Operational probabilistic theories and cellular automata: how I learned to stop worrying and love C* algebras

School on Advanced Topics in Quantum Information and Foundations
Quantum Information Unit and the Yukawa Institute for Theoretical Physics, Kyoto University

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INFN Istituto Nazionale di Fisica Nucleare

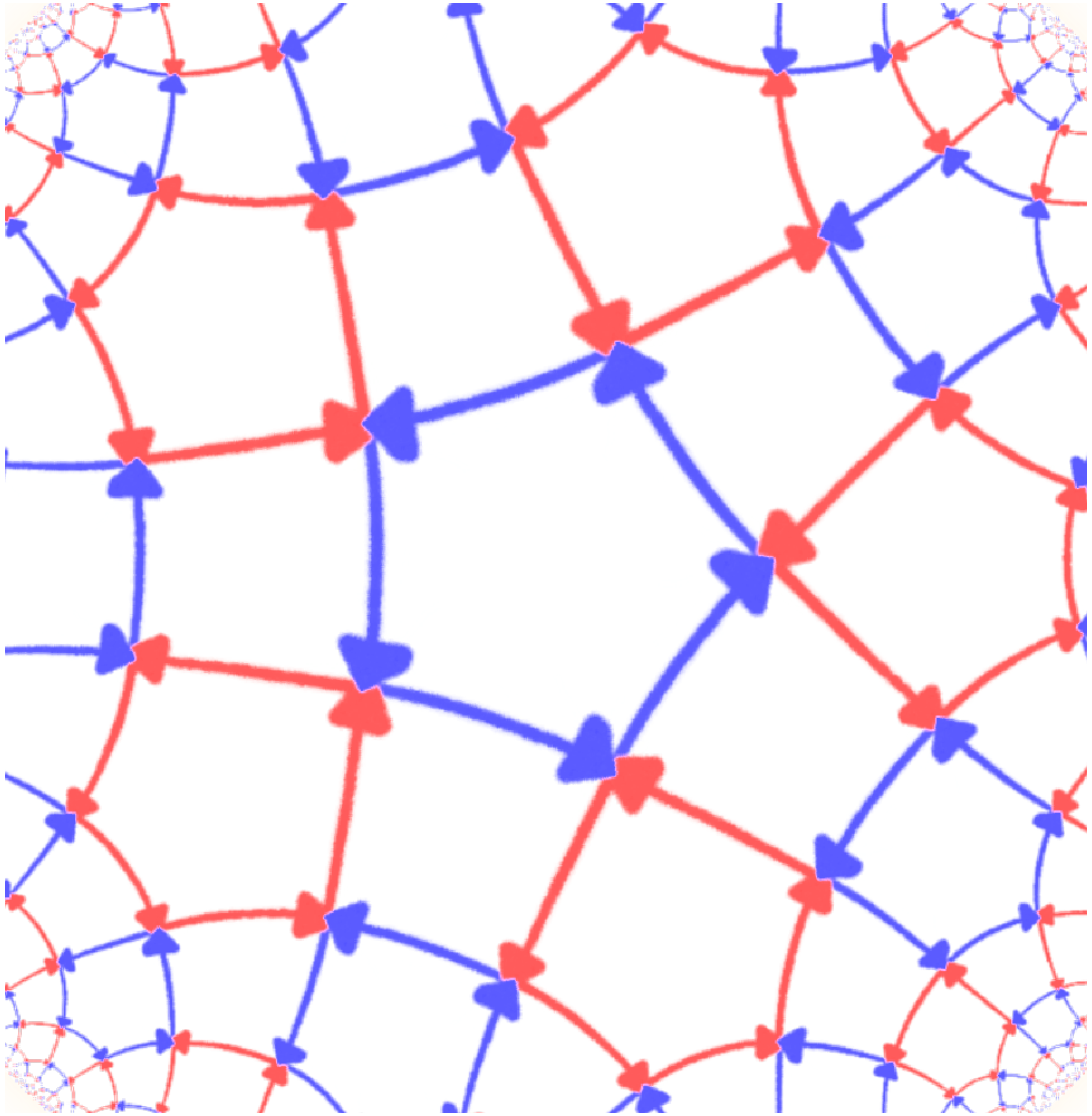
QUit quantum information theory group

Paolo Perinotti - February 8-12 2021

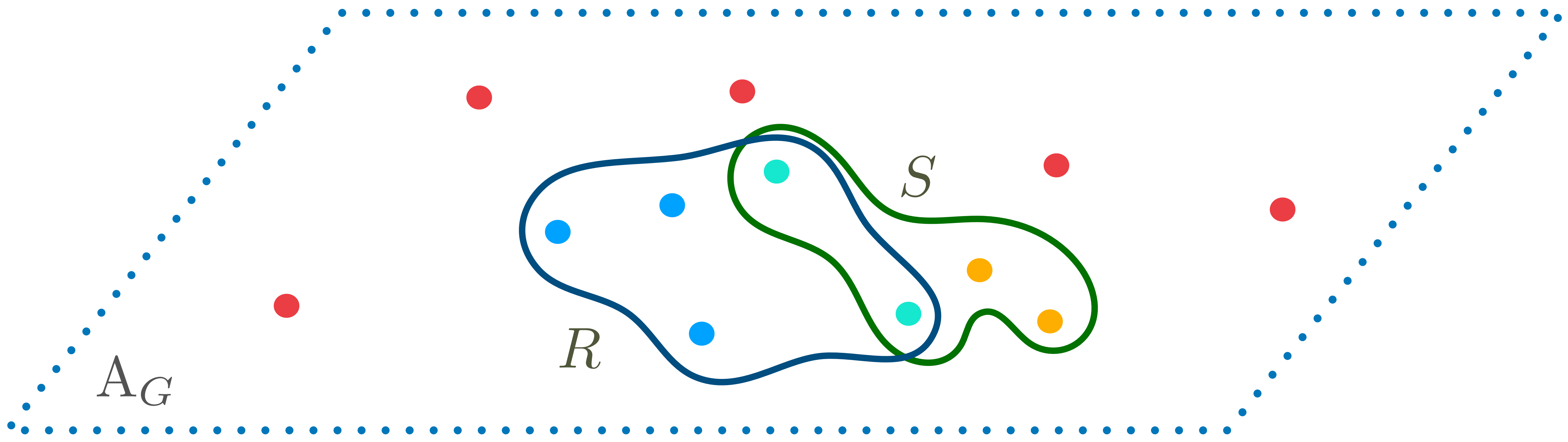
Lecture 5
Homogeneity and Cayley graphs

Summary

- Local processes in one-to one correspondence with CA
- Homogeneity
  - Homologous regions
- Cayley graphs and translations



Local action



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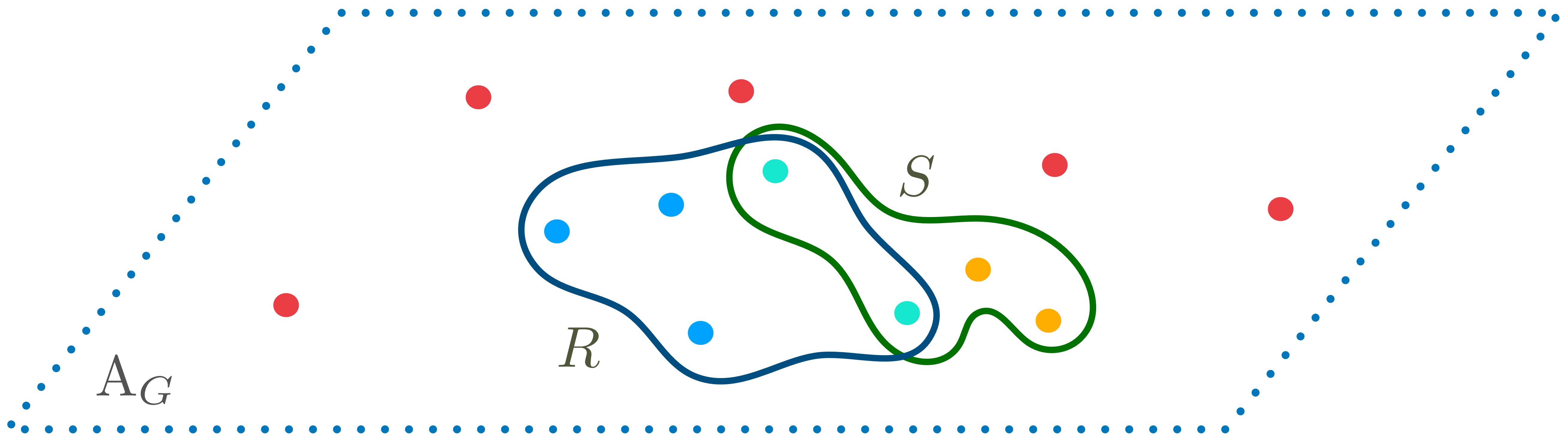
$$\mathscr{V}_S^\dagger = \mathscr{V}_{R\cap S}^\dagger \otimes \mathscr{I}_{S\setminus R}$$



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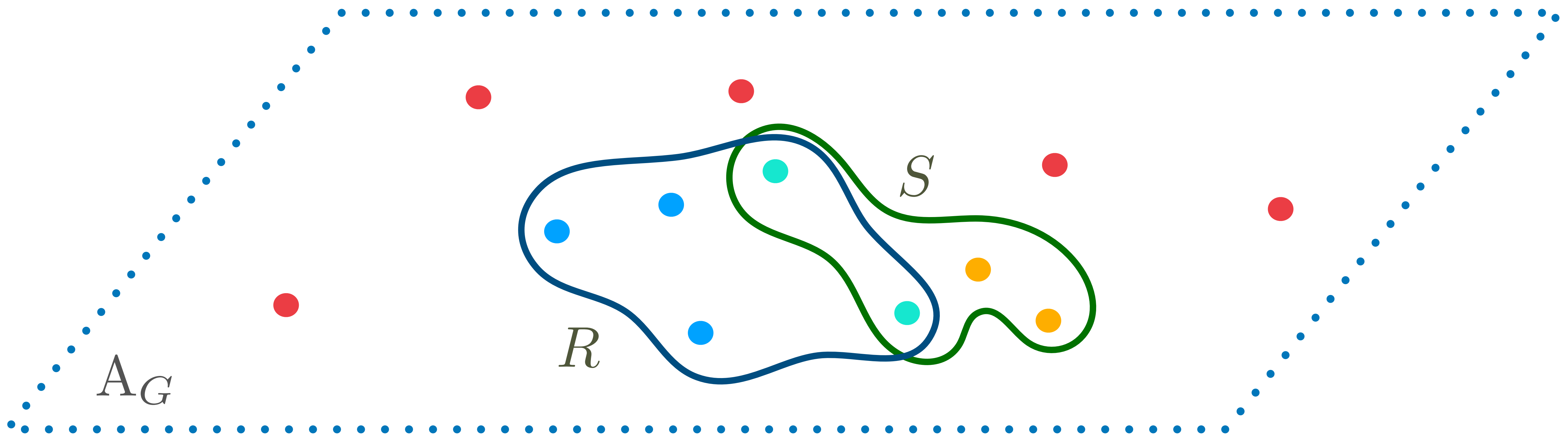

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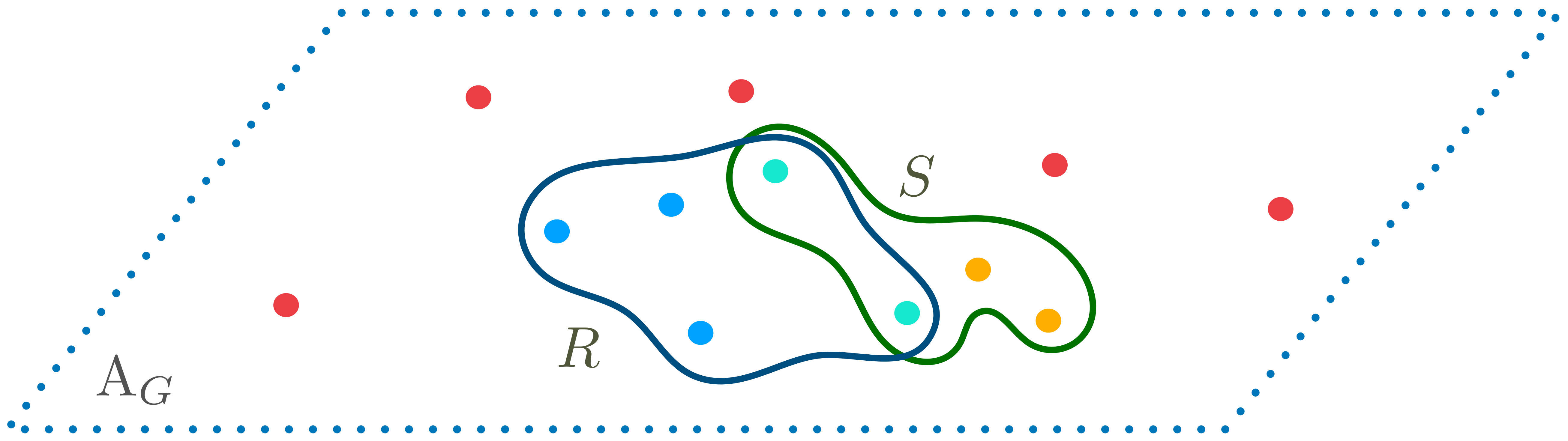
- We write $\mathscr{V}^\dagger = \mathscr{V}'^\dagger \otimes \mathscr{I}_{G\setminus R}^\dagger$

- If $\mathscr{V}^\dagger$ acts locally on R and $\mathscr{W}^\dagger$ acts locally on $G\setminus R$

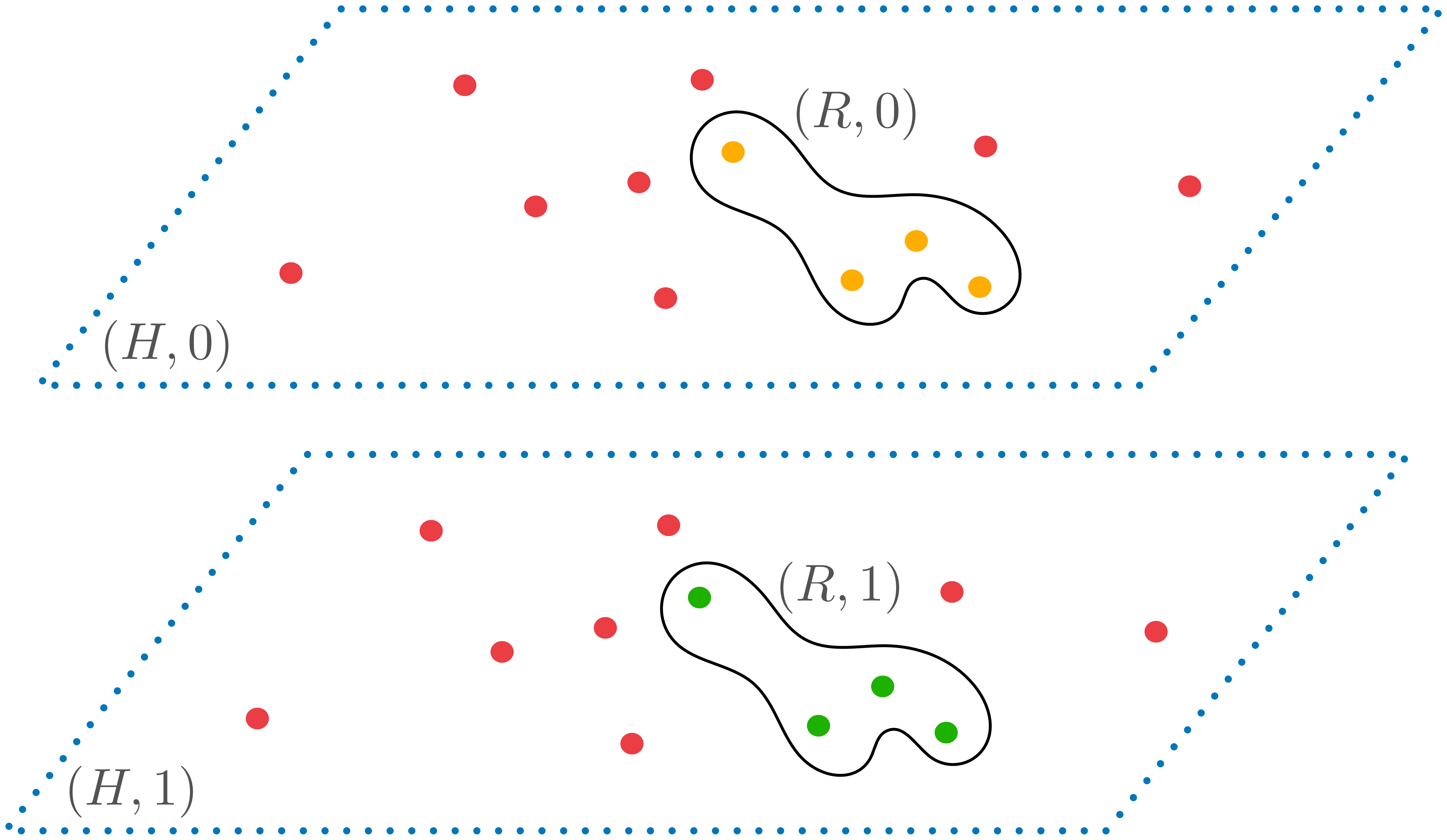
$$\mathscr{V}'^\dagger \otimes \mathscr{W}'^\dagger := \mathscr{V}^\dagger \mathscr{W}^\dagger = \mathscr{W}^\dagger \mathscr{V}^\dagger$$



Example

The swap GUR

- Let $G = H \times \{0, 1\}$



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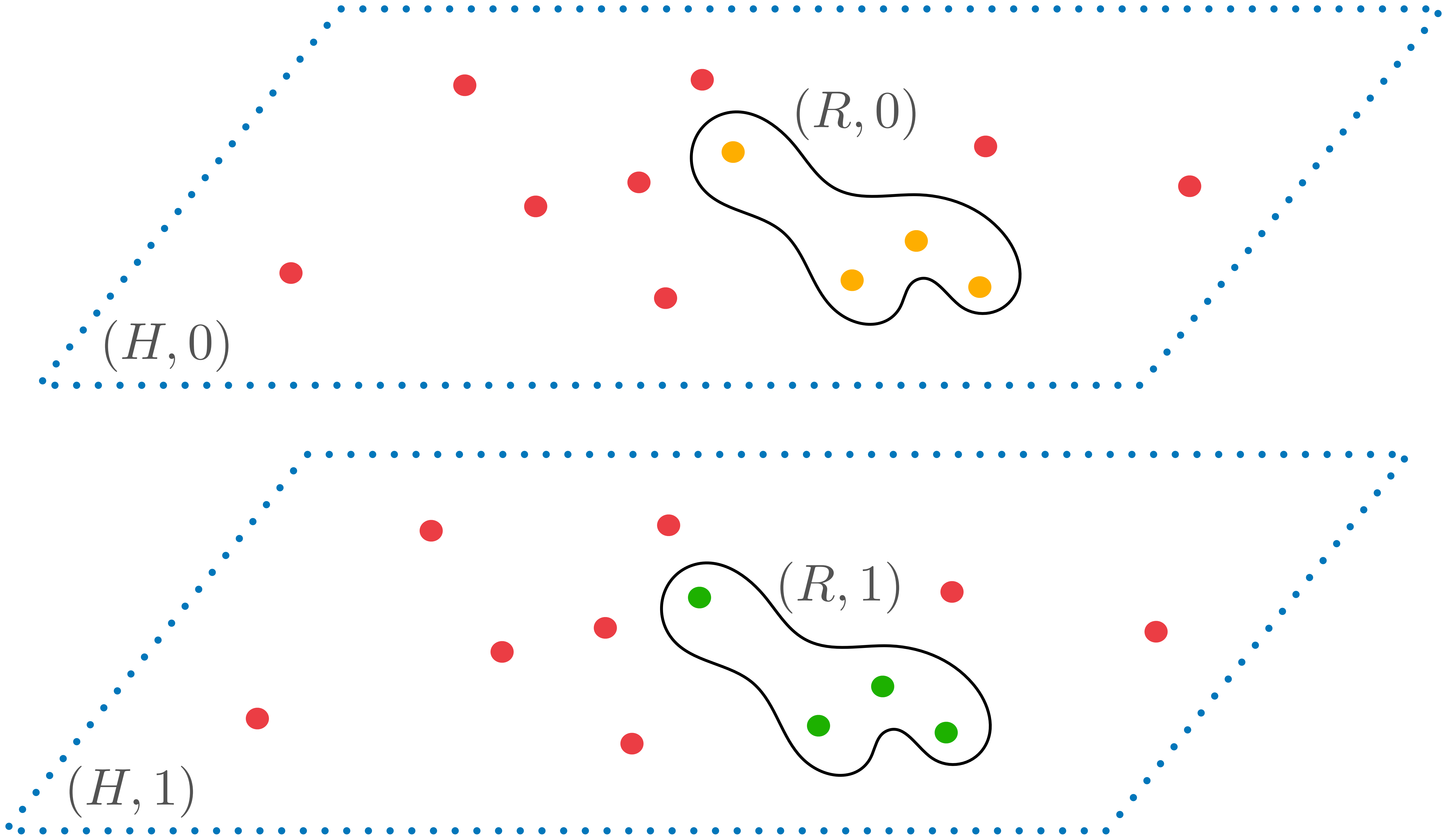

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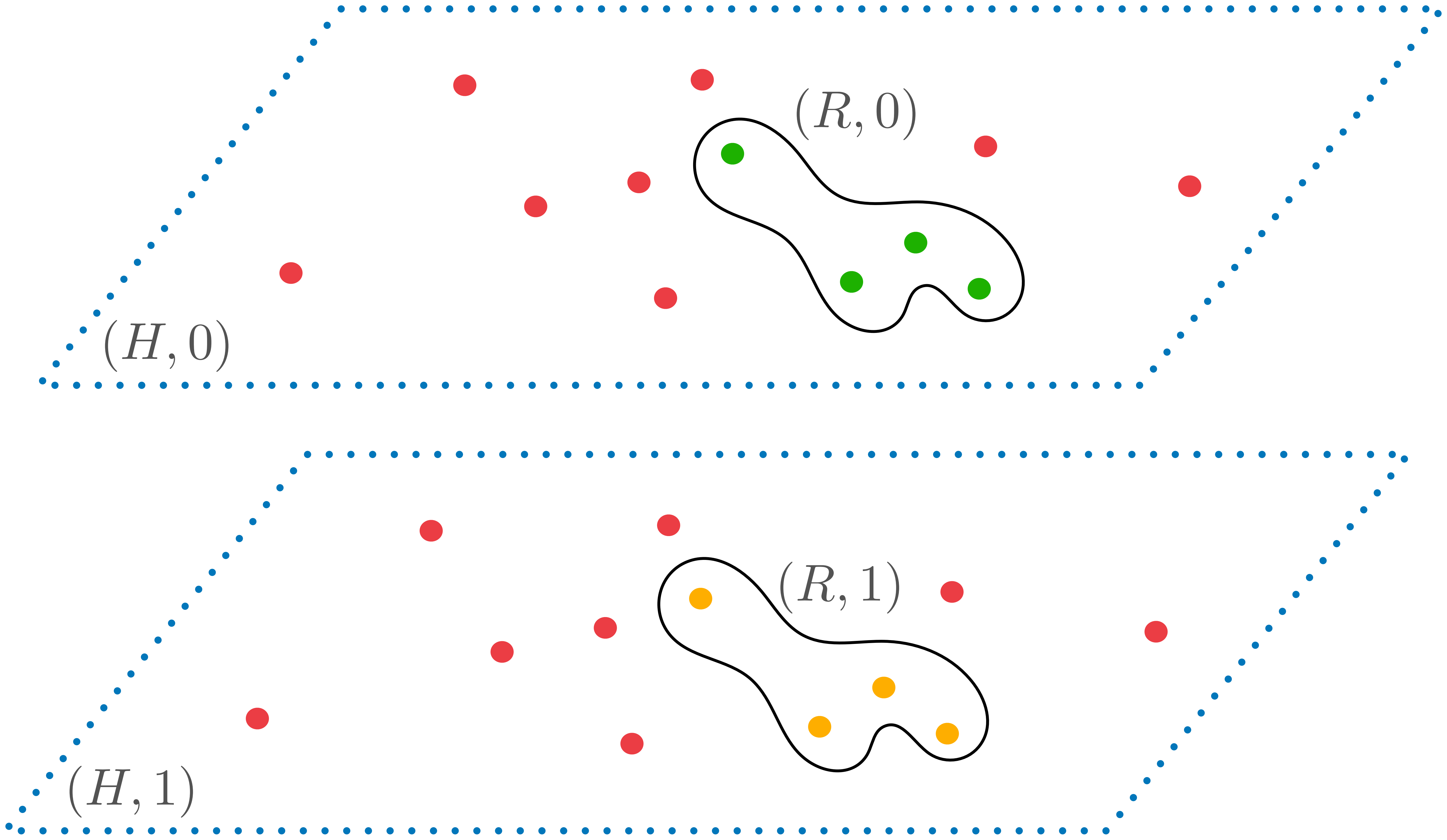

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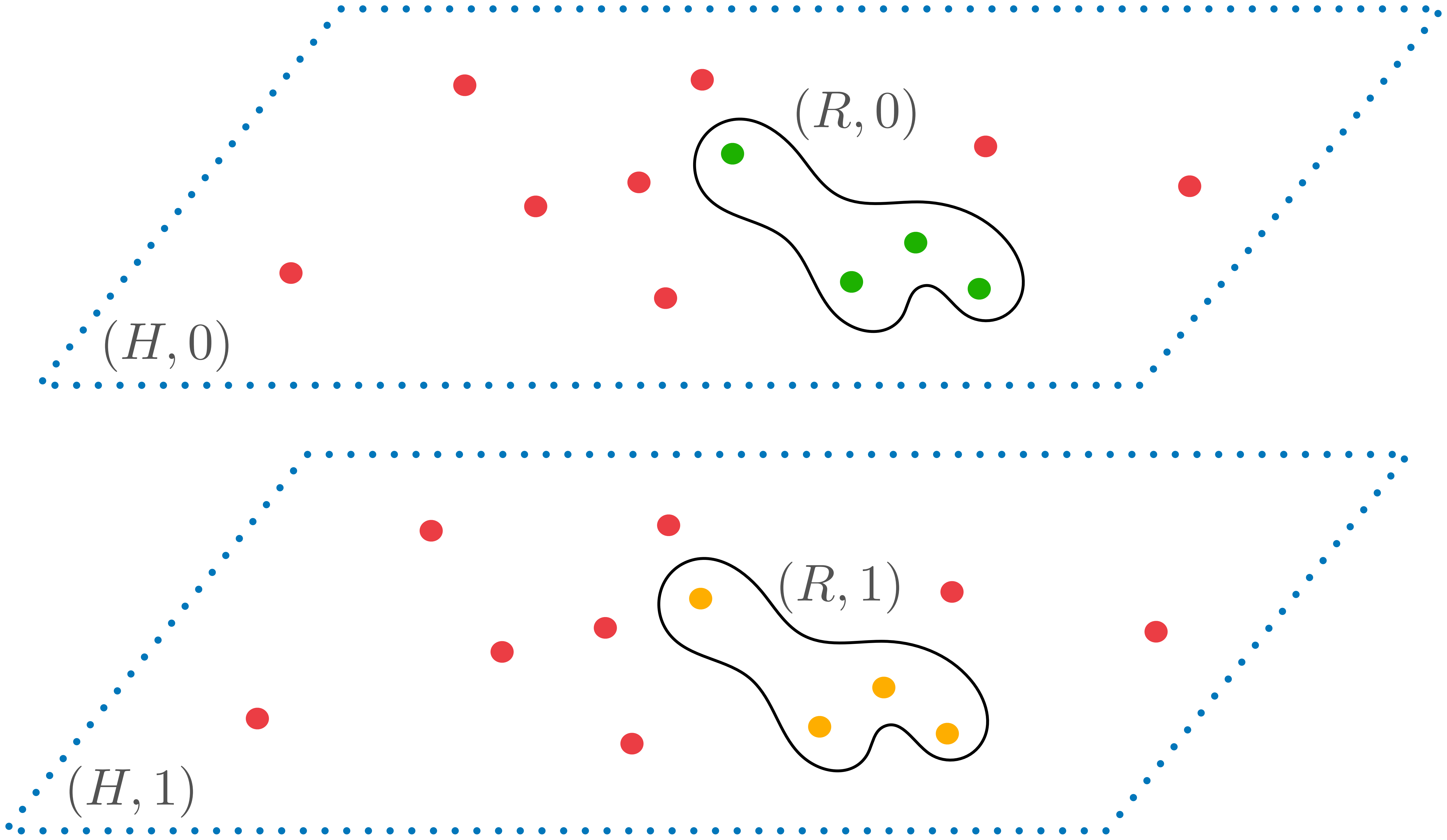

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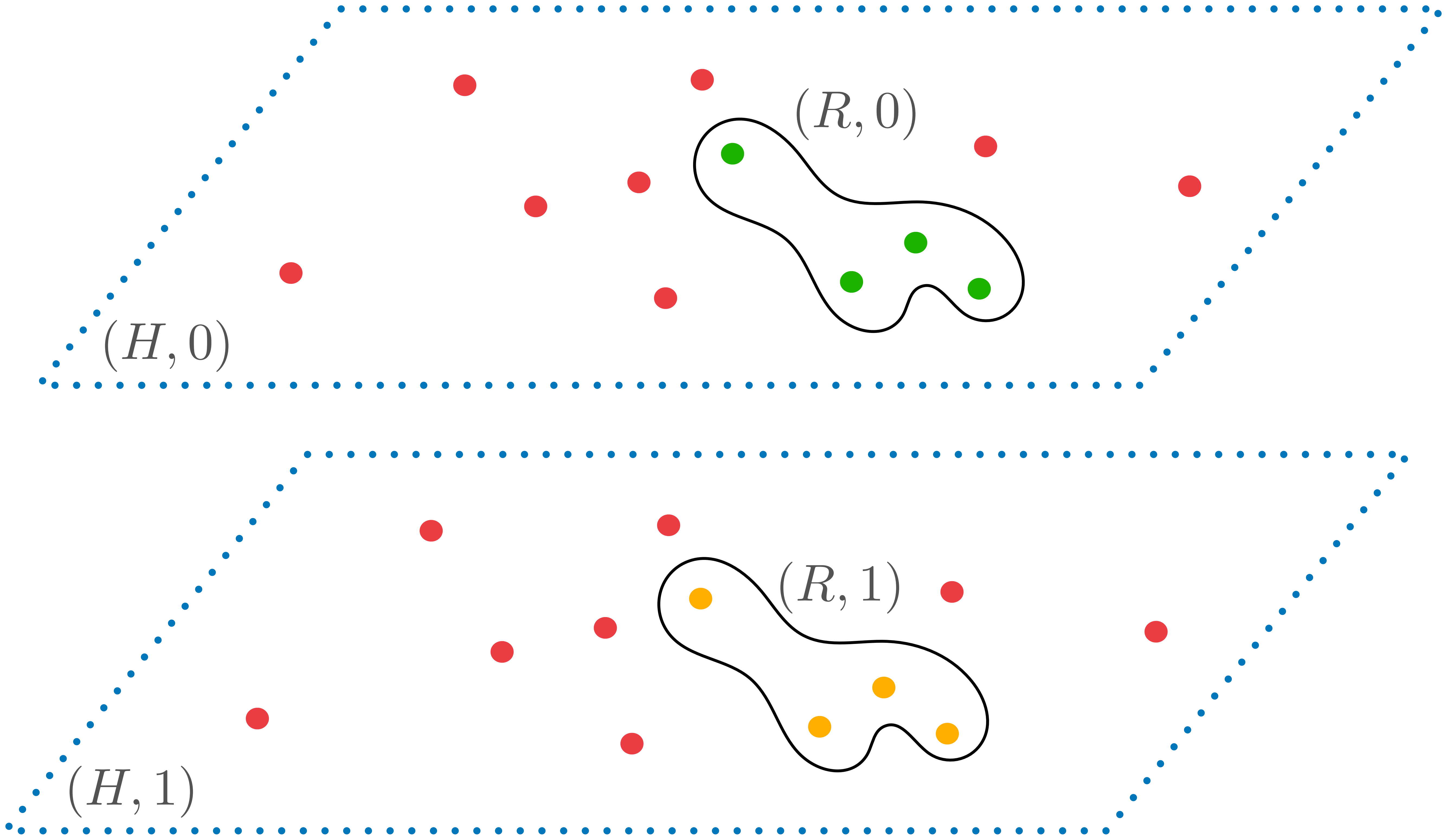
- If $R \cap S = \emptyset$, $\mathscr{V}^\dagger \mathscr{S}_{R\cup S}^\dagger \mathscr{V}^{-1\dagger} = \mathscr{V}^\dagger \mathscr{S}_R^\dagger \mathscr{V}^{-1\dagger} \mathscr{V}^\dagger \mathscr{S}_S^\dagger \mathscr{V}^{-1\dagger}$



The “T” process

Imprinting a GUR in local transformation

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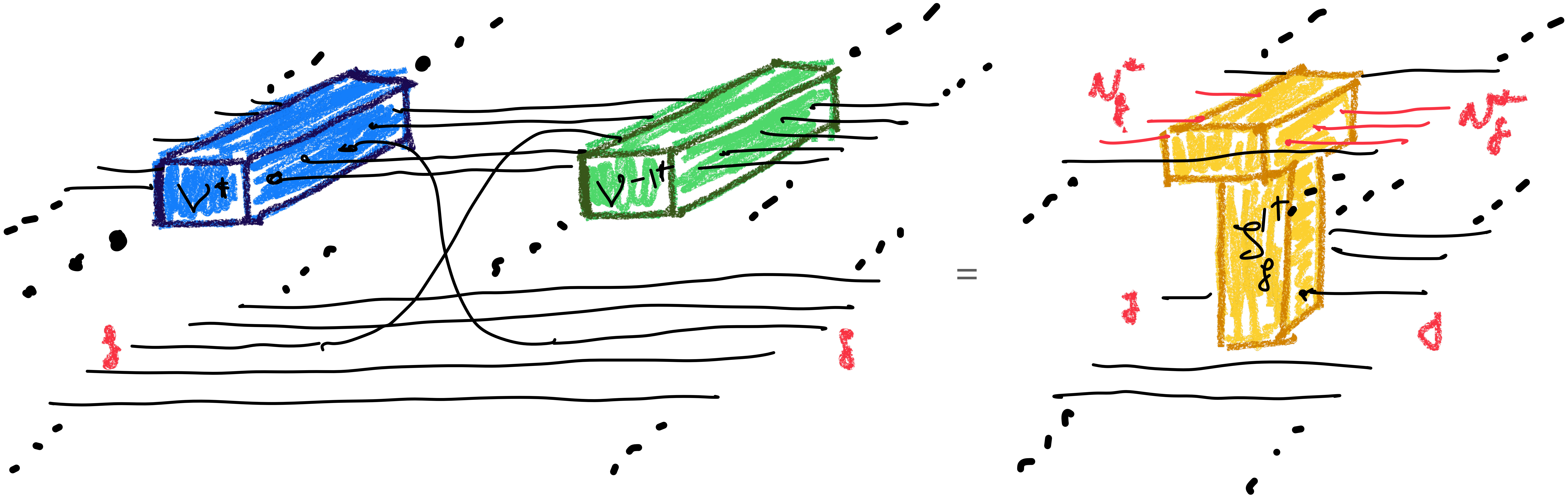
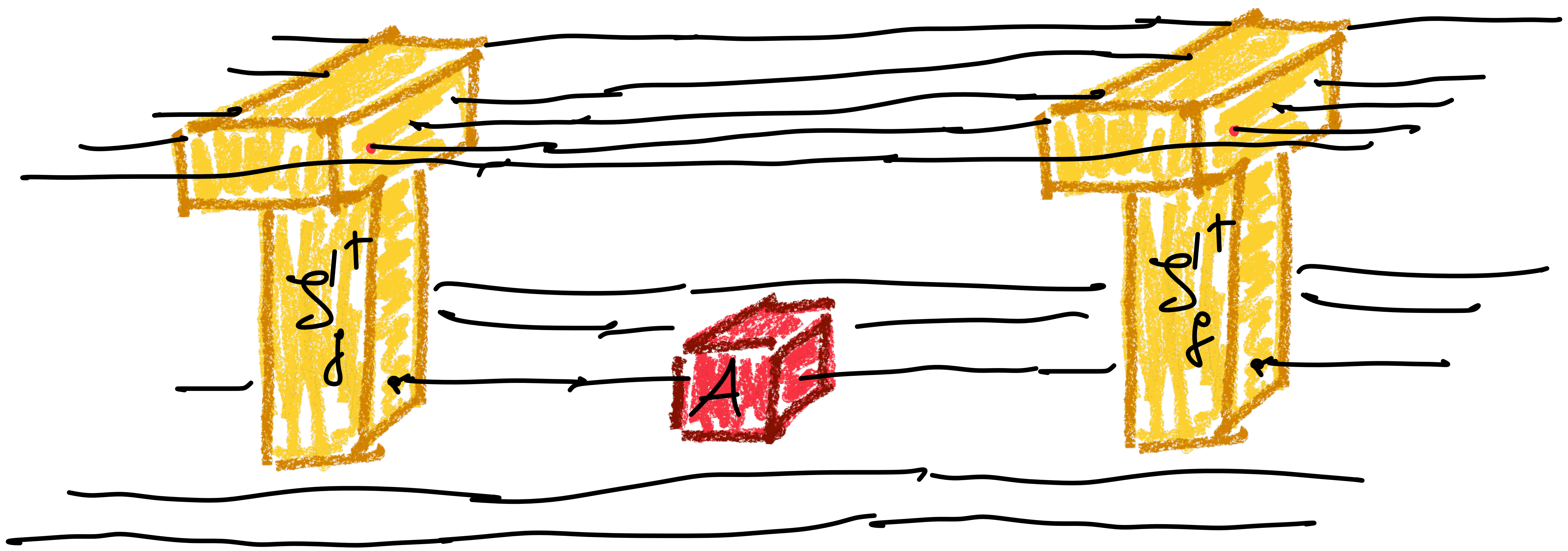
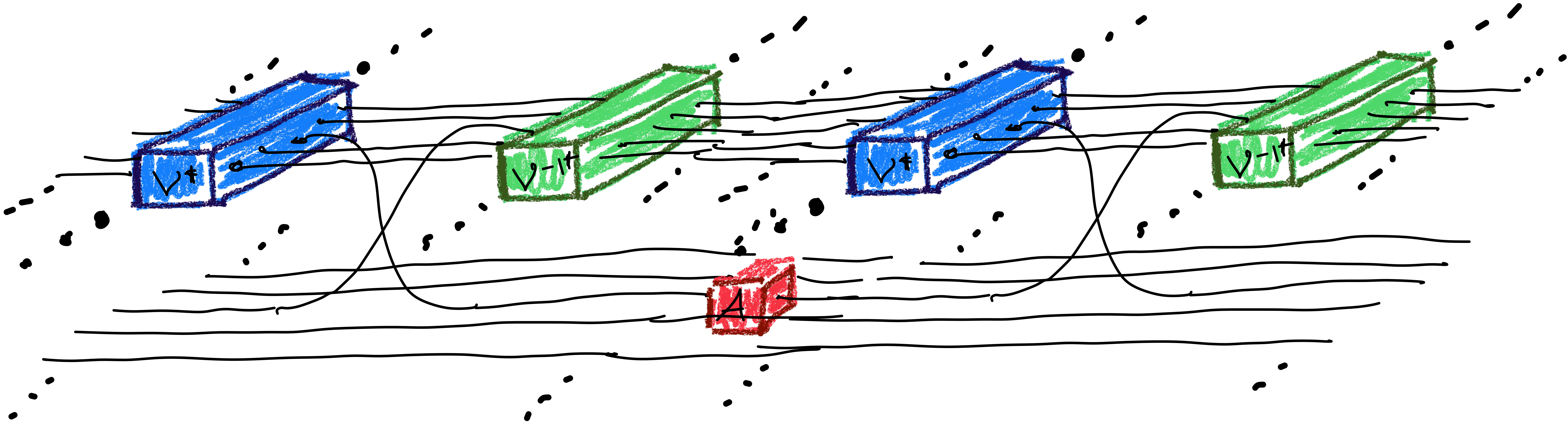
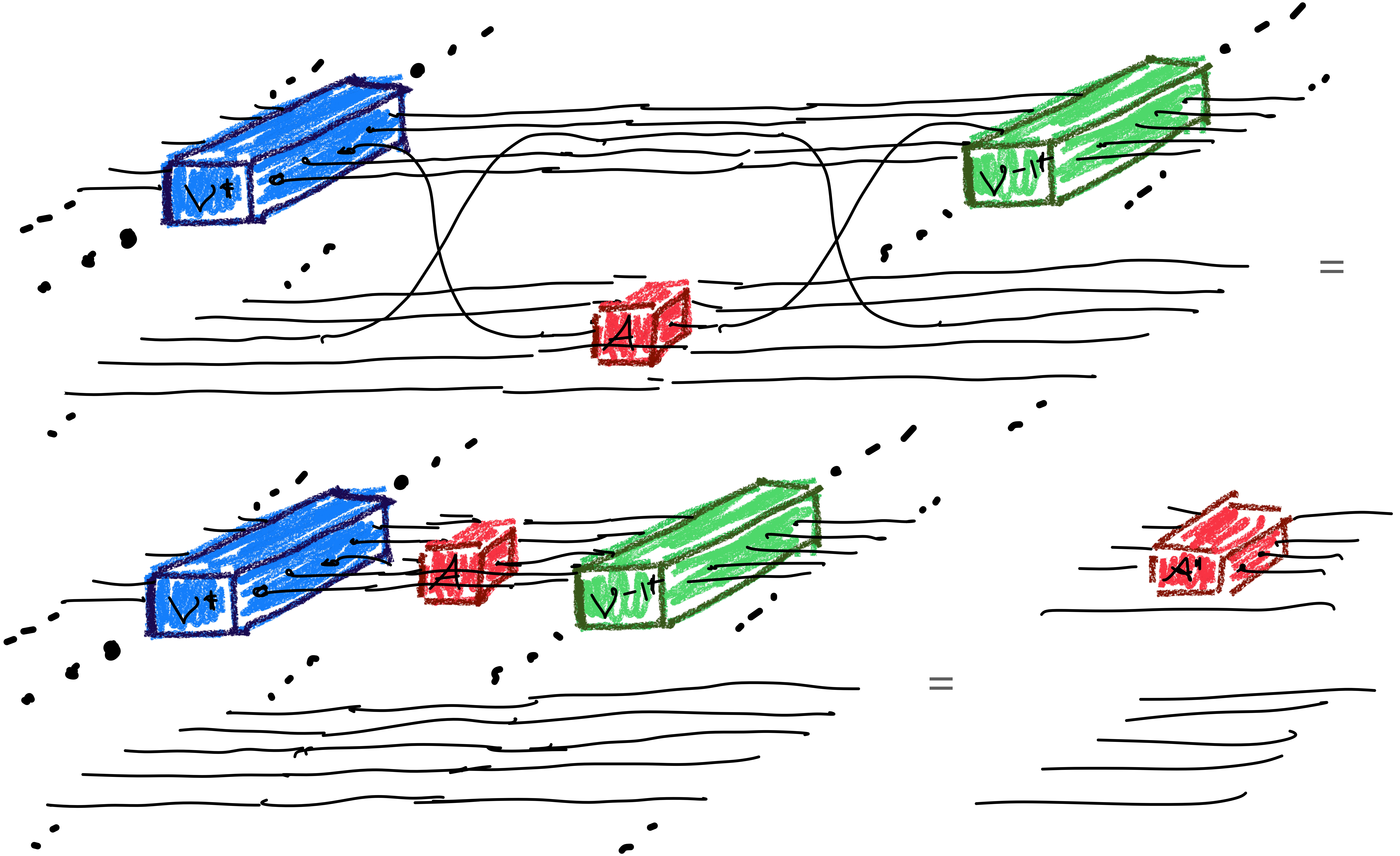
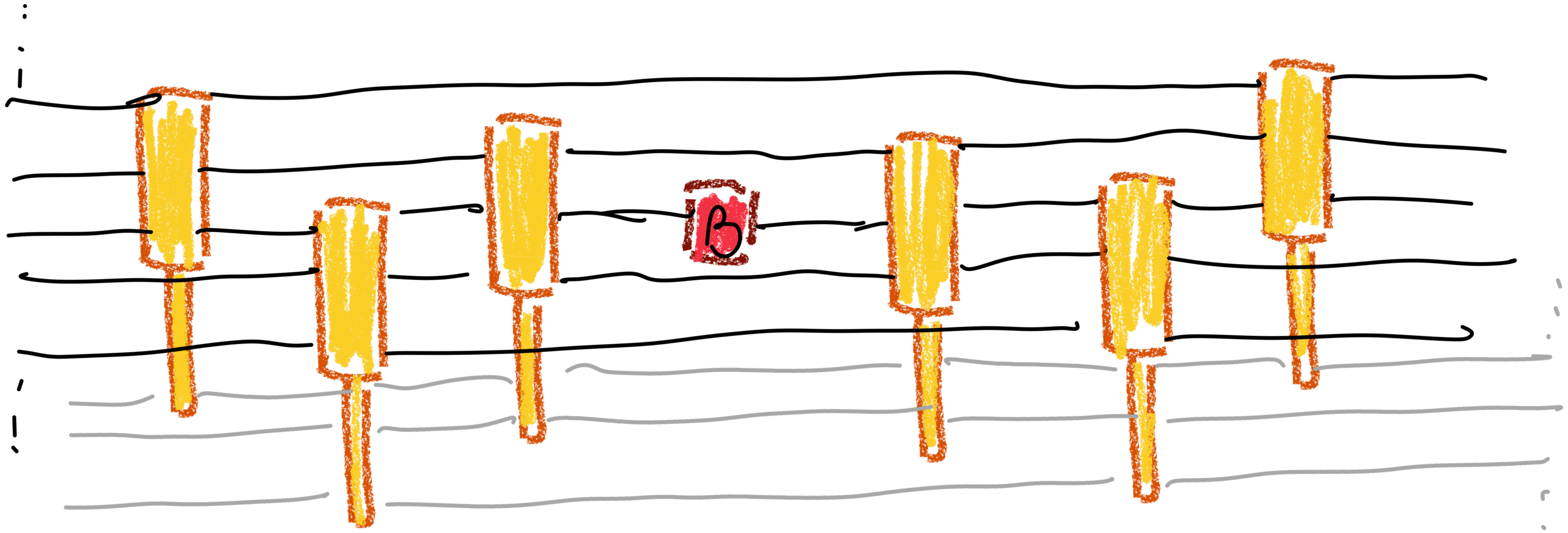
- The transformation ${\mathscr{S}'}_g^\dagger$ identifies both neighbourhoods of g

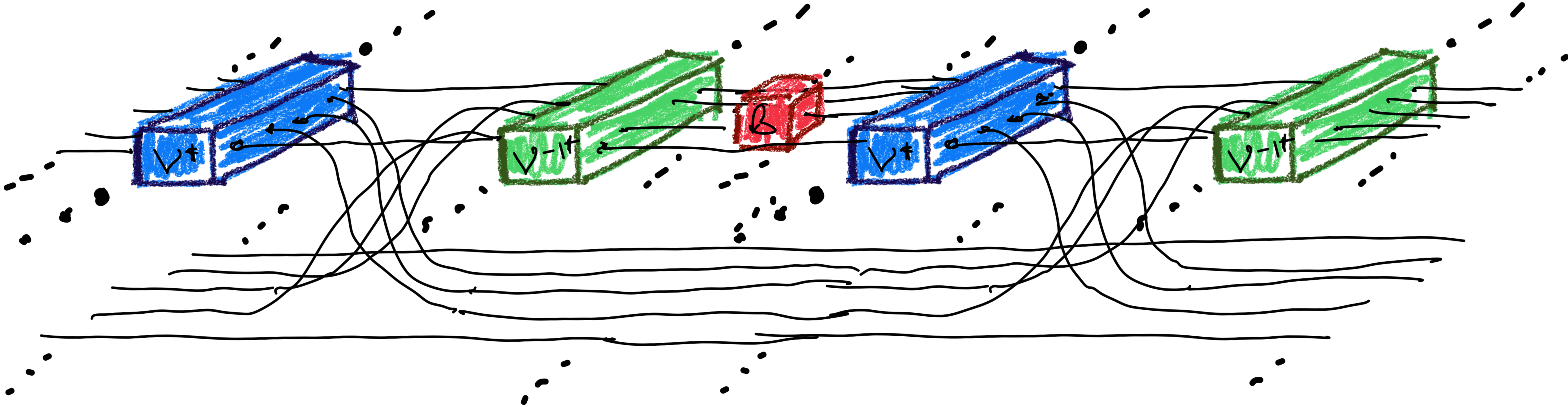


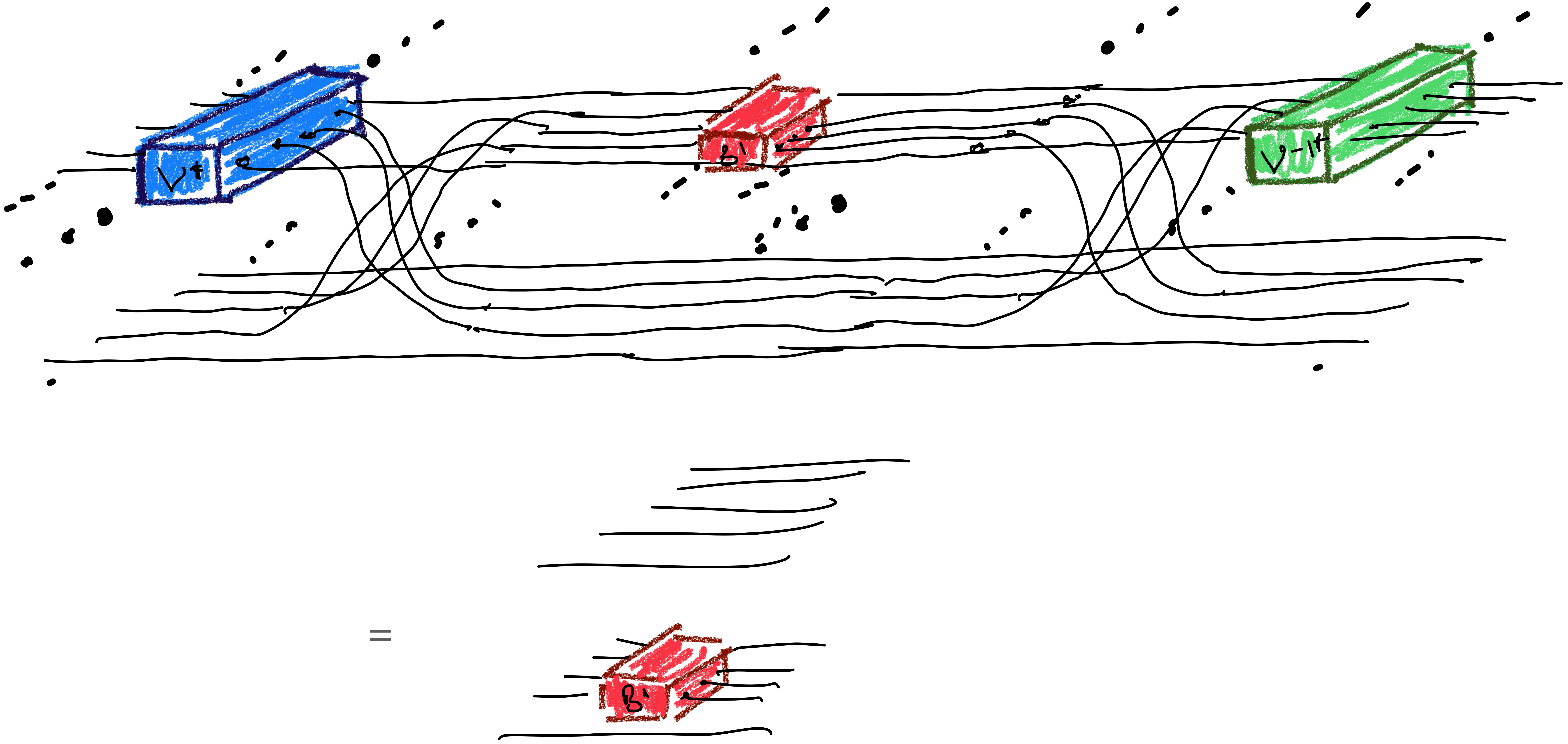

A


V†
V-1†
V†
V-1†
A


V†
A
V-1†
=
V†
A
V-1†
=
A'


B


$V^\dagger$
$V^{-1\dagger}$
B
$V^\dagger$
$V^{-1\dagger}$


V†
B'
V⁻¹†
=
B'

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Reversing the correspondence

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  - ${\mathscr{S}'}^{\dagger}_{g}(\mathscr{I}^{\dagger}_{N^{+}_{g}} \otimes \mathscr{A}^{\dagger}){\mathscr{S}'}^{\dagger}_{g} = {\mathscr{A}'}^{\dagger} \otimes \mathscr{I}^{\dagger}_{g}$
    $\prod_{h \in N^{-}_{g}} {\mathscr{S}'}^{\dagger}_{h}(\mathscr{B}^{\dagger}_{g} \otimes \mathscr{I}^{\dagger}_{N^{-}_{g}}){\mathscr{S}'}^{\dagger}_{h} = \mathscr{I}^{\dagger}_{N^{+}_{N^{-}_{g}}} \otimes {\mathscr{B}'}^{\dagger}$

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- defines a GUR through

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- Intuitively speaking: homogeneity consists in “treating” every cell equally

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Requirements for a precise definition

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- The definition of homogeneity requires first a precise notion of
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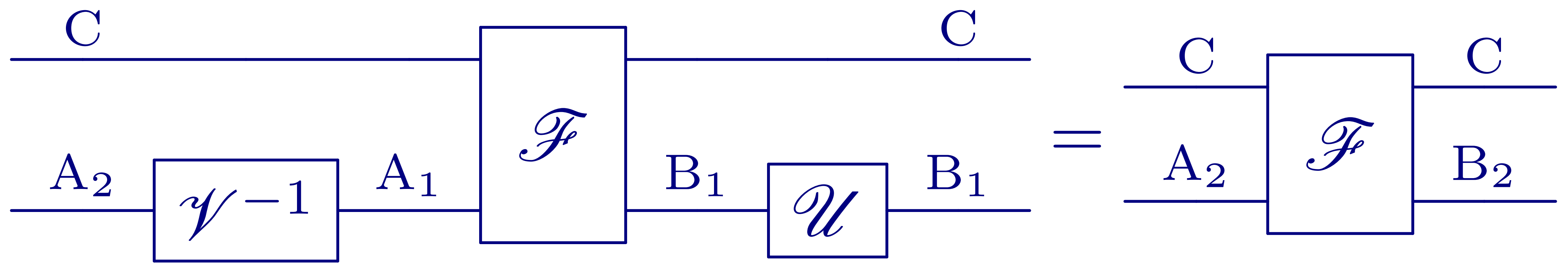
    - exchanging the role of two cells

Operationally equivalent regions

Running the same test



$$\mathscr{V} \in [\![A_1 \to A_2]\!], \quad \mathscr{U} \in [\![B_1 \to B_2]\!]$$

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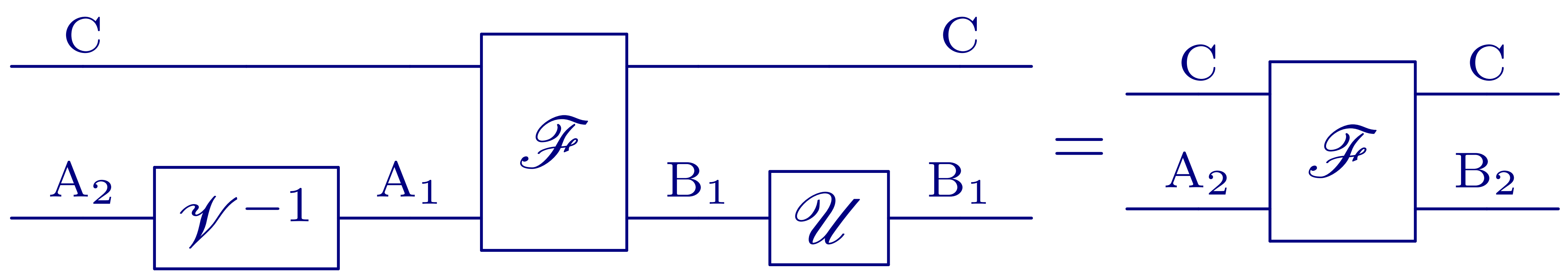
- The reversible transformation defines the notion of “performing the same test”



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Operationally equivalent regions

Accounting for internal structure

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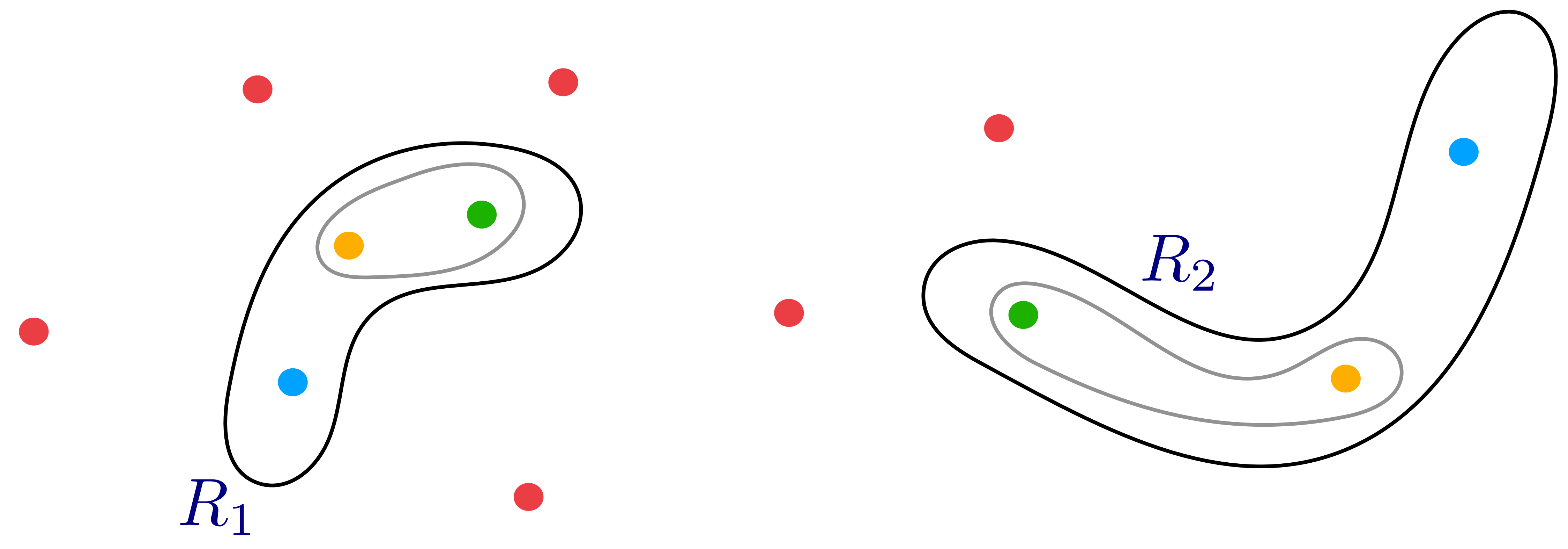

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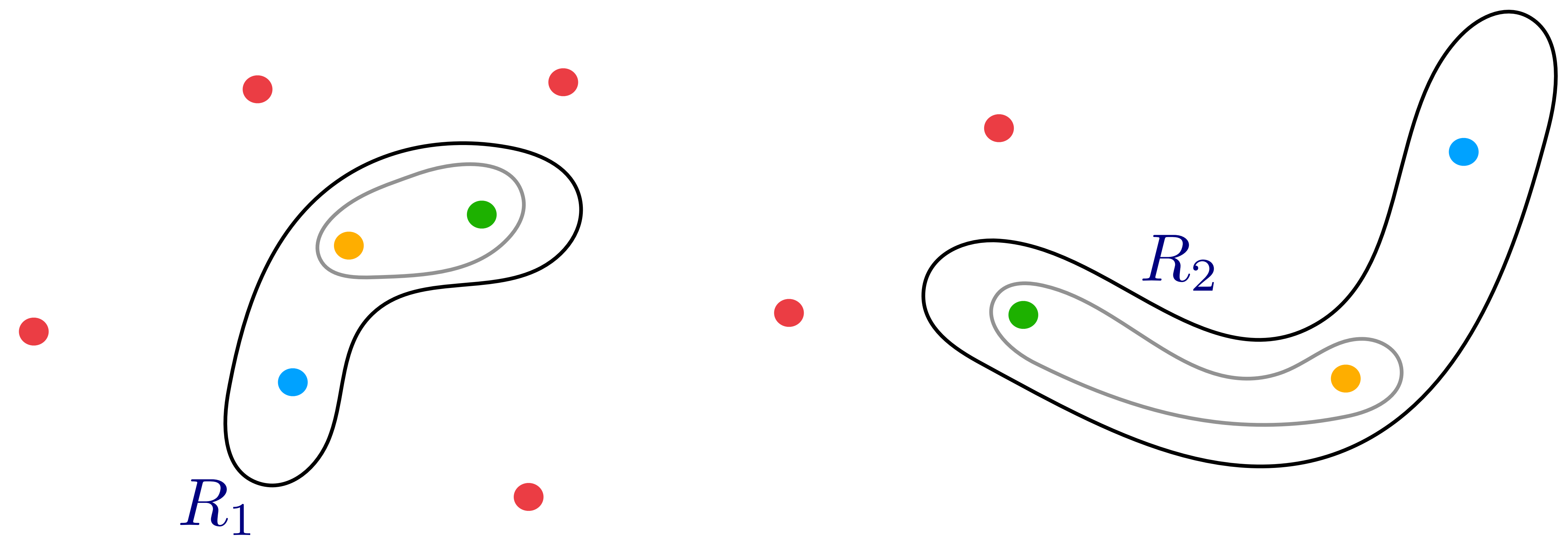

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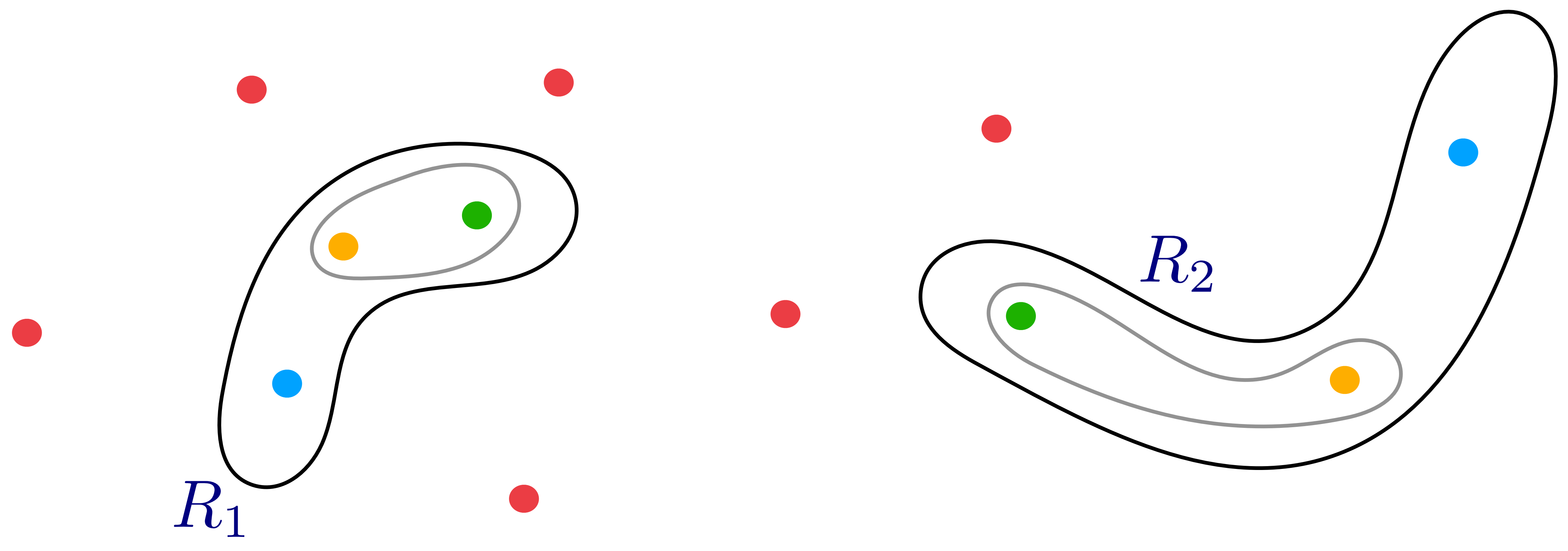

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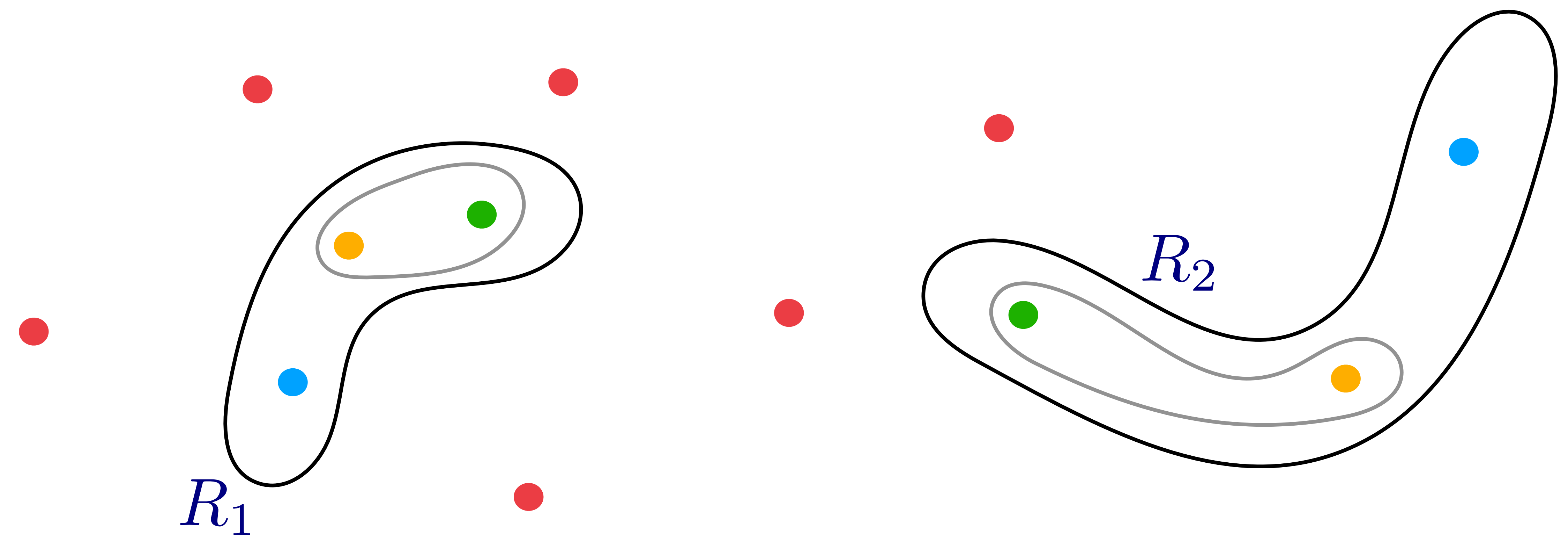

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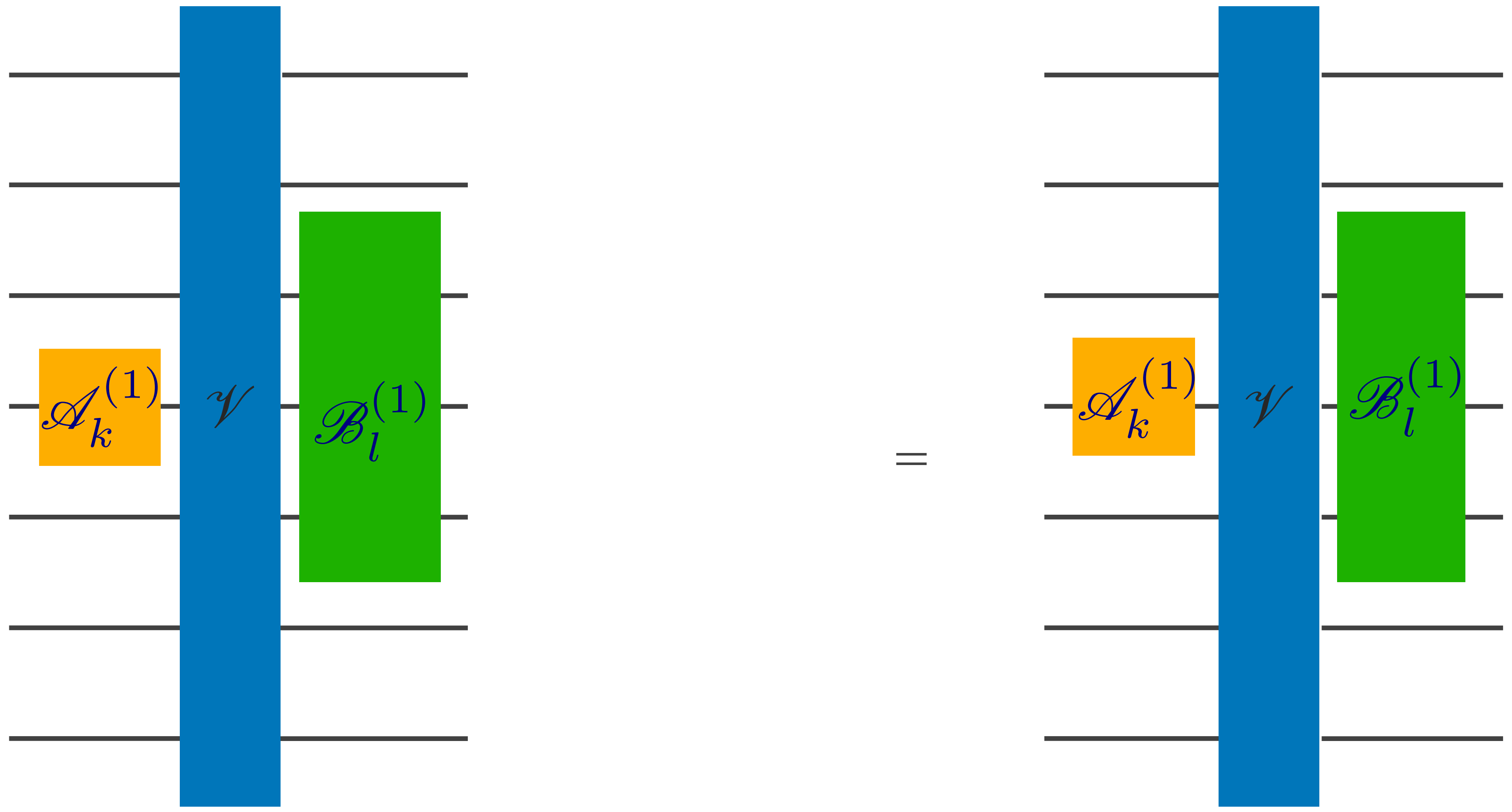
  $R_1 = \{g_1, \ldots, g_l\},\ R_2 = \{h_1, \ldots, h_l\},\ \mathrm{A}_{g_i} \cong \mathrm{A}_{h_i}$
- In this case $\mathscr{U}_{R_1,R_2} = \bigotimes_{g_i \in R_1} \mathscr{U}_{g_1,h_i}$

Homologous regions

"Treated in the same way"

- Given a GUR $(G, \mathrm{A}, \mathscr{V}^\dagger)$, the region R_1 is homologous to an o.e. region R_2 if $N^+_{R_1}$ is o.e. to $N^+_{R_2}$, and there exists a GUR $(G, \mathrm{A}, \mathscr{T}^\dagger)$ such that $\mathscr{B}_l^{(1)}\mathscr{V}\mathscr{A}_k^{(1)} = \mathscr{T}^{-1}\mathscr{B}_l^{(2)}\mathscr{V}\mathscr{A}_k^{(2)}\mathscr{T}$. We write $R_1 \bowtie_{\mathscr{T}} R_2$



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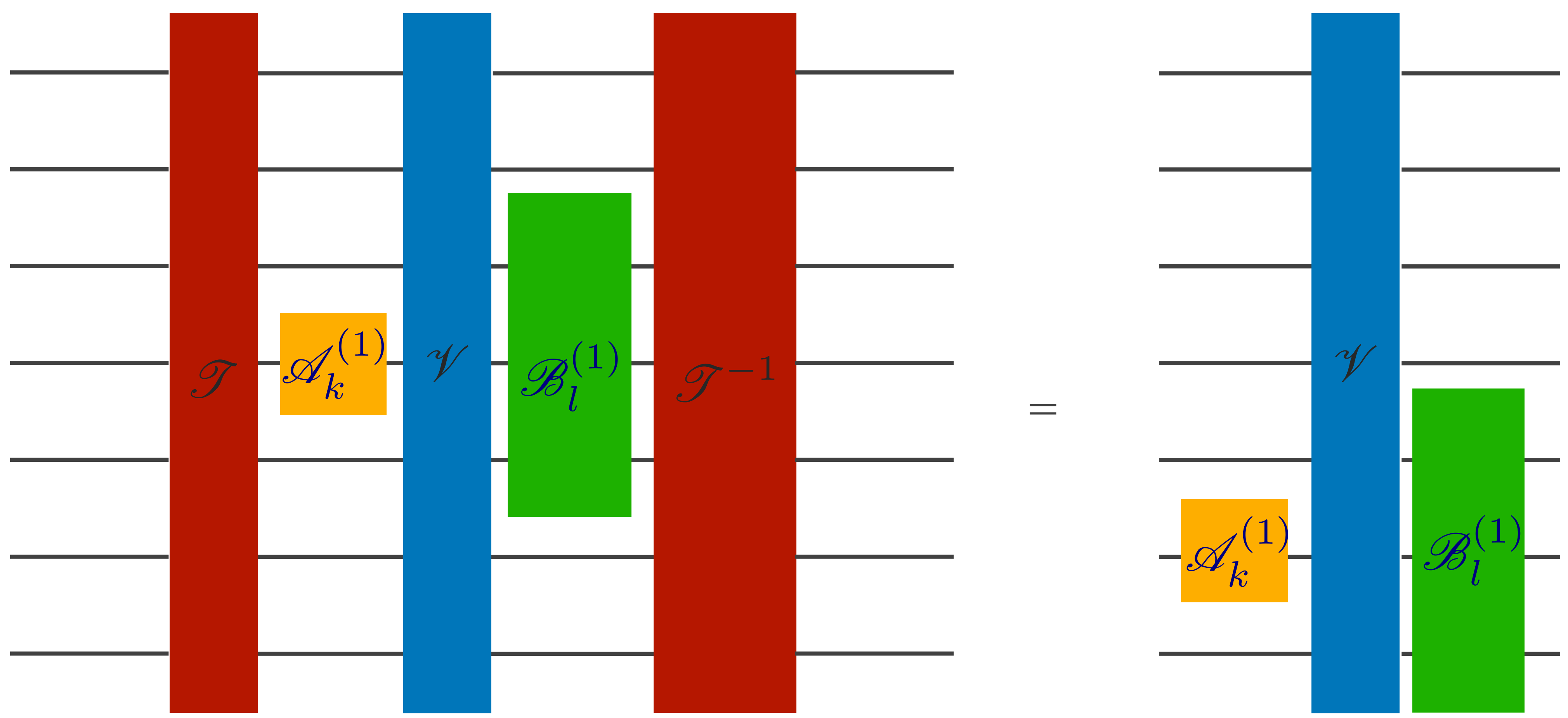

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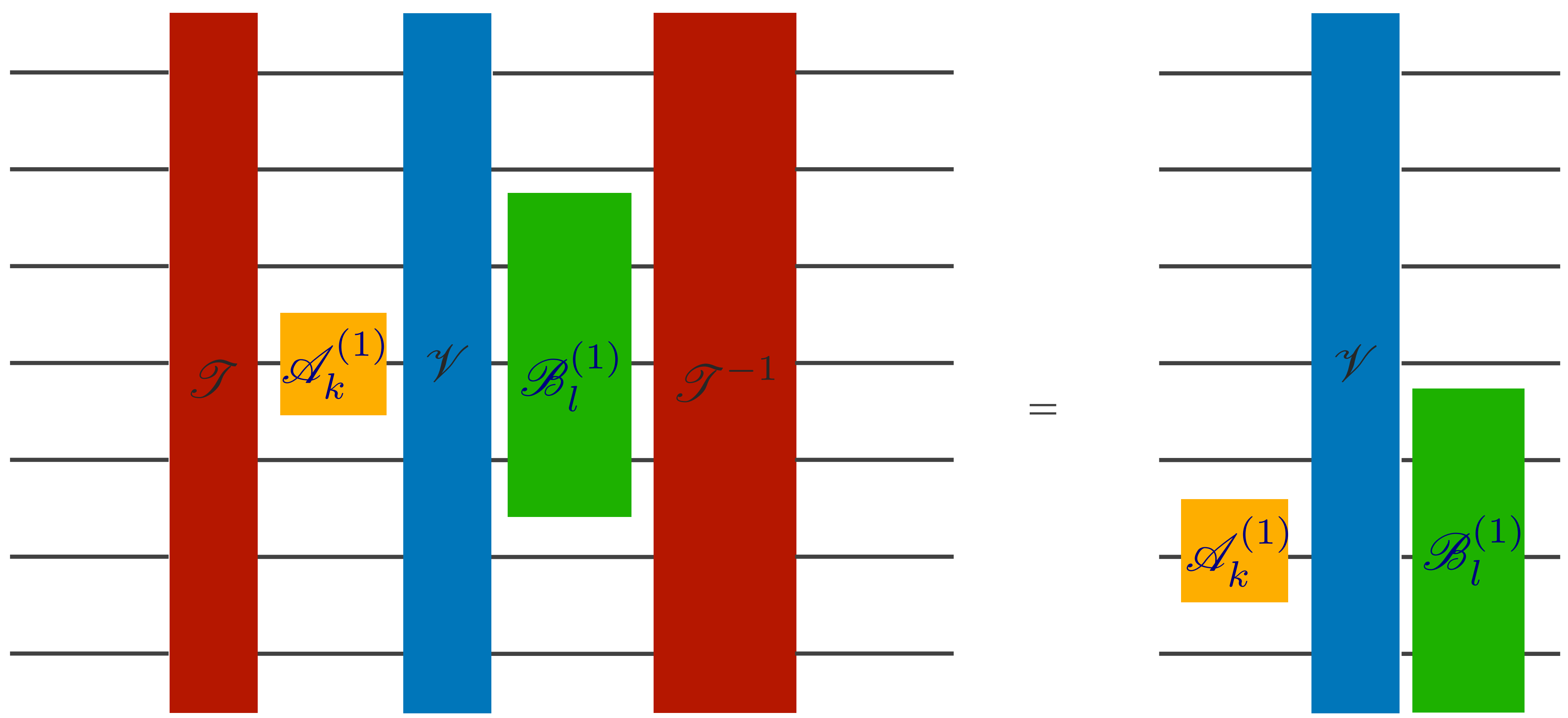

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- If $R_1 \bowtie_{\mathscr{T}} R_2$, one has $\mathscr{T}^{-1}\mathscr{V}\mathscr{T} = \mathscr{V}$

Absolute discrimination

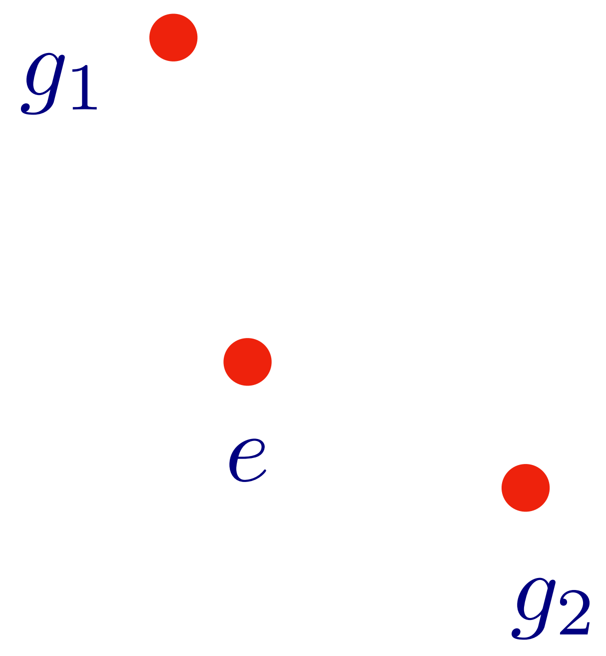
"Treated differently"

- Given a GUR $(G, \mathrm{A}, \mathscr{V}^\dagger)$, the cells g_1 and g_2 are discriminated by $\mathscr{V}^\dagger$ if for every GUR $(G, \mathrm{A}, \mathscr{T}^\dagger)$ there exists a region $R_1 \ni g_1$ that is not homologous to any $R_2 \ni g_2$ through $\mathscr{T}$

Relative discrimination

“Seen differently from e”

- Given a GUR $(G, \mathrm{A}, \mathscr{V}^\dagger)$, the cells g_1 and g_2 are discriminated by $\mathscr{V}^\dagger$ with respect to $e \in G$ if for every GUR $(G, \mathrm{A}, \mathscr{T}^\dagger)$ there exists a region $R_1 \ni \{g_1, e\}$ that is not homologous to any $R_2 \ni \{g_2, e\}$, through $\mathscr{T}$, with $e \bowtie_{\mathscr{T}} e$ and $g_1 \bowtie_{\mathscr{T}} g_2$



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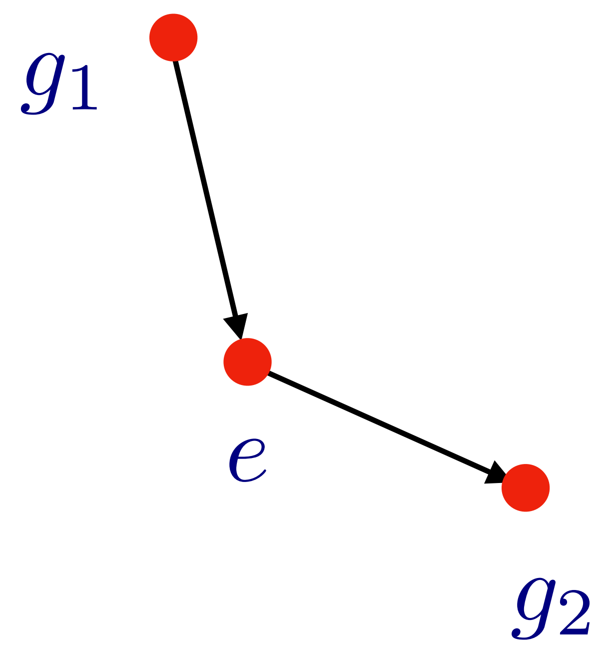

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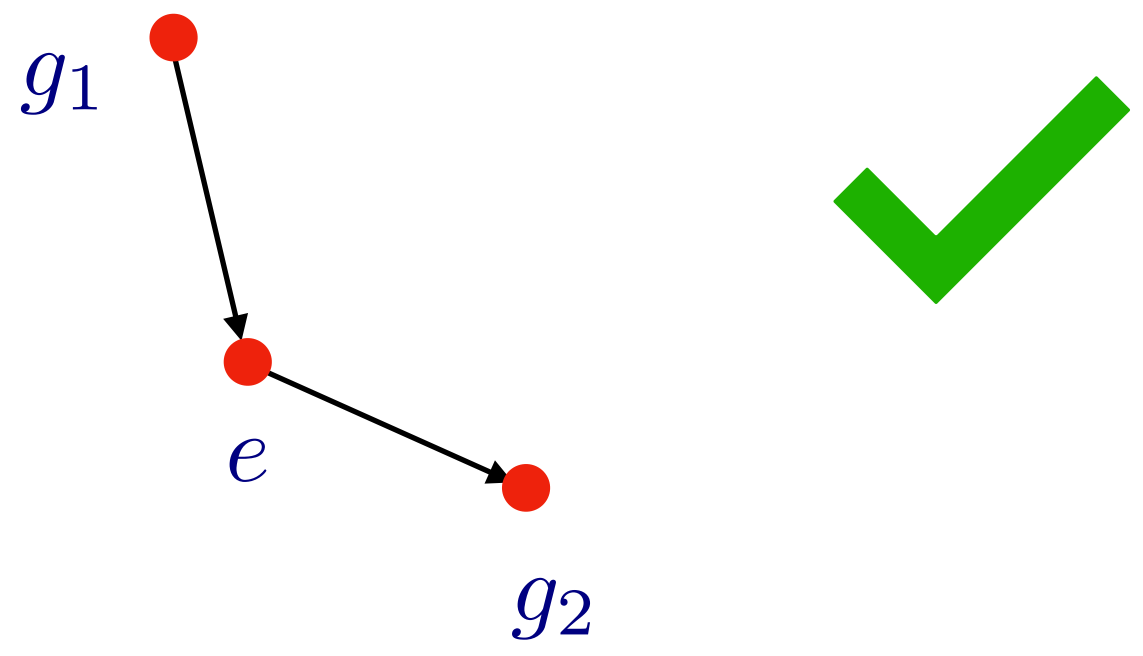

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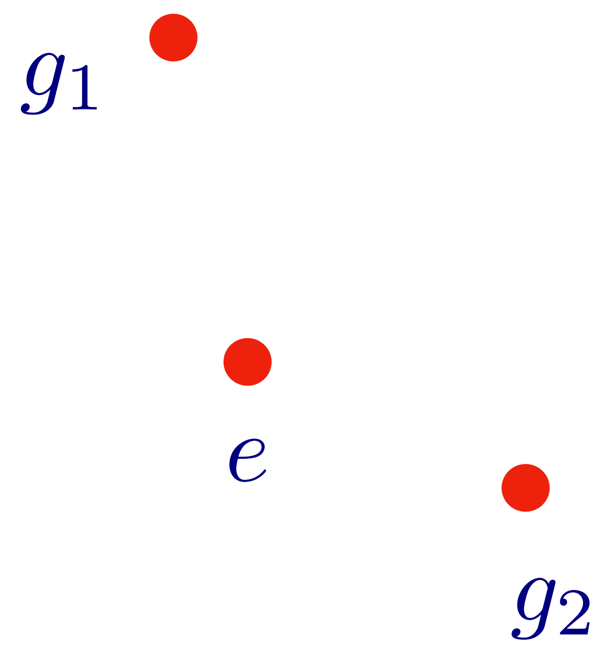

Relative discrimination

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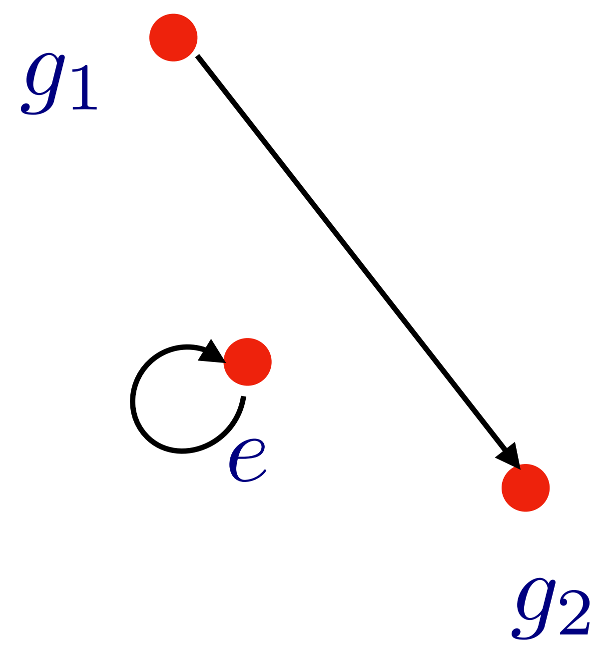

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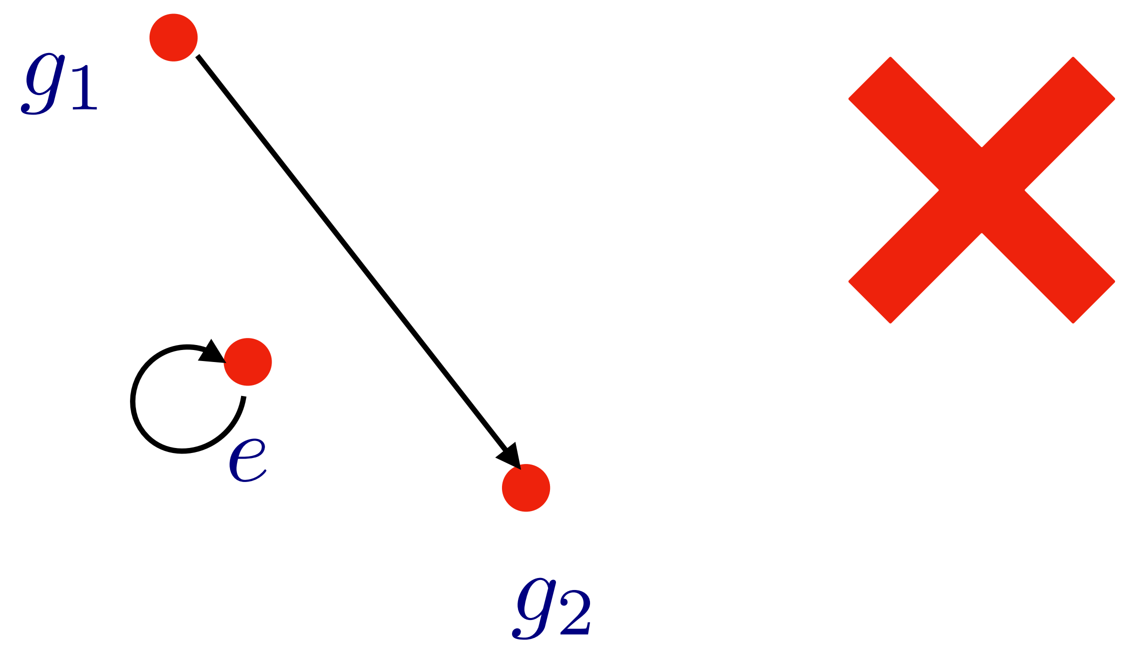

Homogeneity

No two cells are discriminated

- Given a GUR $(G, \mathrm{A}, \mathscr{V}^{\dagger})$, we say that it is homogeneous if for every two cells g_1, g_2 $\mathscr{V}^{\dagger}$ does not discriminate them, but it does relatively to a third cell e

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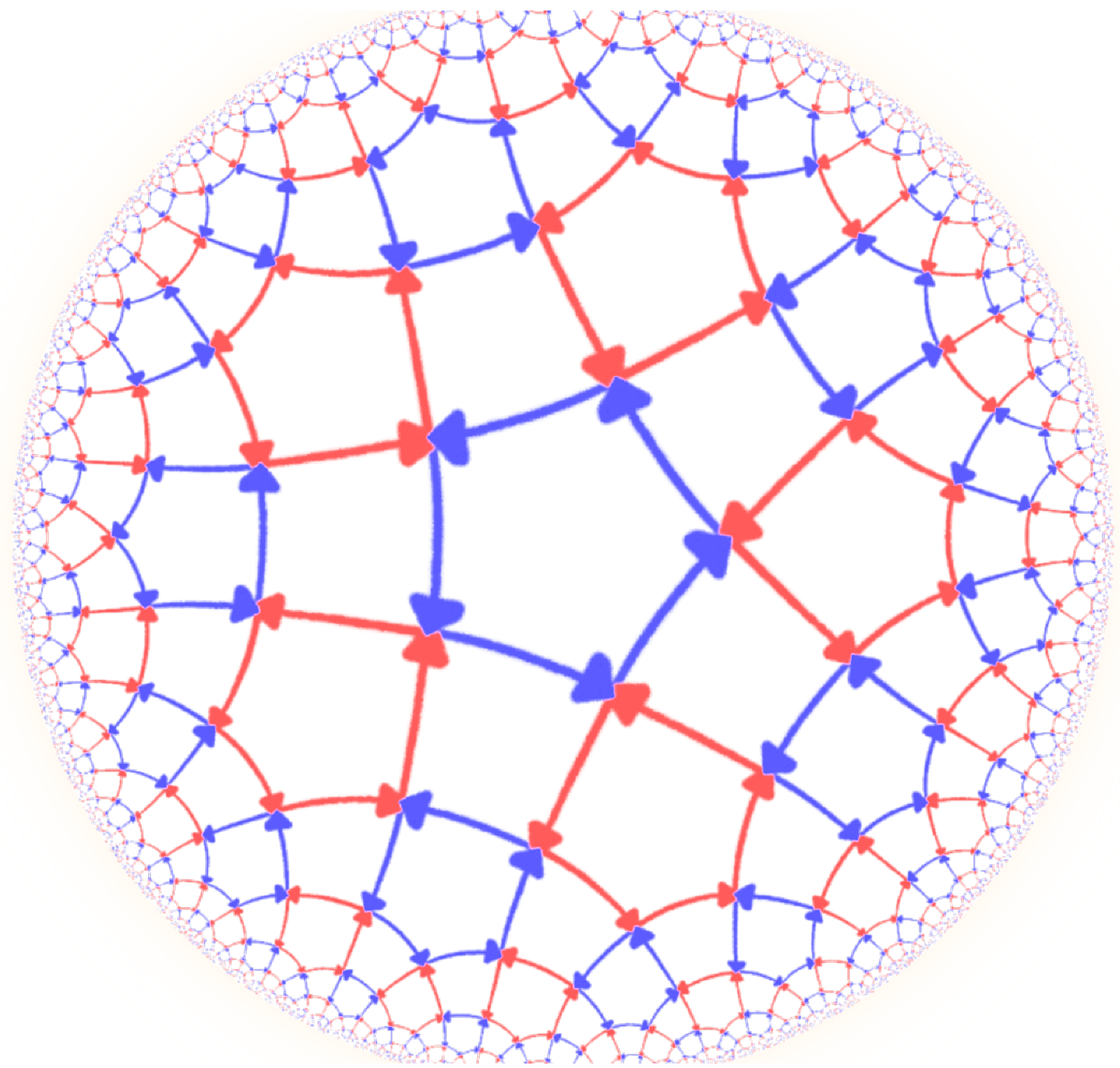
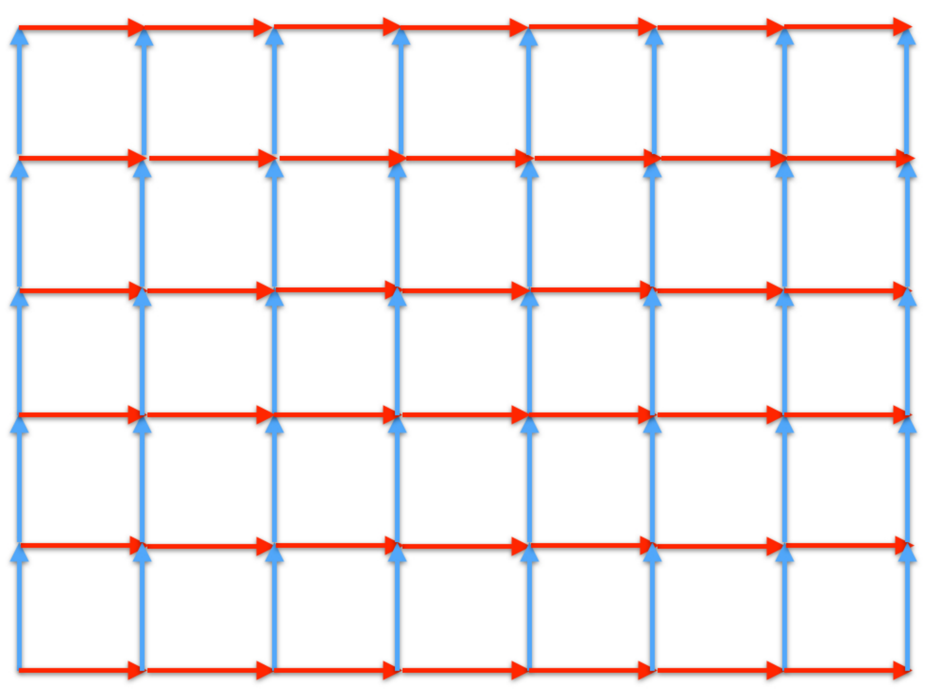
- For every pair (g_1, g_2), there is $\mathscr{T}$ such that for every $R_1 \ni g_1$ one finds $R_2 \ni g_2$ with $R_1 \bowtie_{\mathscr{T}} R_2$

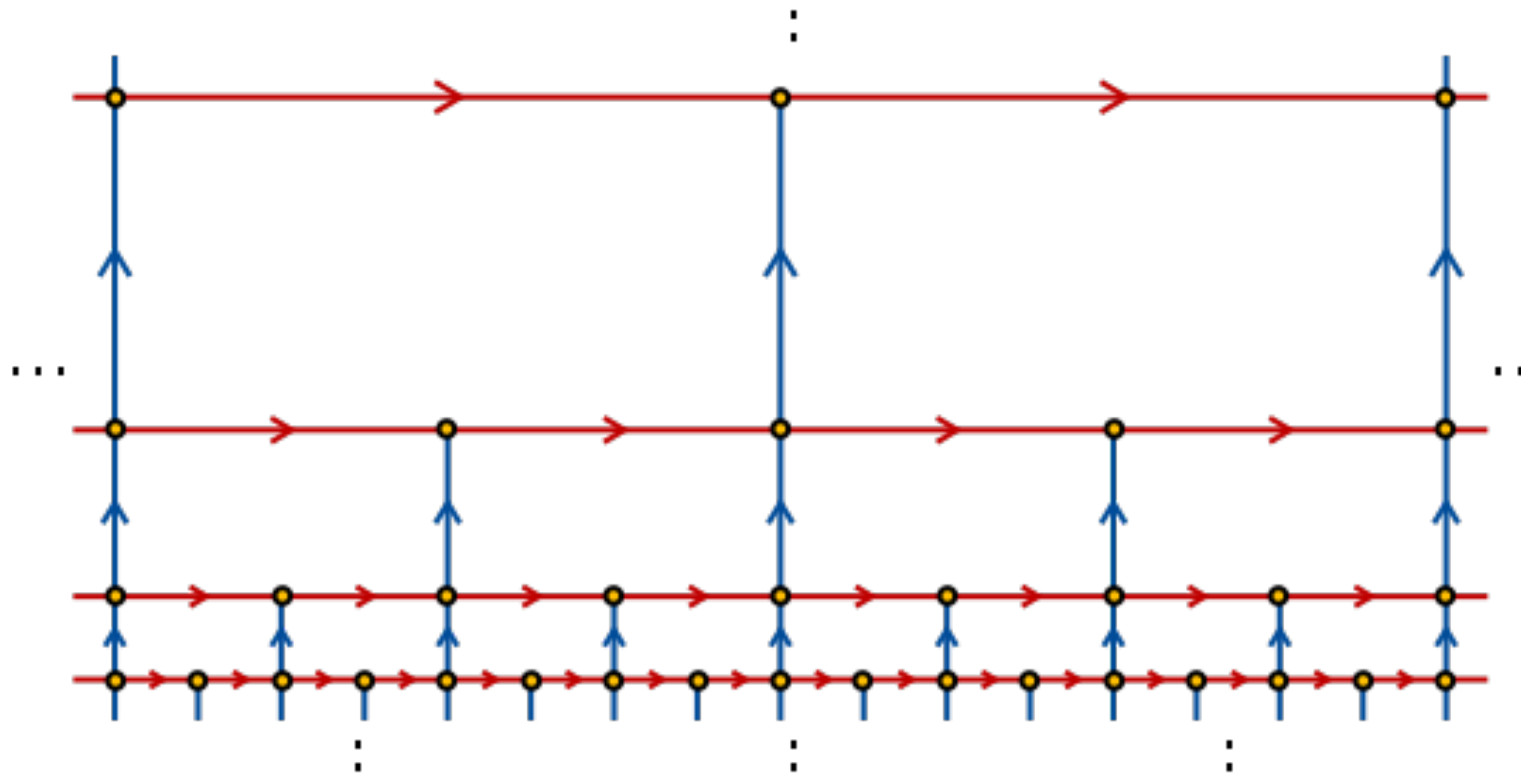
Translation group

From homogeneity

- We collect all the GURs $\mathscr{T}$ such that $R_i \bowtie_{\mathscr{T}} R_j$, for every $R_i \ni g_i$, $R_j \ni g_j$, and $g_i, g_j \in G$







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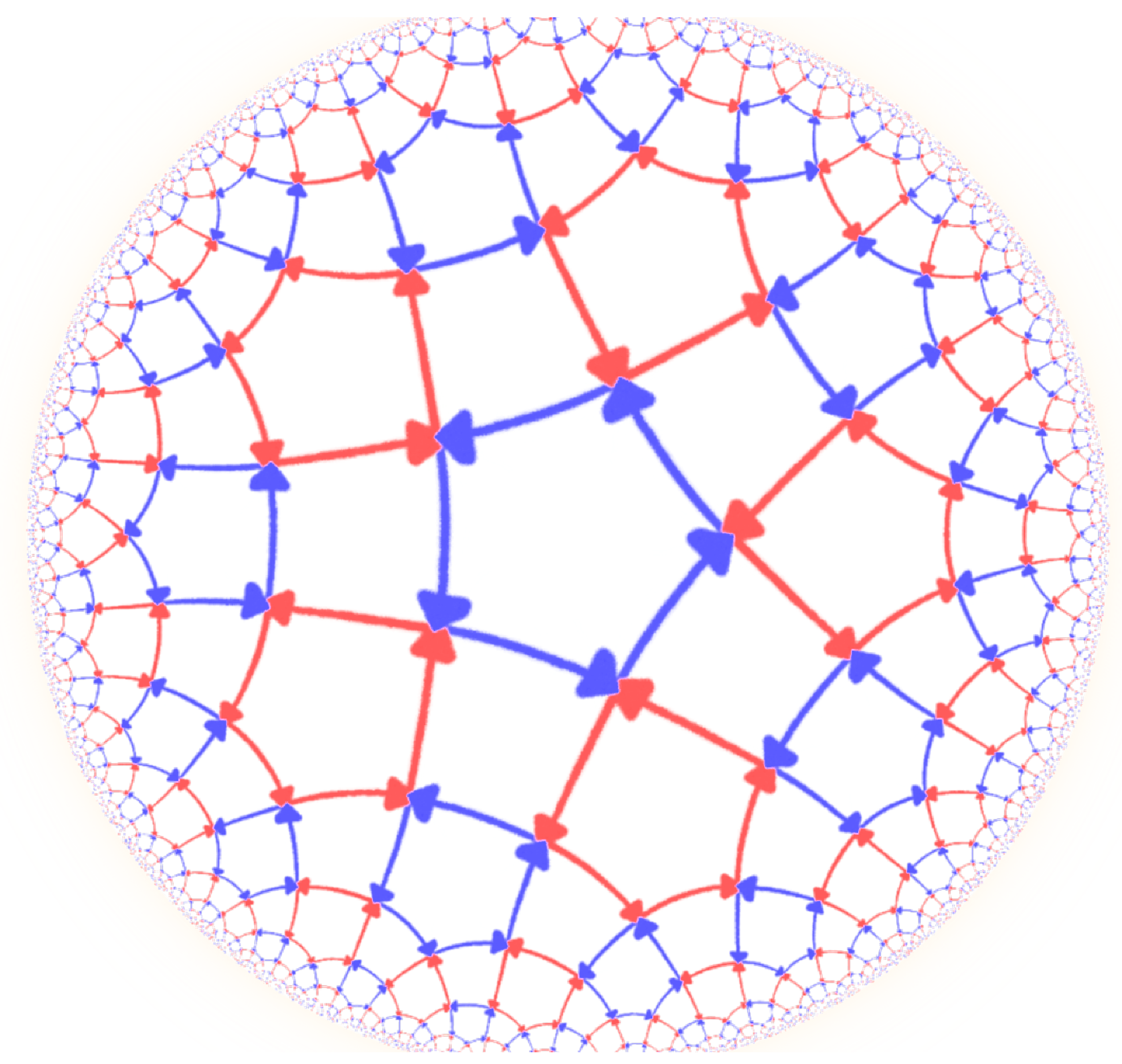
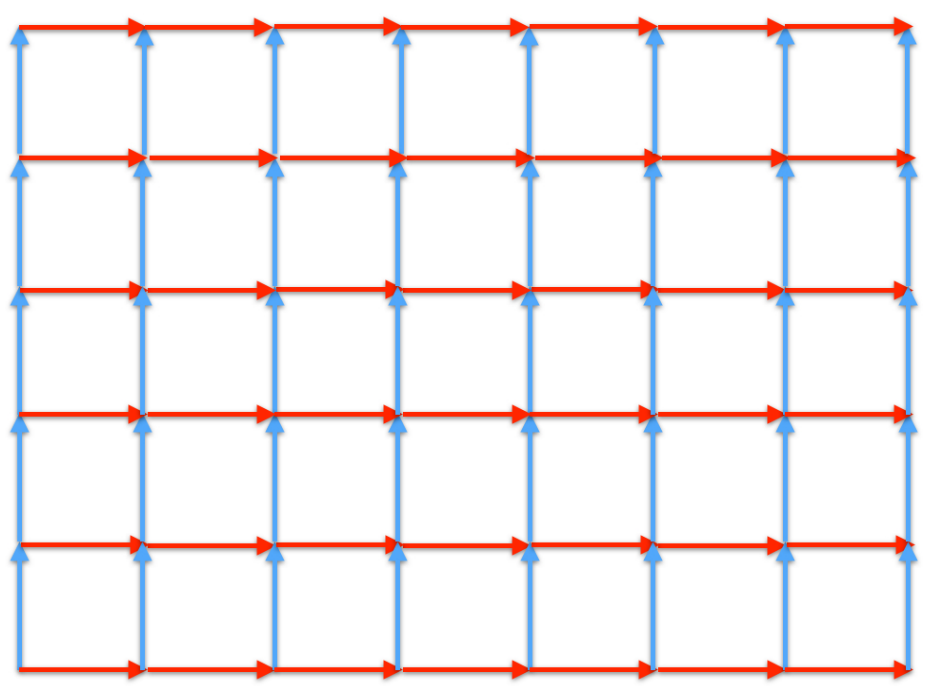
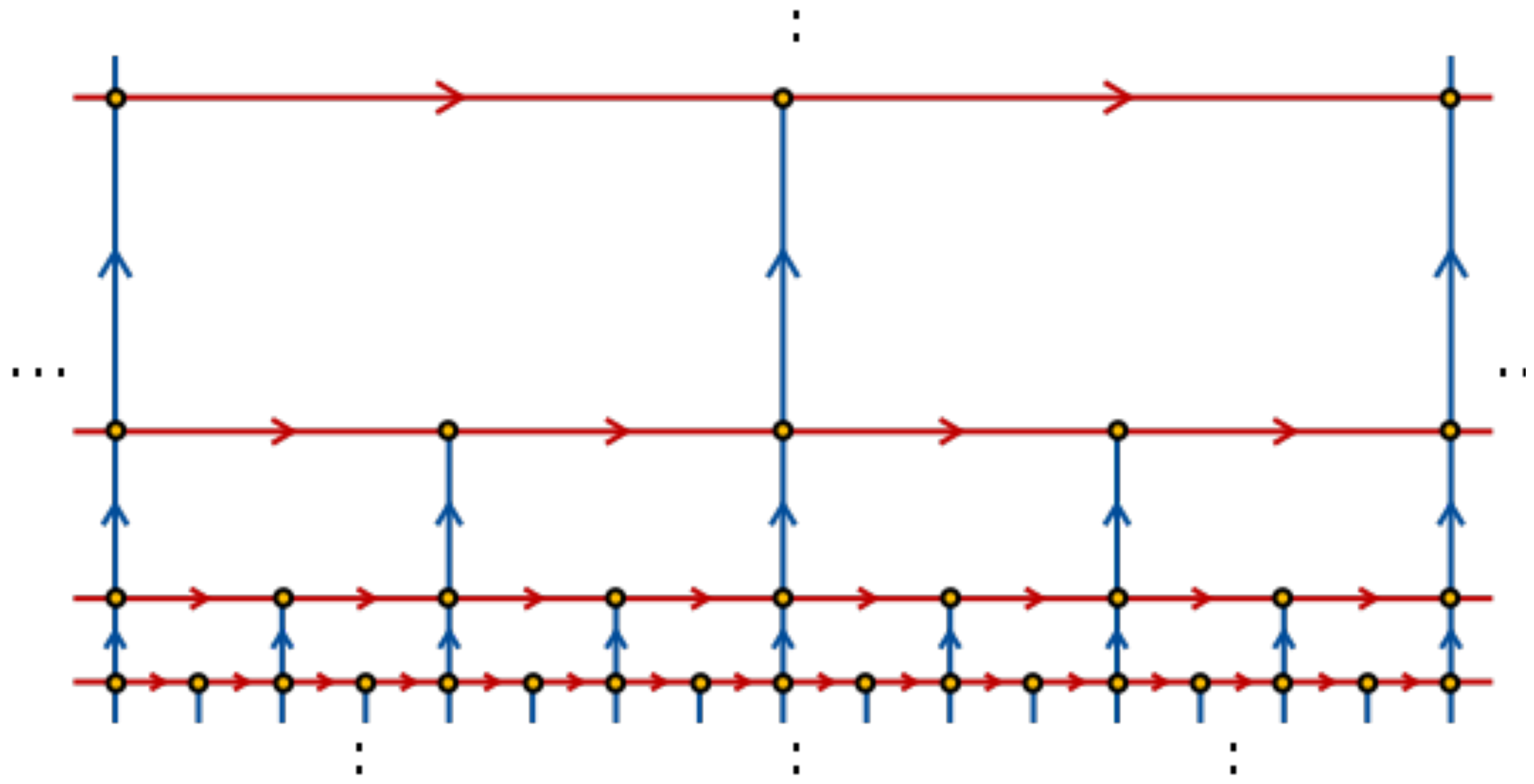
- They form a group of permutations of G







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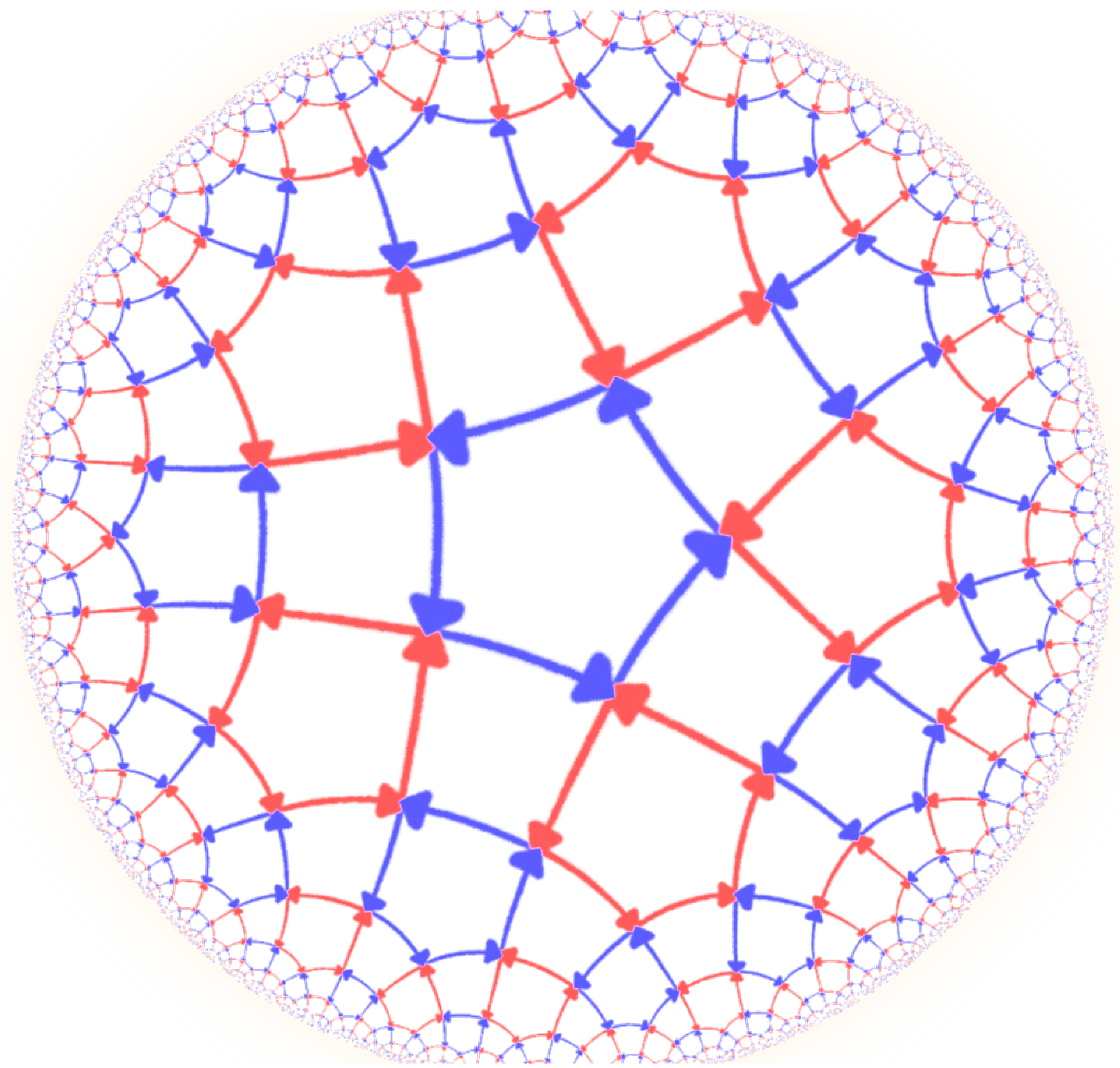

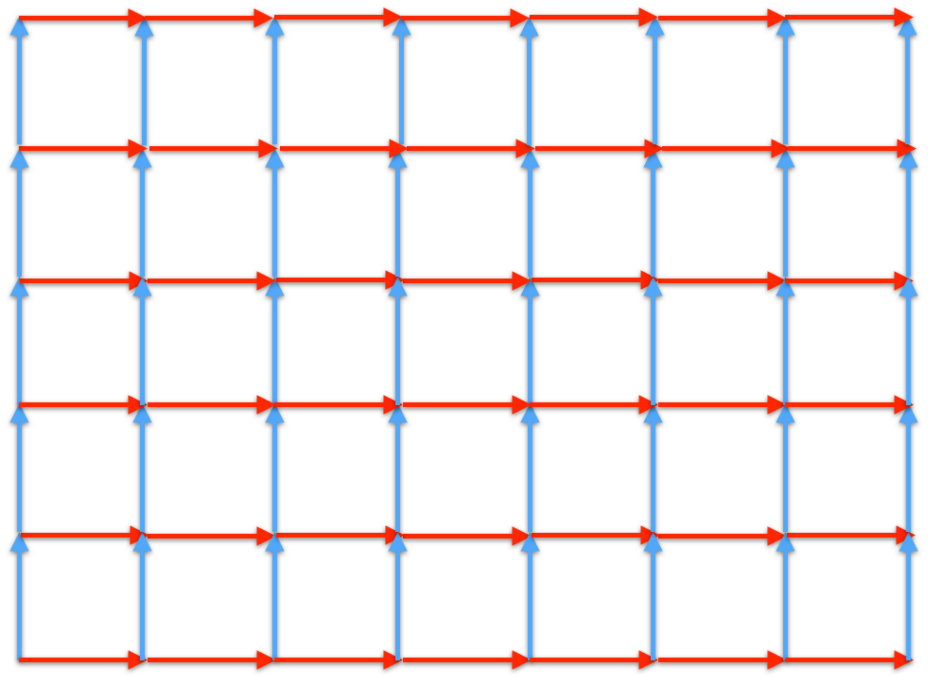

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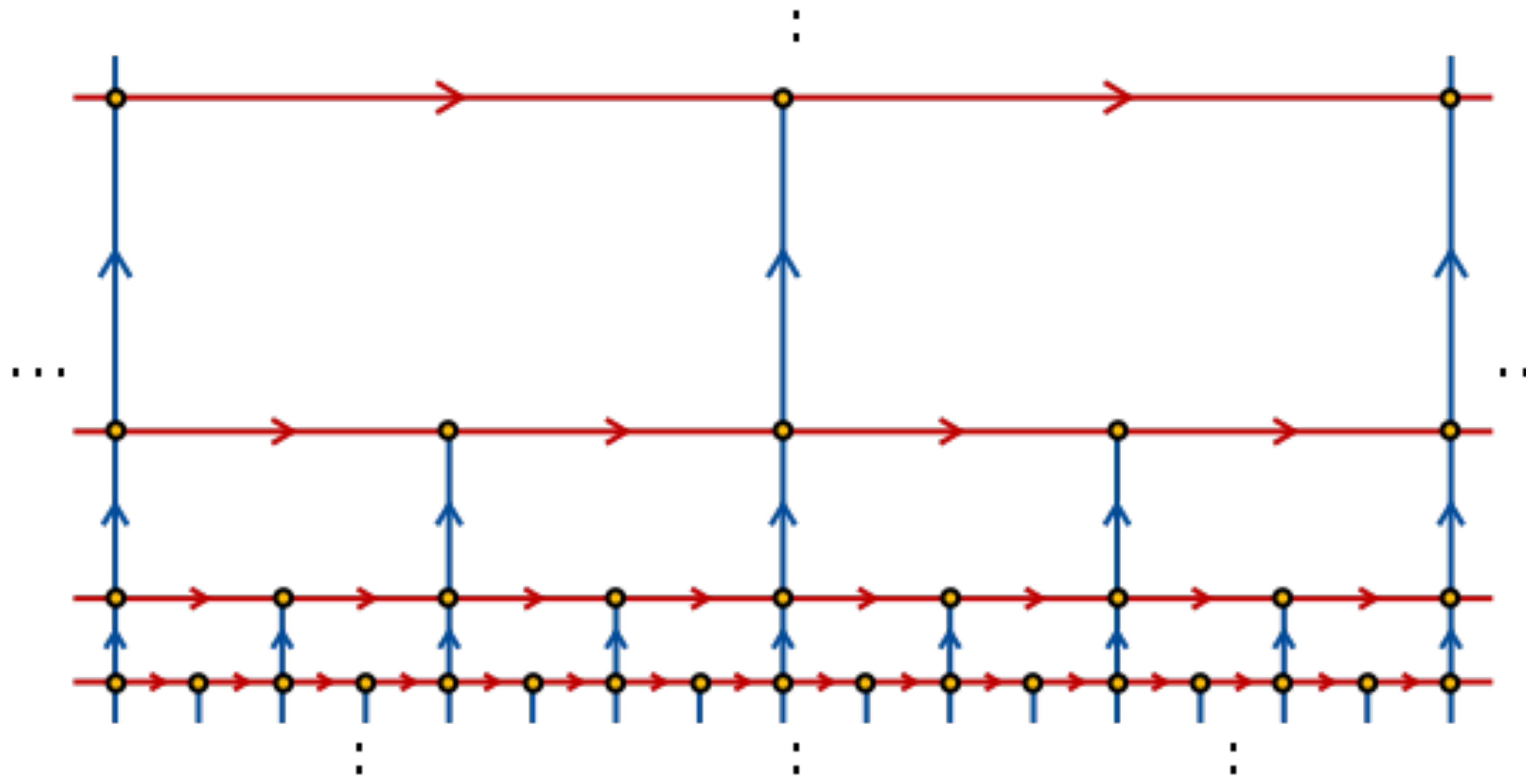

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- The group acts on itself transitively and freely: it is a group of translations