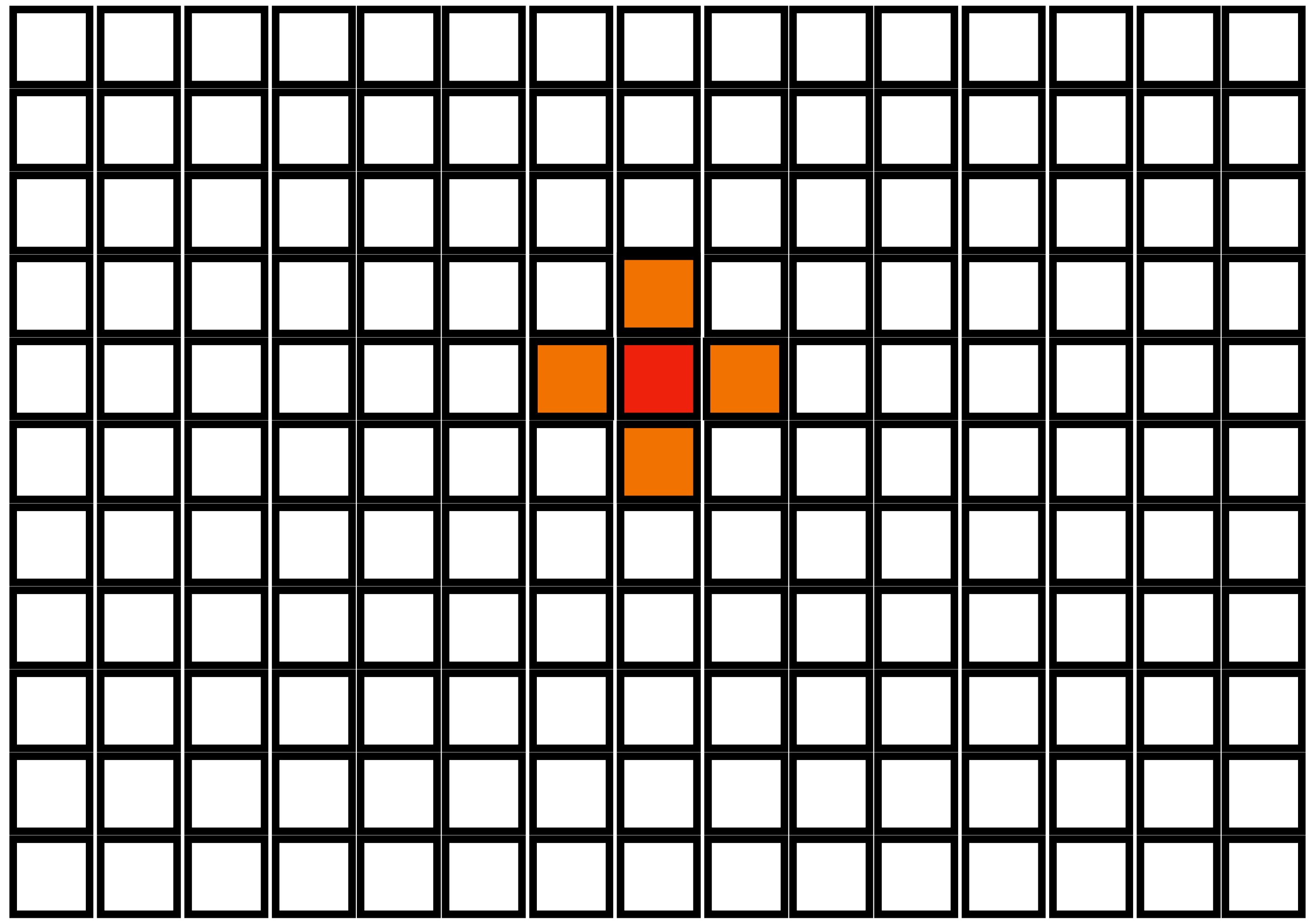

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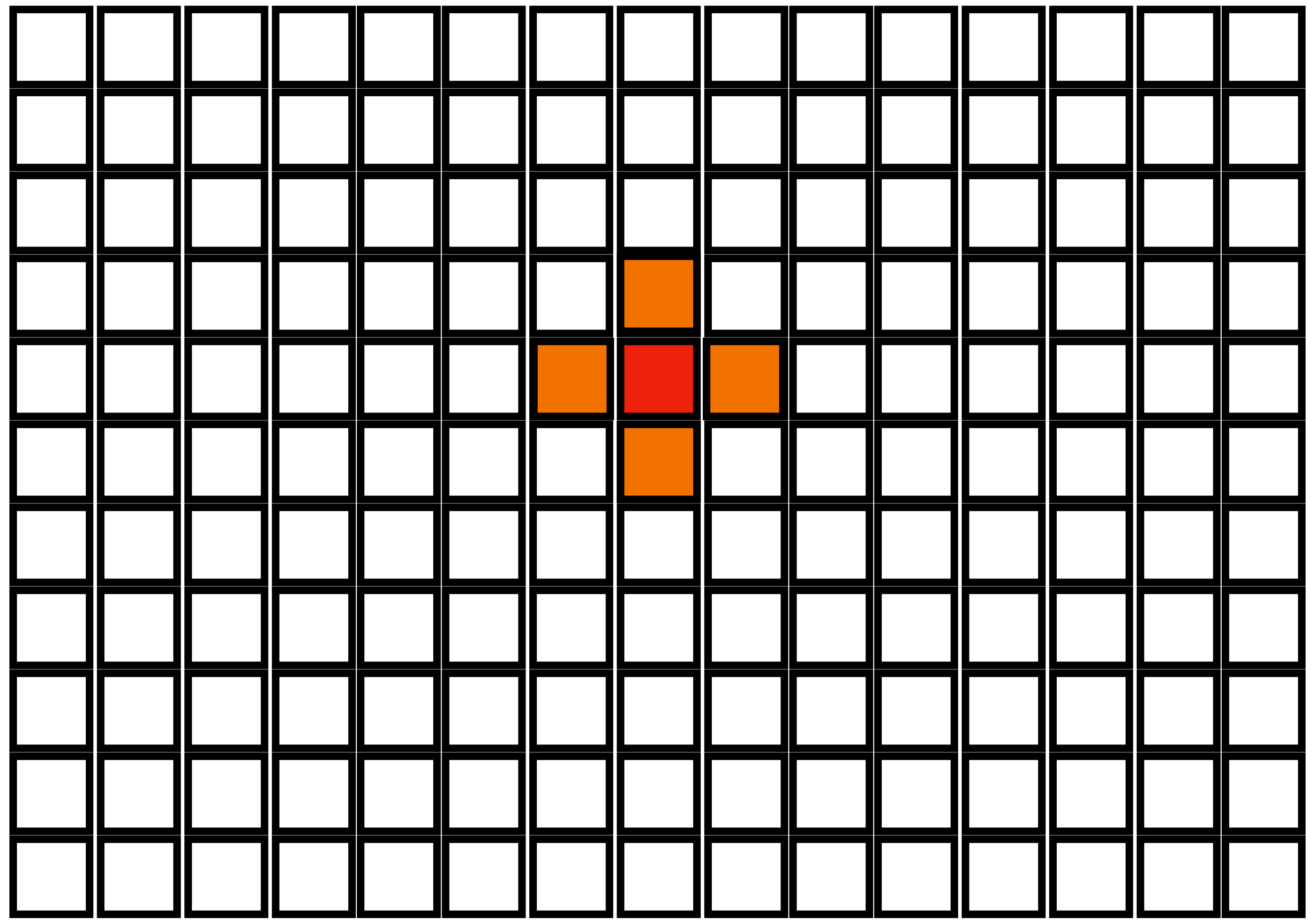

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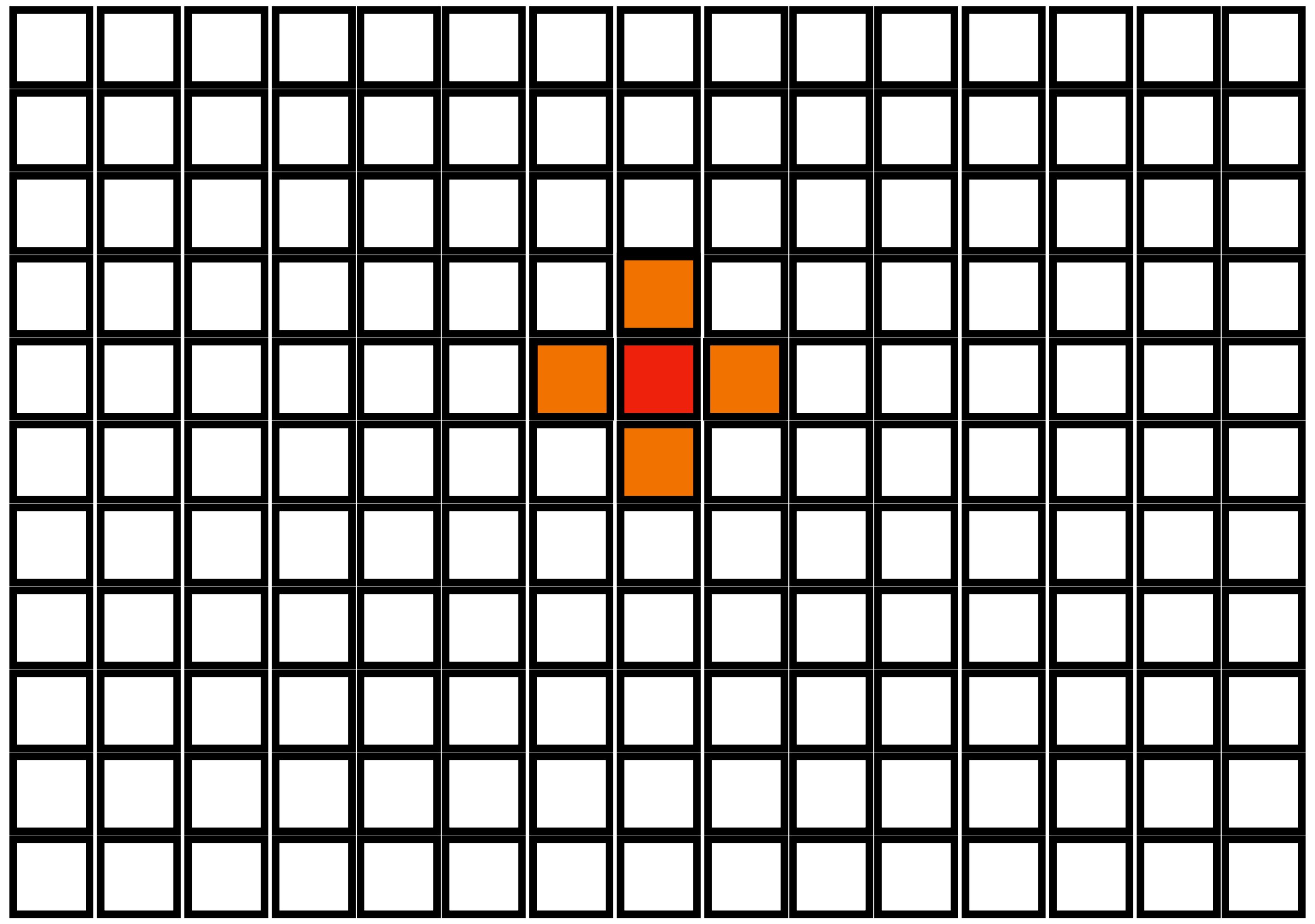

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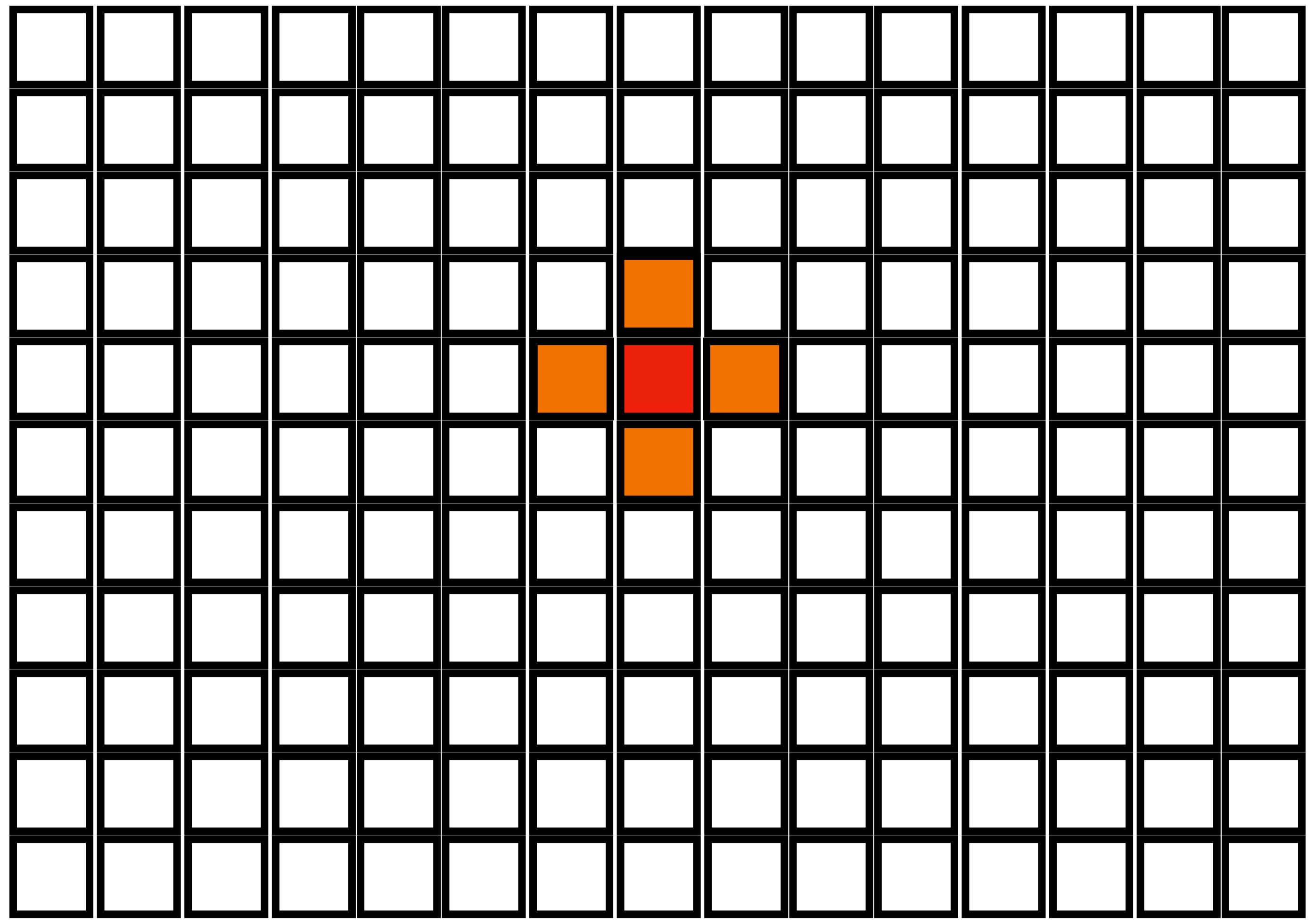

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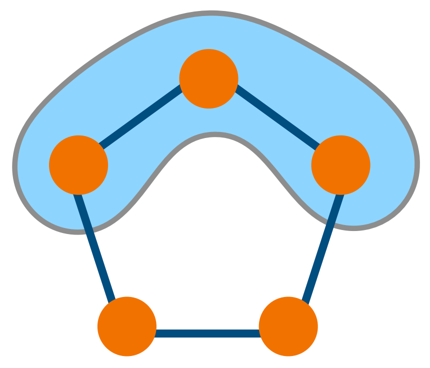

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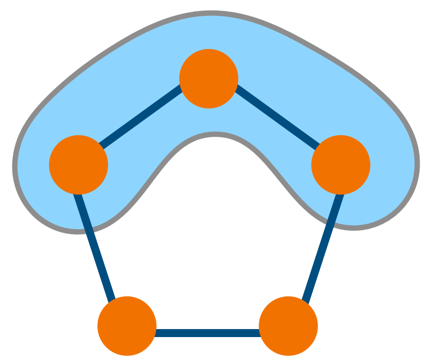

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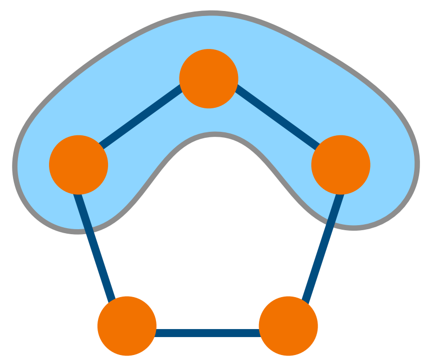

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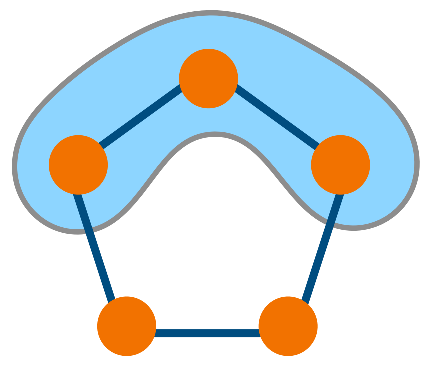

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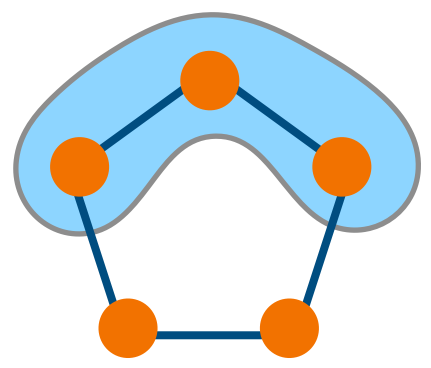
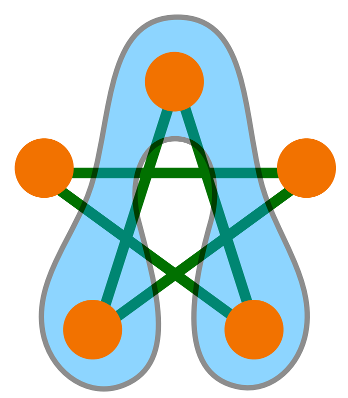


V



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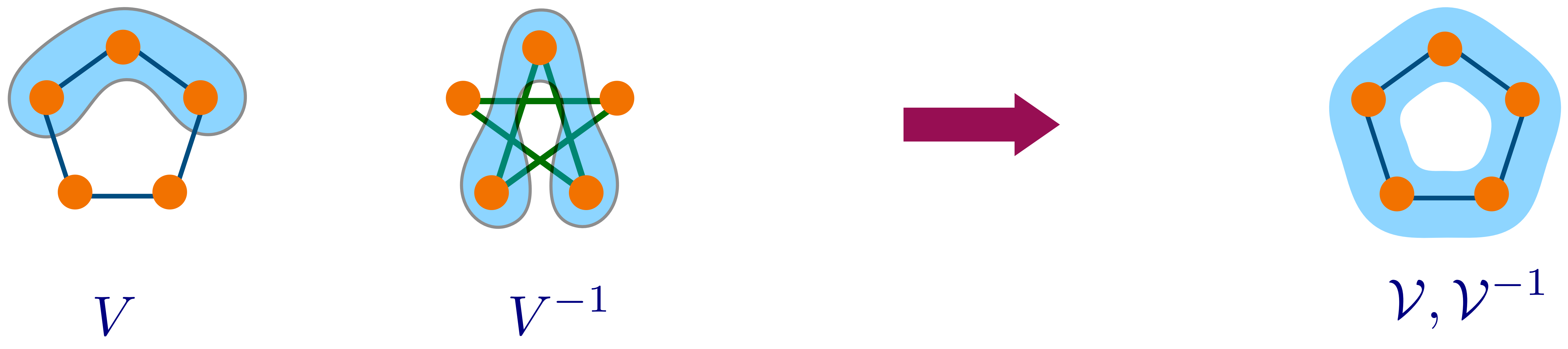

Margolus decomposition

- Let us consider a QCA $\mathcal{V}$ on $\mathbb{Z}$

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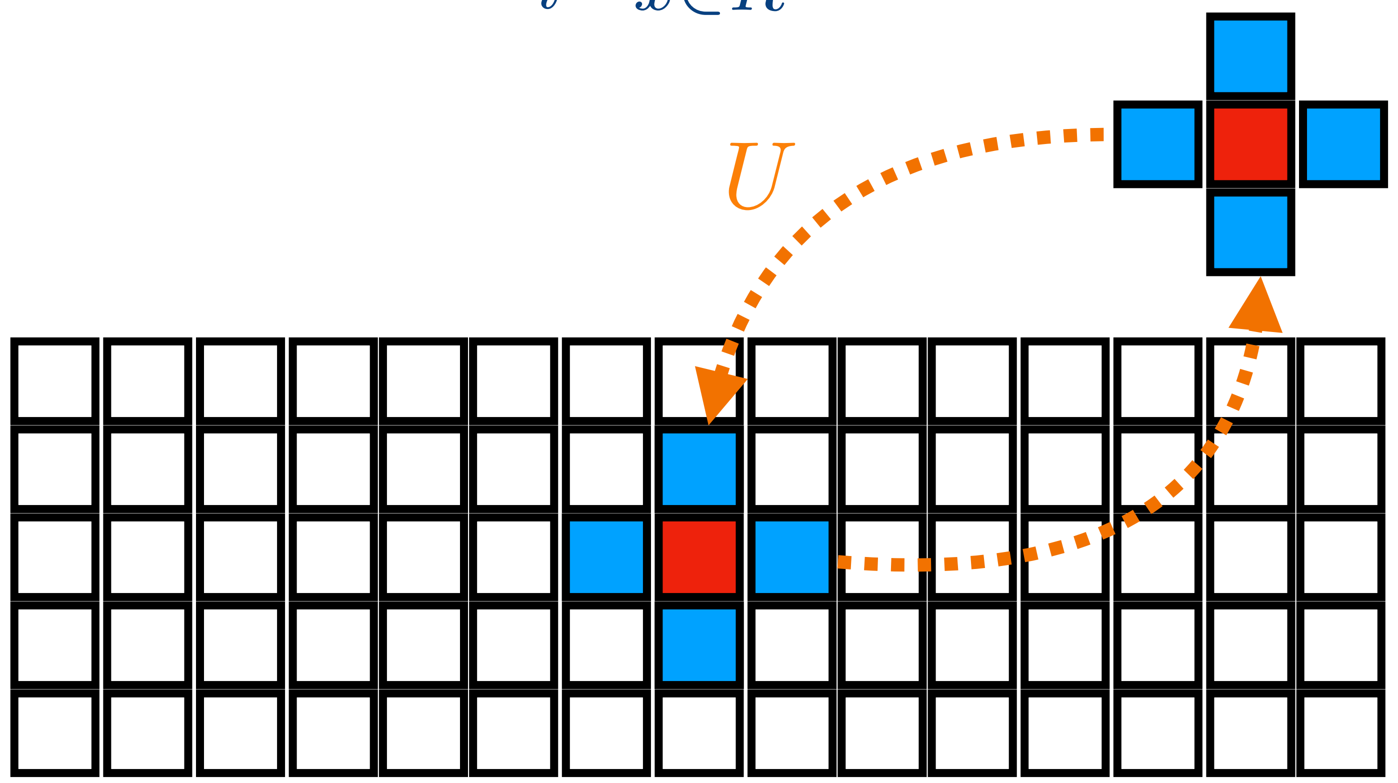

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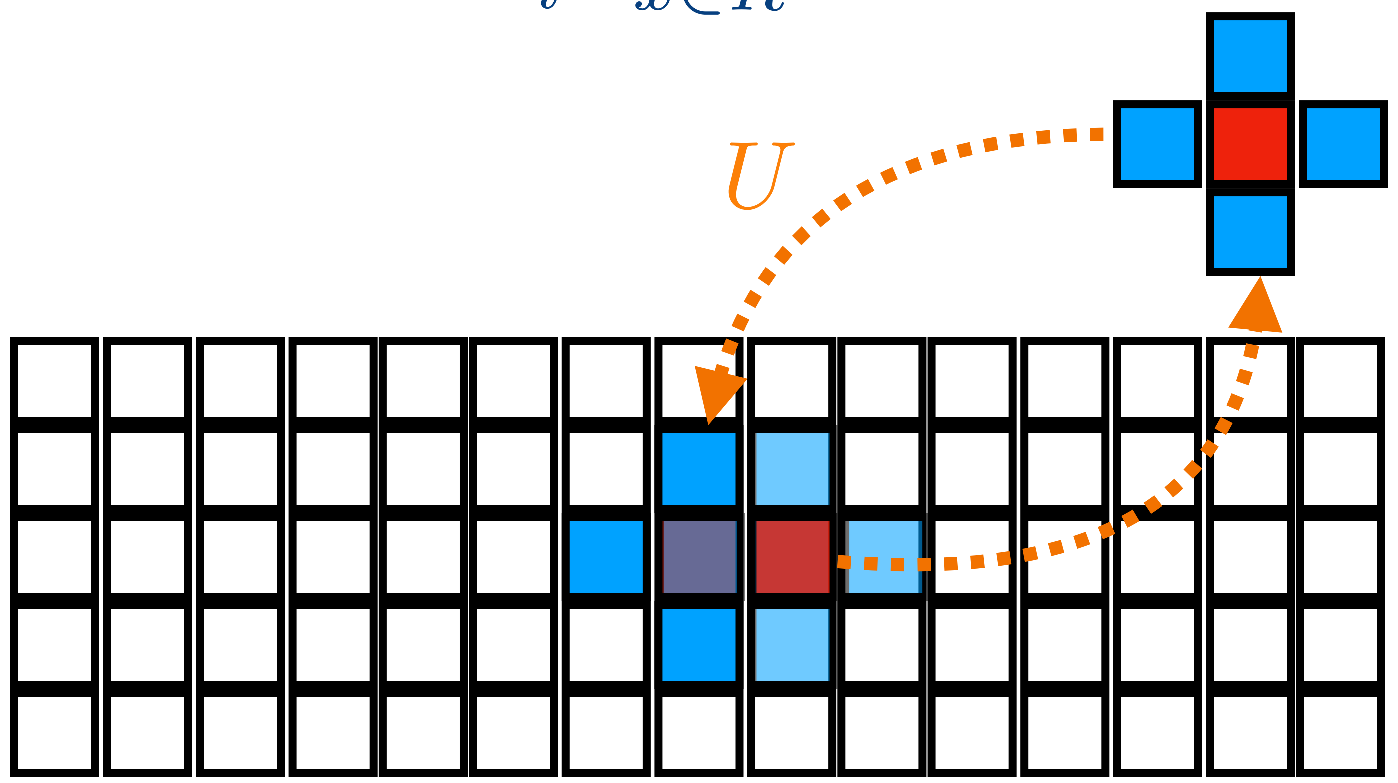

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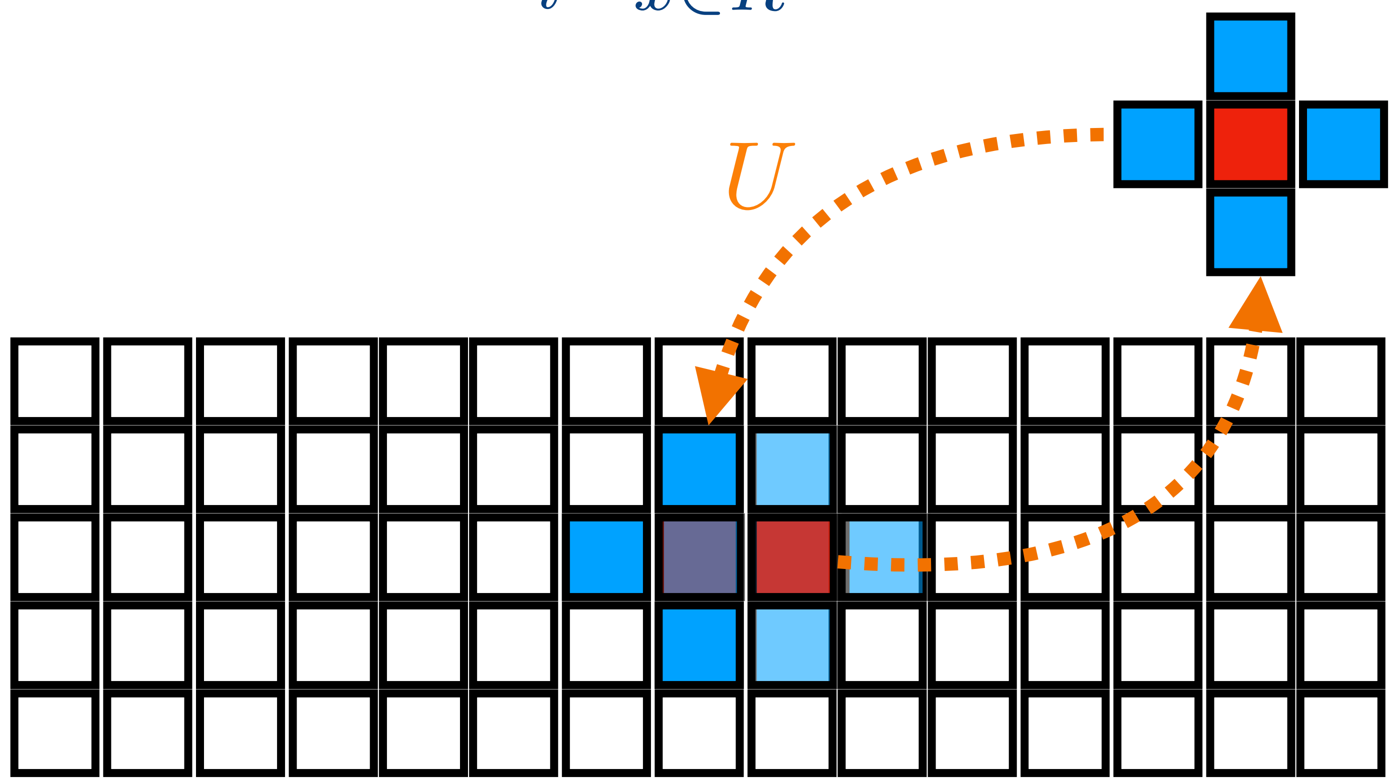

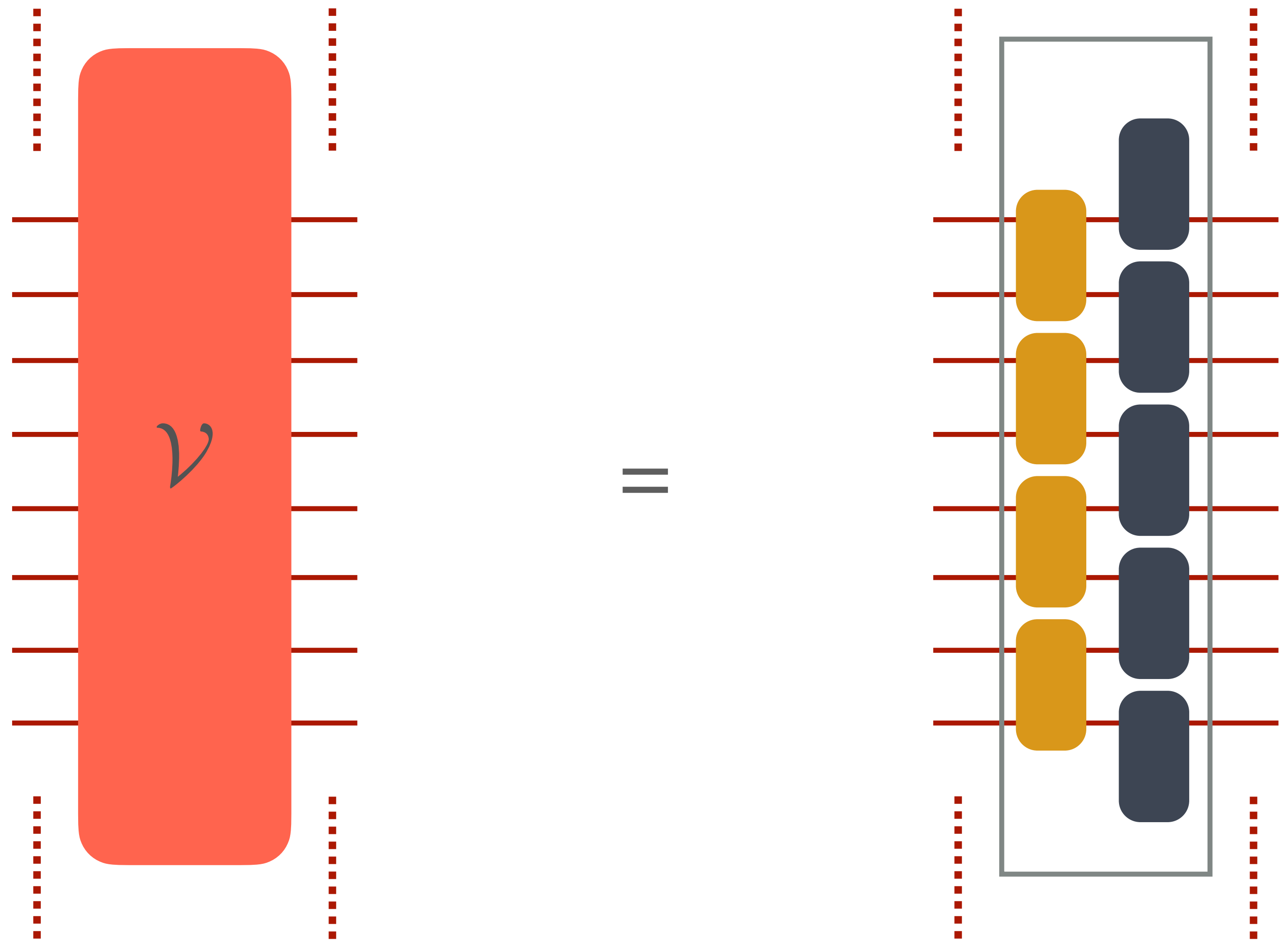

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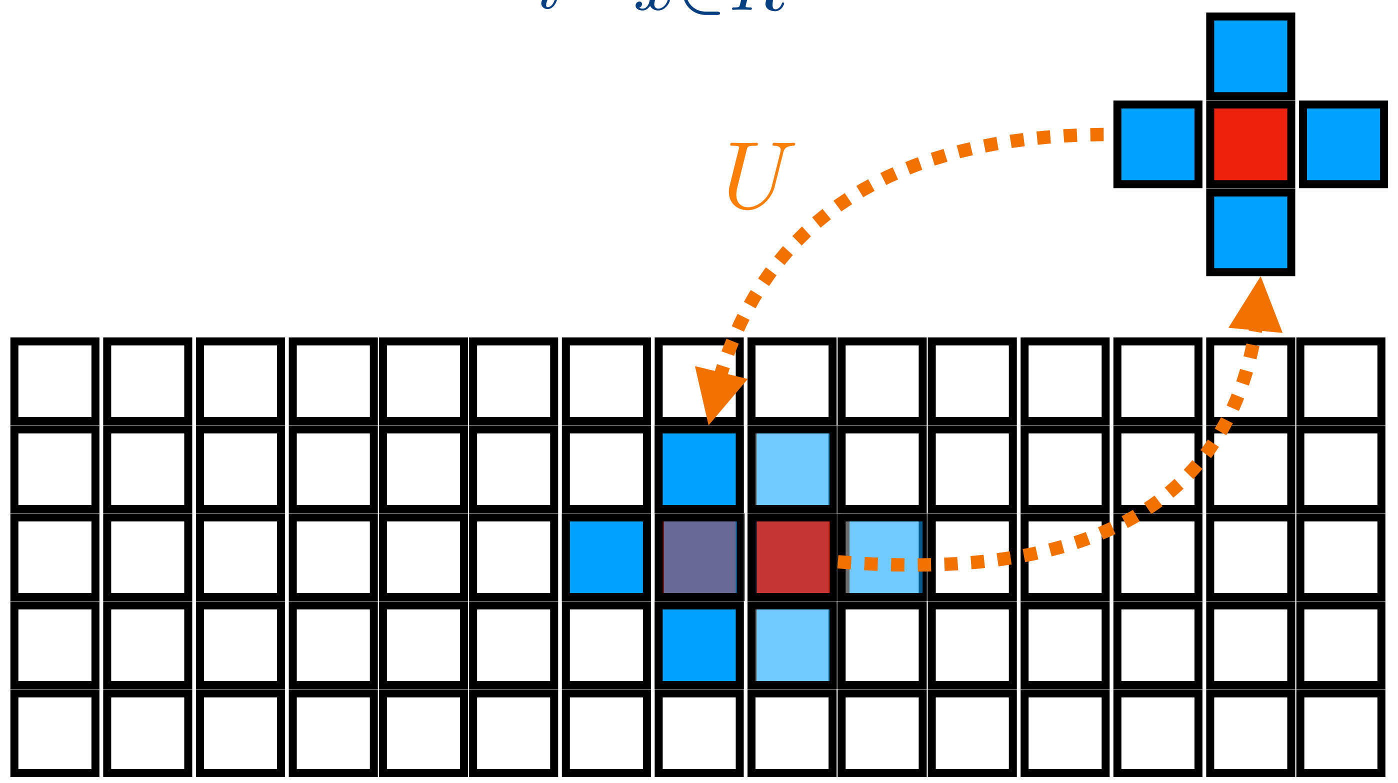

$$[I \otimes U, U \otimes I] = 0$$

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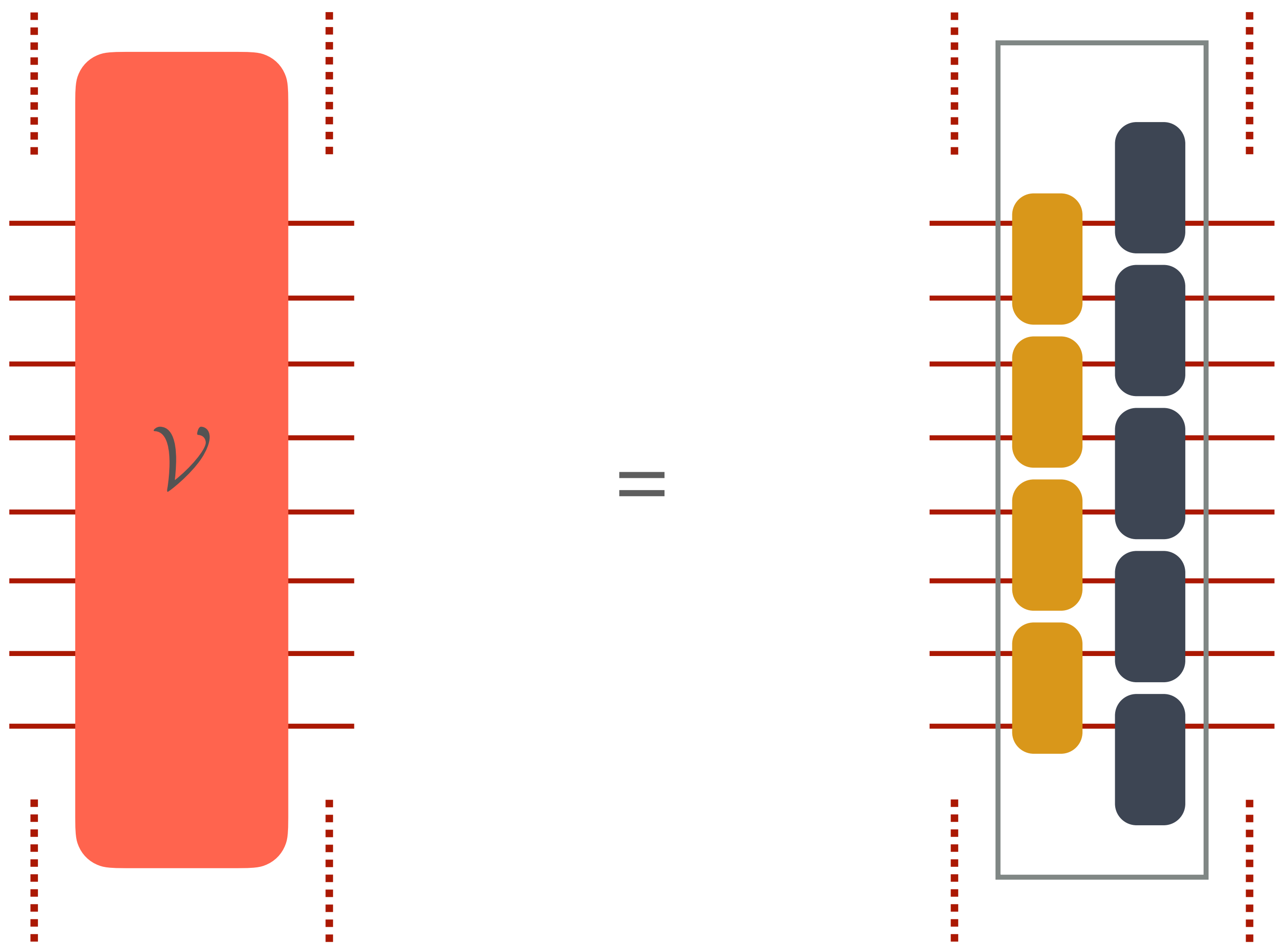
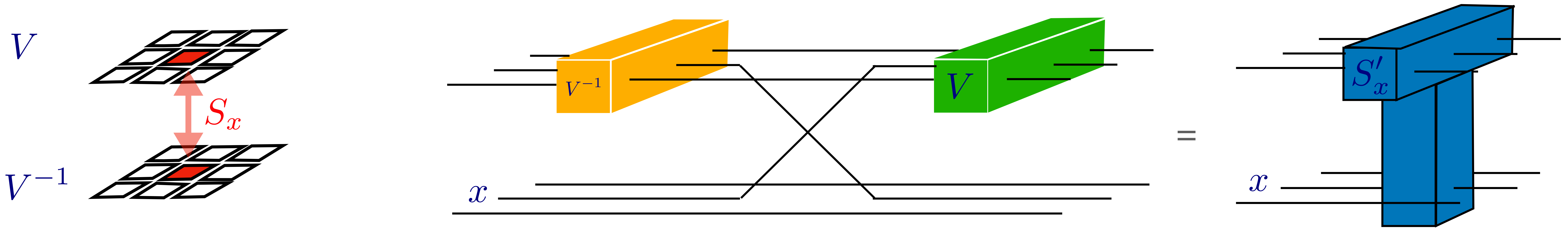


Unitarity and locality imply Margolus decomposability

- Let $\mathcal{V}$ be a QCA on $\mathbb{Z}^d$ with cell $\mathsf{H} \simeq \mathbb{C}^r$
- Consider the QCA $\mathcal{V} \otimes \mathcal{V}^{-1}$ on $\mathbb{Z}^d$ with cell $\mathsf{H} \simeq \mathbb{C}^r \otimes \mathbb{C}^r$
- Since $V \otimes V^{-1} = (V \otimes I)S(V^{-1} \otimes I)S$, where $S|\phi\rangle|\psi\rangle := |\psi\rangle|\phi\rangle$, and $S = \bigotimes_x S_x$

$$V \otimes V^{-1} = \prod_x (S'_x) \bigotimes_y S_y \qquad S'_x := (V \otimes I)S_x(V^{-1} \otimes I)$$



P. Arrighi, V. Nesme, R.F. Werner, J. Comput. Syst. Sci. **77**, 372–378 (2011)

Fermionic theory

- The theory is meant to provide a realisation of the fermion algebra

$$\{\varphi_i^\dagger, \varphi_j\} = \delta_{ij} I, \quad \{\varphi_i, \varphi_j\} = 0$$

for an arbitrary finite number N of Local Fermionic Modes (LFM) $1 \leq i, j \leq N$

Bravy and Kitaev, Annals of Physics **298**, 210–226 (2002)
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- Example of basis state: $|0010110\rangle = \varphi_3^\dagger \varphi_5^\dagger \varphi_6^\dagger |0000000\rangle$

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Theories without local discriminability

Example 2: Fermionic quantum theory

- The representation depends on the chosen ordering of the LFMs

$$J(\varphi_i) := I_1 \otimes \cdots \otimes I_{i-1} \otimes \sigma_i^- \otimes {\sigma^z}_{i+1} \cdots \otimes {\sigma^z}_N$$

$$J(XY) := J(X)J(Y) \qquad J(X^\dagger) := J(X)^\dagger$$

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- Used to prove computational equivalence of Fermionic and standard quantum computation*

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Theories without local discriminability

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- Kraus operators must be combination of either even or odd products

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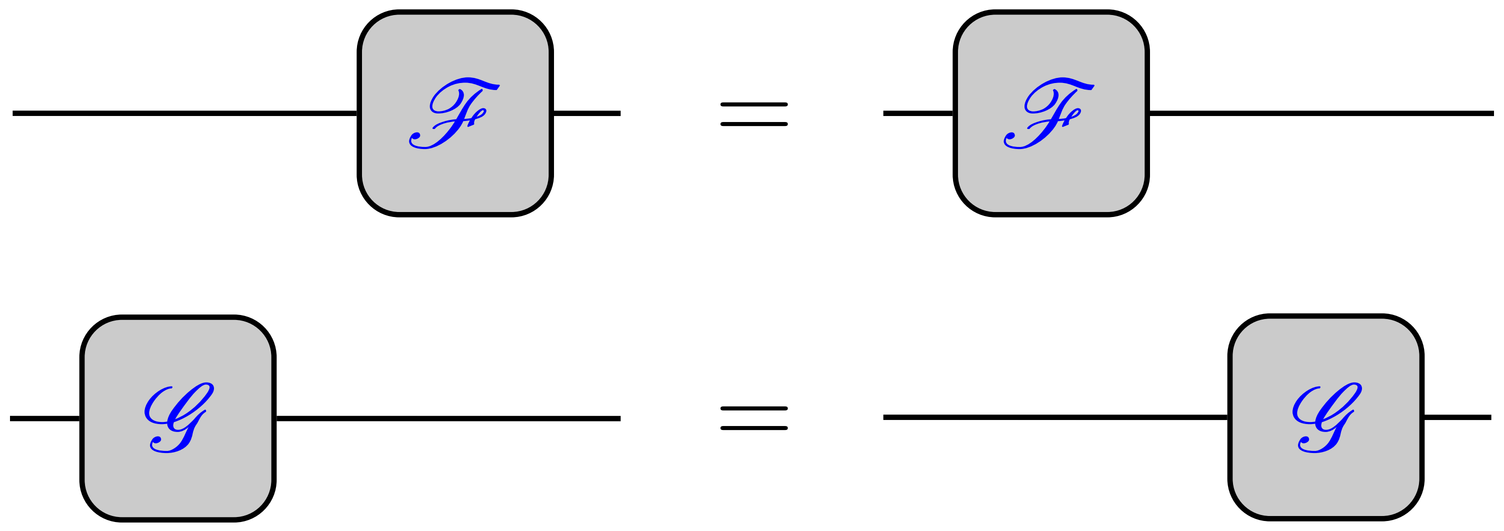

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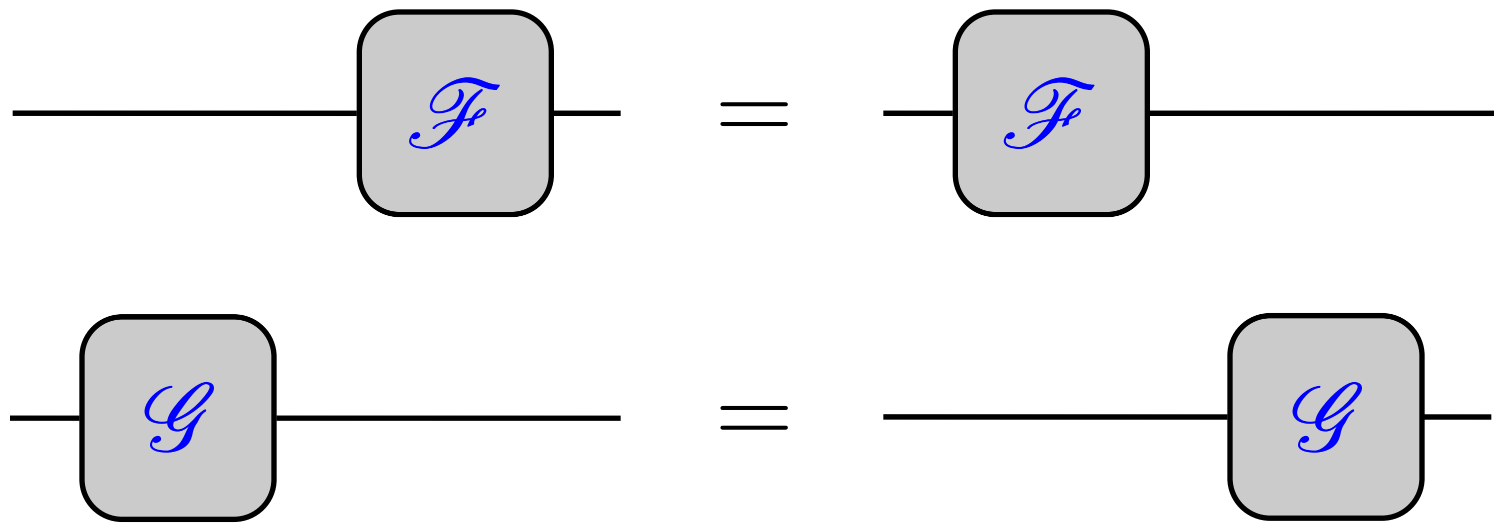

$\varphi_j \; \varphi_i$

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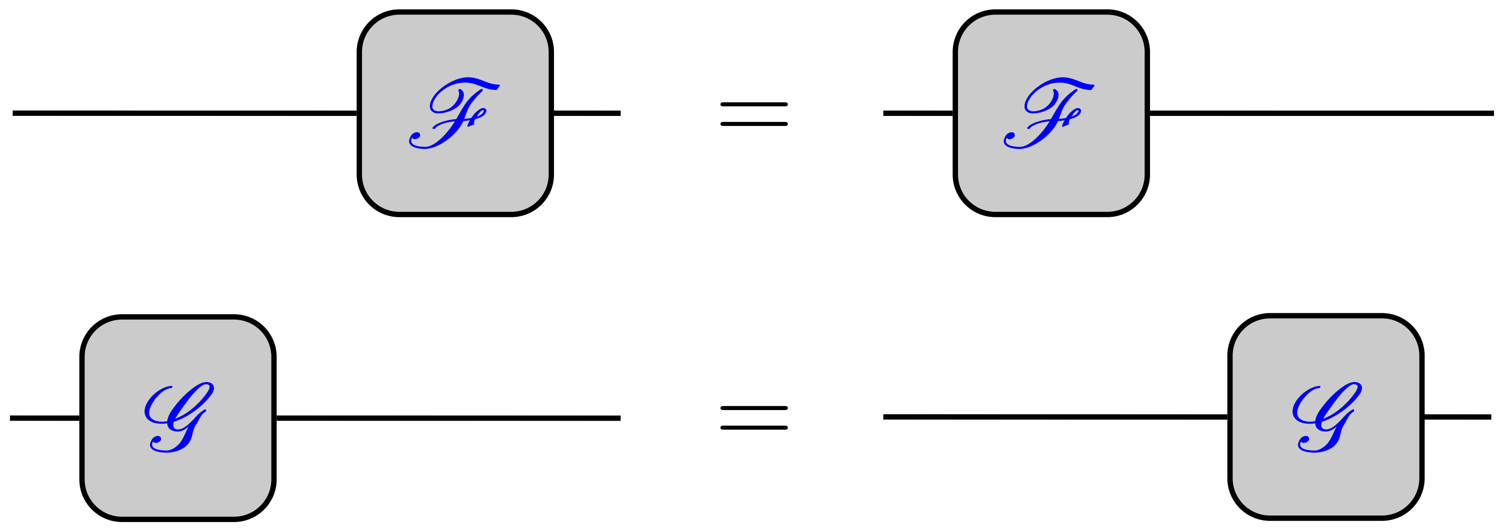

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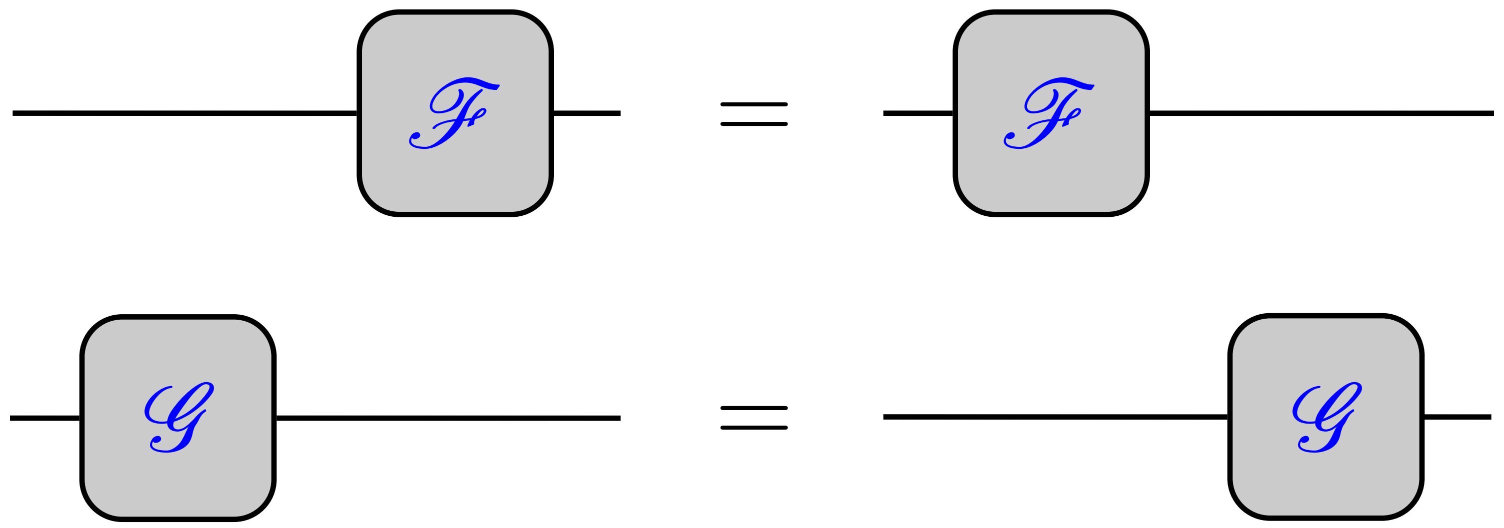

$-\varphi_i \, \varphi_j$ $\qquad K\rho K^\dagger = (-K)\rho(-K)^\dagger$

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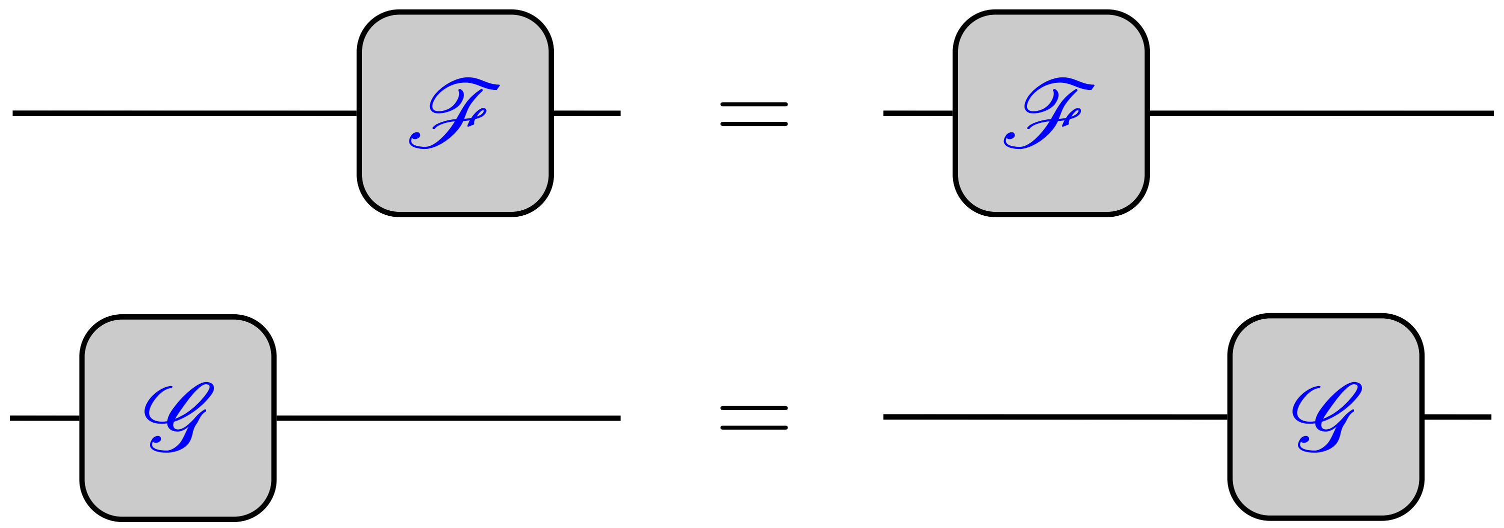

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$\varphi_j(\varphi_i^\dagger \varphi_i + \varphi_i)$

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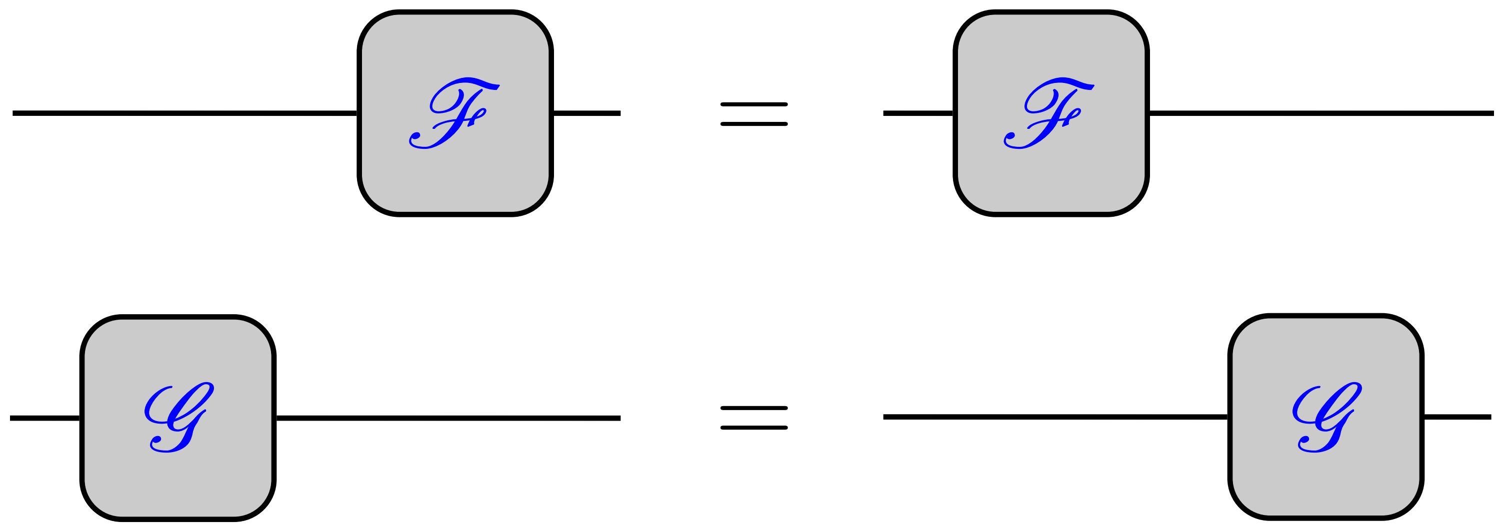

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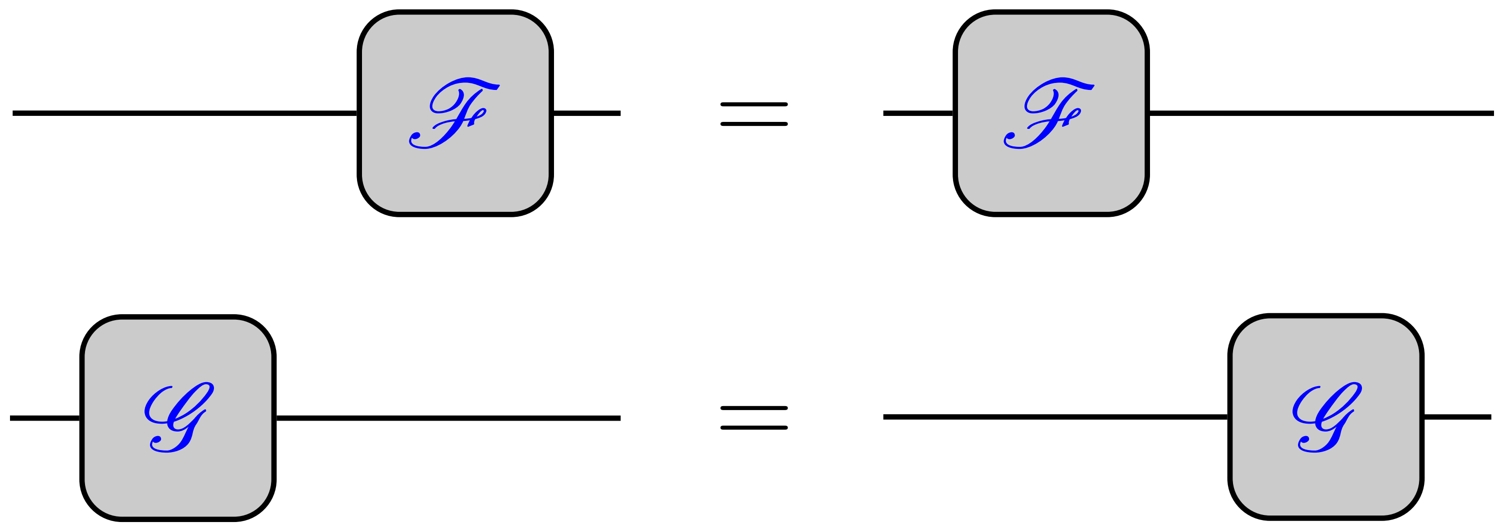

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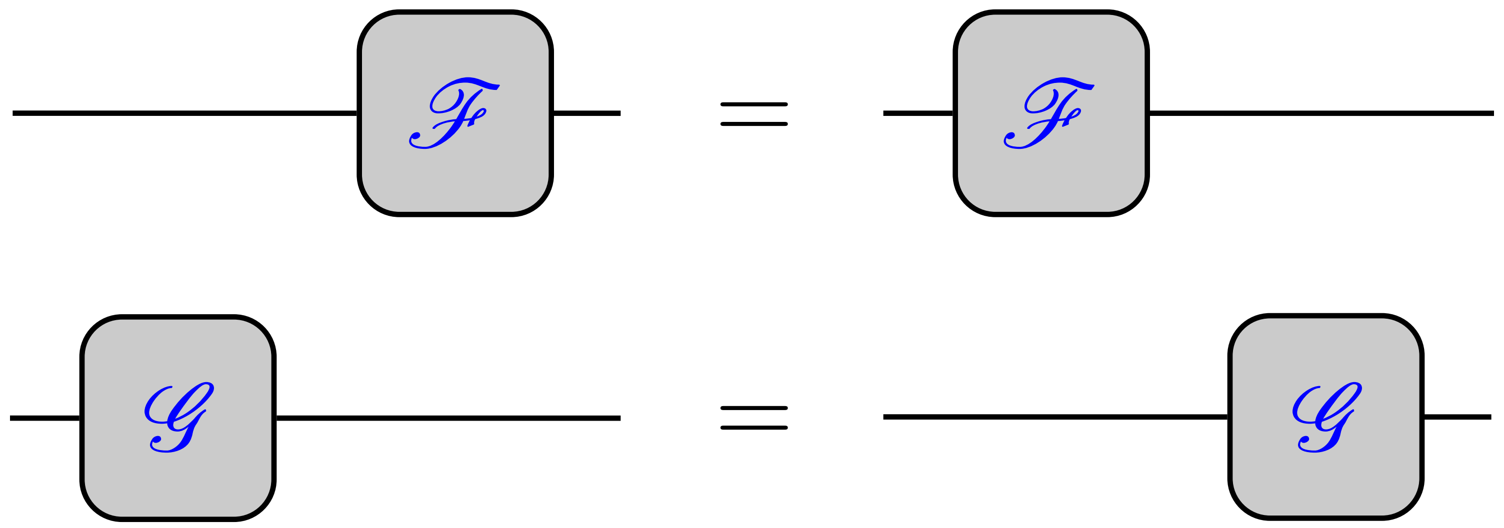

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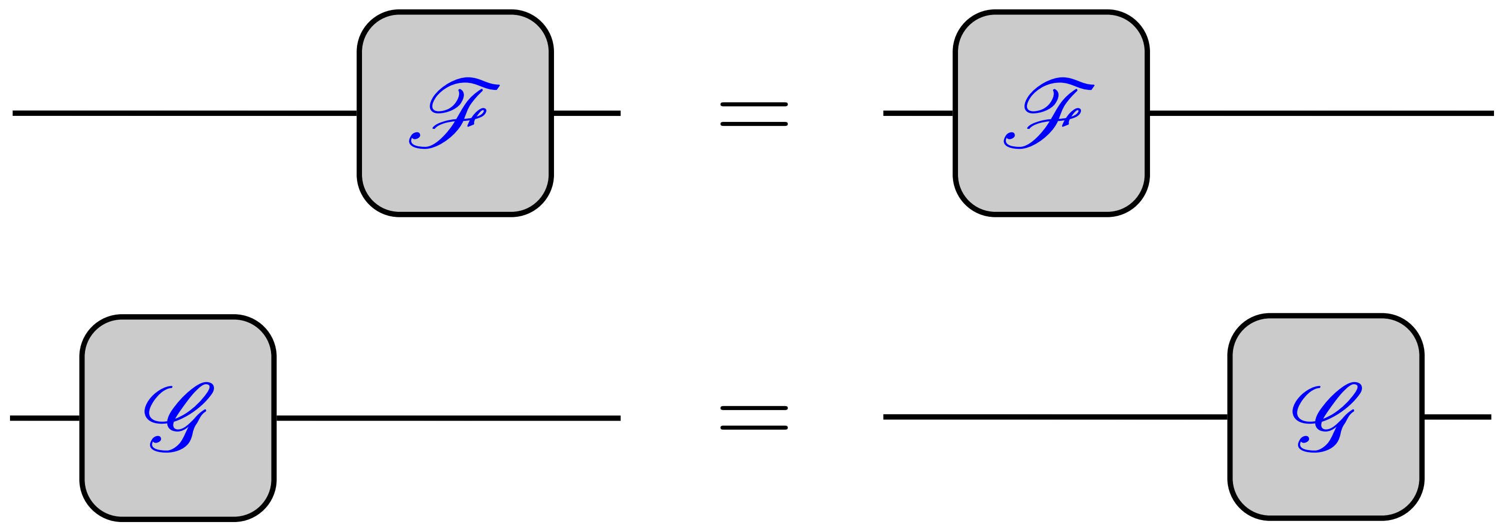

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- States and effects are combinations even products of field operators

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- This corresponds to a parity superselection rule

$$|\psi\rangle = |00\rangle,\ |10\rangle,\ a|10\rangle + b|01\rangle, \ldots$$

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- Parity superselection → block-diagonal structure for states

$$J(\rho) = \left(\begin{array}{c|c} p\rho_O & 0 \\ \hline 0 & (1-p)\rho_E \end{array}\right)$$

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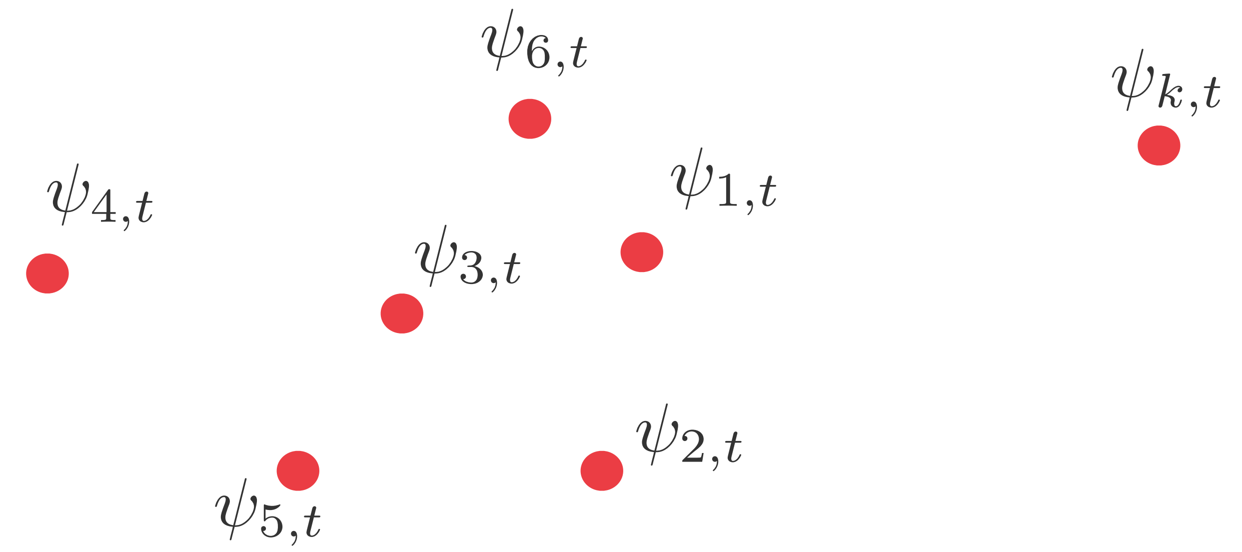
- In the finite case this is an iff condition
    - Wrapping lemma: true also in the infinite case

Linear FCAs and quantum walks



$$\psi_{i,t+1} = \sum_{j \in N_i} A_{i,j} \psi_{j,t} \qquad A_{ij} \in M_{s_i \times s_j}$$

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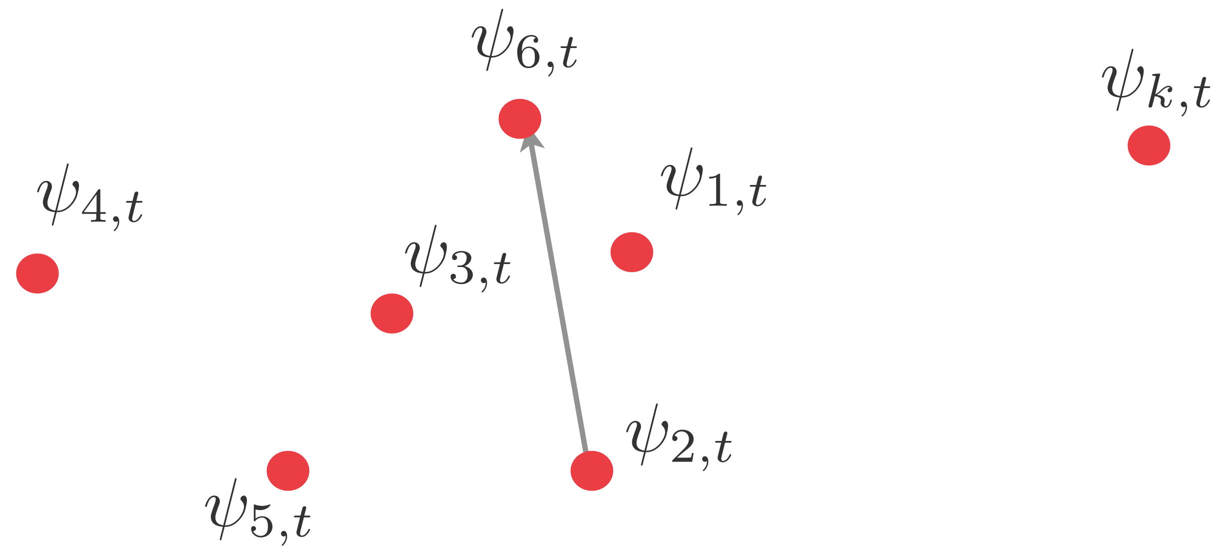

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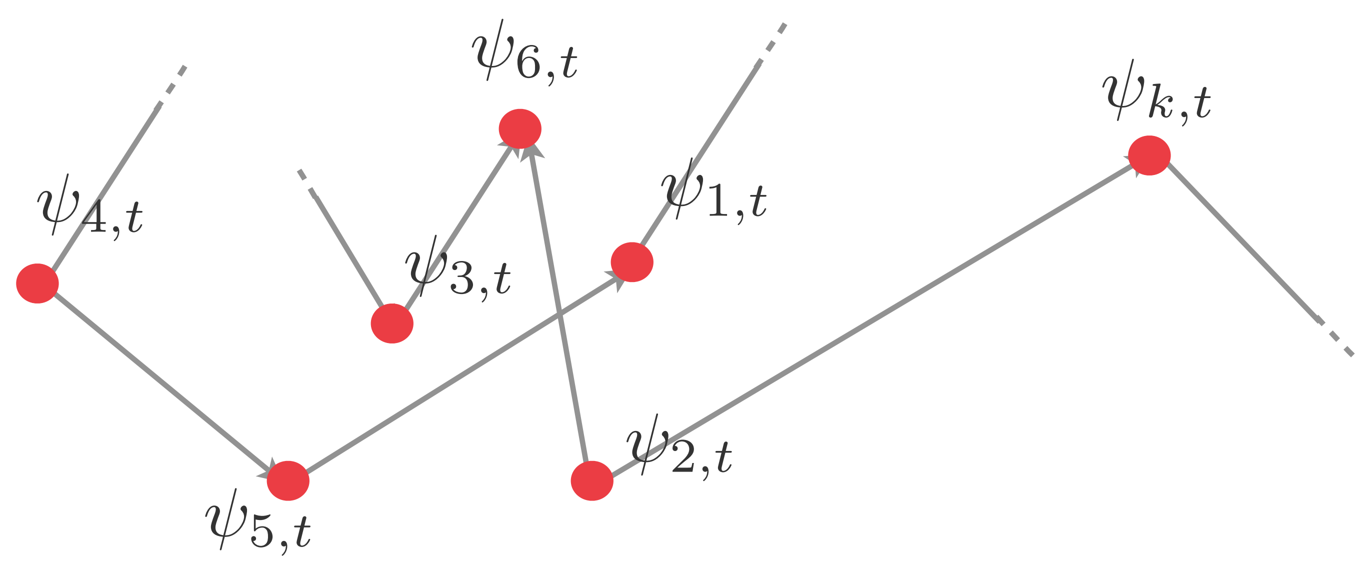

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